# ONLINE SUPPLEMENT
# Optimal Markdown Pricing and Inventory Allocation for Retail Chains with Inventory Dependent Demand

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## Appendix A: Solution for the Optimal Price Trajectory

Consider the following modified form of the optimal control problem (2) that includes an inventory holding cost term

$$\max \int_{t_0}^{t_e} \Big\{ p(t)D(t,I(t),p(t)) - c_I I(t) \Big\} dt + c_e I_e, \quad \text{subject to} \quad I'(t) = -D(t,I(t),p(t))$$

$$\text{and} \quad I_e = I_0 - \int_{t_0}^{t_e} D(t,I(t),p(t))dt \geq 0,$$

The optimality conditions for the Hamiltonian $H = (p-\lambda)D - c_I I$ are

$$\frac{\partial H}{\partial \mathrm{I}} = [p-\lambda]\frac{\partial D}{\partial \mathrm{I}} - c_I = -\lambda' \quad \text{and} \quad \frac{\partial H}{\partial p} = [p-\lambda]\frac{\partial D}{\partial p} + D = 0,$$

with the boundary condition $\lambda(t_e) = c_e + \theta$, where $\theta I(t_e) = 0$.

Using the demand function in (1) and substituting $I' = -D$, we obtain

$$\lambda' = D\frac{\partial D}{\partial \mathrm{I}} / \frac{\partial D}{\partial p} + c_I = \frac{c_e}{\gamma}\frac{y'(I)I'}{y(I)} + c_I \quad \text{and} \quad p + D / \frac{\partial D}{\partial p} = p - \frac{c_e}{\gamma} = \lambda.$$

Thus it follows that $p' = \lambda'$, and integrating the first expression above gives

$$p(t) = \frac{c_e}{\gamma} ln\left(y(I(t))\right) + c_I t + \text{constant}.$$

The constant is obtained by evaluating the Hamiltonian conditions at time $t = t_e$ and subtracting the resulting equation, which gives

$$p(t) = p(t_e) + c_I(t - t_e) + \frac{c_e}{\gamma} ln\left(\frac{y(I)}{y(I_e)}\right).$$

This shows that the price trajectory can be expressed as

$$p(t) = P(I(t)) + c_I(t - t_e), \text{ where } P(I) = p(t_e) + \frac{c_e}{\gamma} ln\left(\frac{y(I)}{y(I_e)}\right).$$

With $c_I = 0$, the optimal trajectory in (3) results. The terminal price $p_e = p(t_e) = P(I_e)$ is obtained from the boundary condition at $t = t_e$. When $y(0) = 0$, it follows that $I_e > 0$. The boundary condition implies that $\theta = 0$ when $I_e > 0$. Thus, the boundary condition becomes $\lambda(t_e) = c_e$, which together with $p - \frac{c_e}{\gamma} = \lambda$, implies that

$$p_e = \lambda(t_e) + c_e/\gamma = c_e(1 + 1/\gamma).$$

The term $c_I(t - t_e)$ can be incorporated into the seasonality factor $k(t)$ to obtain $k^*(t) = k(t)e^{-\gamma c_I(t-t_e)/c_e}$. The demand rate with the optimal price trajectory can therefore be written as

$$D(t, I(t), p(t)) = k^*(t)y(I_e)e^{-\gamma p_e/c_e}.$$

This allows the optimal inventory trajectory $I(t)$ to be determined directly by integration

$$I(t) = I_0 - y(I_e)e^{-\gamma p_e/c_e} \int_{t_0}^{t} k^*(t)dt.$$

The cost $c_I$ lowers the price for smaller $t$ values since the term $c_I(t - t_e)$ is negative. There is also the variable cost term to be evaluated in the profit calculation. Since $I(t)$ declines in proportion to $k^*(t)$, this term can be evaluated as follows

$$c_I \int_{t_0}^{t_e} I(t)dt = c_I\Big\{I_0 - y(I_e)K_e^{**}\Big\}, \text{ where } K_e^{**} = e^{-\gamma p_e/c_e} \int_{t_0}^{t_e} \int_{t_0}^{t} k^*(t)dt. \tag{A.1}$$

For the case of a constant price $p_0$, (8) still holds and the variable cost term $-c_I \int_{t_0}^{t_e} I(t)dt$ is added to the objective function. The optimal price $p_0$ can still be determined by a one dimensional search, but there is no closed form expression for the total sales during $[t_0, t_e]$.

## Appendix B: Proof of Theorem 1 and Lemma 1.

*Proof of Theorem 1:*

To simplify notation in this derivation, we will suppress the dependence on $x$ and use $I_e$ for $I_e(x)$. Substituting for $I_0(x)$ using (12), we have the following terms in $L$ that depend on $I_e = I_e(x)$

$$c_e\left[(1-\mu)[I_e + I_e^{\alpha}K_e] + \frac{1-\alpha}{\gamma}I_e^{\alpha}K_e + \frac{\alpha}{\gamma}[I_e + I_e^{\alpha}K_e]ln(1 + I_e^{\alpha-1}K_e)\right].$$

Therefore, the FONC $\frac{\partial L}{\partial I_e(x)} = 0$ yields

$$\frac{\partial L}{\partial I_e(x)} = (1-\mu)(1 + \alpha I_e^{\alpha-1}K_e) + \alpha\frac{1-\alpha}{\gamma}I_e^{\alpha-1}K_e$$

$$+\frac{\alpha}{\gamma}[1+\alpha I_e^{\alpha-1}K_e]ln(1+I_e^{\alpha-1}K_e)+\frac{\alpha}{\gamma}\frac{I_e+I_e^{\alpha}K_e}{1+I_e^{\alpha-1}K_e}(\alpha-1)I_e^{\alpha-2}K_e$$

$$=\Big[1+\alpha I_e^{\alpha-1}K_e\Big]\Big[1-\mu+\frac{\alpha}{\gamma}ln\{1+I_e^{\alpha-1}K_e\}\Big]=0, \tag{A.2}$$

after canceling terms and combining the multiplicative factors. Since the first multiplicative factor is always positive, we have the equivalent FONC shown with the dependence on $x$

$$1-\mu+\frac{\alpha}{\gamma}ln\{1+I_e^{\alpha-1}(x)K_e(x)\}=0,$$

which implies that

$$I_e(x)=K_e^{\frac{1}{1-\alpha}}(x)[e^{\frac{\gamma}{\alpha}(\mu-1)}-1]^{\frac{1}{\alpha-1}}.$$

To simplify notation, define the parameter $\psi$

$$\psi=\psi(\mu)=[e^{\frac{\gamma}{\alpha}(\mu-1)}-1]^{\frac{1}{\alpha-1}}.$$

For the store with index $x$, we therefore have that

$$I_e(x)=K_e(x)^{\frac{1}{1-\alpha}}\psi(\mu)$$

and from (12) that

$$I_0(x)=K_e(x)^{\frac{1}{1-\alpha}}\psi(\mu)+K_e(x)^{1+\frac{\alpha}{1-\alpha}}\psi(\mu)^{\alpha}=K_e(x)^{\frac{1}{1-\alpha}}[\psi(\mu)+\psi(\mu)^{\alpha}].$$

It can be verified that the second order condition is also satisfied for $\alpha<1$ since

$$\frac{\partial^2 L}{\partial I_e(x)^2}=\alpha(\alpha-1)I_e^{\alpha-2}K_e\Big\{1-\mu+\frac{\alpha}{\gamma}ln(1+I_e^{\alpha-1}K_e)\Big\}$$

$$+(1+\alpha I_e^{\alpha-1}K_e)\frac{\alpha}{\gamma}\frac{(\alpha-1)I_e^{\alpha-1}K_e}{1+I_e^{\alpha-1}K_e}<0.$$

Suppose that there is a variable inventory cost $c_I$ per unit, and we use the optimal price trajectory (3) at each store $x$. Define

$$K^*(x,t)=\int_{t_0}^{t}k(x,t)e^{-\gamma c_I(t-t_e)/c_e}dt \quad \text{with} \qquad K_e^*(x)=K^*(x,t_e)e^{-\gamma p_e/c_e}I_r^{-\alpha}$$

$$\text{and} \quad K_e^{**}(x)=e^{-\gamma p_e/c_e}I_r^{-\alpha}\int_{t_0}^{t_e}K^*(x,t)dt.$$

Thus, if there is a variable cost $c_I$ of holding inventory, the factors $K_e(x)$ are replaced by $K_e^*(x)$, which includes the additional price trajectory term in (3). Also, a term

$$c_I\Big\{I_0(x)-y(I_e(x))K_e^{**}(x)\Big\}=c_I\Big\{I_e(x)+K_e^*I_e^{\alpha}(x)-I_e^{\alpha}(x)K_e^{**}(x)\Big\}$$

corresponding to (A.1) is subtracted from the objective function. This means that (A.2) becomes

$$\Big[1+\alpha I_e^{\alpha-1}K_e^*\Big]\Big[1-\mu-c_I+\frac{\alpha}{\gamma}ln\{1+I_e^{\alpha-1}K_e^*\}\Big]+\alpha c_I I_e^{\alpha-1}K_e^{**}=0,$$

where the last term can be written as $[1+\alpha I_e^{\alpha-1}K_e^*]\frac{K_e^{**}}{K_e^*}-\frac{K_e^{**}}{K_e^*}$.

The proof is based on showing that $1+\alpha I_e^{\alpha-1}(x)K_e(x)$ is a constant that is independent of $x$. This is consistent with the FONC above, provided that $K_e^{**}(x)/K_e^*(x)$ is independent of $x$. A sufficient condition for this to be true is that the ratio $G^*(t)=K_e^*(x,t)/K_e^*(x)$ corresponding to $G(t)$ in the proof of Lemma 1 is independent of $x$. That is, all stores have the same seasonal variation. However, this solution to the FONC may not be unique, as it was in the case with no variable inventory cost.

If the ratio $K_e^*(x)/K_e^{**}(x)$ is independent of $x$, then there is a constant $\psi^*$ such that the FONC for optimal inventory allocation are satisfied by

$$I_e(x)=K_e^*(x)^{\frac{1}{1-\alpha}}\psi^* \quad \text{and}$$

$$I_0(x)=K_e^*(x)^{\frac{1}{1-\alpha}}[\psi^*+(\psi^*)^\alpha].$$

In this case, the parameter $\psi^*$ can still be determined from the total inventory constraint as in (19), but it cannot be related to $\mu$ as in (16) in this case. A sufficient condition for the inventory allocation result to hold is that all stores have the same seasonal variation, i.e., $K(x,t)$ is separable. The form of the optimal inventory allocation does not appear to extend to the constant price case, however.

*Proof of Lemma 1:*

If $K(x,t)=K(x,t_e)G(t)$, we integrate the relationship (4) for store $x$ to express the optimal inventory trajectory at each time $t$ as

$$I(t,x)=I_0(x)-I_e(x)^\alpha K_e(x)G(t).$$

It can be seen from (3) that the optimal price for store $x$ at time $t$ satisfies

$$P(x,I(x,t))=p_e+\frac{\alpha c_e}{\gamma}ln\left(\frac{I(x,t)}{I_e(x)}\right). \tag{A.3}$$

From (21) using Theorem 1 to substitute for $I_0(x)$ and $I_e(x)$, we have

$$\frac{I(t,x)}{I_e(x)}=\frac{I_0(x)}{I_e(x)}-I_e(x)^{\alpha-1}K_e(x)G(t)=1+\psi^{\alpha-1}-\psi^{\alpha-1}G(t).$$

Thus, the optimal price trajectory (A.3) in this case is

$$P(x,I(x,t))=p_e+\frac{c_e\alpha}{\gamma}ln\left(1+\psi^{\alpha-1}[1-G(t)]\right),$$

which is independent of $x$. The second part of the Lemma follows since

$$I(t,x)/I_0(x)=(I(t,x)/I_e(x))*(I_e(x)/I_0(x))=1-G(t)\psi^{\alpha-1}/(1+\psi^{\alpha-1}). \text{ QED.}$$

## Appendix C: Proof of Theorem 2

Assign the Lagrange multiplier $c_e\mu$ to the inventory constraint as before and form the Lagrangian $L$ for (23). Taking the partial derivative of the Lagrangian, the following FONC for $I_0(x)$ is obtained

$$\frac{\partial L}{\partial I_0(x)}=-\mu c_e+p_0(x)+(c_e-p_0(x))I_0^{-\alpha}(x)I_e^\alpha(x)=0,$$

after substituting from (22). This FONC can be rewritten in terms of the fraction of unsold inventory

$$U_e(x)^\alpha=\frac{p_0(x)-\mu c_e}{p_0(x)-c_e} \quad \text{for all } x, \quad \text{where} \quad U_e(x)=I_e(x)/I_0(x). \tag{A.4}$$

The partial derivative with respect to price yields

$$\frac{\partial L}{\partial p_0(x)} = I_0(x) - I_e(x) + [p_0(x) - \mu c_e] K_1(x) f'(p_0(x)) I_0(x)^{\alpha} = 0,$$

after substituting the expression (A.4) above. This can be rearranged to obtain

$$K_1(x) I_0(x)^{\alpha-1} = -\frac{1 - U_e(x)}{[p_0(x) - \mu c_e] f'(p_0(x))} \ .$$

Rearranging the expression (22) for $I_e(x)^{1-\alpha}$, we obtain

$$K_1(x) I_0(x)^{\alpha-1} = \frac{1 - U_e(x)^{1-\alpha}}{[1-\alpha] f(p_0(x))} \ . \tag{A.5}$$

Equating the right hand sides of the two expressions above gives a FONC for $p_0(x)$. Since $U_e(x)$ depends on $x$ only through $p_0(x)$, the right hand side of this FONC can be expressed in terms of values $p_0$ and $U_e$ that are independent of $x$. Also, (A.4) gives the fraction of inventory $U_e(x) = U_e$ that is salvaged at all stores under the optimal allocation, and (A.5) shows that $I_0(x)$ is proportional to $K_1(x)^{\frac{1}{1-\alpha}}$ as follows

$$I_0(x) = \rho K_1(x)^{\frac{1}{1-\alpha}} \text{ for all } x, \text{ where } \rho = \left( \frac{(1-\alpha) f(p_0)}{1 - U_e^{1-\alpha}} \right)^{\frac{1}{1-\alpha}},$$

and $\rho$ is independent of $x$. The equation for $\rho$ allows us to write $U_e$ as a function of $p_0$ as follows

$$U_e = \left[ 1 - \frac{(1-\alpha)}{\rho^{1-\alpha}} f(p_0) \right]^{\frac{1}{1-\alpha}} .$$

For a given $I_1$ and $X$, $\rho$ can be obtained from the inventory constraint

$$I_1 = \sum_{x \le X} I_0(x) = \rho \sum_{x \le X} K_1(x)^{\frac{1}{1-\alpha}}, \text{ or } \rho = \frac{I_1}{\sum_{x \le X} K_1(x)^{\frac{1}{1-\alpha}}}. \text{ QED.}$$

## Appendix D: Proof of Lemma 2

To simplify notation, let $K_e = K(t_e) I_r^{-\alpha} e^{-\gamma p_e / c_e}$ and define

$U(n) =$ the total ending inventory when $n$ stores stocked.

We wish to determine the sign of $U'(n)$ because the total number of units sold increases if and only if $U(n)$ decreases. If $n$ stores are stocked, the relationship for inventory and sales in (5) for any store that is stocked can be written as

$$\frac{I_1}{n} = \frac{U(n)}{n} + y\left( \frac{U(n)}{n} \right) K_e \text{ or } I_1 = U(n) + ny\left( \frac{U(n)}{n} \right) K_e.$$

Taking the derivative with respect to $n$, we have

$$0 = U'(n) + y\left( \frac{U(n)}{n} \right) K_e + ny'\left( \frac{U(n)}{n} \right) \left[ \frac{U'(n)}{n} - \frac{U(n)}{n^2} \right] K_e.$$

Now, let $u = U(n)/n$ and solve for $U'(n)$ to get

$$U'(n) = \frac{[uy'(u) - y(u)] K_e}{1 + K_e y'(u)}.$$

Since $y(0) = 0$ and $y'(u) \ge 0$ for all $u$, we see that $U'(n) > 0$ if and only if

$$y(u) - y(0) < uy'(u).$$

For $u > 0$, this last condition holds if $y(I)$ is convex and the sign is reversed if $y(I)$ is concave. Since the total number of units sold is $I_1 - U(n)$, we see that the total number sold increases with $n$ if $y(I)$ is concave and decreases with $n$ if $y(I)$ is convex. QED.

## Appendix E: Proof of Theorem 3

We focus on revenue maximization since the inventory costs are sunk costs. The total inventory $I_1$ will be allocated equally to a subset of $n$ stores. For the optimal price trajectory case with exponential price sensitivity, the total revenue can be obtained from (6) as follows

$$R(I_0) = c_e\left[I_0 + \frac{I_0 - I_e}{\gamma} + \frac{1}{\gamma}\int_{I_e}^{I_0} ln\left(\frac{y(I)}{y(I_e)}\right)dI\right].$$

If inventory $I_1$ is allocated equally to $n$ identical stores, this can be written as

$$nR\left(\frac{I_1}{n}\right) = c_e\left[I_1 + \frac{I_1 - U(n)}{\gamma} + \frac{n}{\gamma}\int_{U(n)/n}^{I_1/n}\left\{ln(y(I)) - ln\left(y\left(\frac{U(n)}{n}\right)\right)\right\}dI\right].$$

This integral can be simplified by using the change of variable $z = nI$ and substituting the functional form $y(I) = (I/I_r)^\alpha$ to obtain

$$c_e\left[I_1 + \frac{I_1 - U(n)}{\gamma} + \frac{\alpha}{\gamma}\int_{U(n)}^{I_1}[ln(z) - ln(U(n)]dz\right].$$

This expression can be integrated to obtain

$$nR\left(\frac{I_1}{n}\right) = c_e\left[I_1 + \frac{1-\alpha}{\gamma}[I_1 - U(n)] + \frac{\alpha I_1}{\gamma}ln\left(\frac{I_1}{U(n)}\right)\right]. \tag{A.6}$$

Differentiating with respect to $n$, we have

$$\frac{\partial}{\partial n} = -c_e\left[\frac{(1-\alpha)U'(n)}{\gamma} + \frac{\alpha I_1}{\gamma}\frac{U'(n)}{U(n)}\right] = -\frac{c_e}{\gamma}U'(n)\left\{1 - \alpha + \alpha\frac{I_1}{U(n)}\right\}.$$

Since $1 < I_1/U(n)$ always holds, this shows that the derivative of total revenue with respect to $n$ has the opposite sign of $U'(n)$ and the same sign as the derivative of the total number sold.

For constant pricing, we know that the total revenue is

$$R(I_1, n) = n\left\{p_0\left[\frac{I_1}{n} - I_e(n)\right] + c_e I_e(n)\right\} = I_1 p_0 + n(c_e - p_0)I_e(n).$$

Thus we wish to show that

$$\frac{\partial R(I_1, n)}{\partial n} = (c_e - p_0)I_e(n) + n(c_e - p_0)I_e'(n) > 0.$$

Since $(c_e - p_0) < 0$, we need to show that $nI_e'(n)/I_e(n) < -1$. From (8) and the fact that each store receives $I_0/n$, we see that

$$\frac{nI_e'(n)}{I_e(n)} = \frac{-I_1/n}{I_1/n - (1-\alpha)K_1 f(p_0)} < -1.$$

This holds as long as $p_0$ is chosen so that demand $K_1 f(p_0) \le I_1/n$ and $\alpha < 1$. QED.

## Appendix F: Proof of Theorem 4

This proof is only for the optimal price trajectory and exponential price sensitivity. Let

$n =$ the number of identical stores that are stocked,

$U(n) =$ the total unsold inventory,

and use the total revenue equation $nR\left(\frac{I_1}{n}\right)$ in (A.6). With the fixed cost $F$, we want to optimize $nR\left(\frac{I_1}{n}\right) - nF$ with respect to $n$, subject to the ending inventory constraint in (5). For $n$ identical stores, (5) can be written as

$$\frac{I_1}{n} = \frac{U(n)}{n} + \left(\frac{U(n)}{n}\right)^{\alpha} K_e, \quad \text{where} \quad K_e = K(t_e)e^{-\gamma p_e/c_e} I_r^{-\alpha},$$

$$\text{or} \quad I_1 = U(n) + n^{1-\alpha} U(n)^{\alpha} K_e. \tag{A.7}$$

Now, we want to optimize $nR\left(\frac{I_1}{n}\right) - nF$ with respect to $n$, subject to (A.7). The partial derivative yields the FONC

$$\frac{\partial}{\partial n}\left\{nR\left(\frac{I_1}{n}\right) - nF\right\} = -\frac{c_e}{\gamma} U'(n)\left\{1 + \frac{\alpha}{U(n)}[I_1 - U(n)]\right\} - F = 0.$$

Differentiating the constraint (A.7) shows that the following must hold

$$0 = U'(n) + (1-\alpha)n^{-\alpha}U(n)^{\alpha}K_e + \alpha U'(n)n^{1-\alpha}U(n)^{\alpha-1}K_e,$$

This can be rearranged to obtain

$$U'(n) = -\frac{(1-\alpha)\left(U(n)/n\right)^{\alpha} K_e}{1 + \alpha\left(U(n)/n\right)^{\alpha-1} K_e} = -\frac{(1-\alpha)\left(U(n)/n\right)^{\alpha} K_e}{1 + \alpha[I_1 - U(n)]/U(n)}.$$

Substituting this into the FONC, we obtain

$$\frac{c_e}{\gamma}(1-\alpha)\left(\frac{U(n)}{n}\right)^{\alpha} K_e = F, \tag{A.8}$$

which can be solved for $U(n)/n$ to obtain

$$\frac{U(n)}{n} = \left(\frac{F\gamma/c_e}{K_e(1-\alpha)}\right)^{1/\alpha}.$$

Using (A.7) to substitute in (A.8) yields

$$\frac{I_1 - U(n)}{n} = \frac{F\gamma/c_e}{1-\alpha},$$

which gives a formula for the amount sold at each store. Combining this with the amount of inventory unsold above implies that the optimal inventory at each store is

$$\frac{I_1}{n} = \left(\frac{F\gamma/c_e}{K_e(1-\alpha)}\right)^{1/\alpha} + \frac{F\gamma/c_e}{1-\alpha}.$$

QED.

It can also be verified that the inventory allocation in Theorem 4 is consistent with the optimal inventory allocations for nonidentical stores in Theorem 1 by setting

$$\psi = K_e^{-\frac{1}{\alpha(1-\alpha)}}\left(\frac{F\gamma/c_e}{(1-\alpha)}\right)^{1/\alpha}.$$

This is also analogous to (31) for the case in which $I_1$ is a decision variable and $\mu = c/c_e$ and $\psi$ are constants.

*Comparison to the Case with No Inventory Effect*

It is interesting to compare the above results in the identical store case to the case $\alpha = 0$. These formulas can be derived directly from a new optimization, or by simply taking the limit in the expressions above. Letting $\alpha$ approach 0, we see that the units sold per store goes to $F\gamma/c_e$. It can verified that this is the optimal sales per store when there is no inventory effect, provided that the demand per store is sufficiently high to make this an achievable sales target. The Unsold Units in (27) should approach 0 as $\alpha$ approaches 0, which is the optimal result when there is no inventory effect. This will happen if and only if

$$\frac{F\gamma/c_e}{K_e(1-\alpha)} < 1 \quad \text{or equivalently if the Sold Units } = \frac{F\gamma/c_e}{1-\alpha} < K_e.$$

But we note that $K_e$ is the number of units demanded at the price $p_e$, when there is no inventory effect. Since the optimal trajectory with the inventory effect uses prices that are at least this large, the inequality must hold for any set of parameter values that allows the Sold Units $= \frac{F\gamma/c_e}{1-\alpha}$ to be achieved. Clearly, if $F$ is large enough, the inequalities above will be reversed. This would mean that it is optimal to stock no stores at all and simply salvage all units.

## Appendix G: Two Proofs Related to Optimizing the Set of Stores Stocked

*Extending the Optimal Search to Include Bounds on the Inventories*

Let us consider only the upper bound for simplicity, since both bounds are handled in a similar way. First find the optimal $X$ without the bounds. Then define

$$X_{max} = \text{Max}\{x | K_e(x)^{\frac{1}{1-\alpha}}(\psi + \psi^\alpha) \geq I_{max}\},$$

recalling that the indices $x$ are arranged in decreasing order of $K_e(x)$. Let

$$I_0(x) = I_{max} \text{ for all } x \leq X_{max}.$$

Then reoptimize the $X$ using the revised inventory constraint

$$\sum_{X_{max} \leq x \leq X} K_e(x)^{\frac{1}{1-\alpha}}(\psi + \psi^\alpha) = I_1 - \sum_{x \leq X_{max}} I_{max}.$$

This may cause some additional inventory allocations to exceed $I_{max}$. If so, repeat the process until no inventory allocations violate the upper bound constraint. For $x \leq X_{max}$, the profit equation (17) still holds, but now we must use $\psi = \psi(x)$, where $\psi(x)$ is obtained from

$$K_e(x)^{\frac{1}{1-\alpha}}(\psi(x) + \psi(x)^\alpha) = I_{max} \text{ for all } x \leq X_{max}.$$

This $\psi(x)$ also determines the revised ending inventory $I_e(x) = K_e(x)^{\frac{1}{1-\alpha}}\psi(x)$, which then allows the price trajectory to be determined from (A.3). Because of this, the optimal price trajectories will need to be determined separately for each store with a constrained inventory level. As a result, the price trajectories will no longer be synchronized in time because some $I_e(x)$ have been modified.

*Showing that there is a Unique Local Maximum in the Continuous Case*

If we treat the store indices $x$ as a continuous variable, it can be shown by calculus that a local optimum must be unique. With discrete indices $x$, this proof is also sufficient to imply that there are no separated local maxima. To prove this, let the index $x$ be defined as a continuous percentile ranking of $K_e(x)$, and consider the set of stores $\{x \le X\}$, where $X \le 1$ must hold. This implies that the Lagrangian for optimizing the total net profit with respect to $\{I_0(x)\}$ and $X$ can be written as

$$L = \int_{x \le X} R(I_0(x))dx + c_e\mu\left[I_1 - \int_{x \le X} I_0(x)dx\right] - XF + \xi(1 - X),$$

which is also subject to (12). Optimization with respect to $I_0(x)$ leads to the allocation results in Theorem 1. In addition, the Kuhn-Tucker conditions give the FONCs for $X$

$$R(I_0(X)) - c_e\mu I_0(X) - F - \xi = 0, \quad \text{and} \quad \xi(1 - X) = 0.$$

These FONCs can be solved as two separate cases [a] and [b].

Case a. For $X = 1$, find the unique value of $\psi$ such that

$$\int_{x \le 1} I_0(x)dx = [\psi + \psi^\alpha] \int_{x \le 1} K_e(x)^{\frac{1}{1-\alpha}} dx = I_1.$$

If $R(I_0(1)) - c_e\mu I_0(1) \ge F$ using this value of $\psi$ and the corresponding $\mu$, then $X = 1$ must hold. This completes the solution, because $\mu$ is determined from $\psi$ and $\xi > 0$ is determined from the FONC with $X = 1$.

Case b. Otherwise, we have $X < 1$, which implies that $\xi = 0$. We therefore need to solve the two FONCs for $\mu$ and $X$ simultaneously. To simplify notation, let

$$\kappa(X) = \int_{x \le X} K_e(x)^{\frac{1}{1-\alpha}} dx,$$

which allows the inventory constraint equation to be written as

$$Y_1(X, \mu) = [\psi(\mu) + \psi(\mu)^\alpha]\kappa(X) - I_1 = 0.$$

To obtain the second equation, we use $\mu = 1 + \frac{\alpha}{\gamma} ln\left(1 + \psi^{\alpha-1}\right)$ to obtain

$$R(I_0(X)) - c_e\mu I_0(X) - F = c_e I_0(X)\frac{1-\alpha}{\gamma}\frac{\psi(\mu)^\alpha}{\psi(\mu) + \psi(\mu)^\alpha} - F = 0,$$

which simplifies to

$$Y_2(X, \mu) = c_e K_e(X)^{\frac{1}{1-\alpha}}\frac{1-\alpha}{\gamma}\psi(\mu)^\alpha - F = 0.$$

These two equations can be solved simultaneously for $X$ and $\psi = \psi(\mu)$ since there is a one to one relationship between $\mu$ and $\psi$.

To understand the behavior of the solution above, define the two implicit functions $X_1(\psi)$ and $X_2(\psi)$ such that $Y_1(X_1(\psi), \psi) \equiv 0$ and $Y_2(X_2(\psi), \psi) \equiv 0$. By taking the total derivative of the two FONCs, it can be verified that

$$X_1'(\psi) = -\frac{\partial Y_1(X_1, \psi)/\partial\psi}{\partial Y_1(X_1, \psi)/\partial X_1} = -\frac{(1 + \alpha\psi^{\alpha-1})\kappa(X)}{(\psi + \psi^\alpha)K_e(X)^{\frac{1}{1-\alpha}}} < 0 \qquad \text{and,}$$

$$X_2'(\psi) = -\frac{\partial Y_2(X_2, \psi)/\partial\psi}{\partial Y_2(X_2, \psi)/\partial X_2} = -\frac{K_e(X)^{\frac{1}{1-\alpha}}\alpha\psi^{\alpha-1}}{\frac{\partial K_e(X)^{\frac{1}{1-\alpha}}}{\partial X}\psi^\alpha} > 0,$$

because $K_e(X)$ is decreasing in $X$.

Thus, as $\psi$ is adjusted, there can be at most one value $X$ such that $X = X_1(\mu) = X_2(\mu)$. Existence follows because $0 \le X < 1$ holds as well. This shows that there is at most one solution of the FONC in the continuous case, because each $X$ determines a unique value of $\psi$.

This uniqueness result applies to the search on $X$ in the discrete case as well, in that the discrete case can be approximated by the continuous case. But in the discrete case, there may be ties between the optimal number of stores $n$ to stock and a neighboring value $n - 1$ or $n + 1$.