Electronic Companion to "Note on Cournot Competition under Yield Uncertainty"

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Appendix A: Proofs of results

Proof of Proposition 1. Problem (2) is a concave maximization problem in $\bar{q}_i$ subject to linear constraints. Let λ_i and π_i be the dual multipliers associated with the constraints $\bar{q}_i \geq 0$ and $\bar{q}_i \leq A_i$, respectively. Then, the necessary and sufficient optimality conditions are given by:

$$\begin{aligned}
&\frac{\partial \Pi_i(\bar{q}_i, \bar{q}^*_{-i})}{\partial \bar{q}_i} + \lambda_i - \pi_i = (a - c_i)\mathbb{E}[\alpha_i] - 2b\bar{q}_i\mathbb{E}[\alpha_i^2] - b\mathbb{E}\left[\alpha_i \sum_{j \neq i} \alpha_j \bar{q}^*_j\right] + \lambda_i - \pi_i = 0, \\
&\lambda_i \bar{q}_i = 0, \;\; \pi_i(A_i - \bar{q}_i) = 0, \\
&\lambda_i \geq 0, \;\; \pi_i \geq 0, \;\; 0 \leq \bar{q}_i \leq A_i, \;\; i = 1, \ldots, n. \qquad (1)
\end{aligned}$$

Suppose there exists an optimal solution where $\bar{q}^*_j < A_j$ for some $j \in \{i+1, \ldots, n\}$. From (1), $\pi_j = 0$ and $\partial \Pi_j(\bar{q}_j, \bar{q}^*_{-j})/\partial \bar{q}_j \leq 0$. Moreover, $\lambda_i = 0$ and $\partial \Pi_i(\bar{q}_i, \bar{q}^*_{-i})/\partial \bar{q}_i \geq 0$.

$$a - c_j - 2b\bar{q}^*_j(\sigma_j^2 + \mu_j^2)/\mu_j - b\sum_{k \neq j} \mu_k \bar{q}^*_k \leq 0 \qquad (2a)$$

$$a - c_i - 2b\bar{q}^*_i(\sigma_i^2 + \mu_i^2)/\mu_i - b\sum_{k \neq i} \mu_k \bar{q}^*_k \geq 0. \qquad (2b)$$

After simplifying (2a) and (2b), we obtain

$$bB_i = 2b\bar{q}_i^*(\sigma_i^2+\mu_i^2)/\mu_i - b\mu_i\bar{q}_i^* + c_i \le a - b\sum_k \mu_k\bar{q}_k^* \le 2b\bar{q}_j^*(\sigma_j^2+\mu_j^2)/\mu_j - b\mu_j\bar{q}_j^* + c_j < bB_j,$$

which contradicts $B_i \ge B_j$. The last inequality follows since $\bar{q}_j^* < A_j$. $\square$

Next, we study some structural properties of β_k and $\Gamma_k(\beta)$ that will be used when characterizing the market equilibrium in Theorem 1.

LEMMA 1. *$\Gamma_k(\beta)$ is decreasing in β for $k \in \{0,\dots,n\}$.*

Proof of Lemma 1 Lemma 1 directly follows from the definition of $\Gamma_k(\beta)$. $\square$

LEMMA 2. *$\Gamma_k(B_k) \le \Gamma_k(B_{k+1}) = \Gamma_{k+1}(B_{k+1})$ for $k \in \{0,\dots,n-1\}$.*

Proof of Lemma 2 $\Gamma_k(\beta)$ decreases in β, and $B_k \ge B_{k+1}$. Thus, $\Gamma_k(B_k) \le \Gamma_k(B_{k+1})$.

$$\begin{aligned}\Gamma_{k+1}(B_{k+1}) - \Gamma_k(B_{k+1}) =& b\mu_{k+1}A_{k+1} - bB_{k+1}\frac{\mu_{k+1}^2}{2\sigma_{k+1}^2+\mu_{k+1}^2} + \frac{c_{k+1}\mu_{k+1}^2}{2\sigma_{k+1}^2+\mu_{k+1}^2}\\ =& b\mu_{k+1}A_{k+1} - b\left(A_{k+1}\left(\frac{2\sigma_{k+1}^2+\mu_{k+1}^2}{\mu_{k+1}}\right) + \frac{c_{k+1}}{b}\right)\frac{\mu_{k+1}^2}{2\sigma_{k+1}^2+\mu_{k+1}^2} + \frac{c_{k+1}\mu_{k+1}^2}{2\sigma_{k+1}^2+\mu_{k+1}^2} = 0\end{aligned}$$

As a result, $\Gamma_k(B_k) \le \Gamma_k(B_{k+1}) = \Gamma_{k+1}(B_{k+1})$. $\square$

LEMMA 3. *We have that $B_{k^*+1} < \beta_{k^*} \le B_{k^*}$.*

Proof of Lemma 3 Note that $\Gamma_{k^*}(\beta_{k^*}) = 0$ and $\Gamma_{k^*}(B_{k^*}) \le 0$ from the definition of k^*. $\Gamma_k(\beta)$ is decreasing in β, thus $\beta_{k^*} \le B_{k^*}$. From Lemma 2, $\Gamma_{k^*}(B_{k^*+1}) = \Gamma_{k^*+1}(B_{k^*+1}) > 0$, where the inequality follows from the definition of k^*. As a result, $\beta_{k^*} > B_{k^*+1}$. $\square$

LEMMA 4. *(i) For each firm $i = 1,\dots,k^*$, $\bar{q}_i^* \le A_i$, and the marginal profit is equal to zero. (ii) The marginal profit of each firm $i = k^*+1,\dots,n$ at $\bar{q}_i^*$ is positive.*

Proof of Proposition 4 We first prove the first claim. It follows from Lemma 3 that,

$$\bar{q}_i^* = (\beta_{k^*} - c_i/b)\left(\frac{\mu_i}{2\sigma_i^2+\mu_i^2}\right) \le (B_i - c_i/b)\left(\frac{\mu_i}{2\sigma_i^2+\mu_i^2}\right) = A_i, \quad i = 1,\dots,k^*.$$

Hence, $\bar{q}_i^*$ is feasible to the capacity constraint. Next, we show that $\dfrac{\Pi_i(\bar{q}_i^*,\bar{q}_{-i}^*)}{\partial \bar{q}_i}=0$ for $i=1,\dots,k^*$.

$$\begin{aligned}
\frac{\Pi_i(\bar{q}_i^*,\bar{q}_{-i}^*)}{\partial \bar{q}_i} &= \mu_i\left(a-c_i-2b\bar{q}_i^*(\sigma_i^2+\mu_i^2)/\mu_i-b\sum_{j\neq i}\mu_j\bar{q}_j^*\right),\\
&= \mu_i\left(a-c_i-2b\bar{q}_i^*\sigma_i^2/\mu_i-b\bar{q}_i^*\mu_i-b\sum_{j=1}^{n}\mu_j\bar{q}_j^*\right),\\
&= \mu_i\left(a-c_i-b\left(\beta_{k^*}-c_i/b\right)-b\beta_{k^*}\sum_{j=1}^{k^*}\frac{\mu_j^2}{2\sigma_j^2+\mu_j^2}+\sum_{j=1}^{k^*}\frac{c_j\mu_j^2}{2\sigma_j^2+\mu_j^2}-b\sum_{j=k^*+1}^{n}\mu_jA_j\right),\\
&= \mu_i\left(a-b\beta_{k^*}-b\beta_{k^*}\sum_{j=1}^{k^*}\frac{\mu_j^2}{2\sigma_j^2+\mu_j^2}+\sum_{j=1}^{k^*}\frac{c_j\mu_j^2}{2\sigma_j^2+\mu_j^2}-b\sum_{j=k^*+1}^{n}\mu_jA_j\right),\\
&= \mu_i\Gamma_{k^*}(\beta_{k^*})=0.
\end{aligned}$$

Next, we prove the second claim for $i=k^*+1,\dots,n$.

$$\begin{aligned}
\frac{\Pi_i(\bar{q}_i^*,\bar{q}_{-i}^*)}{\partial \bar{q}_i} &= \mu_i\left(a-c_i-2b\bar{q}_i^*(\sigma_i^2+\mu_i^2)/\mu_i-b\sum_{j\neq i}\mu_j\bar{q}_j^*\right),\\
&= \mu_i\left(a-c_i-2bA_i\sigma_i^2/\mu_i-bA_i\mu_i-b\sum_{j=1}^{n}\mu_j\bar{q}_j^*\right),\\
&= \mu_i\left(a-bB_i-b\beta_{k^*}\sum_{j=1}^{k^*}\frac{\mu_j^2}{2\sigma_j^2+\mu_j^2}+\sum_{j=1}^{k^*}\frac{c_j\mu_j^2}{2\sigma_j^2+\mu_j^2}-b\sum_{j=k^*+1}^{n}\mu_jA_j\right),\\
&> \mu_i\left(a-b\beta_{k^*}-b\beta_{k^*}\sum_{j=1}^{k^*}\frac{\mu_j^2}{2\sigma_j^2+\mu_j^2}+\sum_{j=1}^{k^*}\frac{c_j\mu_j^2}{2\sigma_j^2+\mu_j^2}-b\sum_{j=k^*+1}^{n}\mu_jA_j\right),\\
&= \mu_i\Gamma_{k^*}(\beta_{k^*})=0,
\end{aligned}$$

where the inequality follows since $B_n\le\ldots\le B_{k^*+1}<\beta_{k^*}$ from Lemma 3. $\square$

Proof of Corollary 1 From Theorem 1, $\bar{q}_1^*=\ldots=\bar{q}_{k^*}^*=(\beta_{k^*}-c/b)\left(\frac{\mu}{2\sigma^2+\mu^2}\right)$, so $\Pi_1^*=\ldots=\Pi_{k^*}^*$. It follows from Lemma 3 that,

$$(\beta_{k^*}-c/b)\left(\frac{\mu}{2\sigma^2+\mu^2}\right)>\underbrace{(B_{k^*+1}-c/b)\left(\frac{\mu}{2\sigma^2+\mu^2}\right)}_{A_{k^*+1}=\bar{q}_{k^*+1}^*}\ge\ldots\ge\underbrace{(B_n-c/b)\left(\frac{\mu}{2\sigma^2+\mu^2}\right)}_{A_n=\bar{q}_n^*}$$

Thus, $\bar{q}_1^*=\ldots=\bar{q}_{k^*}^*>\bar{q}_{k^*+1}^*\ge\bar{q}_{k^*+2}^*\ge\ldots\ge\bar{q}_n^*$ and $\Pi_1^*=\ldots=\Pi_{k^*}^*>\Pi_{k^*+1}^*\ge\Pi_{k^*+2}^*\ge\ldots\ge\Pi_n^*$. $\square$

Appendix B: Corrected analysis of Deo and Corbett (2009)

Deo and Corbett (2009) assumed that firms are uncapacitated and identical. That is, $\mu_i=\mu$, $\sigma_i=\sigma$, and $c_i=c$ for all $i\in I$. They investigated the problem of a central planner who maximizes the expected social welfare by choosing the number of firms (n), and the target production quantity of each firm (q_i). Each producing firm must pay a fixed cost of entry f. The expected social welfare equals the expected consumer utility minus the expected production cost minus the cost of entry:

$$W(Q,n) = E\left[\int_0^Q (a-bQ)du - cQ\right] - nf = (a-c)E(Q) - \frac{b}{2}E(Q^2) - nf, \tag{3}$$

where $Q = \sum_{i=1}^n \alpha_i q_i$ is the total market supply. The optimal solution to the central planner's problem is referred to as the "first-best" solution. Let q^{fb} denote the target production quantity of each firm that maximizes the expected social welfare in the first-best solution. Deo and Corbett (2009) showed that q^{fb} for each firm is twice that under competition. (see Lemma 3 in Deo and Corbett (2009)). However, there is an algebraic mistake in their proof. We show that the correct value of q^{fb} is equal to $\dfrac{(a-c)}{b\mu(n+\delta^2)}$.

Corrected Proof of Lemma 3 in Deo and Corbett (2009). Note that

$$E(Q) = \mu\sum_{i=1}^n q_i \text{ and } E(Q^2) = E\left[\sum_{i=1}^n \alpha_i^2 q_i^2\right] + E\left[\sum_{i\neq j}(\alpha_i q_i)(\alpha_j q_j)\right] = (\sigma^2+\mu^2)\sum_{i=1}^n q_i^2 + \mu^2\sum_{i\neq j} q_i q_j$$

Substituting $E(Q)$ and $E(Q^2)$ into the expected social welfare function

$$W(Q,n) = (a-c)\mu\sum_{i=1}^n q_i - \frac{b}{2}\left[(\sigma^2+\mu^2)\sum_{i=1}^n q_i^2 + \mu^2\sum_{i\neq j} q_i q_j\right] - nf$$

Taking the derivative with respect to q_i

$$\frac{\partial W(Q,n)}{\partial q_i} = (a-c)\mu - \frac{b}{2}\left[2(\sigma^2+\mu^2)q_i + 2\mu^2\sum_{j\neq i} q_j\right] = (a-c)\mu - b\left[\sigma^2 q_i + \mu^2\sum_{j=1}^n q_j\right]$$

The solution to $\dfrac{\partial W(Q,n)}{\partial q_i} = 0$ for all $i = 1,\dots,n$ is $\bar{q}^{fb} = \dfrac{(a-c)}{b\mu(n+\delta^2)}$.

Deo and Corbett (2009) failed to account for symmetric terms in $\sum_{i\neq j} q_i q_j$ when taking the derivative of $W(Q,n)$ with respect to q_i. $\square$

The next step is to characterize the socially optimal number of firms in the first-best solution denoted by n^{fb}. Deo and Corbett (2009) found that $1 \le n^{fb} < 1+2\delta^2$, and also $n^{fb} = 1$ if $\dfrac{2(1+\delta^2)}{\delta} \ge \dfrac{a-c}{\sqrt{bf}}$ (see Proposition 4 in Deo and Corbett (2009)). These results are again inaccurate, because they were derived using the incorrect value for q^{fb}.

Substituting $q^{fb} = \dfrac{(a-c)}{b\mu(n+\delta^2)}$ in the expected social welfare function (3) and simplifying, we obtain the following optimization problem:

$$n^{fb} \in \arg\max \left\{ W(n) = \frac{n(a-c)^2}{2b(n+\delta^2)} - nf \right\}. \tag{4}$$

When $\delta = 0$, the first term in $W(n)$ is constant, and the second term is decreasing in n. As a result, intuitively, the first-best solution in a deterministic setting is to have a benevolent monopoly that produces the socially optimal quantity because society then incurs the fixed cost of entry only once. Deo and Corbett (2009) was able to make the same observation from their incorrect Proposition 4 when $\delta = 0$. On the other hand, when $\delta > 0$, we show that

$$n^{fb} \in \left\{ \left\lfloor \delta \left(\frac{a-c}{\sqrt{2bf}} - \delta \right) \right\rfloor, \left\lceil \delta \left(\frac{a-c}{\sqrt{2bf}} - \delta \right) \right\rceil \right\}.$$

Corrected Proof of Proposition 4 in Deo and Corbett (2009). Relaxing n to $x \in \mathbb{R}_+$, it can be verified that $W(x)$ is strictly concave in x, and so it can have at most one maximizer. Then, the first-order condition is given by:

$$\frac{\partial W(x)}{\partial x} = \frac{(a-c)^2}{2b} \frac{\delta^2}{(x+\delta^2)^2} - f = 0 \Rightarrow x = \frac{a-c}{\sqrt{2bf}}\delta - \delta^2. \quad \square$$

It follows from the correct value of n^{fb} that under yield uncertainty ($\delta > 0$), if $\dfrac{a-c}{\sqrt{2bf}} \geq \dfrac{1}{\delta} + \delta$, then the society benefits from supplier diversification, i.e., $n^{fb} \geq 1$; otherwise, $n^{fb} \in \{0, 1\}$. Note that $\dfrac{1}{\delta} + \delta \geq 1$ for any $\delta \geq 0$. Thus, if the market is not very "attractive", e.g., $\dfrac{a-c}{\sqrt{2bf}} < 1$, it is socially optimal to have either a monopoly ($n^{fb} = 1$), or no supplier at all ($n^{fb} = 0$), regardless of the level of uncertainty. Deo and Corbett (2009) incorrectly claimed that it is socially optimal to have a monopoly regardless of the level of uncertainty when $\dfrac{a-c}{\sqrt{bf}} < 4$. They also stated that $\delta \geq 1/2$ is a necessary condition for $n^{fb} > 1$ (see Section 4.5 in Deo and Corbett (2009)). However, based on the corrected results, it is possible that society benefits from supplier diversification, i.e., $n^{fb} > 1$, even when $\delta < 1/2$ as long as $\dfrac{a-c}{\sqrt{2bf}} > \dfrac{1}{\delta} + \delta$.

References

Deo, S., C. J. Corbett. 2009. Cournot competition under yield uncertainty: The case of the U.S. influenza vaccine market. *Manufacturing & Service Operations Management* **11**(4) 563–576.