Online Supplement for "Managing Posterior Price Matching: The Role of Customer Boundedly Rational Expectations"

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This Online Supplement has six sections: §1 includes all the technical proofs to all propositions and lemmas. §2 provides technical lemmas. §3 provides the equilibrium analysis when there is a capacity constraint, and §4 for budget constraint. In §5, we show that imperfect memory cannot explain why the PM policy should be adopted. In §6, we present the dynamic markdown results supplementary to those in the main paper.

Online Supplement 1. Proofs

Proof of Proposition 1. Suppose the firm uses PM. A rational customer purchases in period 1 if $u_1 \equiv v - p_1 + \xi(p_1 - p_2) \geq u_2 \equiv (1-\xi)\max\{v_L - p_1, v_0\} + \xi \max\{v_L - p_2, v_0\}$, which dictates that $v - p_1 + \xi(p_1 - p_2) \geq v_0$. The firm expected profit is $\Pi_R \equiv p_1 - \xi(p_1 - p_2) - c \leq v - v_0 - c$. Hence, the optimal strategy is $p_{1R}^* = p_{2R}^* = v - v_0$ and $\xi_R^* = 0$. If the firm chooses NP, it is clear that it can uses the same $p_{1R}^* = p_{2R}^* = v - v_0$ and $\xi_R^* = 0$ to obtain the same profit. Hence, using PM is strictly less profitable than not using it. □

Proof of Lemma 1. Suppose that the firm marks down with some strictly positive probability $\xi > 0$, then a customer gets the sample $\{p_1 > p_2\}$ with probability ξ. It is clear that, customers who get the markdown signal purchase in the second period and those who do not get a markdown signal purchase in the first period, provided that they are not worse than the outside option. Hence, the firm profit $\Pi_{NP} \equiv (1-\xi)p_1 + \xi p_2 - c \leq v - v_0 - c$ because $p_1, p_2 \leq v - v_0$. Therefore, the firm rationally chooses never to markdown $\xi_{NP}^* = 0$ and charges the fixed price $p_1 = p_2 = v - v_0$ throughout. □

Proof of Lemma A-1. We sketch the proof by depicting the feasible pricing region in Figure A-1. According to Figure A-1 and compare the slope of the line $p_1 - c - \xi(p_1 - p_2) = 0$ and the slope of the line $p_1 - (\lambda\xi_0 + 1 - \lambda)(p_1 - p_2) = v - v_0$, we obtain the results of Lemma A-1. □

Proof of Proposition 2. (i) From Lemma A-1, we know that if the firm uses PM policy, the prices should be set as $p_1 = \frac{v - v_0 - (\lambda\xi_0 + 1 - \lambda)c}{\lambda - \lambda\xi_0}$ and $p_2 = c$. Given this, we can then formulate the firm's expected profit function as follows: $\Pi(\xi) = \xi(1-\xi)\left[\frac{v - v_0 - (\lambda\xi_0 + 1 - \lambda)c}{\lambda - \lambda\xi_0} - c\right]$. According to first order condition, we obtain that $\xi^* = \frac{1}{2}$. Then, we have that $\Pi(\xi^*) = \frac{1}{4}\frac{v - v_0 - c}{\lambda - \lambda\xi_0}$. (ii) When the firm adopts the regular selling policy, the value of the firm is $v - v_0 - c$. Thus, the PM policy is strictly better than the regular selling policy when $\lambda - \lambda\xi_0 < \frac{1}{4}$. □

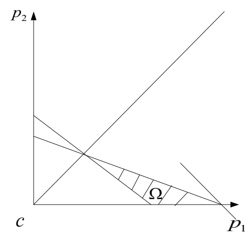

Figure A-1 **Illustration of the feasible decision region and optimal pricing decisions.**

Proof of Corollary 1. Comparing the profit of the PM policy with p_1, p_2 and ξ to the profit of the NP policy with $p_1 = p_2 = v - v_0$, we have the difference Δ satisfies $\Delta = -(p_1 - p_2)\xi^2 + (p_1 - c)\xi - (v - v_0 - c)$. To carry out PM policy, $(p_1, p_2) \in \Omega$ must hold. Moreover, we need to find a ξ with $\xi \in (0,1)$ that makes $\Delta > 0$. Thus, the optimal probability $\xi^* = \frac{p_1 - c}{2(p_1 - p_2)}$ should satisfy $0 < \frac{p_1 - c}{2(p_1 - p_2)} < 1$ and the equation $\Delta = 0$ on ξ must have two roots. Thus, we have $p_1 > 2p_2 - c$ and $(p_1 - c)^2 - 4(v - v_0 - c)(p_1 - p_2) > 0$. It can be easily examined that the larger root of the equation $\Delta = 0$ is less than 1. Thus, if $(p_1, p_2) \in S_1$ and $\xi \in S_2$, the profit of the PM policy is greater than the profit of the NP policy. □

Proof of Corollary 2. When the firm adopts the NP policy, according to Lemma 1, we have $\Pi_{NP} = v - v_0 - c$. When the firm adopts the PM policy, according to Proposition 2, we have $\Pi_{PM} = \frac{1}{4}\frac{v - v_0 - c}{\lambda - \lambda\xi_0}$. Based on the definition of $\Delta\pi_L$, we have the result. □

Proof of Proposition 3. (i) If the firm uses the OES strategy, the result is the same as that in Proposition 2 in Section 3.4. (ii) We now consider the case of the TES strategy. Following a similar argument as in Lemma A-1, we illustrate the feasible region and optimal pricing decisions in Figure A-2. We know that if the firm uses PM policy, the prices should be set as $p_1^* = \frac{v - (\lambda\xi_0 + 1 - \lambda)v_L}{\lambda - \lambda\xi_0} - v_0$, $p_2^* = v_L - v_0$. Given this, we can then formulate the firm's expected profit function as follows: $\Pi(\xi, p_1^*, p_2^*) = -(p_1^* - c)\xi^2 + (p_1^* + p_2^* - 2c)\xi$. According to the first-order condition, we obtain that $\xi^* = \frac{p_1^* + p_2^* - 2c}{2(p_1^* - c)} \in [\frac{1}{2}, 1)$. Then, we have that $\Pi^* = \frac{1}{4}\frac{(p_1^* + p_2^* - 2c)^2}{(p_1^* - c)}$. Substituting the expressions of the optimal prices and markdown probability, we obtain the profit expression. We have also checked that this profit is higher than the profit $v - v_0 - c$ under NP policy, if $\lambda - \lambda\xi_0 < \frac{\sqrt{(v - v_L)^2 + (v - v_L)(v_L - v_0 - c)} - (v - v_L)}{2(v_L - v_0 - c)}$. (iii) Comparing the OES strategy to the TES strategy, we have the OES strategy is better if $\lambda - \lambda\xi_0 < \frac{v - v_0 - c - 4(v - v_L) + \sqrt{8(v - v_L)^2 + 8(v_L - v_0 - c)(v - v_L) + (v - v_0 - c)^2}}{8(v_L - v_0 - c)}$ and the TES strategy is better if $\lambda - \lambda\xi_0 > \frac{v - v_0 - c - 4(v - v_L) + \sqrt{8(v - v_L)^2 + 8(v_L - v_0 - c)(v - v_L) + (v - v_0 - c)^2}}{8(v_L - v_0 - c)}$. It is easy to check that $\frac{\sqrt{(v - v_L)^2 + (v - v_L)(v_L - v_0 - c)} - (v - v_L)}{2(v_L - v_0 - c)} \le \frac{1}{4} \le \frac{v - v_0 - c - 4(v - v_L) + \sqrt{8(v - v_L)^2 + 8(v_L - v_0 - c)(v - v_L) + (v - v_0 - c)^2}}{8(v_L - v_0 - c)}$. Thus, the PM policy with OES strategy is optimal if $\lambda - \lambda\xi_0 < \frac{1}{4}$ and the NP policy is optimal if $\lambda - \lambda\xi_0 \ge \frac{1}{4}$. □

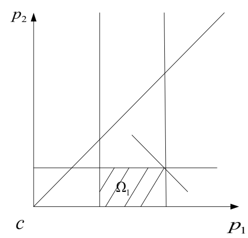

Figure A-2 **Illustration of the feasible decision region and optimal pricing decisions.**

Proof of Lemma A-2. Under the OES strategy, from inequality (4), we can obtain that the customer purchases in the first period if and only if $v - p_1 + \xi_i(m)(p_1 - p_2) \ge v_0$. According to the definition of $\xi_i(m)$, we have $\sum_{j=1}^{m} I_{\{i,j\}} \ge \frac{p_1 - (v - v_0) - \lambda\xi_0(p_1 - p_2)}{(1 - \lambda)(p_1 - p_2)} m$. Rearranging this inequality, we obtain the result. Because PM policy can be potentially effective in this case when $0 \leq \frac{p_1-(v-v_0)-\lambda\xi_0(p_1-p_2)}{(1-\lambda)(p_1-p_2)} \leq 1$ and $p_1 \geq p_2 \geq c$, we have the decision variables $(p_1,p_2) \in \Omega$. Under the TES strategy, according to inequality (5) and the definition of $\xi_i(m)$, we have $\sum_{j=1}^m I_{\{i,j\}} \geq \frac{\frac{p_1+v_0-v}{p_1+v_0-v_L}-\lambda\xi_0}{1-\lambda}m$. Rearranging this inequality, we obtain the result. Because PM policy can be potentially effective in this case when $0 \leq \frac{\frac{p_1+v_0-v}{p_1+v_0-v_L}-\lambda\xi_0}{1-\lambda} \leq 1$ and $v_L - v_0 \geq p_2 \geq c$, we have the decision variables $(p_1,p_2) \in \Omega_1$. □

Proof of Proposition 4. Under the OES strategy, if $p_1 = \frac{v-v_0-(\lambda\xi_0+1-\lambda)c}{\lambda-\lambda\xi_0}$, $p_2 = c$, then $\gamma_1(\xi,m) = \xi^m$. Hence, $\Pi(\xi,p_1,p_2) = \gamma_1(\xi,m)[p_1 - c - \xi(p_1-p_2)] = \frac{v-v_0-c}{\lambda-\lambda\xi_0}(\xi^m - \xi^{m+1})$. Taking the first and second derivative of Π on ξ, we have $\frac{\partial\Pi}{\partial\xi} = \frac{v-v_0-c}{\lambda-\lambda\xi_0}\xi^{m-1}[m-(m+1)\xi]$ and $\frac{\partial^2\Pi}{\partial\xi^2} = \frac{v-v_0-c}{\lambda-\lambda\xi_0}m\xi^{m-2}[m-1-(m+1)\xi]$. Then, we have Π is increasing convex on ξ when $\xi \in [0, \frac{m-1}{m+1}]$ and concave on ξ when $\xi \in [\frac{m-1}{m+1}, 1]$. Thus, the optimal probability ξ^* satisfies the first order condition $\xi^* = \frac{m}{m+1}$. The optimal profit is $\frac{v-v_0-c}{\lambda-\lambda\xi_0}\left(\frac{m}{m+1}\right)^m \frac{1}{m+1}$. Because the profit of the firm is $v - v_0 - c$ if the firm adopts the NP policy, the PM policy is strictly better than the NP policy if $\lambda - \lambda\xi_0 < \left(\frac{m}{m+1}\right)^m \frac{1}{m+1}$.

We next consider the TES strategy. Recall that this strategy could be adopted when $c \leq v_L - v_0$. If $p_1 = \frac{v-(\lambda\xi_0+1-\lambda)v_L}{\lambda-\lambda\xi_0} - v_0$, $p_2 = v_L - v_0$, then $\gamma_2(\xi,m) = \xi^m$. Hence,

$$\begin{aligned}\Pi(\xi,p_1,p_2) &= \gamma_2(\xi,m)[p_1 - c - \xi(p_1-p_2)] + [1-\gamma_2(\xi,m)]\xi(p_2-c)\\ &= (1-\xi)\xi^m\frac{v-v_L}{\lambda-\lambda\xi_0} + [\xi^m + (1-\xi^m)\xi](v_L - v_0 - c).\end{aligned}$$

Taking the first and second derivative of Π on ξ, we have $\frac{\partial\Pi}{\partial\xi} = (m\xi^{m-1} - \xi^m - m\xi^m)\left(\frac{v-v_L}{\lambda-\lambda\xi_0} + v_L - v_0 - c\right) + v_L - v_0 - c$ and $\frac{\partial^2\Pi}{\partial\xi^2} = \left(\frac{v-v_L}{\lambda-\lambda\xi_0} + v_L - v_0 - c\right)m\xi^{m-2}[m-1-(m+1)\xi]$. Then, we have Π is increasing convex on ξ when $\xi \in \left[0, \frac{m-1}{m+1}\right]$ and concave on ξ when $\xi \in \left[\frac{m-1}{m+1}, 1\right]$. Thus, by the first order condition, we have $\xi^* = \widehat{\xi}$, where $\widehat{\xi} \in \left[\frac{m-1}{m+1}, 1\right)$ is the unique solution of $\xi^m + m\xi^m - m\xi^{m-1} = \frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}$. The optimal profit is $(1-\widehat{\xi})\widehat{\xi}^m\frac{v-v_L}{\lambda-\lambda\xi_0} + [\widehat{\xi}^m + (1-\widehat{\xi}^m)\widehat{\xi}](v_L - v_0 - c)$. Because the profit of the firm is $v - v_0 - c$ if the firm adopts the NP policy, the PM policy is strictly better than the NP policy if $\frac{(1-\widehat{\xi})\widehat{\xi}^m-(\lambda-\lambda\xi_0)}{\lambda(1-\xi_0)(1-\widehat{\xi})(1-\widehat{\xi}^m)} > \frac{v_L-v_0-c}{v-v_L}$. □

Proof of Lemma A-3. Observing $\gamma_1(\xi,m)$ in the objective function (6), we notice that $\frac{p_1-(v-v_0)-\lambda\xi_0(p_1-p_2)}{(1-\lambda)(p_1-p_2)}m$ strictly increases in p_1. However, $\lfloor\frac{p_1-(v-v_0)-\lambda\xi_0(p_1-p_2)}{(1-\lambda)(p_1-p_2)}m - \epsilon\rfloor$ is a step function in p_1. Also, $p_1 - c - \xi(p_1-p_2)$ strictly increases in p_1. Hence, it is clear that the firm will rationally choose (p_1,p_2) such that $\frac{p_1-(v-v_0)-\lambda\xi_0(p_1-p_2)}{(1-\lambda)(p_1-p_2)}m = i$, for some $i = 1,2,3,...,m$. Otherwise, the firm can increase p_1 to increase its expected profit until this equality holds. Given this equality, we can express p_2 in terms of p_1 and i:

$$p_2 = p_1 - \frac{m[p_1-(v-v_0)]}{i(1-\lambda)+\lambda\xi_0 m}. \tag{A-1}$$

From Figure A-1, we have $v - v_0 \leq p_1 \leq \frac{v-v_0-(\lambda\xi_0+1-\lambda)c}{\lambda-\lambda\xi_0}$ and $c \leq p_2 \leq v - v_0$ if $(p_1,p_2) \in \Omega$. By equation (A-1), $p_1 \in \left[v - v_0, \frac{m(v-v_0)-[i(1-\lambda)+\lambda\xi_0 m]c}{(1-\lambda\xi_0)m-i(1-\lambda)}\right]$ must hold. Therefore, the optimization problems are equivalent between (6) and (8). □

Proof of Proposition 5. This result directly follows from Lemma A-3. If $\xi^* < \frac{i^*(1-\lambda)+\lambda\xi_0 m}{m}$, then equation (8) suggests that the maximum p_1 is optimal. If not, the minimum p_1 is optimal. □

Proof of Proposition 6. Rewriting the equation (7) as

$$\Pi(\xi,p_1,p_2) = \gamma_2(\xi,m)(1-\xi)(p_1-c) + \xi(p_2-c). \tag{A-2}$$

It can be found that the profit function is increasing in p_2. Because $c \leq p_2 \leq v_L - v_0$, $p_2^* = v_L - v_0$. Observing $\gamma_2(\xi,m)$ in this profit function, we can find that $\frac{\frac{p_1+v_0-v}{p_1+v_0-v_L}-\lambda\xi_0}{1-\lambda}m$ strictly increases in p_1. However, $\lfloor\frac{\frac{p_1+v_0-v}{p_1+v_0-v_L}-\lambda\xi_0}{1-\lambda}m - \epsilon\rfloor$ is a step function in p_1. Also, $p_1 - c$ strictly increases in p_1. Hence, it is clear that the firm will rationally choose p_1 such that $\frac{\frac{p_1+v_0-v}{p_1+v_0-v_L}-\lambda\xi_0}{1-\lambda}m = i$, for some $i = 1,2,3,...,m$. Otherwise, the firm can increase p_1 to increase its expected profit until this equality holds. Then, we know that p_1 should be $\frac{v-v_0+\left[(1-\lambda)\frac{i}{m}+\lambda\xi_0\right](v_0-v_L)}{1-(1-\lambda)\frac{i}{m}-\lambda\xi_0}$. Substituting the two prices in equation (A-2), we could get the equivalent problem. □

Proof of Proposition 7. Because $\frac{\partial^2 \pi}{\partial \xi_{t-1}\partial \xi_t} = -(p_1 - p_2) \leq 0$ under the OES strategy and $\frac{\partial^2 \pi}{\partial \xi_{t-1}\partial \xi_t} = -(p_1 - p_2) - (p_2 - c) \leq 0$ under the TES strategy, $\pi(\xi_{t-1}, \xi_t)$ is submodular and so is the term on the right-hand side of the Bellman equations (13) and (14). Moreover, the feasible set is fixed as $[0,1]$. Thus, based on Topkis's Theorem (Topkis 1998, Thm. 2.8.2) the policy function $\xi^*(\cdot)$ is decreasing on $[0,1]$. It means that if $\xi_t > \xi_{t-1}$ or $\xi_t < \xi_{t-1}$ $(t \geq 1)$, we then have $\xi_{t+1} < \xi_t$ or $\xi_{t+1} > \xi_t$. We assume that $\{\xi_t\}$ is the optimal policy path and $\{\xi_{2t-1}\}$ and $\{\xi_{2t}\}$ are two sub-paths. Discuss four cases, i.e., $\xi_3 \leq \xi_1 \leq \xi_2$, $\xi_2 \leq \xi_1 \leq \xi_3$, $\xi_1 \leq \xi_3 \leq \xi_2$, and $\xi_1 \leq \xi_3 \leq \xi_1$, and by induction, we can find that both $\{\xi_{2t-1}\}$ and $\{\xi_{2t}\}$ are monotonous but in opposite directions. According to the monotone convergence theorem, we know that $\{\xi_{2t-1}\}$ and $\{\xi_{2t}\}$ converge to two steady points. If these two points are different, then the optimal policy is cyclic. If they turn out to be the same, then the optimal policy converges to a single stationary point. □

Proof of Proposition 8. (i) We start with the case of the OES strategy. We first study the optimal markdown probability path when $\underline{\xi}$ is sufficiently small. Let ξ_1 and ξ_2 be the two steady points. Based on the Euler equation and the Karush-Kuhn-Tucker (KKT) conditions, we know that there exist non-negative constants λ_1, λ_2, η_1 and η_2, such that ξ_1 and ξ_2 are the solutions of

$$\begin{cases} [-\xi_2 + \delta_1(1-\xi_2)]\frac{v-v_0-c}{\lambda-\lambda\xi_0} - \lambda_1 + \eta_1 = 0, \\ \delta_1[-\xi_1 + \delta_1(1-\xi_1)]\frac{v-v_0-c}{\lambda-\lambda\xi_0} - \lambda_2 + \eta_2 = 0, \\ \lambda_1(1-\xi_1) = 0,\ \lambda_2(1-\xi_2) = 0,\ \eta_1(\xi_1 - \underline{\xi}) = 0,\ \eta_2(\xi_2 - \underline{\xi}) = 0. \end{cases} \tag{A-3}$$

We consider six cases: (a) If $\underline{\xi} < \xi_1 < 1$ and $\underline{\xi} < \xi_2 < 1$, from equation (A-3), we have $\lambda_1 = 0$, $\lambda_2 = 0$, $\eta_1 = 0$, $\eta_2 = 0$ and $\xi_1 = \xi_2 = \frac{\delta_1}{1+\delta_1}$; (b) If $\xi_1 = \underline{\xi}$ and $\xi_2 = 1$, from equation (A-3), we have $\lambda_1 = 0$, $\lambda_2 \geq 0$, $\eta_1 \geq 0$, and $\eta_2 = 0$. It can be examined that all the conditions are satisfied; (c) If $\xi_1 = 1$ and $\xi_2 = 1$, from equation (A-3), we have $\eta_1 = 0$, $\eta_2 = 0$, $\lambda_1 = -\frac{v-v_0-c}{\lambda-\lambda\xi_0} \geq 0$, and $\lambda_2 = -\delta_1\frac{v-v_0-c}{\lambda-\lambda\xi_0} \geq 0$. We obtain two contradictions. This case is never optimal; (d) If $\underline{\xi} < \xi_1 < 1$ and $\xi_2 = 1$, from equation (A-3), we have $\lambda_1 = 0$, $\lambda_2 \geq 0$, $\eta_1 = 0$, $\eta_2 = 0$, and $-\frac{v-v_0-c}{\lambda-\lambda\xi_0} = 0$. Thus, we arrive at a contradiction. This case is never optimal; (e) If $\xi_1 = \underline{\xi}$ and $\xi_2 = \underline{\xi}$, from equation (A-3), we have $\lambda_1 = 0$, $\lambda_2 = 0$, $\eta_1 = -[-\underline{\xi} + \delta_1(1-\underline{\xi})]\frac{v-v_0-c}{\lambda-\lambda\xi_0} \geq 0$, and $\eta_2 = -\delta_1[-\underline{\xi} + \delta_1(1-\underline{\xi})]\frac{v-v_0-c}{\lambda-\lambda\xi_0} \geq 0$. Because $\underline{\xi}$ is sufficiently small, we obtain two contradictions. This case is never optimal; (f) If $\xi_1 = \underline{\xi}$ and $\underline{\xi} < \xi_2 < 1$, from equation (A-3), we have $\lambda_1 = 0$, $\lambda_2 = 0$, $\eta_1 \geq 0$, $\eta_2 = 0$, and $[-\underline{\xi} + \delta_1(1-\underline{\xi})]\frac{v-v_0-c}{\lambda-\lambda\xi_0} = 0$. Because $\underline{\xi}$ is sufficiently small, this case cannot hold.

From the above analysis, we know that the potential optimal policy is $(\frac{\delta_1}{1+\delta_1}, \frac{\delta_1}{1+\delta_1})$ and $(\underline{\xi}, 1)$. We now compare the firm's long-term profit under these two policies. The long-term profit when $\xi_1 = \xi_2 = \frac{\delta_1}{1+\delta_1}$ is $\frac{1}{1-\delta_1}\frac{\delta_1}{1+\delta_1}\frac{1}{1+\delta_1}\frac{v-v_0-c}{\lambda-\lambda\xi_0}$, and the long-term profit when $\xi_1 = \underline{\xi}$ and $\xi_2 = 1$ is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_0-c}{\lambda-\lambda\xi_0}$. Comparing these two profits, we have that the policy $(\underline{\xi}, 1)$ is always better since $\underline{\xi}$ is sufficiently small.

We then relax the assumption that $\underline{\xi}$ is sufficiently small. Based on the above analysis, we know that the policy $(\underline{\xi}, 1)$ yields a larger profit than the policy $(\frac{\delta_1}{1+\delta_1}, \frac{\delta_1}{1+\delta_1})$ if $\underline{\xi}$ is less than $\frac{1}{1+\delta_1}$. In addition, we can show that the policy $(\underline{\xi}, \underline{\xi})$ may be optimal if $\underline{\xi}$ increases to satisfy $\underline{\xi} \geq \frac{\delta_1}{1+\delta_1}$. The firm's long-term profit under the policy $(\underline{\xi}, \underline{\xi})$ is $\frac{1}{1-\delta_1}\underline{\xi}(1-\underline{\xi})\frac{v-v_0-c}{\lambda-\lambda\xi_0}$. Comparing the policy $(\underline{\xi}, 1)$ and the policy $(\underline{\xi}, \underline{\xi})$, we know that the policy $(\underline{\xi}, 1)$ is better if $\underline{\xi} < \frac{1}{1+\delta_1}$. Because we consider the case of $\delta_1 > \frac{1}{2-\underline{\xi}}$ (i.e., $\underline{\xi} < 2 - \frac{1}{\delta 1}$) to guarantee that the long-term selling yields a larger profit, $\underline{\xi}$ can at most increase to satisfy $\underline{\xi} < \min\{\frac{1}{1+\delta_1}, 2 - \frac{1}{\delta 1}\}$. The other policies are still never optimal when $\underline{\xi}$ increases. Thus, based on the above analysis, we know that the policy $(\underline{\xi}, 1)$ is still optimal if $\underline{\xi} < \min\{\frac{1}{1+\delta_1}, 2 - \frac{1}{\delta 1}\}$. Now, we compare the long-term profit under OES strategy to the long-term profit under the regular NP policy. The long-term profit under the NP policy is $\frac{1}{1-\delta_1}(v - v_0 - c)$. The long-term profit under the OES strategy is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_0-c}{\lambda-\lambda\xi_0}$. Comparing the two profits, we obtain the results. (ii) Next, we consider the case of the TES strategy. Let ξ_1 and ξ_2 be the two steady points. Based on the Euler equation and the Karush-Kuhn-Tucker (KKT) conditions, we know that there exist non-negative constants λ_1, λ_2, η_1 and η_2, such that ξ_1 and ξ_2 are the solutions of

$$\begin{cases} [-\xi_2 + \delta_1(1-\xi_2)]\frac{v-v_L}{\lambda-\lambda\xi_0} + (1+\delta_1)(1-\xi_2)(v_L - v_0 - c) - \lambda_1 + \eta_1 = 0, \\ \delta_1[-\xi_1 + \delta_1(1-\xi_1)]\frac{v-v_L}{\lambda-\lambda\xi_0} + \delta_1(1+\delta_1)(1-\xi_1)(v_L - v_0 - c) - \lambda_2 + \eta_2 = 0, \\ \lambda_1(1-\xi_1) = 0,\ \lambda_2(1-\xi_2) = 0,\ \eta_1(\xi_1 - \underline{\xi}) = 0,\ \eta_2(\xi_2 - \underline{\xi}) = 0. \end{cases} \tag{A-4}$$

We consider six cases: (a) If $\underline{\xi} < \xi_1 < 1$ and $\underline{\xi} < \xi_2 < 1$, from equation (A-4), we have $\lambda_1 = 0$, $\lambda_2 = 0$, $\eta_1 = 0$, $\eta_2 = 0$ and $\xi_1 = \xi_2 = \overline{\xi} \equiv \frac{\delta_1 \frac{v-v_L}{\lambda-\lambda\xi_0} + (1+\delta_1)(v_L - v_0 - c)}{(1+\delta_1)\left(\frac{v-v_L}{\lambda-\lambda\xi_0} + v_L - v_0 - c\right)}$; (b) If $\xi_1 = \underline{\xi}$ and $\xi_2 = 1$, from equation (A-4), we have $\lambda_1 = 0$, $\lambda_2 \geq 0$, $\eta_1 \geq 0$, and $\eta_2 = 0$. It can be examined that all the conditions are satisfied; (c) If $\xi_1 = 1$ and $\xi_2 = 1$, from equation (A-4), we have $\eta_1 = 0$, $\eta_2 = 0$, $\lambda_1 = -\frac{v-v_L}{\lambda-\lambda\xi_0} \geq 0$, and $\lambda_2 = -\delta_1 \frac{v-v_L}{\lambda-\lambda\xi_0} \geq 0$. We obtain two contradictions. This case is never optimal; (d) If $\underline{\xi} < \xi_1 < 1$ and $\xi_2 = 1$, from equation (A-4), we have $\lambda_1 = 0$, $\lambda_2 \geq 0$, $\eta_1 = 0$, $\eta_2 = 0$, and $-\frac{v-v_L}{\lambda-\lambda\xi_0} = 0$. Thus, we arrive at a contradiction. This case is never optimal; (e) If $\xi_1 = \underline{\xi}$ and $\xi_2 = \underline{\xi}$, from equation (A-4), we have $\lambda_1 = 0$, $\lambda_2 = 0$,

$$\eta_1 = -\left[\delta_1 \frac{v-v_L}{\lambda-\lambda\xi_0} + (1+\delta_1)(v_L - v_0 - c) - (1+\delta_1)\left(\frac{v-v_L}{\lambda-\lambda\xi_0} + v_L - v_0 - c\right)\underline{\xi}\right] \geq 0,$$

and

$$\eta_2 = -\delta_1\left[\delta_1 \frac{v-v_L}{\lambda-\lambda\xi_0} + (1+\delta_1)(v_L - v_0 - c) - (1+\delta_1)\left(\frac{v-v_L}{\lambda-\lambda\xi_0} + v_L - v_0 - c\right)\underline{\xi}\right] \geq 0.$$

Because $\underline{\xi}$ is sufficiently small, we obtain two contradictions. This case is never optimal; (f) If $\xi_1 = \underline{\xi}$ and $\underline{\xi} < \xi_2 < 1$, from equation (A-4), we have $\lambda_1 = 0$, $\lambda_2 = 0$, $\eta_1 \geq 0$, $\eta_2 = 0$, and $\delta_1\left[\delta_1 \frac{v-v_L}{\lambda-\lambda\xi_0} + (1+\delta_1)(v_L - v_0 - c) - (1+\delta_1)\left(\frac{v-v_L}{\lambda-\lambda\xi_0} + v_L - v_0 - c\right)\underline{\xi}\right] = 0$. Because $\underline{\xi}$ is sufficiently small, we arrive at a contradiction. This case is never optimal.

From the above analysis, we know that the potential optimal policy is $(\overline{\xi}, \overline{\xi})$ and $(\underline{\xi}, 1)$. We now compare the firm's long-term profit under these two policies. The long-term profit when $\xi_1 = \xi_2 = \overline{\xi}$ is $\frac{1}{1-\delta_1}\overline{\xi}(1-\overline{\xi})\frac{v-v_L}{\lambda-\lambda\xi_0} + \frac{1}{1-\delta_1}\overline{\xi}(2-\overline{\xi})(v_L - v_0 - c)$, and the long-term profit when $\xi_1 = \underline{\xi}$ and $\xi_2 = 1$ is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_L}{\lambda-\lambda\xi_0} + \frac{1}{1-\delta_1}(v_L - v_0 - c)$. It can be easily examined that $\overline{\xi}(1-\overline{\xi}) < \frac{1-\underline{\xi}}{1+\delta_1}$ when $\underline{\xi}$ is sufficiently small and $\overline{\xi}(2-\overline{\xi}) < 1$ hold. Thus, the policy $(\underline{\xi}, 1)$ is always better. We then relax the assumption that $\underline{\xi}$ is sufficiently small. Based on the above analysis, we know that the policy $(\underline{\xi}, 1)$ is optimal if $\overline{\xi}(1-\overline{\xi}) < \frac{1-\underline{\xi}}{1+\delta_1}$ and $\underline{\xi} < \overline{\xi}$. Based on these two conditions, we know that $\underline{\xi}$ can be increased to $\min\{1-\overline{\xi}(1-\overline{\xi})(1+\delta_1), \overline{\xi}\}$ at most to guarantee that the optimal policy is still $(\underline{\xi}, 1)$. The long-term profit under the regular NP policy is $\frac{1}{1-\delta_1}(v - v_0 - c)$. The long-term profit under the TES strategy is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_L}{\lambda-\lambda\xi_0} + \frac{1}{1-\delta_1}(v_L - v_0 - c)$. Comparing the two profits, we can obtain the result. (iii) We finally compare the OES strategy and the TES strategy. From the above analysis, we know that the OES strategy or the TES strategy is better than the NP policy if $\lambda - \lambda\xi_0 < \frac{1-\underline{\xi}}{1+\delta_1}$. Comparing the profits under the OES and the TES strategy, we have that the OES strategy is better if $\lambda - \lambda\xi_0 < \frac{1-\underline{\xi}}{1+\delta_1}$ and the TES strategy is better if $\lambda - \lambda\xi_0 > \frac{1-\underline{\xi}}{1+\delta_1}$. Furthermore, we have the OES strategy is optimal if $\lambda - \lambda\xi_0 < \frac{1-\underline{\xi}}{1+\delta_1}$ and the NP policy is optimal if $\lambda - \lambda\xi_0 \geq \frac{1-\underline{\xi}}{1+\delta_1}$. □

Proof of Lemma A-4. Because $\frac{\partial^2 \pi}{\partial \xi_{t-1}\partial \xi_t} = -(p_1 - p_2)B\left(\lfloor \frac{p_1 - (v-v_0) - \lambda\xi_0(p_1-p_2)}{(1-\lambda)(p_1-p_2)} m - \epsilon \rfloor; m-1, \xi_{t-1}\right) < 0$ under the OES strategy and $\frac{\partial^2 \pi}{\partial \xi_{t-1}\partial \xi_t} = -(p_1 - c)B\left(\lfloor \frac{\frac{p_1+v_0-v}{p_1+v_0-v_L} - \lambda\xi_0}{1-\lambda} m - \epsilon \rfloor; m-1, \xi_{t-1}\right) < 0$ under the TES strategy, $\pi(\xi_{t-1}, \xi_t)$ is submodular and so is the term on the right-hand side of the Bellman equation (20). Then, using the same method as in Proposition 7, we obtain the result. □

Proof of Proposition 9. (i) We start with the case of the OES strategy. We first study the optimal markdown probability path when $\underline{\xi}$ is sufficiently small. Let ξ_1 and ξ_2 be the two steady points. Based on the Euler equation and the Karush-Kuhn-Tucker (KKT) conditions, we know that there exist non-negative constants λ_1, λ_2, η_1 and η_2, such that ξ_1 and ξ_2 are the solutions of

$$\begin{cases} [-\xi_2^m + \delta_1 m \xi_1^{m-1}(1-\xi_2)]\frac{v-v_0-c}{\lambda-\lambda\xi_0} - \lambda_1 + \eta_1 = 0, \\ \delta_1[-\xi_1^m + \delta_1 m \xi_2^{m-1}(1-\xi_1)]\frac{v-v_0-c}{\lambda-\lambda\xi_0} - \lambda_2 + \eta_2 = 0, \\ \lambda_1(1-\xi_1) = 0,\ \lambda_2(1-\xi_2) = 0,\ \eta_1(\xi_1 - \underline{\xi}) = 0,\ \eta_2(\xi_2 - \underline{\xi}) = 0. \end{cases} \tag{A-5}$$

We consider six cases: (a) If $\underline{\xi} < \xi_1 < 1$ and $\underline{\xi} < \xi_2 < 1$, from equation (A-5), we have $\lambda_1 = 0$, $\lambda_2 = 0$, $\eta_1 = 0$, $\eta_2 = 0$ and ξ_1 and ξ_2 are the solutions of

$$\begin{cases} -\xi_2^m + \delta_1 m \xi_1^{m-1}(1-\xi_2) = 0, \\ -\xi_1^m + \delta_1 m \xi_2^{m-1}(1-\xi_1) = 0. \end{cases} \tag{A-6}$$

Combining these two equations, we have

$$(\xi_1^m - \xi_2^m) + \delta_1 m(\xi_1^{m-1} - \xi_2^{m-1}) + \delta_1 m\xi_1\xi_2(\xi_2^{m-2} - \xi_1^{m-2}) = 0. \tag{A-7}$$

If $K=2$, equation (A-7) can be rewritten as $(\xi_1-\xi_2)[(\xi_1+\xi_2)+2\delta_1]=0$. Thus, we have $\xi_1=\xi_2$. If $K=3$, equation (A-7) can be rewritten as $(\xi_1-\xi_2)[(\xi_1^2+\xi_1\xi_2+\xi_2^2)+3\delta_1(\xi_1+\xi_2-\xi_1\xi_2)]=0$. Because $\xi_1+\xi_2-\xi_1\xi_2>0$, we have $\xi_1=\xi_2$. If $K\geq 4$, equation (A-7) can be rewritten as

$$\begin{aligned}
&(\xi_1-\xi_2)[(\xi_1^{m-1}+\xi_1^{m-2}\xi_2+\cdots+\xi_2^{m-1})\\
&+m\delta_1(\xi_1^{m-2}+\xi_1^{m-3}\xi_2+\cdots+\xi_2^{m-2})\\
&-m\delta_1\xi_1\xi_2(\xi_1^{m-3}+\xi_1^{m-4}\xi_2+\cdots+\xi_2^{m-3})]\\
=\ &(\xi_1-\xi_2)\{(\xi_1^{m-1}+\xi_1^{m-2}\xi_2+\cdots+\xi_2^{m-1})\\
&+m\delta_1[\xi_1^{m-2}(1-\xi_2)+\xi_1^{m-3}\xi_2(1-\xi_2)+\cdots+\xi_1\xi_2^{m-3}(1-\xi_2)+\xi_2^{m-2}]\}\\
=\ &0.
\end{aligned}$$

We thus have $\xi_1=\xi_2$. Then, solving the equation (A-6), we obtain $\xi_1=\xi_2=\frac{\delta_1 m}{1+\delta_1 m}$. (b) If $\xi_1=\underline{\xi}$ and $\xi_2=1$, from equation (A-5), we have $\lambda_1=0$, $\lambda_2\geq 0$, $\eta_1\geq 0$, and $\eta_2=0$. It can be examined that all the conditions are satisfied; (c) If $\xi_1=1$ and $\xi_2=1$, from equation (A-5), we have $\eta_1=0$, $\eta_2=0$, $\lambda_1=-\frac{v-v_0-c}{\lambda-\lambda\xi_0}\geq 0$, and $\lambda_2=-\delta_1\frac{v-v_0-c}{\lambda-\lambda\xi_0}\geq 0$. We obtain two contradictions. This case is never optimal; (d) If $\underline{\xi}<\xi_1<1$ and $\xi_2=1$, from equation (A-5), we have $\lambda_1=0$, $\lambda_2\geq 0$, $\eta_1=0$, $\eta_2=0$, and $-\frac{v-v_0-c}{\lambda-\lambda\xi_0}=0$. Thus, we arrive at a contradiction. This case is never optimal; (e) If $\xi_1=\underline{\xi}$ and $\xi_2=\underline{\xi}$, from equation (A-5), we have $\lambda_1=0$, $\lambda_2=0$, $\eta_1=-\underline{\xi}^{m-1}[\delta_1 m(1-\underline{\xi})-\underline{\xi}]\frac{v-v_0-c}{\lambda-\lambda\xi_0}\geq 0$, and $\eta_2=-\delta_1\underline{\xi}^{m-1}[\delta_1 m(1-\underline{\xi})-\underline{\xi}]\frac{v-v_0-c}{\lambda-\lambda\xi_0}\geq 0$. Because $\underline{\xi}$ is sufficiently small, we obtain two contradictions. This case is never optimal; (f) If $\xi_1=\underline{\xi}$ and $\underline{\xi}<\xi_2<1$, from equation (A-5), we have $\lambda_1=0$, $\lambda_2=0$, $\eta_1\geq 0$, $\eta_2=0$, and $[-\underline{\xi}^m+\delta_1 m\xi_2^{m-1}(1-\underline{\xi})]\frac{v-v_0-c}{\lambda-\lambda\xi_0}=0$. Because $\underline{\xi}$ is sufficiently small, this case cannot hold.

From the above analysis, we know that the potential optimal policy is $(\frac{\delta_1 m}{1+\delta_1 m},\frac{\delta_1 m}{1+\delta_1 m})$ and $(\underline{\xi},1)$. We now compare the firm's long-term profit under these two policies. The long-term profit when $\xi_1=\xi_2=\frac{\delta_1 m}{1+\delta_1 m}$ is $\frac{1}{1-\delta_1}\left(\frac{\delta_1 m}{1+\delta_1 m}\right)^m\frac{1}{1+\delta_1 m}\frac{v-v_0-c}{\lambda-\lambda\xi_0}$, and the long-term profit when $\xi_1=\underline{\xi}$ and $\xi_2=1$ is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_0-c}{\lambda-\lambda\xi_0}$. Comparing these two profits, we have that the policy $(\underline{\xi},1)$ is always better since $\underline{\xi}$ is sufficiently small.

We now relax the assumption that $\underline{\xi}$ is sufficiently small. Based on the above analysis, we know that the policy $(\underline{\xi},1)$ yields a larger profit than the policy $(\frac{\delta_1 m}{1+\delta_1 m},\frac{\delta_1 m}{1+\delta_1 m})$ if $\underline{\xi}$ is less than $1-\frac{(\delta_1 m)^m}{(1+\delta_1 m)^{m+1}}(1+\delta_1)$.

In addition, the policy $(\underline{\xi},\underline{\xi})$ may be optimal if $\underline{\xi}$ increases to satisfy $\underline{\xi}\geq\frac{\delta_1 m}{1+\delta_1 m}$. The firm's long-term profit under the policy $(\underline{\xi},\underline{\xi})$ is $\frac{1}{1-\delta_1}\underline{\xi}^m(1-\underline{\xi})\frac{v-v_0-c}{\lambda-\lambda\xi_0}$. Comparing the policy $(\underline{\xi},1)$ and the policy $(\underline{\xi},\underline{\xi})$, we know that the policy $(\underline{\xi},1)$ is better if $\underline{\xi}<\sqrt[m]{\frac{1}{1+\delta_1}}$. The other policies are still never optimal when $\underline{\xi}$ increases.

Recall that to guarantee the profitability for the long-term dynamic selling, we assume that $\delta_1>\frac{1}{1+(1-\underline{\xi})\underline{\xi}^{m-1}}$. When $\underline{\xi}$ increases, we have that $(1-\underline{\xi})\underline{\xi}^{m-1}>\frac{1}{\delta_1}-1$ must hold. It can be shown that the function $(1-\xi)\xi^{m-1}$ is increasing in ξ when $0<\xi\leq\frac{m-2}{m}$ and concave in ξ when $\frac{m-2}{m}\leq\xi<1$. According to the first-order condition, we know that $\xi=\frac{m-1}{m}$ maximizes the function $(1-\xi)\xi^{m-1}$. Thus, $\frac{(m-1)^{m-1}}{m^m}>\frac{1}{\delta_1}-1$ must hold to guarantee that the set $(1-\underline{\xi})\underline{\xi}^{m-1}>\frac{1}{\delta_1}-1$ on $\underline{\xi}$ is not empty. Because $\frac{(m-1)^{m-1}}{m^m}$ is decreasing in m, we denote m_0 as an integer that satisfies $\frac{(m_0)^{m_0}}{(m_0+1)^{m_0+1}}<\frac{1}{\delta_1}-1<\frac{(m_0-1)^{m_0-1}}{m_0^{m_0}}$. Let $\underline{a}(m)$ and $\overline{a}(m)$ satisfying $\underline{a}(m)<\overline{a}(m)$ be two solutions of the equation $(1-\underline{\xi})\underline{\xi}^{m-1}=\frac{1}{\delta_1}-1$. Then, we know that when $m\leq m_0$, $(1-\underline{\xi})\underline{\xi}^{m-1}>\frac{1}{\delta_1}-1$ holds if $\underline{a}(m)<\underline{\xi}<\overline{a}(m)$.

Hence, based on the above analysis, we know that the policy $(\underline{\xi},1)$ is still optimal if $a(m)<\underline{\xi}<b(m)$. The long-term profit under the regular NP policy is $\frac{1}{1-\delta_1}(v-v_0-c)$. The long-term profit under the dynamic markdown PM policy is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_0-c}{\lambda-\lambda\xi_0}$. Comparing the two profits, we obtain the results. (ii) Next, we study the case of the TES strategy. Let ξ_1 and ξ_2 be the two steady points. Based on the Euler equation and the Karush-Kuhn-Tucker (KKT)

conditions, we know that there exist non-negative constants λ_1, λ_2, η_1 and η_2, such that ξ_1 and ξ_2 are the solutions of

$$\begin{cases} [-\xi_2^m+\delta_1 m\xi_1^{m-1}(1-\xi_2)]\left[\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c\right]+(v_L-v_0-c)-\lambda_1+\eta_1=0, \\ \delta_1[-\xi_1^m+\delta_1 m\xi_2^{m-1}(1-\xi_1)]\left[\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c\right]+\delta_1(v_L-v_0-c)-\lambda_2+\eta_2=0, \\ \lambda_1(1-\xi_1)=0,\ \lambda_2(1-\xi_2)=0,\ \eta_1(\xi_1-\underline{\xi})=0,\ \eta_2(\xi_2-\underline{\xi})=0. \end{cases} \tag{A-8}$$

We consider six cases: (a) If $\underline{\xi}<\xi_1<1$ and $\underline{\xi}<\xi_2<1$, from equation (A-8), we have $\lambda_1=0$, $\lambda_2=0$, $\eta_1=0$, $\eta_2=0$ and ξ_1 and ξ_2 are the solutions of

$$\begin{cases} -\xi_2^m+\delta_1 m\xi_1^{m-1}(1-\xi_2)=-\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}, \\ -\xi_1^m+\delta_1 m\xi_2^{m-1}(1-\xi_1)=-\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}. \end{cases}$$

Using the same method in the case of the OES strategy above, we can prove that $\xi_1=\xi_2$. Then, solving equation (Online Supplement 1), we obtain $\xi_1=\xi_2=\overline{\xi}$, where $\overline{\xi}\in(\underline{\xi},1)$ satisfies $\overline{\xi}^m-\delta_1 m\overline{\xi}^{m-1}(1-\overline{\xi})=\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}$. (b) If $\xi_1=\underline{\xi}$ and $\xi_2=1$, from equation (A-8), we have $\lambda_1=0$, $\lambda_2\geq 0$, $\eta_1\geq 0$, and $\eta_2=0$. It can be examined that all the conditions are satisfied; (c) If $\xi_1=1$ and $\xi_2=1$, from equation (A-8), we have $\eta_1=0$, $\eta_2=0$, $\lambda_1=-\frac{v-v_L}{\lambda-\lambda\xi_0}\geq 0$, and $\lambda_2=-\delta_1\frac{v-v_L}{\lambda-\lambda\xi_0}\geq 0$. We obtain two contradictions. This case is never optimal; (d) If $\underline{\xi}<\xi_1<1$ and $\xi_2=1$, from equation (A-8), we have $\lambda_1=0$, $\lambda_2\geq 0$, $\eta_1=0$, $\eta_2=0$, and $-\frac{v-v_L}{\lambda-\lambda\xi_0}=0$. Thus, we arrive at a contradiction. This case is never optimal; (e) If $\xi_1=\underline{\xi}$ and $\xi_2=\underline{\xi}$, from equation (A-8), we have $\lambda_1=0$, $\lambda_2=0$,

$$\eta_1=-(v_L-v_0-c)+\underline{\xi}^{m-1}[\underline{\xi}-\delta_1 m(1-\underline{\xi})]\left[\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c\right]\geq 0,$$

and

$$\eta_2=-\delta_1(v_L-v_0-c)+\delta_1\underline{\xi}^{m-1}[\underline{\xi}-\delta_1 m(1-\underline{\xi})]\left[\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c\right]\geq 0.$$

Because $\underline{\xi}$ is sufficiently small, we obtain two contradictions. This case is never optimal; (f) If $\xi_1=\underline{\xi}$ and $\underline{\xi}<\xi_2<1$, from equation (A-8), we have $\lambda_1=0$, $\lambda_2=0$, $\eta_1\geq 0$, $\eta_2=0$, and $\delta_1[-\underline{\xi}^m+\delta_1 m\xi_2^{m-1}(1-\underline{\xi})]\left[\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c\right]+\delta_1(v_L-v_0-c)=0$. Because $\underline{\xi}$ is sufficiently small, we obtain a contradiction. This case cannot hold.

From the above analysis, we know that the potential optimal policy is $(\overline{\xi},\overline{\xi})$ and $(\underline{\xi},1)$. We now compare the firm's long-term profit under these two policies. The long-term profit when $\xi_1=\xi_2=\overline{\xi}$ is $\frac{1}{1-\delta_1}\overline{\xi}^m(1-\overline{\xi})\frac{v-v_L}{\lambda-\lambda\xi_0}+\frac{1}{1-\delta_1}[\overline{\xi}^m(1-\overline{\xi})+\overline{\xi}](v_L-v_0-c)$, and the long-term profit when $\xi_1=\underline{\xi}$ and $\xi_2=1$ is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_L}{\lambda-\lambda\xi_0}+\frac{1}{1-\delta_1}(v_L-v_0-c)$. According to the definition of $\overline{\xi}$, we have

$$\begin{aligned}\overline{\xi}^m(1-\overline{\xi}) &= \frac{\left(\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}+\delta_1 m\overline{\xi}^{m-1}\right)(1-\overline{\xi})}{1+\delta_1 m} \\ &= \frac{\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}(1-\overline{\xi})+\delta_1 m\overline{\xi}^{m-1}(1-\overline{\xi})}{1+\delta_1 m} \\ &= \frac{\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}(1-\overline{\xi})+\overline{\xi}^m-\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}}{1+\delta_1 m} \\ &= \frac{\overline{\xi}\left(\overline{\xi}^{m-1}-\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}\right)}{1+\delta_1 m}\end{aligned}$$

Because $1+\delta_1 m>1+\delta_1$ and $\overline{\xi}\left(\overline{\xi}^{m-1}-\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}\right)<1$, we have $\overline{\xi}^m(1-\overline{\xi})<\frac{1-\underline{\xi}}{1+\delta_1}$ when $\underline{\xi}$ is sufficiently small. In addition, we have $\overline{\xi}^m(1-\overline{\xi})+\overline{\xi}<1$. Therefore, the profit under the policy $(\underline{\xi},1)$ is always larger than the profit under the policy $(\overline{\xi},\overline{\xi})$.

We then relax the assumption that $\underline{\xi}$ is sufficiently small. Based on the above analysis, we know that the policy $(\underline{\xi},1)$ is optimal if $\overline{\xi}(1-\overline{\xi})<\frac{1-\underline{\xi}}{1+\delta_1}$ and $\underline{\xi}^m-\delta_1 m\underline{\xi}^{m-1}(1-\underline{\xi})<\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}$. It can be checked that the function $f(\xi)=\xi^m-\delta_1 m\xi^{m-1}(1-\xi)$ is increasing in ξ if $\xi\geq\frac{\delta_1(m-1)}{1+\delta_1 m}$ and decreasing in ξ if $\xi\leq\frac{\delta_1(m-1)}{1+\delta_1 m}$. In addition, we can find that $f(0)=f(\frac{\delta_1 m}{1+\delta_1 m})=0$ and $f(1)=1$. Thus, we have $\underline{\xi}^m-\delta_1 m\underline{\xi}^{m-1}(1-\underline{\xi})<\frac{v_L-v_0-c}{\frac{v-v_L}{\lambda-\lambda\xi_0}+v_L-v_0-c}$ holds if $\underline{\xi}<\overline{\xi}$. Because $\overline{\xi}(1-\overline{\xi})<\frac{1-\underline{\xi}}{1+\delta_1}$ is equivalent to $\underline{\xi}<1-\overline{\xi}^m(1-\overline{\xi})(1+\delta_1)$, we know that $\underline{\xi}$ can be increased to $\min\{1-\overline{\xi}^m(1-\overline{\xi})(1+\delta_1),\overline{\xi}\}$ at most to guarantee that the optimal policy is still $(\underline{\xi},1)$. The long-term profit under the regular NP policy is $\frac{1}{1-\delta_1}(v-v_0-c)$. The long-term profit under the TES strategy is $\frac{1}{1-\delta_1^2}(1-\underline{\xi})\frac{v-v_L}{\lambda-\lambda\xi_0}+\frac{1}{1-\delta_1}(v_L-v_0-c)$. Comparing the two profits, we obtain the result. (iii) We finally compare the OES strategy and the TES strategy. Because the profits under these two strategies are the same as the case of the $S(1)$ model, we have the same results as that in Proposition 8. □

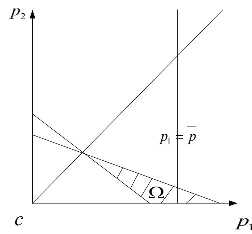

Figure A-3 Illustration of the feasible decision region and optimal pricing decisions with budget-constrained consumers.

Proof of Proposition A-1. (i) According to the new feasible region illustrated in Figure A-3 and Lemma A-1, we know that if the firm uses PM policy, the prices should be set as $p_1=\overline{p}$ and $p_2=\frac{v-v_0-(\lambda-\lambda\xi_0)\overline{p}}{\lambda\xi_0+1-\lambda}$. (ii) Given the pricing policy, we can then formulate the firm's expected profit function as follows: $\Pi(\xi)=\xi\left\{\overline{p}-c-\xi\left[\overline{p}-\frac{v-v_0-(\lambda-\lambda\xi_0)\overline{p}}{\lambda\xi_0+1-\lambda}\right]\right\}$. According to first-order condition, we obtain that $\xi^*=\frac{\lambda\xi_0+1-\lambda}{2}\frac{\overline{p}-c}{\overline{p}-(v-v_0)}$ and $\Pi(\xi^*)=\frac{\lambda\xi_0+1-\lambda}{4}\frac{(\overline{p}-c)^2}{\overline{p}-(v-v_0)}$. If $\lambda-\lambda\xi_0>1-2\frac{\overline{p}-(v-v_0)}{\overline{p}-c}$, then $\xi^*<1$. If $\lambda-\lambda\xi_0<1-4\frac{\overline{p}-(v-v_0)}{(\overline{p}-c)^2}(v-v_0-c)$, then $\Pi(\xi^*)$ is strictly larger than the firm's profit under NP policy, i.e., $v-v_0-c$. It can be easily examined that $1-2\frac{\overline{p}-(v-v_0)}{\overline{p}-c}<\frac{v-v_0-c}{\overline{p}-c}$ holds. To guarantee that $1-2\frac{\overline{p}-(v-v_0)}{\overline{p}-c}<1-4\frac{\overline{p}-(v-v_0)}{(\overline{p}-c)^2}(v-v_0-c)$, $\overline{p}>2(v-v_0)-c$ must hold. Furthermore, $1-2\frac{\overline{p}-(v-v_0)}{\overline{p}-c}<0$ holds. Then, we complete the proof of Proposition A-1. □

Online Supplement 2. Technical Lemmas

Lemma A-1. *Given a markdown probability ξ, the firm's optimal pricing decisions satisfy*

(i) $p_1^*=p_2^*=v-v_0$ if $\xi\geq\lambda\xi_0+1-\lambda$;

(ii) $p_1^*=\frac{v-v_0-(\lambda\xi_0+1-\lambda)c}{\lambda-\lambda\xi_0}$ and $p_2^*=c$ if $\xi\leq\lambda\xi_0+1-\lambda$.

Lemma A-2. *(i) Under the OES strategy, the fraction of the customers that purchase in the regular period is*

$$\gamma_1(\xi,m)=1-\Sigma_{n=0}^{\lfloor\frac{p_1-(v-v_0)-\lambda\xi_0(p_1-p_2)}{(1-\lambda)(p_1-p_2)}m-\epsilon\rfloor}B(n;m,\xi),\tag{A-9}$$

where $(p_1,p_2)\in\Omega$, $\epsilon>0$ is an arbitrarily small number, and $B(n;m,\xi)=\binom{m}{n}\xi^n(1-\xi)^{m-n}$ is the probability function of the binomial distribution with parameters m and ξ.

(ii) Under the TES strategy, the fraction of the customers that purchase in the regular period is

$$\gamma_2(\xi,m)=1-\Sigma_{n=0}^{\lfloor\frac{\frac{p_1+v_0-v}{p_1+v_0-v_L}-\lambda\xi_0}{1-\lambda}m-\epsilon\rfloor}B(n;m,\xi),\tag{A-10}$$

where $(p_1,p_2)\in\Omega_1$, $\epsilon>0$ is an arbitrarily small number, and $B(n;m,\xi)=\binom{m}{n}\xi^n(1-\xi)^{m-n}$ is the probability function of the binomial distribution with parameters m and ξ.

Lemma A-3. *The firm's optimization problem (6) can be equivalently reduced to problem (8).*

Lemma A-4. *When each customer has multiple samples, the optimal long-run stationary policy is a cyclic policy or converges to a single point.*

Online Supplement 3. Capacity Constraint

In this section, we examine the impact of limited capacity/inventory on the firm's PM policy. Following Whang (2014), we assume that the firm can expedite the order at a higher cost and zero margin to guarantee that each consumer who decides to buy in the first period can obtain one product. However, such a supply capability cannot be achieved in the second period. Thus, if a consumer decides to purchase in the second period, she faces a stockout probability $q \in (0,1]$. Then, if consumer i purchases in period 1, her utility U_{i1} satisfies $U_{i1} = v - p_1 + \xi_i(p_1 - p_2)$, where $\xi_i = \lambda\xi_0 + (1-\lambda)I_i$. If consumer i purchases in period 2, her utility U_{i2} is $U_{i2} = (1-q)[(1-\xi_i)(v_L - p_1) + \xi_i(v_L - p_2)] = (1-q)[v_L - p_1 + \xi_i(p_1 - p_2)]$. It is clear that $U_{i1} > U_{i2}$ because $q \in (0,1]$ and $v_L < v$. It indicates that each consumer only purchases the product in the first period if she decides to buy. This phenomenon is caused by the PM policy that induces consumers to purchase in the first period, which is consistent with Lai et al. (2010) and the same as the basic model in Section 3. Therefore, the result will be the same as what we have in the basic model.

Online Supplement 4. Budget Constraint

We assume that $\overline{p} > v - v_0$. It is clear that $\overline{p}$ has no effects on the firm's PM policy if $\overline{p} \geq \frac{v - v_0 - (\lambda\xi_0 + 1 - \lambda)c}{\lambda - \lambda\xi_0}$. Thus, we will focus on the case of $\overline{p} < \frac{v - v_0 - (\lambda\xi_0 + 1 - \lambda)c}{\lambda - \lambda\xi_0}$ (i.e., $\lambda - \lambda\xi_0 < \frac{v - v_0 - c}{\overline{p} - c}$). The following proposition characterizes the influence of $\overline{p}$ on the optimal PM policy.

Proposition A-1. *Suppose $\lambda - \lambda\xi_0 < \frac{v - v_0 - c}{\overline{p} - c}$, we have*
(i) the optimal prices are $p_1^* = \overline{p}$ and $p_2^* = \frac{v - v_0 - (\lambda - \lambda\xi_0)\overline{p}}{\lambda\xi_0 + 1 - \lambda}$;
(ii) if $\overline{p} > 2(v - v_0) - c$ and $\lambda - \lambda\xi_0 < 1 - 4\frac{\overline{p} - (v - v_0)}{(\overline{p} - c)^2}(v - v_0 - c)$, it is optimal for the firm to use PM policy and the optimal markdown strategy is $\xi^* = \frac{\lambda\xi_0 + 1 - \lambda}{2}\frac{\overline{p} - c}{\overline{p} - (v - v_0)}$. The value of PM policy is $\frac{\lambda\xi_0 + 1 - \lambda}{4}\frac{(\overline{p} - c)^2}{\overline{p} - (v - v_0)}$.

Proposition A-1 shows that if the budget-constraint is not sufficiently high (i.e., $\overline{p} < \frac{v - v_0 - (\lambda\xi_0 + 1 - \lambda)c}{\lambda - \lambda\xi_0}$), the firm can only set the price in the first period as $\overline{p}$. Correspondingly, the firm should increase the price in the second period from c to $\frac{v - v_0 - (\lambda - \lambda\xi_0)\overline{p}}{\lambda\xi_0 + 1 - \lambda}$. In addition, this proposition shows that it is profitable to carry out the PM policy when $\lambda - \lambda\xi_0$ is not sufficiently high, which is similar to the case without budget constraint analyzed earlier.

Online Supplement 5. Consumer Imperfect Memory

In this section, we assume that customers may simply forget or do not bother to make the refund claim if there is indeed markdown in the second period. This type of consumer bounded rationality comes from their imperfect memory. Suppose that there is a probability β, a consumer forgets (or does not bother) to make the refund/reimbursement claim if there is a markdown in the second period, where $\beta \in [0,1]$. Consistent with our basic model and the literature (e.g., Lai et al. (2010) and reference therein), we maintain the assumption that consumers are still strategic/forward-looking, i.e., they are aware and anticipate that their memory is not perfect when making their purchase decisions in the first period. To isolate and focus on the role of imperfect memory, we assume that consumers have rational expectations about the markdown probability. We are interested in whether a different type of consumer bounded rationality (e.g., imperfect memory) can induce profitability of the PM policy.

A customer with imperfect memory purchases in period 1 if $u_1 \equiv v - p_1 + (1-\beta)\xi(p_1 - p_2) \geq u_2 \equiv (1-\xi)\max\{v_L - p_1, v_0\} + \xi\max\{v_L - p_2, v_0\}$. Hence, we must have $v - p_1 + (1-\beta)\xi(p_1 - p_2) \geq v_0$. Therefore, the firm expected profit under consumer imperfect memory is $\Pi_{IM} \equiv p_1 - (1-\beta)\xi(p_1 - p_2) - c \leq v - v_0 - c = \Pi_{NP}$. Hence, using the PM policy cannot strictly improve the firm's profit under the NP policy in this setting. This suggests that imperfect memory as a type of bounded rationality itself cannot explain why the PM policy should be adopted.

Online Supplement 6. Dynamic Markdown When $c > v_L - v_0$

In this section, we present the results supplementary to Section 4 in the main paper, when $c > v_L - v_0$ holds. All the results are parallel to those in Section 4, but we present them below for completeness. All the proofs are similar to those in Section 4 and hence omitted for brevity (they are available from the authors).

In this setting, only the OES strategy is possible. The Bellman equation is

$$V(\xi_{t-1}) = \max_{\xi_t \in [0,1]} \left\{ \xi_{t-1}(1-\xi_t)\frac{v - v_0 - c}{\lambda - \lambda\xi_0} + \delta_1 V(\xi_t) \right\}. \tag{A-11}$$

We have the main result below under the $S(1)$ framework.

PROPOSITION A-2. *Suppose $c > v_L - v_0$ holds, we have:*
*(i) (**Optimal markdown probability path for PM policy**) The optimal markdown probability path converges to the cyclic policy $(\underline{\xi}, 1)$, where $\underline{\xi}$ is strictly positive but sufficiently small; As long as $\underline{\xi}$ satisfies $\underline{\xi} < \min\{\frac{1}{1+\delta_1}, 2 - \frac{1}{\delta_1}\}$, the optimal dynamic markdown probability path is still $(\underline{\xi}, 1)$.*
*(ii) (**Dynamic markdown PM policy v.s. NP policy**) If $\lambda - \lambda\xi_0 < \frac{1-\underline{\xi}}{1+\delta_1}$, the dynamic markdown PM policy is better than the NP policy (i.e., $p_1 = p_2 = v - v_0$). Furthermore, the upper bound of $\lambda - \lambda\xi_0$ is higher than that for the fixed-probability PM policy in the basic model, i.e., $\frac{1-\underline{\xi}}{1+\delta_1} > \frac{1}{4}$, if $\underline{\xi} < \frac{3-\delta_1}{4}$.*

Under the $S(m)$ framework, we also find that the optimal markdown probability policy is robust to the number of samples. Similar to the $S(1)$ framework, we can show that the firm has flexibility to increase the markdown probability $\underline{\xi}$ in a range while still benefiting from customer bounded rationality.

PROPOSITION A-3. *Suppose $c > v_L - v_0$ holds, we have:*
(i) The optimal markdown probability path converges to the cyclic policy $(\underline{\xi}, 1)$ when each customer has m samples, where $\underline{\xi}$ is strictly positive but sufficiently small.
(ii) If there exists m_0 and $m \le m_0$, then as long as $\underline{\xi}$ satisfies $a(m) \equiv \underline{a}(m) < \underline{\xi} < b(m) \equiv \min\left\{\overline{a}(m), 1 - \frac{(\delta_1 m)^m}{(1+\delta_1 m)^{m+1}}(1+\delta_1), \sqrt[m]{\frac{1}{1+\delta_1}}\right\}$, the optimal dynamic markdown probability path is still $(\underline{\xi}, 1)$.

Proposition A-3 demonstrates the robustness of the cyclic policy with respect to the consumers' level of rationality m.

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