Appendix to "Simple Contracts to Assure Supply under Noncontractible Capacity and Asymmetric Cost Information": Proofs of Lemmas and Propositions

Proof of Proposition 1: The first-order condition for truth-telling in (3) is

$$\pi_s'(x;c) = T'(x) - cK'(x) = 0 \tag{1}$$

at $x = c$ for all c. Note in $\pi_s'(x;c)$, the derivative is taken with respect to the decision variable x for given parameter c. This implies

$$T'(x) - xK'(x) = 0 \tag{2}$$

for all x. The second-order condition for truth-telling is

$$T''(x) - xK''(x) \le 0 \tag{3}$$

for all x. Differentiating the first-order condition in (2) again, we have

$$T''(x) - K'(x) - xK''(x) = 0. \tag{4}$$

From (3) and (4) we have

$$K'(x) \le 0. \tag{5}$$

Conditions in (2) and (5) constitute the local incentive constraints, which ensure that the supplier will truthfully announce his cost locally. Next we check the global incentive constraint, that is, $\pi_s(c;c) \ge \pi_s(x;c)$ for all $x \ne c$. If the Spence-Mirrlees single-crossing condition is satisfied, then the local incentive constraints are sufficient for global optimality (Bolton and Dewatripont 2005, Chapter 2). It is straightforward to show $\frac{\partial}{\partial x}\left(\frac{\partial \pi_s/\partial K}{\partial \pi_s/\partial T}\right) = -1 < 0$, which indicates that the single-crossing condition holds in our model. Thus we have shown that (2) and (5) together guarantee truth-telling.

Under truth-telling, we may write notation $(x;x)$ as (x) for concision. For example, the supplier's profit function $\pi_s(x;x)$ becomes $\pi_s(x)$, which is given by

$$\pi_s(x) = T(x) - xK(x). \tag{6}$$

Taking derivative of $\pi_s(x)$ and by (2), we have

$$\pi_s'(x) = -K(x).$$

If we replace the payment term $T(x)$ by $\pi_s(x) + xK(x)$, then the problem in (5) can be rewritten as

$$\max_{\{\pi_s(\cdot),K(\cdot)\}} \int_0^{\bar{c}} \{pE_D[\min(K(x), D)] - xK(x) - \pi_s(x)\} f(x)dx$$

$$\begin{aligned} s.t.\ \ \pi_s'(x) &= -K(x), && (7)\\ K'(x) &\le 0, \\ K(x) &\ge 0, \\ \pi_s(\bar{c}) &\ge 0. && (8) \end{aligned}$$

The supplier's capacity must be non-negative, so there is $K(x) \ge 0$; the condition $\pi_s(\bar{c}) \ge 0$ guarantees supplier participation. We may drop the constraint $K'(x) \le 0$ for now; later we will check for it after characterizing the optimal $K(x)$. In this optimization problem, π_s is the state variable and $K(x)$ is the control variable. Define

$$J(K, \pi_s, \lambda, \beta, x) = \{pE_D[\min(K(x), D)] - xK(x) - \pi_s(x)\}f(x) - \lambda(x)K(x) + \beta(x)K(x)$$

as the Hamiltonian, where $\lambda(x)$ and $\beta(x)$ are multipliers to be determined. By the Pontryagin principle (p. 138 in Laffont and Martimort, 2002) we know

$$\lambda^{'} = -\frac{\partial J}{\partial \pi_s} = f(x).$$

From the transversality condition (since there is no constraint on $\pi_s(x)$ at 0), we know $\lambda = F(x)$. Necessary optimality conditions include

$$\begin{aligned}\frac{\partial J}{\partial K} &= \{(p-x) - p\Phi(K(x))\}f(x) - F(x) + \beta = 0,\\ \beta &\geq 0,\ K(x) \geq 0,\ \beta K(x) = 0.\end{aligned}$$

If $\beta > 0$, then there must be $K(x) = 0$ and $\beta = -f(x)\left[p - (x + F(x)/f(x))\right] > 0$; otherwise $K(x)$ must satisfy

$$[(p-x) - p\Phi(K(x))]f(x) - F(x) = 0.$$

Therefore, the optimal solution is

$$\begin{aligned}K^o(x) &= 0,\ \text{if } p - (x + F(x)/f(x)) < 0;\\ K^o(x) &= \Phi^{-1}\left(\frac{p - (x + F(x)/f(x))}{p}\right),\ \text{otherwise.}\end{aligned}$$

Due to log-concavity of F, $p - (x + F(x)/f(x))$ is a decreasing function of x, which confirms the previous assumption $K'(x) \leq 0$. Let $\hat{c}^o$ be the solution to $p - (x + F(x)/f(x)) = 0$ if it exists or set $\hat{c}^o = \bar{c}$ otherwise (note that $p - (x + F(x)/f(x)) < 0$ for $x > \hat{c}^o$). Thus we have

$$\begin{aligned}K^o(x) &= 0,\ \text{if } x > \hat{c}^o,\\ K^o(x) &= \Phi^{-1}\left(\frac{p - (x + F(x)/f(x))}{p}\right),\ \text{if } x \leq \hat{c}^o.\end{aligned}$$

We can derive the profit function $\pi_s(x)$ from (7) and (8):

$$\pi_s(x) = \int_x^{\bar{c}} K^o(u)du.$$

This clearly satisfies the participation constraints. Finally, from (6) and (7), the side payment $T(x)$ must satisfy

$$T^o(x) = xK^o(x) + \int_x^{\bar{c}} K^o(u)du. \ \square$$

Proof of Lemma 1: (i) We can write the supplier's profit under the contract $\{T(x,z)\}$ as

$$\pi_s(x, K; c) = E_D\left[T(x,z)\right] - cK = \int_0^K T(x,y)\phi(y)dy + \int_K^\infty T(x,K)\phi(y)dy - cK.$$

The supplier's optimal K for given c and x must satisfy:

$$\frac{\partial \pi_s(x,K;c)}{\partial K} = \int_K^\infty \frac{\partial T(x,K)}{\partial K}\phi(y)dy - c = 0. \tag{9}$$

Let $K(x;c)$ denote the solution from (9), i.e., the optimal capacity induced by $T(x,z)$.

For truth-telling, we need the following first-order condition:

$$\frac{\partial \pi_s(x,K;c)}{\partial x} = \int_0^K \frac{\partial T(x,y)}{\partial x}\phi(y)dy + \int_K^\infty \frac{\partial T(x,K)}{\partial x}\phi(y)dy = 0 \tag{10}$$

at $x = c$ for all c, where we have used $\int_K^\infty \frac{\partial T(x,K)}{\partial K}\phi(y)dy = c$ from (9).

The second-order condition for truth-telling is

$$\frac{\partial^2 \pi_s(x,K;c)}{\partial x^2} = \int_0^K \frac{\partial^2 T(x,y)}{\partial x^2}\phi(y)dy + \int_K^\infty \left[\frac{\partial T^2(x,K)}{\partial x^2} + \frac{\partial T^2(x,K)}{\partial x \partial K}K'(x;c)\right]\phi(y)dy \leq 0$$

at $x = c$ for all c, where

$$K'(x;c) = \frac{\partial K(x;c)}{\partial x} = -\frac{\int_K^\infty \frac{\partial T^2(x,K)}{\partial x \partial K}\phi(y)dy}{\int_K^\infty \frac{\partial^2 T(x,K)}{\partial K^2}\phi(y)dy - \frac{\partial T(x,K)}{\partial K}\phi(K)}.$$

Thus we have

$$\int_0^K \frac{\partial^2 T(x,y)}{\partial x^2}\phi(y)dy + \int_K^\infty \left[\frac{\partial T^2(x,K)}{\partial x^2} + \frac{\partial T^2(x,K)}{\partial x \partial K}K'(x;x)\right]\phi(y)dy \leq 0 \tag{11}$$

for all x.

Differentiating the truth-telling condition in (10) gives:

$$\int_0^K \frac{\partial^2 T(x,y)}{\partial x^2}\phi(y)dy + \int_K^\infty \left[\frac{\partial T^2(x,K)}{\partial x^2} + \frac{\partial T^2(x,K)}{\partial x \partial K}K'(x;x)\right]\phi(y)dy = 0, \tag{12}$$

where

$$K'(x;x) = -\frac{\int_K^\infty \frac{\partial^2 T(x,K)}{\partial x \partial K}\phi(y)dy - 1}{\int_K^\infty \frac{\partial^2 T(x,K)}{\partial K^2}\phi(y)dy - \frac{\partial T(x,K)}{\partial K}\phi(K)}.$$

From (11) and (12) we have:

$$\int_K^\infty \phi(y)dy \left(\frac{\partial T^2(x,K)}{\partial x \partial K}\right) \frac{-1}{\int_K^\infty \frac{\partial^2 T(x,K)}{\partial K^2}\phi(y)dy - \frac{\partial T(x,K)}{\partial K}\phi(K)} \leq 0, \tag{13}$$

which simplifies to the condition in Lemma 1(i).

(ii) Again, under truth-telling, we may write notation $(x;x)$ as (x) whenever appropriate, e.g., the supplier's profit function $\pi_s(x,K;x)$ will become $\pi_s(x,K)$, the buyer's profit function $\pi_b(x;x)$ will become $\pi_b(x)$, and the supplier's capacity $K(x;x)$ will become $K(x)$. The firms' profit functions are given by

$$\begin{aligned} \pi_s(x,K) &= E_D\left[T(x,z)\right] - xK(x), \\ \pi_b(x) &= pE_D(z) - xK(x) - \pi_s(x,K). \end{aligned}$$

Taking derivative of $\pi_s(x,K)$ with respect to x and comparing to (10) gives:

$$\frac{\partial \pi_s(x,K)}{\partial x} = \pi_s'(x) = -K(x),$$

where for concision we use $\pi_s'(x)$ for the partial derivative of $\pi_s(x,K)$ with respect to x.

The buyer's problem can be written as

$$\max_{\{\pi_s(\cdot),T(x,\cdot)\}} \int_0^{\bar{c}} [pE_D\left(\min(K,D)\right) - xK(x) - \pi_s(x)]f(x)dx$$

$$\begin{aligned} s.t. \qquad & \pi_s'(x) = -K(x), \\ & K(x) \text{ satisfies } \int_K^\infty \frac{\partial T(x,K)}{\partial K}\phi(y)dy - x = 0, \\ & \pi_s(\bar{c}) \geq 0. \end{aligned}$$

We can form the Hamiltonian as follows:

$$J(K,\pi_s,\lambda) = [pE_D\left(\min(K,D)\right) - xK(x) - \pi_s(x)]f(x) - \lambda(x)K(x),$$

where $\lambda(x)$ is the multiplier to be determined. By the Pontryagin's principle we know $\lambda' = -\frac{\partial J}{\partial \pi_s} = f(x)$. Together with the transversality condition we know $\lambda = F(x)$. For optimality we must have

$$\frac{\partial J}{\partial K} = [p\bar{\Phi}(K(x)) - x]f(x) - F(x) = 0,$$

which implies

$$K^o(x) = \Phi^{-1}\left(\frac{p - (x + F(x)/f(x))}{p}\right).$$

That is, it is necessary for the payment schedule $T(x,z)$ to induce the above capacity $K^o(x)$, which is the condition in Lemma 1(ii). □

Proof of Proposition 2: We can write the supplier's profit under the contract $T(x,z) = w(x)z + \hat{T}(x)$ as:

$$\begin{aligned} \pi_s(x;c) &= w(x)E_D[\min(K(x;c),D)] - cK(x;c) + \hat{T}(x) \\ &= w(x)\int_0^{K(x;c)} \overline{\Phi}(u)du - cK(x;c) + \hat{T}(x), \end{aligned}$$

where $K(x;c) = \Phi^{-1}\left(\frac{w(x)-c}{w(x)}\right)$. For truth-telling, we need the following first-order condition:

$$\begin{aligned} \pi_s'(x;c) &= w'(x)\int_0^{K(x;c)} \overline{\Phi}(u)du + w(x)\overline{\Phi}(K(x;c))K'(x;c) - cK'(x;c) + \hat{T}'(x) \\ &= w'(x)\int_0^{K(x;c)} \overline{\Phi}(u)du + \hat{T}'(x) = 0 \end{aligned}$$

at $x = c$ for all c, where we have used $w(x)\overline{\Phi}(K(x;c)) = c$. This implies

$$w'(x)\int_0^{K(x;x)} \overline{\Phi}(u)du + \hat{T}'(x) = 0 \tag{14}$$

for all x.

Under the linear payment schedule, the second-order condition for truth-telling in (13) simplifies to:

$$\frac{w'(x)}{-w(x)\phi(K)} \geq 0, \tag{15}$$

which is equivalent to $w'(x) \leq 0$. It can be shown $\frac{\partial}{\partial x}\left(\frac{\partial \pi_s/\partial K}{\partial \pi_s/\partial T}\right) = w'(x)\overline{\Phi}(K) - 1 < 0$, where the inequality is from the local incentive constraint $w'(x) \leq 0$. This implies that the Spence-Mirrlees single-crossing condition holds. Thus the local incentive constraint is sufficient for truth-telling.

Again, we write the notation $(x;x)$ as (x) under truth-telling. Then the supplier's profit is given by

$$\pi_s(x;x) = \pi_s(x) = w(x)E_D[\min(K(x), D)] - xK(x) + \hat{T}(x),$$

and the buyer's profit is given by

$$\begin{aligned}\pi_b(x;x) &= \pi_b(x) = (p - w(x))E_D[\min(K(x), D)] - \hat{T}(x)\\ &= pE_D[\min(K(x), D)] - xK(x) - \pi_s(x).\end{aligned}$$

Taking derivative of $\pi_s(x)$ and comparing to the first-order condition (14) give $\pi_s'(x) = -K(x)$. Thus the buyer's problem can be written as:

$$\max_{\{\pi_s(\cdot), w(\cdot)\}} \int_0^{\overline{c}} \{pE_D[\min(K(x), D)] - xK(x) - \pi_s(x)\}f(x)dx \tag{16}$$

$$\begin{aligned} s.t.\ \ \pi_s'(x) &= -K(x), && (17)\\ w'(x) &\leq 0, &&\\ K(x) &= \Phi^{-1}\left(\frac{w(x) - x}{w(x)}\right), &&\\ \pi_s(\overline{c}) &\geq 0. && (18)\end{aligned}$$

First we form the Hamiltonian as follows (for now we ignore the $w'(x) \leq 0$ constraint; later we will show that it is satisfied if and only if $H'(x) \leq 0$):

$$J(K, \pi_s, \lambda) = \{pE_D[\min(K(x), D)] - xK(x) - \pi_s(x)\}f(x) - \lambda(x)K(x),$$

where $\lambda(x)$ is a multiplier to be determined. By the Pontryagin's principle we know $\lambda' = -\frac{\partial J}{\partial \pi_s} = f(x)$. Together with the transversality condition we know $\lambda = F(x)$. For optimality we must have:

$$\frac{\partial J}{\partial K} = \{p\overline{\Phi}(K(x)) - x\}f(x) - F(x) = 0,$$

which implies

$$K^o(x) = \Phi^{-1}\left(\frac{p - (x + F(x)/f(x))}{p}\right).$$

Since F is log-concave, $p - \left(x + \frac{F(x)}{f(x)}\right)$ is a decreasing function of x. Define $\hat{c}^o$ as in (6). Then there is $K^o(x) = 0$ for $x > \hat{c}^o$, i.e., the buyer does not transact with the supplier. From $K^o(x) = \Phi^{-1}\left(\frac{p-(x+F(x)/f(x))}{p}\right) = \Phi^{-1}\left(\frac{w^o(x)-x}{w^o(x)}\right)$, we can solve $w^o(x)$ as:

$$w^o(x) = \frac{pH(x)}{1 + H(x)}. \tag{19}$$

Recall the local incentive constraint for truth-telling requires

$$(w^o(x))' = \frac{pH'(x)}{(1 + H(x))^2} \leq 0.$$

Thus there must be $H'(x) \leq 0$, or $H(x) = \frac{xf(x)}{F(x)}$ must be decreasing in x.

Finally, we can derive the profit function $\pi_s(x)$ from (17) and (18): $\pi_s(x) = \int_x^{\bar{c}} K^o(u)du$. Then the optimal transfer payment is:

$$\hat{T}^o(x) = xK^o(x) + \int_x^{\bar{c}} K^o(u)du - w^o(x)E_D[\min(K^o(x), D)]. \square$$

Proof of Proposition 3: For the first part, we only need to verify that the two conditions in Lemma 1 cannot be satisfied simultaneously by a linear payment schedule. Consider $T(x, z) = w(x)z + \hat{T}(x)$. Plugging it into (13) of Lemma 1(i) gives

$$\frac{w'(x)}{-w(x)\phi(K)} \geq 0, \text{ or } w'(x) \leq 0.$$

By Lemma 1(ii), we have

$$w(x) = \frac{px}{x + F(x)/f(x)} = \frac{pH(x)}{1 + H(x)}.$$

Clearly, $w'(x) \leq 0$ if and only if $H'(x) \leq 0$. This implies that the truth-telling (or incentive compatibility constraint) cannot be satisfied by the linear payment schedule when $H'(x) > 0$ for some x.

The proof of the second part is by construction. We need to search for quadratic payment schedules that satisfy the two conditions in Lemma 1. Consider the payment schedule $T(x, z) = w_1 z^2 + w_2(x)z + \hat{T}(x)$, where w_1 is independent of x. Then we have

$$\frac{\partial T(x, K)}{\partial K} = 2w_1 K + w_2(x), \quad \frac{\partial^2 T(x, K)}{\partial x \partial K} = w_2'(x), \text{ and } \frac{\partial^2 T(x, K)}{\partial K^2} = 2w_1.$$

From the first-order condition $\frac{\partial T}{\partial K} = \frac{x}{1-\Phi(K)}$, condition (13) can be written as:

$$\frac{-(1 - \Phi(K))w_2'(x)}{2w_1(1 - \Phi(K)) - \frac{x\phi(K)}{1-\Phi(K)}} \leq 0,$$

or

$$\frac{-w_2'(x)}{2w_1 - \frac{x\phi(K)}{(1-\Phi(K))^2}} \leq 0. \tag{20}$$

To satisfy the condition in Lemma 1(ii), we also need

$$\frac{\partial T(x, K)}{\partial K} = 2w_1 K + w_2(x) = w^o(x) = \frac{pH(x)}{1 + H(x)}, \tag{21}$$

where $w^o(x)$ is the wholesale price that induces the optimal capacity $K^o(x)$. Note from (21) we can write $w_2'(x)$ as

$$w_2'(x) = (w^o(x))' - 2w_1 (K^o(x))'. \tag{22}$$

The denominator of (20) is negative if $w_1 \leq 0$. So for condition (20) to hold, it is sufficient to find $w_1 \leq 0$ such that $w_2'(x) = (w^o(x))' - 2w_1 (K^o(x))' \leq 0$. Define $\underline{w} = \min\left\{\frac{(w^o(x))'}{2(K^o(x))'}, x \in [0, 1]\right\}$. Then we can set $w_1 = \min(0, \underline{w})$ to satisfy (20). Given such w_1, we can readily solve for $w_2(x)$ so that together w_1 and $w_2(x)$ induce $K^o(x)$ from the supplier.

Finally, from the proof of Proposition 2, the supplier's expected profit (information rent) is given by $\pi_s(x) = \int_x^{\bar{c}} K^o(u)du$, which only depends on the optimal capacity function $K^o(x)$. Thus we know that the buyer's optimal profit under the quadratic schedule will be the same as those in Propositions 2. That is, the quadratic payment schedule can achieve the second-best outcome for the buyer. $\square$

Proof of Proposition 4: The first part of the proof is similar to that of Proposition 2. We start with the buyer's problem as in (16). Using $\pi_s(x) = \int_x^{\bar{c}} K^o(u)du$ from (17) and integration by parts, we can rewrite the buyer's problem as

$$\max_{\{w(\cdot)\}} \int_0^{\bar{c}} \left\{ pE_D[\min(K(x), D)] - xK(x) - K(x)\frac{F(x)}{f(x)} \right\} f(x)dx$$

$$\begin{aligned} s.t.\ w'(x) &\leq 0, \\ K(x) &= \Phi^{-1}\left(\frac{w(x)-x}{w(x)}\right), \\ \pi_s(\bar{c}) &\geq 0. \end{aligned}$$

We may replace $w'(x) \leq 0$ with the following two constraints:

$$\begin{aligned} w'(x) &= y(x), \\ y(x) &\leq 0. \end{aligned}$$

Define the Hamiltonian as follows:

$$J(w, \mu) = \{pE_D[\min(K(x), D)] - xK(x)\}f(x) - F(x)K(x) + \mu(x)y(x),$$

where $\mu(x)$ is the Pontryagin's multiplier. By the Pontryagin's principle we know:

$$\mu'(x) = -\frac{\partial J}{\partial w} = -\frac{dK(x)}{dw(x)}\{[p\overline{\Phi}(K(x)) - x]f(x) - F(x)\},$$

where $\frac{dK(x)}{dw(x)} = \frac{x}{\phi(K(x))w^2(x)}$. So we can write $\mu'(x)$ as:

$$\mu'(x) = -\frac{x}{\phi(K(x))\,(w(x))^2}\{[p\overline{\Phi}(K(x)) - x]f(x) - F(x)\}.$$

The transversality condition states $\mu(0) = \mu(\bar{c}) = 0$.

Suppose the monotonicity constraint $w'(x) = y(x) \leq 0$ is not binding. Then $\mu(x)$ must be equal to zero and thus $\mu'(x) = 0$:

$$[p\overline{\Phi}(K(x)) - x]f(x) - F(x) = 0,$$

which gives:

$$K^o(x) = \Phi^{-1}\left(\frac{p - (x + F(x)/f(x))}{p}\right).$$

Again we need $x \leq \hat{c}^o$ for $K^o(x) \geq 0$. Thus when the monotonicity constraint $w'(x) = y(x) \leq 0$ is not binding, the optimal solution must be the same as in Proposition 2. From the proof of Proposition 2, we know $y(x) = w'(x) < 0$ requires $H'(x) < 0$, which does not hold throughout the support.

Now suppose there is an interior interval $[c_1, c_2]$ where the monotonicity constraint $w'(x) \leq 0$ is binding. This means that $w(x)$ must be a constant, say, w^o in $[c_1, c_2]$. Because the constraint monotonicity is not binding to the left of c_1 and to the right of c_2, from the continuity of the Pontryagin's multiplier (Laffont and Martimort 2002, Chapter 3), we must have:

$$\mu(c_1) = \mu(c_2) = 0.$$

Integrating between c_1 and c_2 we get:

$$\int_{c_1}^{c_2} \frac{x}{\phi(K(x))(w^o)^2}[(p\overline{\Phi}(K(x)) - x)f(x) - F(x)]dx = 0. \tag{23}$$

We can solve c_1, c_2, and w^o using (23) and the following two equations from (19):

$$\begin{aligned} w^o &= \frac{pH(c_1)}{1 + H(c_1)}, \\ w^o &= \frac{pH(c_2)}{1 + H(c_2)}. \end{aligned}$$

Note that c_1 (c_2) may coincide with the lower bound (upper bound) of the support; in this case, we replace one of the above two equations with $c_1 = 0$ or $c_2 = \bar{c}$ ($c_2 = \hat{c}^o$ if $\hat{c}^o < \bar{c}$).

To obtain the optimal transfer payment, notice that the supplier's profit function is

$$\pi_s(x) = w(x)E_D[\min(K(x), D)] - xK(x) + T(x) = \int_x^{\bar{c}} K(u)du,$$

and the supplier with cost c_2 must be indifferent between accepting contract $\{w^o, T^o\}$ and $\{w(c_2), T(c_2)\}$. Therefore,

$$T^o = c_2K(c_2) - w^o E_D[\min(K(c_2), D)] + \int_{c_2}^{\bar{c}} K(u)du,$$

where $K(x) = \Phi^{-1}\left(\frac{w^o - x}{w^o}\right)$. □

Proof of Proposition 5: If $H'(x) = 0$, then $w^o(x) = \frac{pH(x)}{1+H(x)}$ must be a constant. This means $T^o(x)$ must also be a constant (otherwise all supplier types will have incentives to report the cost with the highest $T^o(x)$). So the optimal menu of contracts reduces to two numbers $\{w^o, T^o\}$. Suppose $H(x) = \frac{xf(x)}{F(x)} = \eta$, or

$$\frac{f(x)}{F(x)} = \frac{\eta}{x},$$

where η is some positive number. Manipulation of the above equation yields:

$$\frac{d}{dx}\ln F(x) = \eta\left(\frac{d}{dx}\ln \gamma x\right) = \frac{d}{dx}\ln(\gamma x)^\eta,$$

where γ is a constant. Thus we have $F(x) = \gamma^\eta x^\eta$, or $f(x) = \delta x^{\eta-1}$, where $\delta = \eta\gamma^\eta$ is also a positive number.

Suppose $\hat{c}^o \leq \bar{c}$. Then we have

$$K^o(\hat{c}^o) = \Phi^{-1}\left(\frac{p - (\hat{c}^o + F(\hat{c}^o)/f(\hat{c}^o))}{p}\right) = 0$$

by the definition of $\hat{c}^o$. Meanwhile, since $K^o(\hat{c}^o) = \Phi^{-1}\left(\frac{w^o - \hat{c}^o}{w^o}\right) = 0$, there must be $w^o = \hat{c}^o$. Plugging $K^o(\hat{c}^o)$ into (15) gives $T^o = T^o(\hat{c}^o) = 0$. □

Proof of Proposition 6: Substituting the optimal capacities and wholesale prices in Proposition (1) and (4) gives:

$$\begin{aligned}\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S} &= \Bigg\{\theta\bar{c}^3(\eta-1)\{\alpha\left(-2+(\eta-1)(2+3\alpha\bar{c}(-\frac{2}{\alpha\bar{c}+\delta\bar{c}^\eta}+\frac{1}{\alpha\bar{c}+\frac{3\delta\bar{c}^\eta}{2+\eta}}))\right)+\\ &\alpha(\eta+1)Hypergeometric2F1\left[1,\frac{3}{\eta-1},\frac{\eta+2}{\eta-1},-\frac{\delta\bar{c}^{\eta-1}}{\alpha}\right]+\\ &6\delta\bar{c}^{\eta-1}Hypergeometric2F1\left[1,\frac{\eta+2}{\eta-1},2+\frac{3}{\eta-1},-\frac{\delta\bar{c}^{\eta-1}}{\alpha}\right]\}\Bigg\}/\\ &\{3\eta p(\alpha\eta\bar{c}+\delta\bar{c}^\eta)\Bigg(\beta(p-\bar{c})+\frac{\alpha\bar{c}}{6\eta p}(\frac{6\bar{c}(\eta+1)^2}{\eta+2}+\alpha\bar{c}^2(\eta-1)\\ &(\frac{12}{\alpha\bar{c}+\delta\bar{c}^\eta}-\frac{\eta^2}{(\eta+2)(\alpha\bar{c}\eta+\delta\bar{c}^\eta)}-\frac{9(\eta+2)}{\alpha\bar{c}(\eta+2)+3\delta\bar{c}^\eta})-6p\eta)\Bigg)\}.\end{aligned}$$

Where $Hypergeometric2F1$ is defined as

$$Hypergeometric2F1(a,b,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where

$$(q)_i=\begin{cases}1 & i=0\\ q(q+1)...(q+i-1) & i>0\end{cases}.$$

First we show that $\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S}$ is decreasing in β. Notice that the numerator is independent of β. So it suffices to show that the denominator is increasing in β. Take the first-order derivative of the denominator with respect to β, and then we need to show

$$3p\eta(\delta\bar{c}^\eta+\alpha\eta\bar{c})(p-\bar{c})/\bar{c}\geq 0.$$

This inequality can be readily established as a result of the implicit assumption that $p>\bar{c}$. This establishes part (i).

Next we show that the performance of the simple contract deteriorates in θ, i.e., $\frac{d}{d\theta}\left(\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S}\right)>0$. Taking the first-order derivative of $\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S}$ with respect to θ gives:

$$\begin{aligned}\frac{d}{d\theta}\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S} &= \Bigg\{\beta\bar{c}^3(\eta-1)(p-\bar{c})\{\alpha\left(-2+(\eta-1)(2+3\alpha\bar{c}(-\frac{2}{\alpha\bar{c}+\delta\bar{c}^\eta}+\frac{1}{\alpha\bar{c}+\frac{3\delta\bar{c}^\eta}{2+\eta}}))\right)+\\ &\alpha(\eta+1)Hypergeometric2F1\left[1,\frac{3}{\eta-1},\frac{\eta+2}{\eta-1},-\frac{\delta\bar{c}^{\eta-1}}{\alpha}\right]+\\ &6\delta\bar{c}^{\eta-1}Hypergeometric2F1\left[1,\frac{\eta+2}{\eta-1},2+\frac{3}{\eta-1},-\frac{\delta\bar{c}^{\eta-1}}{\alpha}\right]\}\Bigg\}/\\ &\{3\eta p(\alpha\eta\bar{c}+\delta\bar{c}^\eta)\Bigg(\beta(p-\bar{c})+\frac{\alpha\bar{c}}{6\eta p}(\frac{6\bar{c}(\eta+1)^2}{\eta+2}+\alpha\bar{c}^2(\eta-1)\\ &(\frac{12}{\alpha\bar{c}+\delta\bar{c}^\eta}-\frac{\eta^2}{(\eta+2)(\alpha\bar{c}\eta+\delta\bar{c}^\eta)}-\frac{9(\eta+2)}{\alpha\bar{c}(\eta+2)+3\delta\bar{c}^\eta})-6p\eta)\Bigg)^2\}\geq 0.\end{aligned}$$

The inequality is a direct result of the fact $\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S}\geq 0$ and our assumption $p>\bar{c}$. This concludes the proof. □

Proof of Proposition 7: Given the density function $f(x) = \alpha + \delta x$, there is $F(x) = \alpha x + \frac{1}{2}\delta x^2$. From $F(1) = 1$ we have $\alpha = 1 - \frac{\delta}{2}$. Since $\alpha \geq 0$ and $\alpha + \delta > 0$, there must be $-2 < \delta \leq 2$. Note

$$\frac{F(x)}{f(x)} = \frac{\alpha x + \frac{1}{2}\delta x^2}{\alpha + \delta x} \text{ and } H(x) = \frac{x(\alpha + \delta x)}{\alpha x + \frac{1}{2}\delta x^2}.$$

We can verify that F is always log-concave and there is $H'(x) > 0$ if and only if $\delta > 0$. In the following analysis, we assume $\hat{c}^o \geq \bar{c} = 1$, i.e., the buyer does not exclude any supplier type in the optimal mechanism.

Consider the first part of the proposition for $0 \leq \delta \leq 2$. The optimal menu of contracts under forced compliance is characterized in Proposition 1. Through calculations we find the buyer's expected profit to be

$$\pi_b^{OM} = \frac{4\alpha^4 \left(\ln(\alpha+\delta) - \ln\alpha\right) - 4\alpha^3\delta + 2\alpha^2\delta^2 + \left(16p^2\alpha + 20\alpha - 32\alpha p\right)\delta^3 + \left(9 - 16p + 8p^2\right)\delta^4}{32p\delta^3}.$$

To simplify this complex expression, we can write the profit function as

$$\begin{aligned}
\pi_b^{OM} &= \frac{4\alpha^4 \left(\ln(\alpha+\delta) - \ln\alpha\right) - 4\alpha^3\delta + 2\alpha^2\delta^2}{32p\delta^3} + \frac{\left(16p^2\alpha + 20\alpha - 32\alpha p\right)\delta^3 + \left(9 - 16p + 8p^2\right)\delta^4}{32p\delta^3} \\
&= \frac{\alpha}{16p}\left(\frac{\alpha}{\delta} + 2\frac{\alpha^2}{\delta^2}\frac{\ln(1+\delta/\alpha)}{\delta/\alpha} - 2\frac{\alpha^2}{\delta^2}\right) + \frac{\left(16p^2\alpha + 20\alpha - 32\alpha p\right)\delta^3 + \left(9 - 16p + 8p^2\right)\delta^4}{32p\delta^3} \\
&\leq \frac{\alpha}{16p}\left(\frac{2}{3}\right) + \frac{\left(16p^2\alpha + 20\alpha - 32\alpha p\right)\delta^3 + \left(9 - 16p + 8p^2\right)\delta^4}{32p\delta^3} \\
&= \frac{1}{96}\frac{48p^2\alpha + 64\alpha + 24p^2\delta + 27\delta - 48p\delta - 96\alpha p}{p},
\end{aligned}$$

where the inequality follows from $\left(\frac{\alpha}{\delta} + 2\frac{\alpha^2}{\delta^2}\frac{\ln(1+\delta/\alpha)}{\delta/\alpha} - 2\frac{\alpha^2}{\delta^2}\right) \leq \frac{2}{3}$ because we can show the term in the parentheses approaches $\frac{2}{3}$ as $\frac{\delta}{\alpha}$ goes to 0. So the bound $\frac{2}{3}$ is tight at $\delta = 0$. Plugging in $\alpha = 1 - \frac{\delta}{2}$ gives

$$\pi_b^{OM} \leq \frac{1}{96}\frac{48p^2 - 96p + 64 - 5\delta}{p}.$$

Next we evaluate the buyer's optimal profit under the single two-part tariff $\{w, T\}$. It can be shown that the optimal wholesale price is given by

$$w^o = 2p\left(\frac{3\delta + 4\alpha}{9\delta + 16\alpha}\right).$$

Then the buyer's profit can be derived to be

$$\begin{aligned}
\pi_b^S &= \frac{(3\delta + 8\alpha)\left(-256\alpha^2 + 192p^2\alpha^2 - 288\alpha\delta + 240p^2\alpha\delta - 81\delta^2 + 72p^2\delta^2\right)}{96p(3\delta + 4\alpha)(9\delta + 16\alpha)} \\
&\quad - \left(\frac{-36p^2\delta^2 - 96p^2\alpha\delta + 108p\delta^2 + 336p\alpha\delta - 64p^2\alpha^2 + 256p\alpha^2 - 81\delta^2 - 288\alpha\delta - 256\alpha^2}{4p(3\delta + 4\alpha)(9\delta + 16\alpha)}\right).
\end{aligned}$$

Plugging in $\alpha = 1 - \frac{\delta}{2}$ gives

$$\pi_b^S = \frac{\delta^2 - 96p\delta + 48p^2\delta + 32\delta + 192p^2 - 384p + 256}{96p(\delta + 4)}.$$

Define

$$r=\frac{\pi_b^{OM}}{\pi_b^S}=\frac{(\delta+4)\left(48p^2-96p+64-5\delta\right)}{\delta^2-96p\delta+48p^2\delta+32\delta+192p^2-384p+256},$$

i.e., $\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S}=r-1$. By taking derivative of r with respect to δ, we can show that r first increases in δ and then decreases in δ.

Taking derivative of r with respect to p gives

$$\frac{dr}{dp}=\frac{576\delta\,(\delta+4)\,(p-1)\,(\delta-2)}{\left(\delta^2-96p\delta+48p^2\delta+32\delta+192p^2-384p+256\right)^2}.$$

Since we have assumed $\hat{c}^o\geq\bar{c}=1$, there must be $p\geq 1+\frac{1}{\alpha+\delta}=1+\frac{1}{1+\delta/2}\geq\frac{3}{2}$. Thus we know $\frac{dr}{dp}\leq 0$, i.e., $\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S}$ decreases in p (or profit margin). Recall it has been assumed $w^o=2p\left(\frac{3\delta+4\alpha}{9\delta+16\alpha}\right)\geq 1$, from which we can derive $p\geq 2$. To obtain an upper bound, we may consider $p=2$, which is the smallest possible price that guarantees $\hat{c}^o\geq 1$ and $w^o\geq 1$ for all δ. The maximum r under $p=2$ should be an upper bound (for any other δ that gives $p>2$, the ratio would be smaller under $p=2$). With $p=2$, the ratio becomes $r=\frac{(\delta+4)(64-5\delta)}{\delta^2+32\delta+256}$.

Taking derivative with respect to δ gives $\frac{dr}{d\delta}=\frac{12(16-17\delta)}{(\delta+16)\left(\delta^2+32\delta+256\right)}$, which is positive first and then negative. So the ratio r has a unique maximizer that is achieved at $\delta=\frac{16}{17}$. Thus an upper bound for $\frac{\pi_b^{OM}-\pi_b^S}{\pi_b^S}$ is $r-1=\frac{1}{48}=0.02083$.

Now we consider the second part of the proposition for $-1\leq\delta<0$. Recall that for $\hat{c}^o>\bar{c}=1$, there must be $p\geq 1+\frac{1}{\alpha+\delta}$. When δ approaches -2, $\frac{1}{\alpha+\delta}$ approaches infinity, which means p must approach infinity. For simplicity, we derive the bound for $\delta\geq -1$; the same process applies to any $\delta\in(-2,-1)$. Given $\delta\geq -1$, there must be $p\geq 1+\frac{1}{\alpha+\delta}=1+\frac{1}{1+\delta/2}\geq 3$. The buyer's optimal profit under forced compliance is derived to be

$$\begin{aligned}\pi_b^{OM} &= -\frac{1}{32}\frac{-16p^2\delta^3\alpha-20\alpha\delta^3-8p^2\delta^4-9\delta^4+16p\delta^4+32\alpha p\delta^3}{p\delta^3}\\ &\quad+\frac{\alpha}{16p}\left(\frac{\alpha}{\delta}+2\frac{\alpha^2}{\delta^2}\frac{\ln(1+\delta/\alpha)}{\delta/\alpha}-2\frac{\alpha^2}{\delta^2}\right).\end{aligned}$$

Let $z=\frac{\delta}{\alpha}$, then $z=-\frac{2}{3}$ at $\delta=-1$. We can write

$$\frac{\alpha}{\delta}+2\frac{\alpha^2}{\delta^2}\frac{\ln(1+\delta/\alpha)}{\delta/\alpha}-2\frac{\alpha^2}{\delta^2}=\frac{1}{z}+2\frac{1}{z^2}\frac{\ln(1+z)}{z}-2\frac{1}{z^2}$$

and show that the function is decreasing in z for $z<0$. So the maximum of the function is $1.4156<\frac{3}{2}$ achieved at $z=-\frac{2}{3}$. So we can use $\frac{3}{2}$ to replace the function. Thus we have

$$\begin{aligned}\pi_b^{OM} &< -\frac{1}{32}\frac{-16p^2\delta^3\alpha-20\alpha\delta^3-8p^2\delta^4-9\delta^4+16p\delta^4+32\alpha p\delta^3}{p\delta^3}+\frac{\alpha}{16p}\left(\frac{3}{2}\right)\\ &= \frac{1}{64}\frac{32p^2+46-5\delta-64p}{p}.\end{aligned}$$

From earlier analysis, we know under the two-part tariff $\{w,T\}$, the optimal wholesale price is $w^o=2p\left(\frac{3\delta+4\alpha}{9\delta+16\alpha}\right)$ and the buyer's profit is given by

$$\pi_b^S=\frac{\delta^2-96p\delta+48p^2\delta+32\delta+192p^2-384p+256}{96p\,(\delta+4)}.$$

Thus the ratio between the two profits π_b^{AS} and π_b^S is:

$$r = \frac{\pi_b^{OM}}{\pi_b^S} = \frac{3}{2}\frac{\left(32p^2 + 46 - 5\delta - 64p\right)(4+\delta)}{192p^2 + 48p^2\delta - 384p - 96p\delta + 256 + 32\delta + \delta^2}.$$

Taking derivative with respect to p we can show $\frac{dr}{dp} < 0$ for $p \geq 3$. Since the ratio decreases in p, next we consider $p = 3$, which gives the highest ratio for any δ. Then the ratio becomes

$$r = \frac{3}{2}\frac{(4+\delta)(142-5\delta)}{832 + 176\delta + \delta^2}.$$

It is straightforward to show that the ratio achieves its maximum value $r = \frac{213}{208} = 1.02404$ at $\delta = 0$. Thus an upper bound of $\frac{\pi_b^{AS} - \pi_b^S}{\pi_b^S}$ is $r - 1 = \frac{5}{208} = 0.02404$ for $-1 \leq \delta \leq 0$. $\square$

Proof of Proposition 8: Proof is similar to Proposition (7) and therefore omitted.$\square$

Proof of Lemma 2: We can write supplier's expected profit as $\pi_s^S = w^o E_D[\min(K^s(x), D)] - xK^s(x) - T^o = w^o \int_0^{K^s(x)} y\phi(y)dy + w^o(1 - \Phi(K^s(x)))K^s(x) - xK^s(x) - T^o$. Since w^o induces the optimal capacity $K^s(x)$, we have

$$\pi_s^S = w^o \int_0^{K^s(x)} y\phi(y)dy - T^o.$$

Substituting from (19) we can show that $T^o = w^o \int_0^{K^s(\bar{c})} y\phi(y)dy$ and therefore $\pi_s^S = w^o \int_{K^s(\bar{c})}^{K^s(x)} y\phi(y)dy$. Then a simple change of variable, i.e., $y = \Phi^{-1}(1 - \frac{u}{w^o})$, results in $w^o \int_{K^s(\bar{c})}^{K^s(x)} y\phi(y)dy = \int_x^{\bar{c}} K^s(u)du$. This concludes the proof. $\square$