Electronic Companion

This electronic companion contains all the proofs of our theoretical results and other supplementary material.

Appendix A: Proofs

Proof of Lemma 1 Assume $\mathbf{x} = (x_1, x_2, ..., x_n)$ is a schedule with at least one slot with less than s customers assigned to it, and let t_0 be the first such slot, i.e., $t_0 = \min\{t : x_t < s\}$. Let $m = \sum_{t=1}^{n} x_t$. We will show via contradiction that $\mathbf{x}$ cannot be optimal.

Case 1: Assume $x_t \le s$ for all $t = 1, 2, ..., n$. Then $\Pr(A_t \le s) = 1$ for all $t = 1, 2, ..., n+1$, and therefore, from (1), $\Pr(Z_t = 0) = 1$ for all $t = 1, 2, ..., n+1$. From (3), (4) and (6) we get

$$\begin{aligned} &O(\mathbf{x}) = 0 \\ &I(\mathbf{x}) = ns - mp > 0 \\ &W(\mathbf{x}) = 0. \end{aligned} \tag{EC.1}$$

Consider now the schedule (see Figure EC.1)

$$\mathbf{x}' = (x_1, x_2, ..., x_{t_0-1}, x_{t_0} + 1, x_{t_0+1}, ..., x_n)$$

and let A'_t and Z'_t be the associated new arrivals and backlog at the beginning of slot t respectively. Then

$$\begin{aligned} &O(\mathbf{x}') = 0 \\ &I(\mathbf{x}') = ns - (m+1)p \ge 0 \\ &W(\mathbf{x}') = 0. \end{aligned} \tag{EC.2}$$

From (EC.1) and (EC.2), and by the assumption that $p > 0$, schedule $\mathbf{x}'$ is less costly than $\mathbf{x}$.

Case 2: Assume $x_t > s$ for for some $t \ne t_0$, i.e., there is at least one slot with more than s customers assigned to it.

Case 2a: Suppose that the first slot with more than s customers assigned to it appears after slot t_0, and let t_1 be that slot, i.e., $t_1 = \min\{t : x_t > s\}$ with $t_1 > t_0$ (see Figure EC.1). This implies that slots $1, 2, \ldots, t_0 - 1$ have at most s customers assigned to each one of them. Consider now the schedule

$$\hat{\mathbf{x}} = (x_1, x_2, ..., x_{t_0-1}, x_{t_0} + 1, x_{t_0+1}, ..., x_{t_1-1}, x_{t_1} - 1, x_{t_1+1}, ..., x_n).$$

Then

$$\Pr(\hat{Z}_t = 0) = \Pr(Z_t = 0) = 1 \text{ for all } t = 1, 2, ..., t_1,$$

and $\Pr(\hat{A}_{t_1} > a) < \Pr(A_{t_1} > a)$ for all $a \ge 0$, i.e., $\hat{A}_{t_1} <_{st} A_{t_1}$. Therefore, from (1),

$$\hat{Z}_{t_1+1} <_{st} Z_{t_1+1} \text{ and } \hat{Z}_t \le_{st} Z_t \text{ for all } t = t_1 + 2, ..., n+1,$$

concluding that

$$O(\hat{\mathbf{x}}) \le O(\mathbf{x}), \; I(\hat{\mathbf{x}}) \le I(\mathbf{x}), \text{ and } W(\hat{\mathbf{x}}) < W(\mathbf{x}).$$

Case 2b: Suppose that there is a slot prior to t_0 with more than s customers assigned to it. Let t_2 be the last slot before t_0 that has more than s customers assigned to it, i.e., $t_2 = \max\{t : x_t > s, t < t_0\}$ (see Figure EC.1). This implies that slots $t_2+1, \dots, t_0-1$ have exactly s customers assigned to each one of them. Consider now the schedule

$$\tilde{\mathbf{x}} = (x_1, x_2, ..., x_{t_2-1}, x_{t_2}-1, x_{t_2+1}, ..., x_{t_0-1}, x_{t_0}+1, x_{t_0+1}, ..., x_n).$$

Then $\tilde{x}_t = x_t$ for all $t = 1, 2, ..., t_2 - 1$ and therefore

$$\tilde{Z}_t \overset{d}{=} Z_t \text{ for all } t = 1, 2, ..., t_2. \tag{EC.3}$$

Note that $\tilde{x}_{t_2} = x_{t_2} - 1$ and hence $\Pr(\tilde{A}_{t_2} > a) < \Pr(A_{t_2} > a)$ for all $a \geq 0$, i.e., $\hat{A}_{t_2} <_{st} A_{t_2}$. Therefore, from (1),

$$\tilde{Z}_t <_{st} Z_t \text{ for all } t = t_2+1, t_2+2, ..., t_0. \tag{EC.4}$$

Next, we will show that $\tilde{Z}_t \overset{d}{=} Z_t$ for all $t = t_0+1, t_0+2, ..., n$. Since $\tilde{x}_t = x_t = s$ for all $t = t_2+1, t_2+2, ..., t_0-1$, we get that

$$Z_{t_0+1} = \max\{Z_{t_2} + \text{Bino}\left((t_0 - t_2 - 1)s + x_{t_2} + x_{t_0}, p\right) - (t_0 - t_2 + 1)s, 0\}$$
$$\tilde{Z}_{t_0+1} = \max\{\tilde{Z}_{t_2} + \text{Bino}\left((t_0 - t_2 - 1)s + \tilde{x}_{t_2} + \tilde{x}_{t_0}, p\right) - (t_0 - t_2 + 1)s, 0\}.$$

Since $\tilde{Z}_{t_2} \overset{d}{=} Z_{t_2}$ and $\tilde{x}_{t_2} + \tilde{x}_{t_0} = x_{t_2} + x_{t_0}$, we conclude that

$$Z_{t_0+1} \overset{d}{=} \tilde{Z}_{t_0+1}. \tag{EC.5}$$

Finally, since $\tilde{x}_t = x_t$ for all $t = t_0+1, t_0+2, ..., n$ we get that

$$\tilde{Z}_t \overset{d}{=} Z_t \text{ for all } t = t_0+2, ..., n, n+1. \tag{EC.6}$$

From (EC.3), (EC.4), (EC.5) and (EC.6) we conclude that

$$O(\tilde{\mathbf{x}}) = O(\mathbf{x}),\ I(\tilde{\mathbf{x}}) = I(\mathbf{x}),\ \text{and } W(\tilde{\mathbf{x}}) < W(\mathbf{x}).$$

□

Proof of Lemma 2 First, recall that $I(\mathbf{x}) = O(\mathbf{x}) + ns - mp$, and therefore a schedule minimizes the expected idle time if and only if it minimizes the expected overtime. It suffices to show that a schedule from class $\mathcal{A}$ minimizes the total expected overtime.

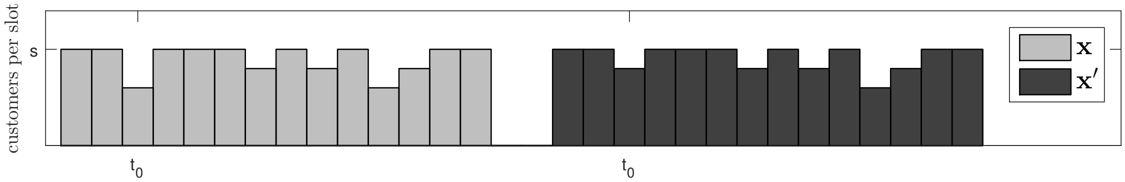
Suppose that for some realization of the arrival process, a out of m customers will actually show up. Then the total number of overtime slots must be at least $\max(a - ns, 0)$, for any schedule $\mathbf{x}$.

Suppose a schedule belongs to class $\mathcal{A}$ and that for some realization of the arrival process a out of m customers will actually show up. Then exactly $\min(a, ns)$ will be served during regular slots. The rest $a - \min(a, ns)$ will be served during overtime slots, and this is true for every a and for every realization of the arrival process with a arrivals. Note that $a - \min(a, ns) = \max(0, a - ns)$, the minimum possible number of overtime slots. Therefore, a schedule from class $\mathcal{A}$ minimizes the total expected overtime and idle time.

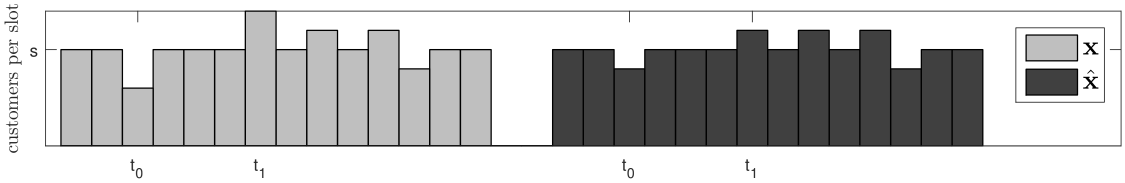
□

Figure EC.1 Proof of Lemma 1.

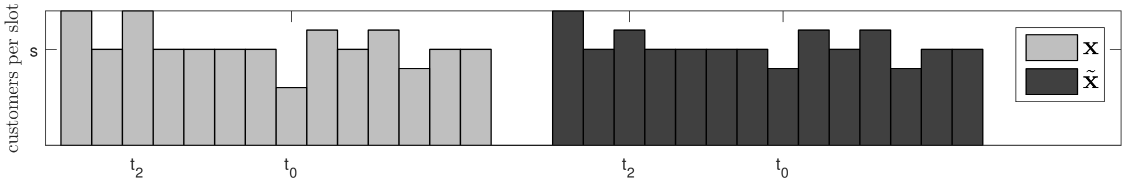
(a) Case 1



(b) Case 2a



(c) Case 2b



Proof of Theorem 1 (i) It is equivalent to show that $I(\mathbf{x}_y)$ is decreasing and discretely convex in $m = ns + y$ on $\{ns, ns+1, ...\}$. Firstly note that

$$\begin{aligned}
I(\mathbf{x}_{y+1}) &= \sum_{k=0}^{ns}(ns-k)\binom{m+1}{k}p^k(1-p)^{m+1-k} \\
&= \sum_{k=0}^{ns}(ns-k)\binom{m}{k}p^k(1-p)^{m+1-k} + \sum_{k=1}^{ns}(ns-k)\binom{m}{k-1}p^k(1-p)^{m+1-k} \\
&= (1-p)\sum_{k=0}^{ns}(ns-k)\binom{m}{k}p^k(1-p)^{m-k} + p\sum_{k=1}^{ns}(ns-k)\binom{m}{k-1}p^{k-1}(1-p)^{m-(k-1)} \\
&\overset{j=k-1}{=} (1-p)I(\mathbf{x}_y) + p\sum_{j=0}^{ns-1}[ns-(j+1)]\binom{m}{j}p^j(1-p)^{m-j} \\
&= (1-p)I(\mathbf{x}_y) + p\sum_{j=0}^{ns-1}(ns-j)\binom{m}{j}p^j(1-p)^{m-j} - p\sum_{j=0}^{ns-1}\binom{m}{j}p^j(1-p)^{m-j} \\
&= (1-p)I(\mathbf{x}_y) + p\sum_{j=0}^{ns}(ns-j)\binom{m}{j}p^j(1-p)^{m-j} - p\sum_{j=0}^{ns-1}\binom{m}{j}p^j(1-p)^{m-j} \\
&= (1-p)I(\mathbf{x}_y) + pI(\mathbf{x}_y) - p\sum_{j=0}^{ns-1}\binom{m}{j}p^j(1-p)^{m-j} \\
&= I(\mathbf{x}_y) - p\sum_{j=0}^{ns-1}\binom{m}{j}p^j(1-p)^{m-j},
\end{aligned}$$

concluding that

$$-p < I(\mathbf{x}_{y+1}) - I(\mathbf{x}_y) = -p\sum_{k=0}^{ns-1}\binom{m}{k}p^k(1-p)^{m-k} < 0. \tag{EC.7}$$

Therefore, $I(\mathbf{x}_y)$ is decreasing in $m = ns + y$.

For the proof of the discrete convexity it suffices to show that $I(\mathbf{x}_y)$ has increasing differences in m on $\{ns, ns+1, ...\}$. Let $P^m = \sum_{k=0}^{ns-1}\binom{m}{k}p^k(1-p)^{m-k}$ and note that

$$\begin{aligned}
P^{m+1} &= \sum_{k=0}^{ns-1}\binom{m+1}{k}p^k(1-p)^{m+1-k}\\
&= \sum_{j=0}^{ns-1}\binom{m}{k}p^k(1-p)^{m+1-k} + \sum_{k=1}^{ns-1}\binom{m}{k-1}p^k(1-p)^{m+1-k}\\
&= (1-p)\sum_{j=0}^{ns-1}\binom{m}{k}p^k(1-p)^{m-k} + p\sum_{k=1}^{ns-1}\binom{m}{k-1}p^{k-1}(1-p)^{m-(k-1)}\\
&\overset{j=k-1}{=} (1-p)P^m + p\sum_{j=0}^{ns-2}\binom{m}{j}p^j(1-p)^{m-j}\\
&= (1-p)P^m + pP^m - p\binom{m}{ns-1}p^{ns-1}(1-p)^{m-(ns-1)}\\
&< P^m.
\end{aligned} \tag{EC.8}$$

From (EC.7) and (EC.8)

$$[I(\mathbf{x}_{y+2}) - I(\mathbf{x}_{y+1})] - [I(\mathbf{x}_{y+1}) - I(\mathbf{x}_y)] = -p[P^{m+1} - P^m] > 0,$$

concluding that $I(\mathbf{x}_y)$ has increasing differences in m on $\{ns, ns+1, ...\}$.

(ii) Recall that $O(\mathbf{x}_y) = pm - ns + I(\mathbf{x}_y)$, and therefore

$$\begin{aligned}
O(\mathbf{x}_{y+1}) - O(\mathbf{x}_y) &= p[(m+1) - m] + [I(\mathbf{x}_{y+1}) - I(\mathbf{x}_y)]\\
&= p - pP^m\\
&= p(1 - P^m)\\
&> 0.
\end{aligned} \tag{EC.9}$$

It is straightforward to verify that $O(\mathbf{x}_y)$ has increasing differences in $m = ns + y$ on $\{ns, ns+1, ...\}$ as well, since $O(\mathbf{x}_y) = pm - ns + I(\mathbf{x}_y)$.

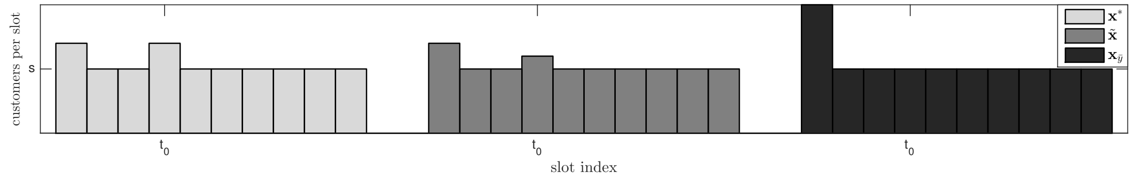
(iii) Let $\bar{y}$ be an optimal solution to $\mathbf{P_2}$ and let $\bar{m} = ns + \bar{y}$. Then $c_i I(\mathbf{x}_{\bar{y}+1}) + c_o O(\mathbf{x}_{\bar{y}+1}) \geq c_i I(\mathbf{x}_{\bar{y}}) + c_o O(\mathbf{x}_{\bar{y}})$, implying (from (EC.7) and (EC.9)) that

$$p\left[(c_i + c_o)P^{\bar{m}} - c_o\right] \leq 0, \tag{EC.10}$$

where $P^{\bar{m}} = \sum_{k=0}^{ns-1}\binom{\bar{m}}{k}p^k(1-p)^{\bar{m}-k}$ is the probability that the waiting time of the last customer of schedule $\mathbf{x}_{\bar{y}+1}$ is equal to zero.

Let $\mathbf{x}^*$ be an optimal solution to $\mathbf{P_1}$ with $m^* = \sum_{t=1}^n x_t^*$, and assume, for contradiction, that $y^* = m^* - ns > \bar{y}$. Let t_0 be the last slot under schedule $\mathbf{x}^*$ that has more than s customers assigned to it, i.e., $t_0 = \max\{t : x_t^* > s\}$. This implies that slots $t_0 + 1, \ldots, n$ have exactly s customers assigned to each one of them.

Figure EC.2 Proof of Theorem 1 (iii).



Consider now the schedule $\tilde{\mathbf{x}} = (x_1^*, x_2^*, ..., x_{t_0-1}^*, x_{t_0}^* - 1, x_{t_0+1}^*, ..., x_n^*)$, refer to Figure EC.2. We will show that $c_iI(\tilde{\mathbf{x}}) + c_oO(\tilde{\mathbf{x}}) + wW(\tilde{\mathbf{x}}) < c_iI(\mathbf{x}^*) + c_oO(\mathbf{x}^*) + wW(\mathbf{x}^*)$, under the assumption that $y^* > \bar{y}$.

Clearly $W(\mathbf{x}^*) > W(\tilde{\mathbf{x}})$. It suffices to show that $I(\tilde{\mathbf{x}}) + c_oO(\tilde{\mathbf{x}}) < I(\mathbf{x}^*) + c_oO(\mathbf{x}^*)$. Let $u = \sum_{t=t_0}^n \tilde{x}_t$ be the number of customers assigned to slots $t_0, ..., n$ under schedule $\tilde{\mathbf{x}}$. Let also B be the random variable denoting the backlog of customers at the beginning of slot t_0, which is stochastically the same under both schedules $\tilde{\mathbf{x}}$ and $\mathbf{x}^*$, and let b be realization of B. Then, using similar arguments as in (7), the expected overtime and idle time costs under schedules $\tilde{\mathbf{x}}$ and $\mathbf{x}^*$, subscripted by b, are

$$\begin{aligned}
O_b(\mathbf{x}^*) &= b + p(u+1) - (n - t_0 + 1)s + \sum_{k=0}^{(n-t_0+1)s-b} [(n - t_0 + 1)s - b - k] f(k; u+1, p) \\
O_b(\tilde{\mathbf{x}}) &= b + pu - (n - t_0 + 1)s + \sum_{k=0}^{(n-t_0+1)s-b} [(n - t_0 + 1)s - b - k] f(k; u, p) \\
I_b(\mathbf{x}^*) &= O_b(\mathbf{x}^*) + ns - pm^* \\
I_b(\tilde{\mathbf{x}}) &= O_b(\tilde{\mathbf{x}}) + ns - p(m^* - 1).
\end{aligned}$$

Therefore

$$c_iI(\tilde{\mathbf{x}}) + c_oO(\tilde{\mathbf{x}}) - c_iI(\mathbf{x}^*) - c_oO(\mathbf{x}^*) = p[(c_i + c_O)\tilde{P} - c_O], \tag{EC.11}$$

where

$$\tilde{P} = \sum_b \Pr(B = b) \sum_{k=0}^{(n-t_0+1)s-b-1} f(k; u, p)$$

is the probability that the waiting time of the last customer of schedule $\tilde{\mathbf{x}}$ is equal to zero. Using similar arguments as in the proof of Lemma 2, and by the assumption that $\sum_{t=1}^n \tilde{x}_t = m^* - 1 \geq \bar{m}$, we conclude that

$$\tilde{P} < P^{\bar{m}}. \tag{EC.12}$$

From (EC.10), (EC.11) and (EC.12) it follows that $c_iI(\tilde{\mathbf{x}}) + c_oO(\tilde{\mathbf{x}}) < c_iI(\mathbf{x}^*) + c_oO(\mathbf{x}^*)$, which is a contradiction. Therefore $y^* \leq \bar{y}$.

(iv) Assume that $c_o \geq \tilde{c}_o$ and let

$$y \in \bar{y}(c_o) := \arg\min_{y \in \mathbb{Z}_+} c_iI(\mathbf{x}_y) + c_oO(\mathbf{x}_y) \text{ and} \tag{EC.13}$$

$$\tilde{y} \in \bar{y}(\tilde{c}_o) := \arg\min_{y \in \mathbb{Z}_+} c_iI(\mathbf{x}_y) + \tilde{c}_oO(\mathbf{x}_y). \tag{EC.14}$$

We will show that $y \leq \tilde{y}$. From (EC.13) and (EC.14) we get

$$c_iI(\mathbf{x}_{\tilde{y}}) + c_oO(\mathbf{x}_{\tilde{y}}) \geq c_iI(\mathbf{x}_y) + c_oO(\mathbf{x}_y) \text{ and} \tag{EC.15}$$

$$c_iI(\mathbf{x}_y) + \tilde{c}_oO(\mathbf{x}_y) \geq c_iI(\mathbf{x}_{\tilde{y}}) + \tilde{c}_oO(\mathbf{x}_{\tilde{y}}). \tag{EC.16}$$

Adding (EC.15) and (EC.16) yields

$$\begin{aligned}
& c_o O(\mathbf{x}_{\tilde{y}}) + \tilde{c}_o O(\mathbf{x}_y) \geq c_o O(\mathbf{x}_y) + \tilde{c}_o O(\mathbf{x}_{\tilde{y}}) \\
\Longleftrightarrow\ & O(\mathbf{x}_{\tilde{y}})[c_o - \tilde{c}_o] \geq O(\mathbf{x}_y)[c_o - \tilde{c}_o] \\
\Longleftrightarrow\ & O(\mathbf{x}_{\tilde{y}}) \geq O(\mathbf{x}_y) && \text{(EC.17)} \\
\Longleftrightarrow\ & \tilde{y} \geq y, && \text{(EC.18)}
\end{aligned}$$

where (EC.17) follows from the assumption that $c_o \geq \tilde{c}_O$, and (EC.18) is a consequence of Theorem 1(ii). We note that alternatively, this result is also a consequence of Topkis's Monotonicity Theorem (see Topkis (1998)), by observing that the the function $g(y, c_o) := I(\mathbf{x}_y) + c_o O(\mathbf{x}_y)$ is submodular in (y, c_o) on $\mathbb{Z}_+ \times \mathbb{R}_+$, where the lattices $\mathbb{Z}_+$ and $\mathbb{R}_+$ are endowed with the usual ordering.

(v) Assume that $c_i \geq \tilde{c}_i$ and let

$$\begin{aligned}
y \in \bar{y}(c_i) &:= \arg\min_{y \in \mathbb{Z}_+} c_i I(\mathbf{x}_y) + c_o O(\mathbf{x}_y) \text{ and} && \text{(EC.19)} \\
\tilde{y} \in \bar{y}(\tilde{c}_i) &:= \arg\min_{y \in \mathbb{Z}_+} \tilde{c}_i I(\mathbf{x}_y) + c_o O(\mathbf{x}_y). && \text{(EC.20)}
\end{aligned}$$

We will show that $y \geq \tilde{y}$. From (EC.19) and (EC.20) we get

$$\begin{aligned}
& c_i I(\mathbf{x}_{\tilde{y}}) + c_o O(\mathbf{x}_{\tilde{y}}) \geq c_i I(\mathbf{x}_y) + c_o O(\mathbf{x}_y) \text{ and} && \text{(EC.21)} \\
& \tilde{c}_i I(\mathbf{x}_y) + c_o O(\mathbf{x}_y) \geq \tilde{c}_i I(\mathbf{x}_{\tilde{y}}) + c_o O(\mathbf{x}_{\tilde{y}}). && \text{(EC.22)}
\end{aligned}$$

Adding (EC.21) and (EC.22) yields

$$\begin{aligned}
& c_i I(\mathbf{x}_{\tilde{y}}) + \tilde{c}_i I(\mathbf{x}_y) \geq c_i I(\mathbf{x}_y) + \tilde{c}_i I(\mathbf{x}_{\tilde{y}}) \\
\Longleftrightarrow\ & I(\mathbf{x}_{\tilde{y}})[c_i - \tilde{c}_i] \geq I(\mathbf{x}_y)[c_i - \tilde{c}_i] \\
\Longleftrightarrow\ & I(\mathbf{x}_{\tilde{y}}) \geq I(\mathbf{x}_y) && \text{(EC.23)} \\
\Longleftrightarrow\ & \tilde{y} \leq y, && \text{(EC.24)}
\end{aligned}$$

where (EC.23) follows from the assumption that $c_o \geq \tilde{c}_o$, and (EC.24) is a consequence of Theorem 1(i). ☐

Proof of Theorem 2 Before we proceed with the proof we introduce the following notation. Let $\mathbf{r}^{\mathbf{x}} = (r_1^{\mathbf{x}}, r_2^{\mathbf{x}}, ..., r_n^{\mathbf{x}}) \in \mathbb{Z}_+^n$ be a realization of the arrival process under schedule $\mathbf{x}$, i.e., $r_t^{\mathbf{x}}$ customers will arrive at the beginning of slot $t = 1, 2, ..., n$. Let also $D(\mathbf{x})$ be the deterministic counterpart of $V(\mathbf{x})$, i.e., $D(\mathbf{x}) = V(\mathbf{x} | \mathbf{r}^{\mathbf{x}} = \mathbf{x})$. We define $\mathbf{z}^{\mathbf{r}} = (z_1^{\mathbf{r}}, z_2^{\mathbf{r}}, ..., z_n^{\mathbf{r}}, z_{n+1}^{\mathbf{r}}) \in \mathbb{Z}_+^{n+1}$ as the backlog vector under some realization $\mathbf{r}$ of the arrival process, i.e., a backlog of $z_t^{\mathbf{r}}$ customers are carried forward to slot $t = 1, 2, ..., n+1$, where $z_{n+1}^{\mathbf{r}}$ represents the customers served overtime. Let also $\mathbf{d}^{\mathbf{r}} = (d_1^{\mathbf{r}}, d_2^{\mathbf{r}}, ..., d_n^{\mathbf{r}}) \in \mathbb{Z}_+^n$ be the idle time vector under some realization $\mathbf{r}$ of the arrival process, i.e., $d_t^{\mathbf{r}}$ servers are idle during slot $t = 1, 2, ..., n$. Then $\mathbf{z}^{\mathbf{r}}$ and $\mathbf{d}^{\mathbf{r}}$ satisfy the (deterministic) recursion

$$\begin{aligned}
& z_1^{\mathbf{r}} = 0, \\
& z_t^{\mathbf{r}} = \max\{z_{t-1}^{\mathbf{r}} + r_{t-1} - s, 0\}, \text{ for } t = 2, 3, ..., n+1, \\
\text{and } & d_t^{\mathbf{r}} = \max\{s - z_t^{\mathbf{r}} - r_t, 0\}, \text{ for } t = 1, 2, ..., n.
\end{aligned}$$

Now let $\mathbf{x}\in\mathbb{Z}_+^n$ and $1\le i\le j\le n$. We will show that

$$\mathrm{E}_{\mathbf{r}^{\mathbf{x}}}\left[[V(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}})-V(\mathbf{x}+\mathbf{e}_i|\mathbf{r}^{\mathbf{x}})]-[V(\mathbf{x}+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}})-V(\mathbf{x}|\mathbf{r}^{\mathbf{x}})]\right]\ge 0. \tag{EC.25}$$

We consider all possible realizations of schedules $\mathbf{e}_i$ and $\mathbf{e}_j$ and the law of total probability to get

$$\begin{aligned}V(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}})-V(\mathbf{x}+\mathbf{e}_i|\mathbf{r}^{\mathbf{x}})=&V(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{r}^{\mathbf{e}_j}=\mathbf{0})(1-p)^2+V(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{e}_i,\mathbf{r}^{\mathbf{e}_j}=\mathbf{0})p(1-p)\\&+V(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{0},\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)(1-p)p+V(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{e}_i,\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)p^2\\&-V(\mathbf{x}+\mathbf{e}_i|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{0})(1-p)-V(\mathbf{x}+\mathbf{e}_i|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{e}_i)p\\=&V(\mathbf{x}|\mathbf{r}^{\mathbf{x}})(1-p)^2+V(\mathbf{x}+\mathbf{e}_i|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{e}_i)p(1-p)+V(\mathbf{x}+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)(1-p)p\\&+V(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{e}_i,\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)p^2-V(\mathbf{x}|\mathbf{r}^{\mathbf{x}})(1-p)-V(\mathbf{x}+\mathbf{e}_i|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_i}=\mathbf{e}_i)p\\=&p(1-p)[D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}})]+p^2[D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i)]. \quad \text{(EC.26)}\end{aligned}$$

Similarly,

$$\begin{aligned}V(\mathbf{x}+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}})-V(\mathbf{x}|\mathbf{r}^{\mathbf{x}})=&V(\mathbf{x}+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)p+V(\mathbf{x}+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_j}=\mathbf{0})(1-p)-V(\mathbf{x}|\mathbf{r}^{\mathbf{x}})\\=&V(\mathbf{x}+\mathbf{e}_j|\mathbf{r}^{\mathbf{x}},\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)p+V(\mathbf{x}|\mathbf{r}^{\mathbf{x}})(1-p)-V(\mathbf{x}|\mathbf{r}^{\mathbf{x}})\\=&p[D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}})]. \quad \text{(EC.27)}\end{aligned}$$

From (EC.26) and (EC.27) and some algebra, inequality (EC.25) holds iff

$$\mathrm{E}_{\mathbf{r}^{\mathbf{x}}}\left[[D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i)]-[D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}})]\right]\ge 0. \tag{EC.28}$$

Next we define the events

$$\begin{aligned}G&=\{\text{No idle time during slots } j,j+1,...,n \text{ under realization } \mathbf{r}^{\mathbf{x}}+\mathbf{e}_i\}\\ \text{and } H&=\{\text{No idle time during slots } j,j+1,...,n \text{ under realization } \mathbf{r}^{\mathbf{x}}\}.\end{aligned}$$

Let also

$$\begin{aligned}\tau&=\mathbb{1}_G(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i}/s\rfloor)+\mathbb{1}_{G^c}\min\{t:d_t^{\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i}>0,j\le t\le n\}\\ \text{and } \kappa&=\mathbb{1}_H(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{x}}}/s\rfloor)+\mathbb{1}_{H^c}\min\{t:d_t^{\mathbf{r}^{\mathbf{x}}}>0,j\le t\le n\}.\end{aligned}$$

Here is an interpretation of the quantities τ and κ. Let customer c_j be the one representing realization $\mathbf{e}_j$. Without loss of generality, since customers are homogeneous, we can consider that customer c_j can be "pushed" towards the end of the queue of customers whenever the queue is nonempty (i.e., the service discipline does not affect the cost function, as long as the queue is work conserving). Then, τ is the time slot (possibly during overtime) that customer c_j will receive service under realization $\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i+\mathbf{e}_j$, and κ is the time slot (possibly during overtime) that customer c_j will receive service under realization $\mathbf{r}^{\mathbf{x}}+\mathbf{e}_j$. Then,

$$\begin{aligned}D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i)&=w(\tau-j)+c_o\mathbb{1}_G-c_i\mathbb{1}_{G^c}\\&=w(\tau-j)+(c_o+c_i)\mathbb{1}_G-c_i \quad \text{(EC.29)}\end{aligned}$$

and

$$\begin{aligned} D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}}) &= w(\kappa-j)+c_o\mathbb{1}_H-c_i\mathbb{1}_{H^c} \\ &= w(\kappa-j)+(c_o+c_i)\mathbb{1}_H-c_i. \end{aligned} \tag{EC.30}$$

It is straightforward to see, inductively, that

$$\begin{aligned} z_t^{\mathbf{r}^{\mathbf{x}}} &= z_t^{\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i} \text{ for } t=1,2,...,i \\ z_t^{\mathbf{r}^{\mathbf{x}}} &\le z_t^{\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i} \text{ for } t=i+1,i+2,...,n+1, \\ d_t^{\mathbf{r}^{\mathbf{x}}} &= d_t^{\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i} \text{ for } t=1,2,...,i-1 \\ d_t^{\mathbf{r}^{\mathbf{x}}} &\ge d_t^{\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i} \text{ for } t=i,i+1,...,n, \end{aligned}$$

concluding that

$$\mathbb{1}_G \ge \mathbb{1}_H \text{ and } \tau \ge \kappa. \tag{EC.31}$$

Finally, from (EC.29), (EC.30) and (EC.31),

$$\begin{aligned} \Delta_{ij}^{\mathbf{r}^{\mathbf{x}}} &:= [D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_i)]-[D(\mathbf{r}^{\mathbf{x}}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{x}})] \\ &= w(\tau-\kappa)+(c_o+c_i)(\mathbb{1}_G-\mathbb{1}_H) \\ &\ge 0 \end{aligned}$$

for all realizations $\mathbf{r}^{\mathbf{x}}$. Therefore, $\mathrm{E}_{\mathbf{r}^{\mathbf{x}}}[\Delta_{ij}^{\mathbf{r}^{\mathbf{x}}}] \ge 0$. □

Proof of Theorem 3 The function $V:\mathbb{Z}_+^n \to \mathbb{R}: \mathbf{x} \mapsto I(\mathbf{x})+c_oO(\mathbf{x})+wW(\mathbf{x})$ is multimodular iff

$$V(\mathbf{x}+\mathbf{u})+V(\mathbf{x}+\mathbf{v}) \ge V(\mathbf{x})+V(\mathbf{x}+\mathbf{u}+\mathbf{v}) \tag{EC.32}$$

for all $\mathbf{x}\in\mathbb{Z}_+^n$ and all $\mathbf{u}\ne\mathbf{v}\in\mathcal{E}$ such that $\mathbf{x}+\mathbf{u},\mathbf{x}+\mathbf{v}\in\mathbb{Z}_+^n$, where

$$\mathcal{E}=\{-\mathbf{e}_1,\mathbf{e}_1-\mathbf{e}_2,\mathbf{e}_2-\mathbf{e}_3,...,\mathbf{e}_{n-1}-\mathbf{e}_n,\mathbf{e}_n\}.$$

An in the proof of Theorem 2, we first introduce the following notation. Let $\mathbf{r}^{\mathbf{x}}=(r_1^{\mathbf{x}},r_2^{\mathbf{x}},...,r_n^{\mathbf{x}})\in\mathbb{Z}_+^n$ be a realization of the arrival process under schedule $\mathbf{x}$, i.e., $r_t^{\mathbf{x}}$ customers will arrive at the beginning of slot $t=1,2,...,n$. Let also $D(\mathbf{x})$ be the deterministic counterpart of $V(\mathbf{x})$, i.e.,

$$D(\mathbf{x})=V(\mathbf{x}|\mathbf{r}^{\mathbf{x}}=\mathbf{x})$$

We define $\mathbf{z}^{\mathbf{r}}=(z_1^{\mathbf{r}},z_2^{\mathbf{r}},...,z_n^{\mathbf{r}},z_{n+1}^{\mathbf{r}})\in\mathbb{Z}_+^{n+1}$ be the backlog vector under some realization $\mathbf{r}$ of the arrival process, i.e., a backlog of $z_t^{\mathbf{r}}$ customers are carried forward to slot $t=1,2,...,n+1$, where $z_{n+1}^{\mathbf{r}}$ represents the customers served overtime. Let also $\mathbf{d}^{\mathbf{r}}=(d_1^{\mathbf{r}},d_2^{\mathbf{r}},...,d_n^{\mathbf{r}})\in\mathbb{Z}_+^n$ be the idle time vector under some realization $\mathbf{r}$ of the arrival process, i.e., $d_t^{\mathbf{r}}$ servers are idle during slot $t=1,2,...,n$. Then $\mathbf{z}^{\mathbf{r}}$ and $\mathbf{d}^{\mathbf{r}}$ satisfy the (deterministic) recursion

$$\begin{aligned} z_1^{\mathbf{r}} &= 0, \\ z_t^{\mathbf{r}} &= \max\{z_{t-1}^{\mathbf{r}}+r_{t-1}-s,0\}, \text{ for } t=2,3,...,n+1, \\ \text{and } d_t^{\mathbf{r}} &= \max\{s-z_t^{\mathbf{r}}-r_t,0\}, \text{ for } t=1,2,...,n. \end{aligned}$$

Now let $\mathbf{x}\in\mathbb{Z}_+^n$ and consider all the possible cases for $\mathbf{u}\neq\mathbf{v}\in\mathcal{D}$ such that $\mathbf{x}+\mathbf{u},\mathbf{x}+\mathbf{v}\in\mathbb{Z}_+^n$.

Case 1: Either $\mathbf{u}=\mathbf{e}_n$ and $\mathbf{v}=-\mathbf{e}_1$, or $\mathbf{v}=\mathbf{e}_n$ and $\mathbf{u}=-\mathbf{e}_1$. Inequality (EC.32) can be written as

$$V(\mathbf{x}+\mathbf{e}_n)-V(\mathbf{x}+\mathbf{e}_n-\mathbf{e}_1)\geq V(\mathbf{x})-V(\mathbf{x}-\mathbf{e}_1),$$

which is true from the supermodularity of V.

Case 2: $\mathbf{u}=\mathbf{e}_i-\mathbf{e}_{i+1}$ and $\mathbf{v}=\mathbf{e}_j-\mathbf{e}_{j+1}$ for some $i\neq j\in\{1,2,...,n-1\}$. Without loss of generality, we can assume that $i<j$. For ease of notation, lets define the vector $\mathbf{w}=\mathbf{x}-\mathbf{e}_{i+1}-\mathbf{e}_{j+1}$. Then, inequality (EC.32) can be written as

$$V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_j)-V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1})\geq V(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_j)-V(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_{j+1}). \tag{EC.33}$$

We will prove that inequality (EC.33) is true by taking expectation over all possible realizations of $\mathbf{w}$, i.e., we will show that

$$\mathrm{E}_{\mathbf{r}^{\mathbf{w}}}\left[\left[V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}})-V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}})\right]-\left[V(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}})-V(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}})\right]\right]\geq 0. \tag{EC.34}$$

First, we consider all possible realizations of schedules $\mathbf{e}_{i+1}$, $\mathbf{e}_j$, $\mathbf{e}_{j+1}$ and use the law of total probability to get

$$\begin{aligned}
V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}})-V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}})=&V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{r}^{\mathbf{e}_j}=\mathbf{0})(1-p)^2\\
&+V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{e}_{i+1},\mathbf{r}^{\mathbf{e}_j}=\mathbf{0})p(1-p)\\
&+V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{0},\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)p(1-p)\\
&+V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{e}_{i+1},\mathbf{r}^{\mathbf{e}_j}=\mathbf{e}_j)p^2\\
&-V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{r}^{\mathbf{e}_{j+1}}=\mathbf{0})(1-p)^2\\
&-V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{e}_{i+1},\mathbf{r}^{\mathbf{e}_{j+1}}=\mathbf{0})p(1-p)\\
&-V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{0},\mathbf{r}^{\mathbf{e}_{j+1}}=\mathbf{e}_{j+1})p(1-p)\\
&-V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}},\mathbf{r}^{\mathbf{e}_{i+1}}=\mathbf{e}_{i+1},\mathbf{r}^{\mathbf{e}_{j+1}}=\mathbf{e}_{j+1})p^2\\
=&D(\mathbf{r}^{\mathbf{w}})(1-p)^2\\
&+D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1})p(1-p)\\
&+D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_j)p(1-p)\\
&+D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j)p^2\\
&-D(\mathbf{r}^{\mathbf{w}})(1-p)^2\\
&-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1})p(1-p)\\
&-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{j+1})p(1-p)\\
&-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1})p^2\\
=&D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_j)p(1-p)\\
&+D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j)p^2\\
&-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{j+1})p(1-p)\\
&-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1})p^2.
\end{aligned} \tag{EC.35}$$

Similarly,

$$\begin{aligned} V(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_j|\mathbf{r}^{\mathbf{w}}) - V(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_{j+1}|\mathbf{r}^{\mathbf{w}}) =& D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_j)p(1-p) \\ &+ D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_j)p^2 \\ &- D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{j+1})p(1-p) \\ &- D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_{j+1})p^2. \end{aligned} \tag{EC.36}$$

From (EC.35) and (EC.36), and some algebra, inequality (EC.34) holds iff

$$\mathrm{E}_{\mathbf{r}^{\mathbf{w}}}\left[\left[D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1})\right] - \left[D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_j) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_{j+1})\right]\right] \geq 0. \tag{EC.37}$$

Next, lets define the events

$$\begin{aligned} G_0 &= \{\text{No idle time during slot } j \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}\} \\ G_1 &= \{\text{No idle time during slots } j, j+1, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}\} \\ G_2 &= \{\text{No idle time during slots } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}\} \\ G_3 &= \{\text{Idle time during slot } j\text{, no idle time during } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}\} \\ H_0 &= \{\text{No idle time during slot } j \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_i\} \\ H_1 &= \{\text{No idle time during slots } j, j+1, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_i\} \\ H_2 &= \{\text{No idle time during slots } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_i\} \\ \text{and } H_3 &= \{\text{Idle time during slot } j\text{, no idle time during } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_i\}. \end{aligned}$$

It is straightforward to see that

$$\mathbb{1}_{G_3} = \mathbb{1}_{G_2} - \mathbb{1}_{G_1} \text{ and } \mathbb{1}_{H_3} = \mathbb{1}_{H_2} - \mathbb{1}_{H_1}. \tag{EC.38}$$

Let also

$$\begin{aligned} \tau_1 &= \mathbb{1}_{G_1}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}}/s\rfloor) + \mathbb{1}_{G_1^c}\min\{t : d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}} > 0, j \leq t \leq n\} \\ \tau_2 &= \mathbb{1}_{G_2}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}}/s\rfloor) + \mathbb{1}_{G_2^c}\min\{t : d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}} > 0, j+1 \leq t \leq n\} \\ \kappa_1 &= \mathbb{1}_{H_1}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i}/s\rfloor) + \mathbb{1}_{H_1^c}\min\{t : d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i} > 0, j \leq t \leq n\} \\ \text{and } \kappa_2 &= \mathbb{1}_{H_2}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i}/s\rfloor) + \mathbb{1}_{H_2^c}\min\{t : d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i} > 0, j+1 \leq t \leq n\}, \end{aligned}$$

where $\mathbb{1}_E$ is the indicator function that event E is true. Here is an intuitive interpretation of the quantities $\tau_1, \tau_2, \kappa_1, \kappa_2$. Let customers c_j and c_{j+1} be the ones representing realizations $\mathbf{e}_j$ and $\mathbf{e}_{j+1}$ respectively. Without loss of generality, since customers are homogeneous, we can consider that customers c_j and c_{j+1} can be "pushed" towards the end of the queue of customers whenever the queue is nonempty (i.e., the service discipline does not affect the cost function, as long as the queue is work conserving). Then, τ_1 is the time slot (possibly during overtime) that customer c_j will receive service under realization $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j$, τ_2 is the time slot (possibly during overtime) that customer that customer c_{j+1} will receive service under realization $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}$, κ_1 is the time slot (possibly during overtime) that customer c_j will receive service under realization $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_j$, and κ_2 is the time slot (possibly during overtime) that customer c_{j+1} will receive service under realization $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_{j+1}$. Let E^c denote the complement of an event E. Then,

$$\begin{aligned}
D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}) =&[D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1})]\\
&-[D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1})-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1})]\\
=&[w(\tau_1-j)+c_o\mathbb{1}_{G_1}-c_i\mathbb{1}_{G_1^c}]-[w(\tau_2-(j+1))+c_o\mathbb{1}_{G_2}-c_i\mathbb{1}_{G_2^c}]\\
=&w(\tau_1-\tau_2+1)+(c_o+c_i)(\mathbb{1}_{G_1}-\mathbb{1}_{G_2})\\
=&w(\tau_1-\tau_2+1)-(c_o+c_i)\mathbb{1}_{G_3}, \qquad \text{(EC.39)}
\end{aligned}$$

where (EC.39) is a consequence of (EC.38).

Then, we express the difference $\tau_1-\tau_2$ in terms of $\mathbb{1}_{G_0}$. When G_0 is true, then customers c_j and c_{j+1} will be served during the same slot t_0 (possibly during overtime) under realizations $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j$ and $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}$ respectively. If G_0^c is true, then customer c_j will be served during slot j under realization $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j$, and customer c_{j+1} will be served during some slot $\tau_2\geq j+1$ (possibly during overtime) under realization $\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}$. Therefore,

$$\begin{aligned}
\tau_1-\tau_2&=\mathbb{1}_{G_0}(t_0-t_0)+\mathbb{1}_{G_0^c}[j-\tau_2]\\
&=\mathbb{1}_{G_0^c}[j-\tau_2]
\end{aligned}$$

and eventually,

$$D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1}) =w-w(\tau_2-j)\mathbb{1}_{G_0^c}-(c_o+c_i)\mathbb{1}_{G_3}. \qquad \text{(EC.40)}$$

Similarly we get that

$$D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_{j+1}) =w-w(\kappa_2-j)\mathbb{1}_{H_0^c}-(c_o+c_i)\mathbb{1}_{H_3}. \qquad \text{(EC.41)}$$

Then, from (EC.40) and (EC.41),

$$\begin{aligned}
\Delta_{ij}^{\mathbf{r}^{\mathbf{w}}}&:=[D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1})]-[D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_j)-D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_{j+1})]\\
&=(c_o+c_i)(\mathbb{1}_{H_3}-\mathbb{1}_{G_3})+w(\kappa_2-j)\mathbb{1}_{H_0^c}-w(\tau_2-j)\mathbb{1}_{G_0^c}. \qquad \text{(EC.42)}
\end{aligned}$$

It is straightforward to see, inductively, that

$$\begin{aligned}
&z_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}}=z_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i} \text{ for } t=1,2,...,i\\
&z_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}}\leq z_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i} \text{ for } t=i+1\\
&z_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i}\leq z_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}} \text{ for } t=i+2,i+3,...,n+1\\
&d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}}=d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i} \text{ for } t=1,2,...,i-1\\
&d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i}\leq d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}} \text{ for } t=i\\
&d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}}\leq d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i} \text{ for } t=i+1,i+2,...,n. \qquad \text{(EC.43)}
\end{aligned}$$

Therefore, from (EC.43),

$$\mathbb{1}_{G_3} \leq \mathbb{1}_{H_3}, \tag{EC.44}$$

$$\text{and } \mathbb{1}_{G_0^c} \leq \mathbb{1}_{H_0^c}. \tag{EC.45}$$

Inequality (EC.45) implies that H_0^c is true whenever G_0^c is true. When G_0^c and H_0^c are true, since $i<j$, we have that $z_t^{\mathbf{r^w}+\mathbf{e}_i} = z_t^{\mathbf{r^w}+\mathbf{e}_{i+1}} = z_t^{\mathbf{r^w}}$ and $d_t^{\mathbf{r^w}+\mathbf{e}_i} = d_t^{\mathbf{r^w}+\mathbf{e}_{i+1}} = d_t^{\mathbf{r^w}}$ for all $t=j+1, j+2, ..., n$. Therefore, $\tau_2 = \kappa_2 = \xi$ for some $\xi \geq j+1$, when G_0^c and H_0^c are true, i.e.,

$$\mathbb{1}_{G_0^c}\tau_2 = \mathbb{1}_{G_0^c}\xi \text{ and } \mathbb{1}_{H_0^c}\kappa_2 = \mathbb{1}_{H_0^c}\xi. \tag{EC.46}$$

Combining (EC.42), (EC.44), (EC.45) and (EC.46) we finally get

$$\begin{aligned}
\Delta_{ij}^{\mathbf{r^w}} &= [D(\mathbf{r^w}+\mathbf{e}_{i+1}+\mathbf{e}_j) - D(\mathbf{r^w}+\mathbf{e}_{i+1}+\mathbf{e}_{j+1})] - [D(\mathbf{r^w}+\mathbf{e}_i+\mathbf{e}_j) - D(\mathbf{r^w}+\mathbf{e}_i+\mathbf{e}_{j+1})] \\
&= (c_o+c_i)(\mathbb{1}_{H_3} - \mathbb{1}_{G_3}) + w(\kappa_2 - j)\mathbb{1}_{H_0^c} - w(\tau_2 - j)\mathbb{1}_{G_0^c} \\
&= (c_o+c_i)(\mathbb{1}_{H_3} - \mathbb{1}_{G_3}) + w(\xi - j)(\mathbb{1}_{H_0^c} - \mathbb{1}_{G_0^c}) \\
&\geq 0
\end{aligned}$$

for all realizations $\mathbf{r^w}$. Therefore, $\mathrm{E}_{\mathbf{r^w}}[\Delta_{ij}^{\mathbf{r^w}}] \geq 0$.

Case 3: Either $\mathbf{u} = -\mathbf{e}_1$ and $\mathbf{v} = \mathbf{e}_j - \mathbf{e}_{j+1}$ for some $j \in \{1, 2, ..., n-1\}$, or $\mathbf{v} = -\mathbf{e}_1$ and $\mathbf{u} = \mathbf{e}_j - \mathbf{e}_{j+1}$ for some $j \in \{1, 2, ..., n-1\}$. This case is treated similarly with Case 2, and thus most of the intermediate steps are omitted. For ease of notation lets define $\mathbf{w} = \mathbf{x} - \mathbf{e}_{j+1} - \mathbf{e}_1$. Then, inequality (EC.32) can be written as

$$V(\mathbf{w}+\mathbf{e}_1+\mathbf{e}_j) - V(\mathbf{w}+\mathbf{e}_1+\mathbf{e}_{j+1}) \geq V(\mathbf{w}+\mathbf{e}_j) - V(\mathbf{w}+\mathbf{e}_{j+1}). \tag{EC.47}$$

We will prove that inequality (EC.47) is true by taking expectation over all possible realizations of $\mathbf{w}$, i.e., we will show that

$$\mathrm{E}_{\mathbf{r^w}}\left[[V(\mathbf{w}+\mathbf{e}_1+\mathbf{e}_j|\mathbf{r^w}) - V(\mathbf{w}+\mathbf{e}_1+\mathbf{e}_{j+1}|\mathbf{r^w})] - [V(\mathbf{w}+\mathbf{e}_j|\mathbf{r^w}) - V(\mathbf{w}+\mathbf{e}_{j+1}|\mathbf{r^w})]\right] \geq 0. \tag{EC.48}$$

First, we consider all possible realizations of schedules $\mathbf{e}_1$, $\mathbf{e}_j$ and $\mathbf{e}_{j+1}$ to get

$$\begin{aligned}
V(\mathbf{w}+\mathbf{e}_1+\mathbf{e}_j|\mathbf{r^w}) - V(\mathbf{w}+\mathbf{e}_1+\mathbf{e}_{j+1}|\mathbf{r^w}) =& D(\mathbf{r^w}+\mathbf{e}_j)p(1-p) \\
&+ D(\mathbf{r^w}+\mathbf{e}_1+\mathbf{e}_j)p^2 \\
&- D(\mathbf{r^w}+\mathbf{e}_{j+1})p(1-p) \\
&- D(\mathbf{r^w}+\mathbf{e}_1+\mathbf{e}_{j+1})p^2.
\end{aligned} \tag{EC.49}$$

Then, we consider all possible realizations of schedules $\mathbf{e}_j$ and $\mathbf{e}_{j+1}$ to get

$$V(\mathbf{w}+\mathbf{e}_j|\mathbf{r^w}) - V(\mathbf{w}+\mathbf{e}_{j+1}|\mathbf{r^w}) = D(\mathbf{r^w}+\mathbf{e}_j)p - D(\mathbf{r^w}+\mathbf{e}_{j+1})p. \tag{EC.50}$$

From (EC.49) and (EC.50), inequality (EC.48) holds iff

$$\mathrm{E}_{\mathbf{r^w}}\left[[D(\mathbf{r^w}+\mathbf{e}_1+\mathbf{e}_j) - D(\mathbf{r^w}+\mathbf{e}_1+\mathbf{e}_{j+1})] - [D(\mathbf{r^w}+\mathbf{e}_j) - D(\mathbf{r^w}+\mathbf{e}_{j+1})]\right] \geq 0.$$

Next, lets define the events

$$\begin{aligned}
&G_0 = \{\text{No idle time during slot } j \text{ under realization } \mathbf{r}^{\mathbf{w}} + \mathbf{e}_1\}\\
&G_1 = \{\text{No idle time during slots } j, j+1, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}} + \mathbf{e}_1\}\\
&G_2 = \{\text{No idle time during slots } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}} + \mathbf{e}_1\}\\
&G_3 = \{\text{Idle time during slot } j, \text{ no idle time during } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}} + \mathbf{e}_1\}\\
&H_0 = \{\text{No idle time during slot } j \text{ under realization } \mathbf{r}^{\mathbf{w}}\}\\
&H_1 = \{\text{No idle time during slots } j, j+1, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}\}\\
&H_2 = \{\text{No idle time during slots } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}\}\\
\text{and } &H_3 = \{\text{Idle time during slot } j, \text{ no idle time during } j+1, j+2, ..., n \text{ under realization } \mathbf{r}^{\mathbf{w}}\}.
\end{aligned}$$

As in Case 2,

$$\begin{aligned}
&\mathbb{1}_{G_3} = \mathbb{1}_{G_2} - \mathbb{1}_{G_1} \leq \mathbb{1}_{H_3} = \mathbb{1}_{H_2} - \mathbb{1}_{H_1},\\
\text{and } &\mathbb{1}_{G_0^c} \leq \mathbb{1}_{H_0^c}.
\end{aligned}$$

Let also

$$\begin{aligned}
\tau_1 &= \mathbb{1}_{G_1}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_1}/s\rfloor) + \mathbb{1}_{G_1^c} \min\{t : d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_1} > 0, j \leq t \leq n\}\\
\tau_2 &= \mathbb{1}_{G_2}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_1}/s\rfloor) + \mathbb{1}_{G_2^c} \min\{t : d_t^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_1} > 0, j+1 \leq t \leq n\}\\
\kappa_1 &= \mathbb{1}_{H_1}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}}/s\rfloor) + \mathbb{1}_{H_1^c} \min\{t : d_t^{\mathbf{r}^{\mathbf{w}}} > 0, j \leq t \leq n\}\\
\text{and } \kappa_2 &= \mathbb{1}_{H_2}(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}}/s\rfloor) + \mathbb{1}_{H_2^c} \min\{t : d_t^{\mathbf{r}^{\mathbf{w}}} > 0, j+1 \leq t \leq n\},
\end{aligned}$$

As in Case 2, we have

$$\begin{aligned}
\Delta_{1j}^{\mathbf{r}^{\mathbf{w}}} &:= [D(\mathbf{r}^{\mathbf{w}} + \mathbf{e}_1 + \mathbf{e}_j) - D(\mathbf{r}^{\mathbf{w}} + \mathbf{e}_1 + \mathbf{e}_{j+1})] - [D(\mathbf{r}^{\mathbf{w}} + \mathbf{e}_j) - D(\mathbf{r}^{\mathbf{w}} + \mathbf{e}_{j+1})]\\
&= (c_o + c_i)(\mathbb{1}_{H_3} - \mathbb{1}_{G_3}) + w(\kappa_2 - j)\mathbb{1}_{H_0^c} - w(\tau_2 - j)\mathbb{1}_{G_0^c}\\
&\geq 0
\end{aligned}$$

for all realizations $\mathbf{r}^{\mathbf{w}}$. Therefore, $\mathrm{E}_{\mathbf{r}^{\mathbf{w}}}[\Delta_{1j}^{\mathbf{r}^{\mathbf{w}}}] \geq 0$.

Case 4: Either $\mathbf{u} = \mathbf{e}_n$ and $\mathbf{v} = \mathbf{e}_i - \mathbf{e}_{i+1}$ for some $i \in \{1, 2, ..., n-1\}$, or $\mathbf{v} = \mathbf{e}_n$ and $\mathbf{u} = \mathbf{e}_i - \mathbf{e}_{i+1}$ for some $i \in \{1, 2, ..., n-1\}$. This case is treated similarly with the proof of Theorem 2, and thus most of the intermediate steps are omitted. For ease of notation lets define $\mathbf{w} = \mathbf{x} - \mathbf{e}_{i+1}$. Then, inequality (EC.32) can be written as

$$V(\mathbf{w} + \mathbf{e}_{i+1} + \mathbf{e}_n) - V(\mathbf{w} + \mathbf{e}_{i+1}) \geq V(\mathbf{w} + \mathbf{e}_i + \mathbf{e}_n) - V(\mathbf{w} + \mathbf{e}_i). \tag{EC.51}$$

We will prove that inequality (EC.51) is true by taking expectation over all possible realizations of $\mathbf{w}$, i.e., we will show that

$$\mathrm{E}_{\mathbf{r}^{\mathbf{w}}}\left[[V(\mathbf{w} + \mathbf{e}_{i+1} + \mathbf{e}_n|\mathbf{r}^{\mathbf{w}}) - V(\mathbf{w} + \mathbf{e}_{i+1}|\mathbf{r}^{\mathbf{w}})] - [V(\mathbf{w} + \mathbf{e}_i + \mathbf{e}_n|\mathbf{r}^{\mathbf{w}}) - V(\mathbf{w} + \mathbf{e}_i|\mathbf{r}^{\mathbf{w}})]\right] \geq 0. \tag{EC.52}$$

First, we consider all possible realizations of schedules $\mathbf{e}_{i+1}$ and $\mathbf{e}_n$ to get

$$\begin{aligned} V(\mathbf{w}+\mathbf{e}_{i+1}+\mathbf{e}_n|\mathbf{r}^{\mathbf{w}}) - V(\mathbf{w}+\mathbf{e}_{i+1}|\mathbf{r}^{\mathbf{w}}) =& D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_n)p(1-p) \\ &+ D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_n)p^2 \\ &- D(\mathbf{r}^{\mathbf{w}})p(1-p) \\ &- D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1})p^2. \end{aligned} \tag{EC.53}$$

Then, we consider all possible realizations of schedules $\mathbf{e}_i$ and $\mathbf{e}_n$ to get

$$\begin{aligned} V(\mathbf{w}+\mathbf{e}_i+\mathbf{e}_n|\mathbf{r}^{\mathbf{w}}) - V(\mathbf{w}+\mathbf{e}_i|\mathbf{r}^{\mathbf{w}}) =& D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_n)p(1-p) \\ &+ D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_n)p^2 \\ &- D(\mathbf{r}^{\mathbf{w}})p(1-p) \\ &- D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i)p^2. \end{aligned} \tag{EC.54}$$

From (EC.53) and (EC.54), inequality (EC.52) holds iff

$$\mathrm{E}_{\mathbf{r}^{\mathbf{w}}}\left[[D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_n) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1})] - [D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_n) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i)]\right] \geq 0.$$

Next, lets define the events

$$\begin{aligned} G &= \{\text{No idle time during slot } n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}\} \\ \text{and } H &= \{\text{No idle time during slot } n \text{ under realization } \mathbf{r}^{\mathbf{w}}+\mathbf{e}_i\}. \end{aligned}$$

Let also

$$\begin{aligned} \tau &= \mathbb{1}_G(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}}/s\rfloor) + \mathbb{1}_{G^c}n \\ \text{and } \kappa &= \mathbb{1}_H(n+1+\lfloor z_{n+1}^{\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i}/s\rfloor) + \mathbb{1}_{H^c}n. \end{aligned}$$

Then,

$$\begin{aligned} D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_n) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}) &= w(\tau-n) + c_o\mathbb{1}_G - c_i\mathbb{1}_{G^c} \\ &= w(\tau-n) + (c_o+c_i)\mathbb{1}_G - c_i \end{aligned}$$

and

$$\begin{aligned} D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_n) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i) &= w(\kappa-n) + c_o\mathbb{1}_H - c_i\mathbb{1}_{H^c} \\ &= w(\kappa-n) + (c_o+c_i)\mathbb{1}_H - c_i. \end{aligned}$$

It is straightforward to see that

$$\mathbb{1}_G \geq \mathbb{1}_H \text{ and } \tau \geq \kappa,$$

concluding that

$$\begin{aligned} \Delta_{in}^{\mathbf{r}^{\mathbf{w}}} &:= [D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1}+\mathbf{e}_n) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_{i+1})] - [D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i+\mathbf{e}_n) - D(\mathbf{r}^{\mathbf{w}}+\mathbf{e}_i)] \\ &= w(\tau-\kappa) + (c_o+c_i)(\mathbb{1}_G - \mathbb{1}_H) \\ &\geq 0 \end{aligned}$$

for all realizations $\mathbf{r}^{\mathbf{w}}$. Therefore, $\mathrm{E}_{\mathbf{r}^{\mathbf{w}}}[\Delta_{in}^{\mathbf{r}^{\mathbf{w}}}] \geq 0$. $\square$

Proof of Lemma 3 The correctness of this lemma is a direct consequence of Lemma 2 in Koole and Van Der Sluis (2003) and Theorem 3.1 in Murota (2005). $\square$

Proof of Theorem 5 (i) For fixed $\tau \geq 1$, and for $z \geq \tau s n_0$, let $H_\tau(z) = \sum_{k=0}^{\tau s n_0} (\tau s n_0 - k)\binom{z}{k} p^k (1-p)^{z-k}$. As in the proof of Theorem 1, $H_\tau(z)$ is decreasing and discretely convex in z on $\{\tau s n_0, \tau s n_0 + 1, \tau s n_0 + 2...\}$. Therefore, $H_\tau(z)$ is decreasing and discretely convex in z on the subset $\{\tau s n_0, \tau s n_0 + \tau, \tau s n_0 + 2\tau, ...\}$, concluding that $H_\tau\big(\tau(s n_0 + y_0)\big)$ is decreasing and discretely convex in y_0 on $\{0,1,2,...\}$.

Then, note that $I(\mathbf{p}_{n_0,y_0}) = \sum_{\tau=1}^{N_0} \frac{1}{\tau} H_\tau\big(\tau(s n_0 + y_0)\big)$, a linear combination of decreasing and discretely convex functions in y_0 on $\{0,1,2,...\}$. Therefore, $I(\mathbf{p}_{n_0,y_0})$ is decreasing and discretely convex in y_0 on $\{0,1,2,...\}$. $\square$

(ii) Recall that $O(\mathbf{p}_{n_0,y_0}) = I(\mathbf{p}_{n_0,y_0}) + p(s n_0 + y_0) N_0 - s n_0 N_0$, and therefore $O(\mathbf{p}_{n_0,y_0})$ is discretely convex in y_0 on $\{0,1,2,...\}$.

For the monotonicity property we need a little more work. First note that

$$O(\mathbf{p}_{n_0,y_0}) = \sum_{\tau=1}^{N_0} \frac{1}{\tau} H_\tau\big(\tau(s n_0 + y_0)\big) + p(s n_0 + y_0) N_0 - s n_0 N_0. \tag{EC.55}$$

Using similar arguments as in the proof of Theorem 1 equation (EC.7), we get that for $z \geq \tau n_0$

$$\begin{array}{rcccl} -p < & H_\tau(z+1) & - & H_\tau(z) & < 0 \\ -p < & H_\tau(z+2) & - & H_\tau(z+1) & < 0 \\ & & \vdots & & \\ -p < & H_\tau(z+\tau) & - & H_\tau(z+\tau-1) & < 0. \end{array}$$

By adding all the inequalities above we get that

$$-p\tau < H_\tau(z+\tau) - H_\tau(z) < 0 \quad \text{for } z \geq \tau n_0. \tag{EC.56}$$

Therefore, from (EC.55) and (EC.56)

$$\begin{aligned} O(\mathbf{p}_{n_0,y_0+1}) - O(\mathbf{p}_{n_0,y_0}) &= \sum_{\tau=1}^{N_0} \frac{1}{\tau} \left[H_\tau\big(\tau(s n_0 + y_0 + 1)\big) - H_\tau\big(\tau(s n_0 + y_0)\big) \right] + p N_0 \\ &> -\sum_{\tau=1}^{N_0} \frac{1}{\tau} p\tau + p N_0 \\ &= 0, \end{aligned}$$

concluding that $O(\mathbf{p}_{n_0,y_0})$ is increasing in y_0 on $\{0,1,2,...\}$. $\square$

Appendix B: Front-Loading Heuristic

Algorithm 3 Front-Loading Heuristic (FLH)

```
procedure FLH($n, s, q, w$)
    $y \leftarrow \left[(1+e^{-(\beta_1 n+\beta_2 s+\beta_3 w+\beta_4 q+\beta_5)})^{-1}\,\bar{y}\right]$    ▷ overbooking level as in §6.1
    $\mathbf{x}_{\text{FLH}} \leftarrow (s+\lfloor \frac{y}{n} \rfloor)\mathbf{e}^n$    ▷ $\mathbf{e}^n$ is the vector of $n$ one's, allocate $\lfloor \frac{y}{n} \rfloor$ overbookings to every slot
    if $y = 0$ then    ▷ trivial case with no-overbooking
        return $\mathbf{x}_{\text{FLH}}$
        end procedure
    end if
    $y_0 \leftarrow y \, mod \, n$    ▷ allocate the remaining $y - n\lfloor \frac{y}{n} \rfloor$ overbookings
    if $y_0 = 0$ then    ▷ if the remainder is 0, then create a spike at the first slot
        $x_1 \leftarrow x_1 + 1$
        $x_n \leftarrow x_n - 1$
        $\mathbf{x}_{\text{FLH}} \leftarrow \mathbf{x}$
        return $\mathbf{x}_{\text{FLH}}$
        end procedure
    end if
    $x_1 \leftarrow x_1 + 1$    ▷ always overbook the first slot if $y_0$ is greater than 0
    $n_0 \leftarrow \lfloor \frac{n}{y_0} \rfloor$
    if $n_0 = 1$ and $y_0 \leq \frac{2}{3}n$ then    ▷ allocate the rest overbookings in a front-loading manner
        $t \leftarrow 0$    ▷ with the first period being shorter
        for $i = 2, ..., \lceil \frac{y_0}{2} \rceil$ do
            $t \leftarrow t + 1$
            $x_t \leftarrow x_t + 1$
        end for
        for $i = \lceil \frac{y_0}{2} \rceil + 1, ..., y_0$ do
            $t \leftarrow t + 2$
            $x_t \leftarrow x_t + 1$
        end for
    else
        for $i = 2, 3, , ..., y_0$ do
            $t \leftarrow (i-1)n_0$
            $x_t \leftarrow x_t + 1$
        end for
    end if
    $\mathbf{x}_{\text{FLH}} \leftarrow \mathbf{x}$
    return $\mathbf{x}_{\text{FLH}}$
end procedure
```

References

Koole, G., E. Van Der Sluis. 2003. Optimal shift scheduling with a global service level constraint. *IIE Transactions* **35**(11) 1049–1055.

Murota, K. 2005. Note on multimodularity and L-convexity. *Mathematics of Operations Research* **30**(3) 658–661.

Topkis, D.M. 1998. *Supermodularity and complementarity*. Princeton university press.