Appendix

In Section A, we give an example to demonstrate that in the DP algorithm given in Section 3, the worst-case demand can be in the middle of the line segment between A and B shown in Figure 1. In Section B, we give an example to show that for the problem formulated in Section 4.1, the worst-case demand can be in the middle of a demand uncertainty space. In Section C, we provide proofs for all the lemmas and theorems given in Sections 3 and 4 (i.e., Lemmas 1 to 6, and Theorems 1 to 3).

A Example for Section 3

We construct an example to show that in the DP algorithm given in Section 3 for the problem with inter-product substitution only, the worst-case demand can be in the middle of the line segment between A and B shown in Figure 1. In this example, there are two time periods and two allowable prices for each product ($p_1^1 = p_2^1 = 100, p_1^2 = p_2^2 = 90$). The initial inventory $i_1 = i_2 = 80$. Table 1 shows the demand uncertainty set for each price pair in each time period. Table 2 shows the revenues under different demand cases in the first period, the possible price pairs and the corresponding worst-case demand cases and revenues in the second period, and the corresponding total revenues over the two periods. As shown, if the first period price $p_1 = p_2 = 100$, picking demand (50,50) in the first period will result in lower total revenue (15500) than picking demand (40,60) or (60,40)(15600).

Table 1: Demand Uncertainty Set

Time Period	Prices	Demand Uncertainty Space
1	(100,100)	$100 \leq D_1 + D_2 \leq 200$, $40 \leq D_1 \leq 100$, $40 \leq D_2 \leq 100$
2	(100,100)	$70 \leq D_1 + D_2 \leq 200$, $25 \leq D_1 \leq 100$, $25 \leq D_2 \leq 100$
2	(100,90)	$80 \leq D_1 + D_2 \leq 200$, $20 \leq D_1 \leq 100$, $40 \leq D_2 \leq 100$
2	(90,100)	$80 \leq D_1 + D_2 \leq 200$, $40 \leq D_1 \leq 100$, $20 \leq D_2 \leq 100$
2	(90,90)	$90 \leq D_1 + D_2 \leq 200$, $35 \leq D_1 \leq 100$, $35 \leq D_2 \leq 100$

Table 2: Analysis of Worst Case Demand

First Period			Second Period			Total
Demand	Revenue	Remaining Inventory	Prices	Worst-Case Demand	Revenue	Revenue
(40,60)	10000	(40, 20)	(100,100)	(25,45)	4500	14500
			(100,90)	(20,60)	3800	13800
			(90,100)(*)	(40,40)	5600	15600(*)
			(90,90)	(35,55)	4950	14950
(60,40)	10000	(20, 40)	(100,100)	(45,25)	4500	14500
			(100,90)(*)	(40,40)	5600	15600(*)
			(90,100)	(60,20)	3800	13800
			(90,90)	(55,35)	4950	14950
(50,50)	10000	(30, 30)	(100,100)(*)	(25,45)	5500	15500(*)
			(100,90)	(20,60)	4700	14700
			(90,100)	(60,20)	4700	14700
			(90,90)	(55,35)	4950	14950

B Example for Section 4

We give an example to show that for the problem formulated in Section 4.1, the worst-case demand can be in the middle of a demand uncertainty space. In this example, there are two time periods, one allowable price for product 1 ($p_1^1 = 50$) and two allowable prices for product 2 ($p_2^1 = 50, p_2^2 = 45$). The initial inventory levels $I_1 = I_2 = 10$. The allowed cumulative demand deviations $B_1 = B_2 = 1$. We will show that if the decision maker chooses price pair (50,50) in the first period, the adversary will choose the middle point of the corresponding demand uncertainty space to achieve the lowest total revenue. The expected demand for price pair (50,50) in the first period is (6,6) and the demand intervals are as follows, i.e., $D_{11}^{11} \in [5,7], D_{21}^{11} \in [5,7], D_{11}^{11} + D_{21}^{11} \in$

$[11, 13]$. The expected demand for price pair (50,50) in the second period is (6,4) and the demand intervals are $D_{12}^{11} \in [5, 7], D_{22}^{11} \in [2, 6], D_{12}^{11} + D_{22}^{11} \in [8, 12]$. Correspondingly, the expected demand for price pair (50,45) in the second period is (4,6) and the demand intervals are $D_{12}^{12} \in [2, 6], D_{22}^{12} \in [5, 7], D_{12}^{12} + D_{22}^{12} \in [8, 12]$.

Corresponding to price pair (50,50) selected in the first period, the demand uncertainty space is $\Omega_{1,0}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, \quad 5 \leq D_2 \leq 7, \quad 11 \leq D_1 + D_2 \leq 13\}$ (note, at the beginning of time period 1, d must be 0). Demand uncertainty spaces for period 2 corresponding to two price pairs (i.e., (50,50),(50,45)) and three cumulative demand deviations (i.e., $d = 1, 0, -1$) are as follows, $\Omega_{2,1}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, 2 \leq D_2 \leq 6, 8 \leq D_1 + D_2 \leq 10\}$, $\Omega_{2,0}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, 2 \leq D_2 \leq 6, 9 \leq D_1 + D_2 \leq 11\}$, $\Omega_{2,-1}^{11} = \{(D_1, D_2) \mid 5 \leq D_1 \leq 7, 2 \leq D_2 \leq 6, 10 \leq D_1 + D_2 \leq 12\}$, $\Omega_{2,1}^{12} = \{(D_1, D_2) \mid 2 \leq D_1 \leq 6, 5 \leq D_2 \leq 7, 8 \leq D_1 + D_2 \leq 10\}$, $\Omega_{2,0}^{12} = \{(D_1, D_2) \mid 2 \leq D_1 \leq 6, 5 \leq D_2 \leq 7, 9 \leq D_1 + D_2 \leq 11\}$, and $\Omega_{2,-1}^{12} = \{(D_1, D_2) \mid 2 \leq D_1 \leq 6, 5 \leq D_2 \leq 7, 10 \leq D_1 + D_2 \leq 12\}$.

Given that the decision maker chooses price pair (50, 50) in the first period, Table 3 demonstrates why choosing demand (6, 6) will lead to the lowest total revenue. For example, if the adversary chooses demand (6, 6) in the first period, it leads to demand deviation of 0, first period revenue of 600 and remaining inventory of 4 for both products. In the second period, if the decision maker chooses price pair (50, 50), which results in the demand uncertainty space $\Omega_{2,0}^{11}$, then the adversary will choose demand realization (7, 2) to minimize the second period revenue (300 in this case). Correspondingly, if the decision maker chooses price pair (50, 45) in the second period, which results in the demand uncertainty space $\Omega_{2,0}^{12}$, then the adversary will choose demand realization (2, 7) to minimize the second period revenue (280 in this case). The decision maker knows exactly what the adversary will choose for each possible price pair. After comparing these two possible price pairs, the decision maker will choose (50, 50) since 300 > 280, which results in a total revenue of 900 for two periods (marked with an asterisk in the table) . For each possible demand in the first period, we can do the same analysis and compute the corresponding total revenue (marked with an asterisk in the table). Since 900 is the smallest among all these total revenue numbers, it means that demand (6, 6) is the worst-case demand if the decision maker chooses price pair (50, 50) in the first period.

Table 3: Worst-case Demand Analysis

First Period				Second Period					
Demand	Demand Deviation	Revenue	Remaining Inventory	Prices	Uncertainty Space	Worst-case Demand	Sales	Revenue	Total Revenue
(6,6)	0	600	(4,4)	(50,50)	$\Omega_{2,0}^{11}$	(7,2)	(4,2)	300	900*
				(50,45)	$\Omega_{2,0}^{12}$	(2,7)	(2,4)	280	880
(6,7)	1	650	(4,3)	(50,50)	$\Omega_{2,1}^{11}$	(6,2)	(4,2)	300	950*
				(50,45)	$\Omega_{2,1}^{12}$	(2,6)	(2,3)	235	885
(6,5)	-1	550	(4,5)	(50,50)	$\Omega_{2,-1}^{11}$	(7,3)	(4,3)	350	900
				(50,45)	$\Omega_{2,-1}^{12}$	(3,7)	(3,5)	375	925*
(5,6)	-1	550	(5,4)	(50,50)	$\Omega_{2,-1}^{11}$	(7,3)	(5,3)	400	950*
				(50,45)	$\Omega_{2,-1}^{12}$	(3,7)	(3,4)	330	880
(7,6)	1	650	(3,4)	(50,50)	$\Omega_{2,1}^{11}$	(6,2)	(3,2)	250	900
				(50,45)	$\Omega_{2,1}^{12}$	(2,6)	(2,4)	280	930*
(7,5)	0	600	(3,5)	(50,50)	$\Omega_{2,0}^{11}$	(7,2)	(3,2)	250	850
				(50,45)	$\Omega_{2,0}^{12}$	(2,7)	(2,5)	325	925*
(5,7)	0	600	(5,3)	(50,50)	$\Omega_{2,0}^{11}$	(7,2)	(5,2)	350	950*
				(50,45)	$\Omega_{2,0}^{12}$	(2,7)	(2,3)	235	835

C Proofs of Lemmas and Theorems

In this section, we prove all the lemmas and theorems given in Sections 3 and 4. We prove each result contained in each lemma or theorem by backward induction. We show that if the result holds for time period $t + 1$, it also holds for time period t. The result for time period T can be proved similarly as a special case, and hence we omit the proof here.

Lemma 1 *In any period t, the following inequalities hold as long as the value of each state variable involved is within its domain:*

$$\begin{aligned} &(i) \quad V_t(i_1-1,i_2,l_1,l_2,r_1,r_2) \geq V_t(i_1,i_2,l_1,l_2,r_1,r_2) - p_1^{l_1}, \\ &(ii) \quad V_t(i_1,i_2-1,l_1,l_2,r_1,r_2) \geq V_t(i_1,i_2,l_1,l_2,r_1,r_2) - p_2^{l_2}. \end{aligned}$$

Proof We prove result (i). Result (ii) can be proved similarly. Denote l_1', l_2' as the optimal price levels for state $(i_1,i_2,l_1,l_2,r_1,r_2)$ in period t and D_1^*, D_2^* as the corresponding worst-case demand. Denote D_1, D_2 as the worst-case demand in period t for state $(i_1-1,i_2,l_1,l_2,r_1,r_2)$ if price levels l_1', l_2' are used in that period. Also denote $i_1' = i_1 - \min(i_1, D_1^*)$, $i_2' = i_2 - \min(i_2, D_2^*)$, $i_1'' = i_1 - \min(i_1, D_1)$, $i_2'' = i_2 - \min(i_2, D_2)$, and $i_1''' = i_1 - 1 - \min(i_1 - 1, D_1)$. **To further simplify notations, in all of the following proofs, we denote** $p_1^* = p_1^{l_1'}, p_2^* = p_2^{l_2'}$. We consider two cases in the following.
Case 1: $i_1 - 1 \geq D_1$.

$$\begin{aligned} & V_t(i_1-1,i_2,l_1,l_2,r_1,r_2) \geq p_1^* \min(i_1-1,D_1) + p_2^* \min(i_2,D_2) + V_{t+1}(i_1''',i_2'',l_1',l_2',r_1',r_2') \\ \geq \;& p_1^* \min(i_1,D_1) + p_2^* \min(i_2,D_2) \\ & + V_{t+1}(i_1'',i_2'',l_1',l_2',r_1',r_2') - p_1^* \text{ (by induction and the fact that } i_1''' = i_1'' - 1) \\ \geq \;& p_1^* \min(i_1,D_1^*) + p_2^* \min(i_2,D_2^*) + V_{t+1}(i_1',i_2',l_1',l_2',r_1',r_2') - p_1^{l_1} \text{ (by definition of } D_1^*, D_2^*) \\ = \;& V_t(i_1,i_2,l_1,l_2,r_1,r_2) - p_1^{l_1} \end{aligned}$$

The same result can be proved for Case 2: $i_1 \leq D_1$. See our technical report for details. ■

Theorem 1 *Given any price pair (p_1, p_2) in any period t, for any two feasible demand realizations (D_1, D_2) and (D_1', D_2') in the demand uncertainty space of the given price pair in the given period, if $D_1' \leq D_1$ and $D_2' \leq D_2$, then the total revenue from period t to T associated with the realization (D_1', D_2') is no greater than that associated with the realization (D_1, D_2).*

Proof To prove this theorem, it is sufficient to show the following results: Given any state $(i_1,i_2,l_1,l_2,r_1,r_2)$ in the beginning of any period t, for any feasible price pair (p_1,p_2) chosen for period t (i.e., $p_1 = p_1^{l_1'}, p_2 = p_2^{l_2'}$, $s.t.\ l_1' \in F_1^{l_1}$, and $l_2' \in F_2^{l_2}$), comparing any three feasible demand realizations $(D_1,D_2), (D_1-1,D_2), (D_1,D_2-1)$ in period t, the following results hold:

$$\begin{aligned} (i) \quad & p_1 \min(i_1, D_1-1) + p_2 \min(i_2, D_2) + V_{t+1}(i_1'',i_2',l_1',l_2',r_1',r_2') \\ & \leq p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2) + V_{t+1}(i_1',i_2',l_1',l_2',r_1',r_2'), \\ (ii) \quad & p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2-1) + V_{t+1}(i_1',i_2'',l_1',l_2',r_1',r_2') \\ & \leq p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2) + V_{t+1}(i_1',i_2',l_1',l_2',r_1',r_2'), \end{aligned}$$

where $i_1' = i_1 - \min(i_1, D_1)$, $i_2' = i_2 - \min(i_2, D_2)$, $i_1'' = i_1 - \min(i_1, D_1 - 1)$, and $i_2'' = i_2 - \min(i_2, D_2 - 1)$.

The left side of (i) (resp. (ii)) represents the total revenue from period t to T given that in period t the price pair used is (l_1', l_2') and the demand realization is $(D_1 - 1, D_2)$ (resp. $(D_1, D_2 - 1)$). The right side of each of these inequalities represents the total revenue from period t to T given that in period t the price pair used is (l_1', l_2') and the demand realization is (D_1, D_2).

We prove result (i). Result (ii) can be proved similarly. Consider two possible cases:
Case 1: $D_1 - 1 \geq i_1$

$$\begin{aligned} & p_1 \min(i_1, D_1-1) + p_2 \min(i_2, D_2) + V_{t+1}(i_1'',i_2',l_1',l_2',r_1',r_2') \\ = \;& p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2) + V_{t+1}(i_1',i_2',l_1',l_2',r_1',r_2') \text{ (note: } i_1'' = i_1' = 0) \end{aligned}$$

Case 2: $D_1 \leq i_1$

$$\begin{aligned} & p_1 \min(i_1, D_1-1) + p_2 \min(i_2, D_2) + V_{t+1}(i_1'',i_2',l_1',l_2',r_1',r_2') \\ \leq \;& p_1 \min(i_1, D_1) - p_1 + p_2 \min(i_2, D_2) \\ & + V_{t+1}(i_1',i_2',l_1',l_2',r_1',r_2') + p_1 \text{ (by Lemma 1 and the fact that } i_1'' = i_1' + 1) \\ = \;& p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2) + V_{t+1}(i_1',i_2',l_1',l_2',r_1',r_2') \end{aligned}$$

This completes the proof. ■

Lemma 2 *In any period t, the following inequalities hold as long as the value of each state variable involved is within its domain in the state space.*

$$(i) \; V_t(i_1+1,i_2,l_1,l_2,r_1,r_2,d) \geq V_t(i_1,i_2,l_1,l_2,r_1,r_2,d)$$

$$(ii)\ V_t(i_1-1,i_2,l_1,l_2,r_1,r_2,d) \geq V_t(i_1,i_2,l_1,l_2,r_1,r_2,d) - p_1^{l_1}$$

$$(iii)\ V_t(i_1,i_2,l_1,l_2,r_1,r_2,d-1) \geq V_t(i_1,i_2,l_1,l_2,r_1,r_2,d)$$

Proof We first prove (i) by backward induction. Denote l'_1, l'_2 as the optimal price levels for state $(i_1,i_2,l_1,l_2,r_1,r_2,d)$ in period t and D_1^*, D_2^* as the corresponding worst-case demand. Denote D_1, D_2 as the worst-case demand in period t for state $(i_1+1,i_2,l_1,l_2,r_1,r_2,d)$ if price levels l'_1, l'_2 are used in period t. Note that $(D_1,D_2),(D_1^*,D_2^*) \in \Omega_{t,d}^{l'_1 l'_2}$. Also denote $i'_1 = i_1 - \min(i_1, D_1^*)$, $i'_2 = i_2 - \min(i_2, D_2^*)$, $i''_1 = i_1 - \min(i_1, D_1)$, $i''_2 = i_2 - \min(i_2, D_2)$, $i'''_1 = i_1 + 1 - \min(i_1+1, D_1)$, $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$, and $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$. Note that $i'''_1 \geq i''_1$ and $i'''_1 \leq i''_1 + 1$.

$$\begin{aligned}
&V_t(i_1+1,i_2,l_1,l_2,r_1,r_2,d) \geq p_1^* \min(i_1+1, D_1) + p_2^* \min(i_2, D_2) + V_{t+1}(i'''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') \\
\geq\ & p_1^* \min(i_1, D_1) + p_2^* \min(i_2, D_2) + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') \text{ (by induction and the fact that } i'''_1 \geq i''_1) \\
\geq\ & p_1^* \min(i_1, D_1^*) + p_2^* \min(i_2, D_2^*) + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') \text{ (by definition of } D_1^*, D_2^*) \\
=\ & V_t(i_1,i_2,l_1,l_2,r_1,r_2,d)
\end{aligned}$$

We then prove result (ii) by backward induction. In the following, we denote l'_1, l'_2 as the optimal price levels for state $(i_1,i_2,l_1,l_2,r_1,r_2,d)$ in period t and D_1^*, D_2^* as the corresponding worst-case demand. We denote D_1, D_2 as the worst-case demand in period t for state $(i_1-1,i_2,l_1,l_2,r_1,r_2,d)$ if price levels l'_1, l'_2 are used in that period. Note that $(D_1,D_2),(D_1^*,D_2^*) \in \Omega_{t,d}^{l'_1 l'_2}$. We also denote $i'_1 = i_1 - \min(i_1, D_1^*)$, $i'_2 = i_2 - \min(i_2, D_2^*)$, $i''_1 = i_1 - \min(i_1, D_1)$, $i''_2 = i_2 - \min(i_2, D_2)$, $i'''_1 = i_1 - 1 - \min(i_1 - 1, D_1)$, $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$, and $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$. We consider the following two cases.

Case 1: $i_1 \leq D_1$.

$$\begin{aligned}
&V_t(i_1-1,i_2,l_1,l_2,r_1,r_2,d) \geq p_1^* \min(i_1-1, D_1) + p_2^* \min(i_2, D_2) + V_{t+1}(i'''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') \\
=\ & p_1^* \min(i_1, D_1) - p_1^* + p_2^* \min(i_2, D_2) + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') \text{ (by the fact that } i'''_1 = i''_1 = 0) \\
\geq\ & p_1^* \min(i_1, D_1^*) + p_2^* \min(i_2, D_2^*) + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') - p_1^{l_1} \text{ (by definition of } D_1^*, D_2^*) \\
=\ & V_t(i_1,i_2,l_1,l_2,r_1,r_2,d) - p_1^{l_1}
\end{aligned}$$

The same result can be proved for Case 2: $i_1 - 1 \geq D_1$.

Result (iii) can be proved similarly. See our technical report for details. ■

Lemma 3 *In any period t, for any $\Delta_1, \Delta_2 \geq 1$, the following inequalities hold as long as the value of each state variable involved is within its domain in the approximate state space.*

$$(i)\ A_t(\tilde{i}_1 + \Delta_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$$

$$(ii)\ A_t(\tilde{i}_1 - \Delta_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^{l_1}\Delta_1$$

$$(iii)\ A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d} + \Delta_2) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p_1^{l_1}\Delta_2 - p_2^{l_2}\Delta_2$$

Proof We first prove result (i) by backward induction. In the following, we denote l'_1, l'_2 as the optimal price levels for state $(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$ in period t and $\tilde{D}_1^*, \tilde{D}_2^*$ as the corresponding worst-case demand. We denote $\tilde{D}_1, \tilde{D}_2$ as the worst-case demand in period t for state $(\tilde{i}_1 + \Delta_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$ if price levels l'_1, l'_2 are used in that period. Note that $(\tilde{D}_1, \tilde{D}_2), (\tilde{D}_1^*, \tilde{D}_2^*) \in \tilde{\Omega}_{t,\tilde{d}}^{l'_1 l'_2}$. We also denote $\tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1^*)\}$, $\tilde{i}'_2 = \Phi_1\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2^*)\}$, $\tilde{i}''_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1)\}$, $\tilde{i}''_2 = \Phi_1\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2)\}$, $\tilde{i}'''_1 = \Phi_1\{\tilde{i}_1 + \Delta_1 - \min(\tilde{i}_1 + \Delta_1, \tilde{D}_1)\}$, $\tilde{d}' = \Phi_2\{\tilde{d} + (\tilde{D}_1^* + \tilde{D}_2^*) - (\bar{D}_1 + \bar{D}_2)\}$, and $\tilde{d}'' = \Phi_2\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)\}$. Note that $\tilde{i}'''_1 \geq \tilde{i}''_1$ and $\tilde{i}'''_1 \leq \tilde{i}''_1 + \Delta_1$.

$$\begin{aligned}
&A_t(\tilde{i}_1 + \Delta_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq p_1^* \min(\tilde{i}_1 + \Delta_1, \tilde{D}_1) + p_2^* \min(\tilde{i}_2, \tilde{D}_2) + A_{t+1}(\tilde{i}'''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \\
\geq\ & p_1^* \min(\tilde{i}_1, \tilde{D}_1) + p_2^* \min(\tilde{i}_2, \tilde{D}_2) + A_{t+1}(\tilde{i}''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \text{ (by induction and the fact that } i'''_1 \geq i''_1) \\
\geq\ & p_1^* \min(\tilde{i}_1, \tilde{D}_1^*) + p_2^* \min(\tilde{i}_2, \tilde{D}_2^*) + A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
=\ & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})
\end{aligned}$$

We then prove result (ii) by backward induction. In the following, we denote l'_1, l'_2 as the optimal price levels for state $(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$ in period t and $\tilde{D}_1^*, \tilde{D}_2^*$ as the corresponding worst-case demand. We denote $\tilde{D}_1, \tilde{D}_2$ as the worst-case demand in period t for state $(\tilde{i}_1 - \Delta_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$ if price levels l'_1, l'_2 are used in that period. Note that $(\tilde{D}_1, \tilde{D}_2), (\tilde{D}^*_1, \tilde{D}^*_2) \in \tilde{\Omega}^{l'_1 l'_2}_{t,\tilde{d}}$. We also denote $\tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}^*_1)\}$, $\tilde{i}'_2 = \Phi_1\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}^*_2)\}$, $\tilde{i}''_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1)\}$, $\tilde{i}''_2 = \Phi_1\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2)\}$, $\tilde{i}'''_1 = \Phi_1\{\tilde{i}_1 - \Delta_1 - \min(\tilde{i}_1 - \Delta_1, \tilde{D}_1)\}$, $\tilde{d}' = \Phi_2\{\tilde{d} + (\tilde{D}^*_1 + \tilde{D}^*_2) - (\bar{D}_1 + \bar{D}_2)\}$, and $\tilde{d}'' = \Phi_2\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)\}$, We consider the following three cases.

Case 1: $\tilde{i}_1 \leq \tilde{D}_1$.

$$\begin{aligned}
& A_t(\tilde{i}_1 - \Delta_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \geq p^*_1 \min(\tilde{i}_1 - \Delta_1, \tilde{D}_1) + \ p^*_2 \min(\tilde{i}_2, \tilde{D}_2) + \ A_{t+1}(\tilde{i}'''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \\
= \ & p^*_1 \min(\tilde{i}_1, \tilde{D}_1) - p^*_1 \Delta_1 + \ p^*_2 \min(\tilde{i}_2, \tilde{D}_2) + \ A_{t+1}(\tilde{i}''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') \text{ (by the fact that } \tilde{i}'''_1 = \tilde{i}''_1 = 0) \\
\geq \ & p^*_1 \min(\tilde{i}_1, \tilde{D}^*_1) + \ p^*_2 \min(\tilde{i}_2, \tilde{D}^*_2) + \ A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') - p^{l_1}_1 \Delta_1 \text{ (by definition of } \tilde{D}^*_1, \tilde{D}^*_2) \\
= \ & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - p^{l_1}_1 \Delta_1
\end{aligned}$$

The same result can be proved for Case 2: $\tilde{i}_1 - \Delta_1 < \tilde{D}_1 < \tilde{i}_1$, and Case 3: $\tilde{i}_1 - \Delta_1 \geq \tilde{D}_1$.

A similar proof can be given for Result (iii). See our technical report for details. ■

Lemma 4 *In any period t, for any $\Delta_1, \Delta_2 \geq 1$, the following inequality holds as long as the value of each state variable involved is within its domain in the corresponding state space,*

$$\begin{aligned}
& A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
\geq \ & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - (T - t + 1)(p^{l_1}_1 + p^{l_2}_2)\{\max(\Delta_1 - 1, \Delta_2 - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)\}
\end{aligned}$$

Proof We prove this lemma by backward induction. We denote l'_1, l'_2 as the optimal price levels corresponding to $V_t(i_1, i_2, l_1, l_2, r_1, r_2, d)$, and D^*_1, D^*_2 as the corresponding worst-case demand. We denote $\tilde{D}_1, \tilde{D}_2$ as the worst-case demand corresponding to $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$ if price levels l'_1, l'_2 are used in that period. Note that $(D^*_1, D^*_2) \in \Omega^{l'_1 l'_2}_{t,d}, (\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}^{l'_1 l'_2}_{t,\tilde{d}}$. According to Observation 3, for any $(\tilde{D}_1, \tilde{D}_2) \in \tilde{\Omega}^{l'_1 l'_2}_{t,\tilde{d}}$, we can find $(D'_1, D'_2) \in \Omega^{l'_1 l'_2}_{t,d}$ such that $D'_1 - (\Delta_2 - 1) \leq \tilde{D}_1 \leq D'_1 + (\Delta_2 - 1)$, $D'_2 - (\Delta_2 - 1) \leq \tilde{D}_2 \leq D'_2 + (\Delta_2 - 1)$ and $(\tilde{D}_1 + \tilde{D}_2) - (D'_1 + D'_2) \leq d - \tilde{d} \leq (\Delta_2 - 1)$. We also denote $i'_1 = i_1 - \min(i_1, D^*_1)$, $i'_2 = i_2 - \min(i_2, D^*_2)$, $i''_1 = i_1 - \min(i_1, D'_1)$, $i''_2 = i_2 - \min(i_2, D'_2)$, $i'''_1 = \tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1)$, $i'''_2 = \tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2)$, $\tilde{i}'''_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1)\}$, $\tilde{i}'''_2 = \Phi_1\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2)\}$, $d' = d + (D^*_1 + D^*_2) - (\bar{D}_1 + \bar{D}_2)$, $d'' = d + (D'_1 + D'_2) - (\bar{D}_1 + \bar{D}_2)$, $d''' = \tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)$, and $\tilde{d}''' = \Phi_2\{\tilde{d} + (\tilde{D}_1 + \tilde{D}_2) - (\bar{D}_1 + \bar{D}_2)\}$. Since $(\tilde{D}_1 + \tilde{D}_2) - (D'_1 + D'_2) \leq d - \tilde{d}$, or $(\tilde{D}_1 + \tilde{D}_2) + \tilde{d} \leq (D'_1 + D'_2) + d$, we have $d''' \leq d''$. It can also be verified that $i''_1 - i'''_1 \leq (\Delta_1 - 1) + (\Delta_2 - 1)$ and $i''_2 - i'''_2 \leq (\Delta_1 - 1) + (\Delta_2 - 1)$.

$$\begin{aligned}
& A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) \\
\geq \ & p^*_1 \min(\tilde{i}_1, \tilde{D}_1) + p^*_2 \min(\tilde{i}_2, \tilde{D}_2) + A_{t+1}(\tilde{i}'''_1, \tilde{i}'''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}''') \\
\geq \ & p^*_1 \min(i_1, D'_1) + p^*_2 \min(i_2, D'_2) - (p^*_1 + p^*_2) \max(\Delta_1 - 1, \Delta_2 - 1) + V_{t+1}(i'''_1, i'''_2, l'_1, l'_2, r'_1, r'_2, d''') \\
& -(T - t)(p^*_1 + p^*_2)\{\max(\Delta_1 - 1, \Delta_2 - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)\} \text{ (by induction)} \\
\geq \ & p^*_1 \min(i_1, D'_1) + p^*_2 \min(i_2, D'_2) - (p^*_1 + p^*_2) \max(\Delta_1 - 1, \Delta_2 - 1) \\
& +V_{t+1}(i'''_1, i'''_2, l'_1, l'_2, r'_1, r'_2, d'') \text{ (by Lemma 2 (iii))} \\
& -(T - t)(p^*_1 + p^*_2)\{\max(\Delta_1 - 1, \Delta_2 - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)\} \\
\geq \ & p^*_1 \min(i_1, D'_1) + p^*_2 \min(i_2, D'_2) - (p^*_1 + p^*_2) \max(\Delta_1 - 1, \Delta_2 - 1)) \\
& +V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') - (p^*_1 + p^*_2)\{(\Delta_1 - 1) + (\Delta_2 - 1)\} \text{ (by Lemma 2 (ii))} \\
& -(T - t)(p^*_1 + p^*_2)\{\max(\Delta_1 - 1, \Delta_2 - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)\} \\
= \ & p^*_1 \min(i_1, D'_1) + p^*_2 \min(i_2, D'_2) + V_{t+1}(i''_1, i''_2, l'_1, l'_2, r'_1, r'_2, d'') \\
& -(T - t + 1)(p^*_1 + p^*_2)\{\max(\Delta_1 - 1, \Delta_2 - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)\} \\
\geq \ & p^*_1 \min(i_1, D^*_1) + p^*_2 \min(i_2, D^*_2) + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') \text{ (by definition of } D^*_1, D^*_2) \\
& -(T - t + 1)(p^{l_1}_1 + p^{l_2}_2)\{\max(\Delta_1 - 1, \Delta_2 - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)\} \\
= \ & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - (T - t + 1)(p^{l_1}_1 + p^{l_2}_2)\{\max(\Delta_1 - 1, \Delta_2 - 1) + (\Delta_1 - 1) + (\Delta_2 - 1)\}.
\end{aligned}$$

■

Lemma 5 *In any period t, for any $\Delta_1, \Delta_2 \geq 1$, the following inequality holds as long as the value of each state variable involved is within its domain in the corresponding state space,*

$$R_t(i_1, i_2, l_1, l_2, r_1, r_2, d) \geq A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - (T - t + 1)(p_1^{l_1} + p_2^{l_2})\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\}$$

Proof We prove this lemma by backward induction. We denote l'_1, l'_2 as the optimal price levels corresponding to $A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d})$, and $\tilde{D}_1^*, \tilde{D}_2^*$ as the corresponding worst-case demand. We denote D_1, D_2 as the worst-case demand corresponding to $R_t(i_1, i_2, r_1, r_2, l_1, l_2, d)$ if price levels l'_1, l'_2 are used in that period. Note that $(\tilde{D}_1^*, \tilde{D}_2^*) \in \tilde{\Omega}_{t,\tilde{d}}^{l'_1 l'_2}, (D_1, D_2) \in \Omega_{t,d}^{l'_1 l'_2}$. Based on Observation 4, for any $(D_1, D_2) \in \Omega_{t,d}^{l'_1 l'_2}$, we have $(\tilde{D}'_1, \tilde{D}'_2) \in \tilde{\Omega}_{t,\tilde{d}}^{l'_1 l'_2}$ such that $\tilde{D}'_1 - (\Delta_2 - 1) \leq D_1 \leq \tilde{D}'_1 + (\Delta_2 - 1), \tilde{D}'_2 - (\Delta_2 - 1) \leq D_2 \leq \tilde{D}'_2 + (\Delta_2 - 1)$ and $(\tilde{D}'_1 + \tilde{D}'_2) - (D_1 + D_2) \leq d - \tilde{d} \leq (\Delta_2 - 1)$. We also denote $\tilde{i}'_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}_1^*)\}$, $\tilde{i}'_2 = \Phi_1\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}_2^*)\}$, $\tilde{i}''_1 = \Phi_1\{\tilde{i}_1 - \min(\tilde{i}_1, \tilde{D}'_1)\}$, $\tilde{i}''_2 = \Phi_1\{\tilde{i}_2 - \min(\tilde{i}_2, \tilde{D}'_2)\}$, $i'''_1 = i_1 - \min(i_1, D_1)$, $i'''_2 = i_2 - \min(i_2, D_2)$, $\tilde{d}' = \Phi_2\{\tilde{d} + (\tilde{D}_1^* + \tilde{D}_2^*) - (\bar{D}_1 + \bar{D}_2)\}$, $\tilde{d}'' = \Phi_2\{\tilde{d} + (\tilde{D}'_1 + \tilde{D}'_2) - (\bar{D}_1 + \bar{D}_2)\}$, and $d''' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$. It can also be verified that $\tilde{i}''_1 - \Phi_1\{i'''_1\} \leq \Phi_1\{(\Delta_1 - 1) + (\Delta_2 - 1)\}$, $\tilde{i}''_2 - \Phi_1\{i'''_2\} \leq \Phi_1\{(\Delta_1 - 1) + (\Delta_2 - 1)\}$, and $\tilde{d}'' \leq \Phi_2\{d'''\} \leq \tilde{d}'' + \Phi_2\{4(\Delta_2 - 1)\}$.

$$\begin{aligned}
& R_t(i_1, i_2, l_1, l_2, r_1, r_2, d) = p_1^* \min(i_1, D_1) + p_2^* \min(i_2, D_2) + R_{t+1}(i'''_1, i'''_2, l'_1, l'_2, r'_1, r'_2, d''') \\
\geq\; & p_1^* \min(\tilde{i}_1, \tilde{D}'_1) + p_2^* \min(\tilde{i}_2, \tilde{D}'_2) - (p_1^* + p_2^*)(\Delta_2 - 1) + A_{t+1}(\Phi_1\{i'''_1\}, \Phi_1\{i'''_2\}, l'_1, l'_2, r'_1, r'_2, \Phi_2\{d'''\}) \\
& -(T - t)(p_1^* + p_2^*)\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\} \quad \text{(by induction)} \\
\geq\; & p_1^* \min(\tilde{i}_1, \tilde{D}'_1) + p_2^* \min(\tilde{i}_2, \tilde{D}'_2) - (p_1^* + p_2^*)(\Delta_2 - 1) \\
& +A_{t+1}(\Phi_1\{i'''_1\}, \Phi_1\{i'''_2\}, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') - (p_1^* + p_2^*)\{4(\Delta_2 - 1)\} \quad \text{(by Lemma 3 (iii))} \\
& -(T - t)(p_1^* + p_2^*)\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\} \\
\geq\; & p_1^* \min(\tilde{i}_1, \tilde{D}'_1) + p_2^* \min(\tilde{i}_2, \tilde{D}'_2) - (p_1^* + p_2^*)(\Delta_2 - 1) + A_{t+1}(\tilde{i}''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') - (p_1^* + p_2^*)\{4(\Delta_2 - 1)\} \\
& -(p_1^* + p_2^*)\{(\Delta_1 - 1) + (\Delta_2 - 1)\} \text{ (by Lemma 3 (ii))} \\
& -(T - t)(p_1^* + p_2^*)\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\} \\
=\; & p_1^* \min(\tilde{i}_1, \tilde{D}'_1) + p_2^* \min(\tilde{i}_2, \tilde{D}'_2) + A_{t+1}(\tilde{i}''_1, \tilde{i}''_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}'') - (T - t + 1)(p_1^* + p_2^*)\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\} \\
\geq\; & p_1^* \min(\tilde{i}_1, \tilde{D}_1^*) + p_2^* \min(\tilde{i}_2, \tilde{D}_2^*) + A_{t+1}(\tilde{i}'_1, \tilde{i}'_2, l'_1, l'_2, r'_1, r'_2, \tilde{d}') \text{ (by definition of } \tilde{D}_1^*, \tilde{D}_2^*) \\
& -(T - t + 1)(p_1^{l_1} + p_2^{l_2})\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\} \\
=\; & A_t(\tilde{i}_1, \tilde{i}_2, l_1, l_2, r_1, r_2, \tilde{d}) - (T - t + 1)(p_1^{l_1} + p_2^{l_2})\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\}.
\end{aligned}$$

■

Theorem 2 *For any $\epsilon > 0$, the approximation algorithm AS with the values of Δ_1, Δ_2 defined in Section 4.2 generates a solution that is within a relative error ϵ from the optimality with running time $O(T^6 m_1^2 m_2^2 R_1 R_2 / \epsilon^5)$.*

Proof We first estimate the running time of AS. In AS, we partition the i_j dimension into equal intervals of length Δ_1 and only one value in each interval is considered, for $j = 1, 2$. Thus in AS, in each period t, at most $\lceil I_j / \Delta_1 \rceil$ different values of i_j are considered, for $j = 1, 2$. Similarly, in each period t, at most $\lceil B_t / \Delta_2 \rceil$ different values of d are considered, and at most $\lceil D_j^{\max} / \Delta_2 \rceil$ different values of D_j are considered, for $j = 1, 2$. Thus, the overall running time of the algorithm AS is bounded by $O(T m_1^2 m_2^2 R_1 R_2 \lceil I_1/\Delta_1 \rceil \lceil I_2/\Delta_1 \rceil \lceil D_1^{\max}/\Delta_2 \rceil \lceil D_2^{\max}/\Delta_2 \rceil \lceil B_{max}/\Delta_2$ By the way Δ_1, Δ_2 are defined, $I_j/\Delta_1 \leq I_j/\theta_{1j} \leq O(T/\epsilon)$ for $j = 1, 2$ and $D_j^{\max}/\Delta_2 \leq D_j^{\max}/\theta_{2j} \leq O(T/\epsilon)$, for $j = 1, 2$. Similarly, $B_{max}/\Delta_2 \leq B_{max}/\theta_{23} \leq O(T/\epsilon)$. This implies that the overall running time of the algorithm is bounded by $O(T^6 m_1^2 m_2^2 R_1 R_2 / \epsilon^5)$, which is polynomial in the problem input size and $1/\epsilon$.

Next we show that AS delivers a solution that is within a relative error ϵ from the optimality, i.e.,

$$V_1(I_1, I_2, 1, 1, R_1, R_2, 0) - R_1(I_1, I_2, 1, 1, R_1, R_2, 0) \leq \epsilon V_1(I_1, I_2, 1, 1, R_1, R_2, 0). \tag{1}$$

By Lemma 4, we have

$$\begin{aligned}
A_1(\tilde{I}_1, \tilde{I}_2, 1, 1, R_1, R_2, 0) \;\geq\; & V_1(I_1, I_2, 1, 1, R_1, R_2, 0) \\
& -T p_1^1\{2(\Delta_1 - 1) + 2(\Delta_2 - 1)\} - T p_2^1\{2(\Delta_1 - 1) + 2(\Delta_2 - 1)\}
\end{aligned} \tag{2}$$

Similarly, by Lemma 5, we have

$$\begin{aligned}
R_1(I_1, I_2, 1, 1, R_1, R_2, 0) \;\geq\; & A_1(\tilde{I}_1, \tilde{I}_2, 1, 1, R_1, R_2, 0) \\
& -T p_1^1\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\} - T p_2^1\{(\Delta_1 - 1) + 6(\Delta_2 - 1)\}
\end{aligned} \tag{3}$$

By (2) and (3), we have

$$\begin{aligned} & V_1(I_1, I_2, 1, 1, R_1, R_2, 0) - R_1(I_1, I_2, 1, 1, R_1, R_2, 0) \\ & \leq Tp_1^1\{3(\Delta_1 - 1) + 8(\Delta_2 - 1)\} + Tp_2^1\{3(\Delta_1 - 1) + 8(\Delta_2 - 1)\}. \end{aligned} \tag{4}$$

Clearly, by definition of D_1^{total} and D_2^{total},

$$V_1(I_1, I_2, 1, 1, R_1, R_2, 0) \geq p_1^{m_1} \min\{I_1, D_1^{total}\}, \tag{5}$$

$$V_1(I_1, I_2, 1, 1, R_1, R_2, 0) \geq p_2^{m_2} \min\{I_2, D_2^{total}\}. \tag{6}$$

By (4), (5) and (6), in order to show (1), it is sufficient to show that

$$\frac{3Tp_1^1(\Delta_1 - 1)}{\max\{p_1^{m_1} \min\{I_1, D_1^{total}\}, p_2^{m_2} \min\{I_2, D_2^{total}\}\}} \leq \epsilon/4, \tag{7}$$

$$\frac{8Tp_1^1(\Delta_2 - 1)}{\max\{p_1^{m_1} \min\{I_1, D_1^{total}\}, p_2^{m_2} \min\{I_2, D_2^{total}\}\}} \leq \epsilon/4, \tag{8}$$

$$\frac{3Tp_2^1(\Delta_1 - 1)}{\max\{p_1^{m_1} \min\{I_1, D_1^{total}\}, p_2^{m_2} \min\{I_2, D_2^{total}\}\}} \leq \epsilon/4, \tag{9}$$

$$\frac{8Tp_2^1(\Delta_2 - 1)}{\max\{p_1^{m_1} \min\{I_1, D_1^{total}\}, p_2^{m_2} \min\{I_2, D_2^{total}\}\}} \leq \epsilon/4, \tag{10}$$

We prove (7) and (8) in the following. The relations (9) and (10) can be proved similarly and hence we omit the proofs for them. To prove (7), it is sufficient to prove the following two results:

$$\frac{3Tp_1^1(\theta_{11} - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \epsilon/4, \tag{11}$$

$$\frac{3Tp_1^1(\theta_{12} - 1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} \leq \epsilon/4, \tag{12}$$

To prove (11), we consider the following two cases.

Case 1: If $I_1 \leq D_1^{total}$, then by the definition of θ_{11} and assumption (i), we have

$$\frac{3Tp_1^1(\theta_{11} - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \frac{3Tp_1^1 I_1 \epsilon}{32p_1^{m_1} I_1 C_0^2 T} \leq \epsilon/4.$$

Case 2: If $I_1 > D_1^{total}$, then by the definition of θ_{11} and assumptions (i) and (ii), we have

$$\frac{3Tp_1^1(\theta_{11} - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \frac{3Tp_1^1 I_1 \epsilon}{32p_1^{m_1} D_1^{total} C_0^2 T} \leq \epsilon/4.$$

To prove (12), we consider the following two cases.

Case 1: If $I_2 \leq D_2^{total}$, then by the definition of θ_{12} and assumption (i), we have

$$\frac{3Tp_1^1(\theta_{12} - 1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} \leq \frac{3Tp_1^1 I_2 \epsilon}{32p_2^{m_2} I_2 C_0^2 T} \leq \epsilon/4.$$

Case 2: If $I_2 > D_2^{total}$, then by the definition of θ_{12} and assumptions (i) and (ii), we have

$$\frac{3Tp_1^1(\theta_{12} - 1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} \leq \frac{3Tp_1^1 I_2 \epsilon}{32p_2^{m_2} D_2^{total} C_0^2 T} \leq \epsilon/4.$$

To prove (8), it is sufficient to prove the following three results:

$$\frac{8Tp_1^1(\theta_{21} - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \epsilon/4, \tag{13}$$

$$\frac{8Tp_1^1(\theta_{22} - 1)}{p_2^{m_2} \min\{I_2, D_2^{total}\}} \leq \epsilon/4, \tag{14}$$

$$\frac{8Tp_1^1(\theta_{23} - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \epsilon/4, \tag{15}$$

To prove (13), we consider the following two cases.

Case 1: If $I_1 \leq D_1^{total}$, then by the definition of θ_{21} and assumptions (i) and (iv), we have

$$\frac{8Tp_1^1(\theta_{21} - 1)}{p_1^{m_1} \min\{I_1, D_1^{total}\}} \leq \frac{8Tp_1^1 D_1^{max} \epsilon}{32p_1^{m_1} I_1 C_0^2 T} \leq \epsilon/4.$$

Case 2: If $I_1 > D_1^{total}$, then by the definition of θ_{21} and assumptions (i) and (iii), we have

$$\frac{8Tp_1^1(\theta_{21}-1)}{p_1^{m_1}\min\{I_1, D_1^{total}\}} \le \frac{8Tp_1^1 D_1^{max}\epsilon}{32p_1^{m_1}D_1^{total}C_0^2T} \le \epsilon/4.$$

To prove (14), we consider the following two cases.

Case 1: If $I_2 \le D_2^{total}$, then by the definition of θ_{22} and assumptions (i) and (iv), we have

$$\frac{8Tp_1^1(\theta_{22}-1)}{p_2^{m_2}\min\{I_2, D_2^{total}\}} \le \frac{8Tp_1^1 D_2^{max}\epsilon}{32p_2^{m_2}I_2C_0^2T} \le \epsilon/4.$$

Case 2: If $I_2 > D_2^{total}$, then by the definition of θ_{22} and assumptions (i) and (iii), we have

$$\frac{8Tp_1^1(\theta_{22}-1)}{p_2^{m_2}\min\{I_2, D_2^{total}\}} \le \frac{8Tp_1^1 D_2^{max}\epsilon}{32p_2^{m_2}D_2^{total}C_0^2T} \le \epsilon/4.$$

To prove (15), we consider the following two cases.

Case 1: If $I_1 \le D_1^{total}$, then by the definition of θ_{23} and assumptions (i) and (vi), we have

$$\frac{8Tp_1^1(\theta_{23}-1)}{p_1^{m_1}\min\{I_1, D_1^{total}\}} \le \frac{8Tp_1^1 B_{max}\epsilon}{32p_1^{m_1}I_1C_0^2T} \le \epsilon/4.$$

Case 2: If $I_1 > D_1^{total}$, then by the definition of θ_{23} and assumptions (i) and (v), we have

$$\frac{8Tp_1^1(\theta_{23}-1)}{p_1^{m_1}\min\{I_1, D_1^{total}\}} \le \frac{8Tp_1^1 B_{max}\epsilon}{32p_1^{m_1}D_1^{total}C_0^2T} \le \epsilon/4.$$

■

Lemma 6 *In any period t, the following inequalities hold as long as the value of each state variable involved is within its domain.*

$$(i)\ V_t(0, i_2, l_1, l_2, r_1, r_2, d-1) = V_t(0, i_2, l_1, l_2, r_1, r_2, d)$$

$$(ii)\ V_t(i_1-1, i_2, l_1, l_2, r_1, r_2, d+1) \ge V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^{l_1}$$

Proof We first prove result (i). As $i_1 = 0$, the revenue only depends on the demand of product 2. Clearly, when $U_{1t}^{ij} - L_{1t}^{ij} \ge U_{2t}^{ij} - L_{2t}^{ij}, \forall\ t, i, j$, the lower and upper bounds of demand (or the uncertainty space) for product 2 will always be L_{2t}^{ij}, U_{2t}^{ij} in any time period for any given prices regardless of the initial states (i.e., $V_t(0, i_2, l_1, l_2, r_1, r_2, d-1)$ or $V_t(0, i_2, l_1, l_2, r_1, r_2, d)$). This completes the proof.

We now prove results (ii) by backward induction. In the following proof, we denote l_1', l_2' as the optimal price levels for state $(i_1, i_2, l_1, l_2, r_1, r_2, d)$ in period t and D_1^*, D_2^* as the corresponding worst-case demand. Note as specified before, $p_1^{l_1'} = p_1^*, p_2^{l_2'} = p_2^*$. We denote D_1, D_2 as the worst-case demand in period t for state $(i_1-1, i_2, l_1, l_2, r_1, r_2, d+1)$ if price levels l_1', l_2' are used in that period. We also denote $i_1' = i_1 - \min(i_1, D_1^*)$, $i_2' = i_2 - \min(i_2, D_2^*)$, $i_1'' = i_1 - \min(i_1, D_1)$, $i_2'' = i_2 - \min(i_2, D_2)$, $i_1''' = i_1 - 1 - \min(i_1 - 1, D_1) = i_1 - \min(i_1, D_1 + 1)$, $d' = d + (D_1^* + D_2^*) - (\bar{D}_1 + \bar{D}_2)$, and $d'' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$. We consider the following three cases.

Case 1: $(D_1, D_2) \in \Omega_{t,d}^{l_1'l_2'}$, and $i_1 \le D_1$.

$$\begin{aligned}
& V_t(i_1-1, i_2, l_1, l_2, r_1, r_2, d+1) \ge p_1^* \min(i_1-1, D_1) + p_2^* \min(i_2, D_2) + V_{t+1}(i_1''', i_2'', l_1', l_2', r_1', r_2', d''+1) \\
= \ & p_1^* \min(i_1, D_1) - p_1^* + p_2^* \min(i_2, D_2) \\
& + V_{t+1}(i_1'', i_2'', l_1', l_2', r_1', r_2', d'') \text{ (by result (i) and the fact that } i_1''' = i_1'' = 0) \\
\ge \ & p_1^* \min(i_1, D_1^*) + p_2^* \min(i_2, D_2^*) + V_{t+1}(i_1', i_2', l_1', l_2', r_1', r_2', d') - p_1^{l_1} \text{ (by definition of } D_1^*, D_2^*) \\
= \ & V_t(i_1, i_2, l_1, l_2, r_1, r_2, d) - p_1^{l_1}
\end{aligned}$$

The same result can be proved for Case 2: $(D_1, D_2) \in \Omega_{t,d}^{l_1'l_2'}$, and $i_1 - 1 \ge D_1$, and Case 3: $(D_1, D_2) \notin \Omega_{t,d}^{l_1'l_2'}$. See our technical report for details. ■

Theorem 3 *Given any price pair (p_1, p_2) in any period t and any cumulative demand deviation d from period 1 through period $t-1$, for any two feasible demand realizations (D_1, D_2) and (D_1', D_2) in the demand uncertainty space corresponding to the given price pair and cumulative demand deviation in period t, if $D_1' \le D_1$, then the total revenue from period t to T associated with the realization (D_1, D_2) is no greater than that associated with the realization (D_1', D_2).*

Proof To prove this theorem, it is sufficient to prove the following result: For any state $(i_1, i_2, l_1, l_2, r_1, r_2, d)$ in any time period t, given any feasible price levels l'_1, l'_2 (or equivalent prices $p_1 = p_1^{l'_2}, p_2 = p_2^{l'_2}$) such that $l'_1 \in F_1^{l_1}, l'_2 \in F_2^{l_2}$, comparing any two feasible demand realizations (D_1, D_2) and $(D_1 - 1, D_2)$, the following inequality holds,

$$\begin{aligned} & p_1 \min(i_1, D_1 - 1) + p_2 \min(i_2, D_2) + V_{t+1}(i''_1, i'_2, l'_1, l'_2, r'_1, r'_2, d' - 1) \\ \leq \;& p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2) + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d'), \end{aligned}$$

where $i'_1 = i_1 - \min(i_1, D_1), i'_2 = i_2 - \min(i_2, D_2), i''_1 = i_1 - \min(i_1, D_1 - 1)$ and $d' = d + (D_1 + D_2) - (\bar{D}_1 + \bar{D}_2)$.

To show this result, we consider the following two cases.

Case 1: $D_1 \leq i_1$

$$\begin{aligned} & p_1 \min(i_1, D_1 - 1) + p_2 \min(i_2, D_2) + V_{t+1}(i''_1, i'_2, l'_1, l'_2, r'_1, r'_2, d' - 1) \\ = \;& p_1 \min(i_1, D_1) - p_1 + p_2 \min(i_2, D_2) + V_{t+1}(i'_1 + 1, i'_2, l'_1, l'_2, r'_1, r'_2, d' - 1) \\ \leq \;& p_1 \min(i_1, D_1) - p_1 + p_2 \min(i_2, D_2) + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') + p_1 \text{ (by Lemma 6 (ii))} \\ = \;& p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2) + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') \end{aligned}$$

Case 2: $D_1 - 1 \geq i_1$

$$\begin{aligned} & p_1 \min(i_1, D_1 - 1) + p_2 \min(i_2, D_2) + V_{t+1}(i''_1, i'_2, l'_1, l'_2, r'_1, r'_2, d' - 1) \\ & = p_1 \min(i_1, D_1) + p_2 \min(i_2, D_2) + V_{t+1}(i'_1, i'_2, l'_1, l'_2, r'_1, r'_2, d') \text{ (by Lemma 6 (i) and the fact that } i''_1 = i'_1 = 0). \end{aligned}$$

■