Optimizing the Profitability and Quality of Service in Carshare Systems under Demand Uncertainty

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Appendix A: A Risk-Averse Model based on CVaR of QoS Penalty

Conditional value-at-risk (CVaR) is a risk measure employed to cope with loss distributions. It is also known as mean excess loss or mean shortfall for continuously distributed random variables. Its value depends on the value-at-risk (VaR) of the same random variable. We consider a reservation-based carshare system with $m = 1$. Given $0 < \epsilon < 1$, the $(1-\epsilon)$-VaR (the VaR at confidence level $1-\epsilon$ of the number of unserved customers, denoted $\sum_{a\in A^{\text{one}}\cup A^{\text{two}}} H_a(\mathbf{w},\mathbf{x}^1)$) is given by

$$\text{VaR}_{1-\epsilon}\left(\sum_{a\in A^{\text{one}}\cup A^{\text{two}}} H_a(\mathbf{w},\mathbf{x}^1)\right) = \min\left\{\eta : \mathbb{P}\left(\sum_{a\in A^{\text{one}}\cup A^{\text{two}}} H_a(\mathbf{w},\mathbf{x}^1) \le \eta\right) \ge 1-\epsilon\right\}.$$

That is, when ranking all scenarios $k \in K$ by the number of unserved customers, the $(1-\epsilon)$-VaR is the best value of the $100\epsilon\%$ worst scenarios. It is important to note that in our model, higher values of unserved customers are worse. Then the $(1-\epsilon)$-CVaR is the expected number of unserved customers given that the number exceeds the $(1-\epsilon)$-VaR, or equivalently the average value of the $100\epsilon\%$ worst scenarios.

We propose a risk-averse model, and impose a penalty G_0 on the $(1-\epsilon)$-CVaR of unserved customers, with ϵ being a given risk parameter. Such a model is appropriate when a company accepts that not providing service to a small number of customers is inevitable but wants to ensure a relatively high QoS on average in the worst-case scenarios. Employing the well-known reformulation of CVaR in Rockafellar and Uryasev (2000), we have

$$G_0\text{CVaR}_{1-\epsilon}\left(\sum_{a\in A^{\text{one}}\cup A^{\text{two}}} H_a(\mathbf{w},\mathbf{x}^1)\right) = G_0 \min_{\eta\ge 0}\left\{\eta + \frac{1}{\epsilon}\mathbb{E}_{\mathbb{P}}\left[\left(\sum_{a\in A^{\text{one}}\cup A^{\text{two}}} H_a(\mathbf{w},\mathbf{x}^1) - \eta\right)^+\right]\right\}, \tag{20}$$

where $\mathbb{E}_{\mathbb{P}}$ measures the expectation given the probability distribution $\mathbb{P}$ of the underlying uncertain demand; the non-negative variable η, when optimized, is equal to $\text{VaR}_{1-\epsilon}\left(\sum_{a\in A^{\text{one}}\cup A^{\text{two}}} H_a(\mathbf{w},\mathbf{x}^1)\right)$. We incorporate (20) to replace $g(\mathbf{y}^1,\ldots,\mathbf{y}^{|K|})$ in the second-stage problem $Q(\mathbf{w},\mathbf{x}^1)$, and define auxiliary variables $\zeta^k,\ k\in K$ such that $\zeta^k = \max\left\{\sum_{a\in A^{\text{one}}\cup A^{\text{two}}}(u_a^k - y_a^k) - \eta, 0\right\}$, representing the number of unserved customers that exceed the threshold value η in each scenario k, for all $k \in K$. As a result, the risk-averse second-stage value function based on the CVaR measure is given by

$$Q(\mathbf{w},\mathbf{x}^1) = \min_{\mathbf{y},\boldsymbol{\zeta},\eta\ge 0} \quad \sum_{k\in K} p^k \sum_{a\in A} f_a y_a^k + G_0\left(\eta + \frac{1}{\epsilon}\sum_{k\in K} p^k \zeta^k\right) \tag{21}$$

$$\text{s.t.} \quad y^k \in Y(\mathbf{w},\mathbf{x}^1,\mathbf{u}^k) \qquad \forall k \in K \tag{22}$$

$$\zeta^k \ge \sum_{a\in A^{\text{one}}\cup A^{\text{two}}} (u_a^k - y_a^k) - \eta,\ \zeta^k \ge 0 \qquad \forall k \in K, \tag{23}$$

where the definition of variables $\zeta^k,\ k \in K$ is ensured by constraint (23).

Appendix B: Rolling-Horizon Vehicle Relocation Model and Benchmark Policy

A detailed description of the rolling-horizon vehicle relocation model is given in Algorithm 1.

Given the number of vehicles that are currently on rent along each route, $\mathbf{v}^s$, and the relocation decision $\mathbf{z}^s$, the total cost plus QoS penalty minus rental revenue in future periods under demand scenario k is

Algorithm 1 A Rolling-Horizon Approach for Vehicle Relocation

Solve initial allocation problem; obtain $\mathbf{x}^1$ and $\mathbf{w}$
Initialize $\mathbf{r}^0 := \mathbf{0}$
Initialize $\mathbf{v}^0 := \mathbf{0}$
for $i \in I$ **do**
    Update $v_{i1}^0 \leftarrow x_i^1$
end for
for $s \in \{1, \ldots, T-1\}$ **do**
    for $i \in I$ **do**
        Intialize number of vehicles available: $r_i^s := v_{is}^{s-1} + \sum_{j \in J(i,1)} z_{ji}^{s-1}$
        for $t \in \{s+1, \ldots, T\}$ **do**
            Initialize number of vehicles that will be available in period t: $v_{it}^s := v_{it}^{s-1} + \sum_{j \in J(i,t-s+1)} z_{ji}^{s-1}$
        end for
        Idealize demand list D_i^s
        while $D_i^s \neq \emptyset$ **do**
            Get the first demand in D_i^s, (j,t) (vehicle will be returned to zone j in period t)
            Update demand list $D_i^s \leftarrow D_i^s \backslash \{(j,t)\}$
            if $r_i^s > 0$ **then**
                Update number of available vehicles $r_i^s \leftarrow r_i^s - 1$
                Update number of vehicles in rent $v_{jt}^s \leftarrow v_{jt}^s + 1$
            end if
        end while
    end for
    Solve vehicle relocation Model (9)–(17) with input $\mathbf{r}^s$ and $\mathbf{v}^s$, obtain $\mathbf{z}^s$
end for

approximated by the function $Q_k^{s+1}(\mathbf{z}^s, \mathbf{v}^s)$ given by (15) to (17). In Model (15)–(17), we have omitted the special case of period T, during which the fleet is rebalanced to its initial allocation for the next day.

$$Q^T(\mathbf{z}^{T-1}, \mathbf{v}^{T-1}) = \min_{\mathbf{z}^T} \quad \sum_{i \in I} \sum_{j \in I} c_{ij}^{\text{rel}} z_{ij}^T \tag{24}$$

$$\text{s.t.} \quad \sum_{j \in I} z_{ij}^T = \sum_{j \in I} z_{ji}^{T-1} + v_{iT}^{T-1}, \quad \forall i \in I \tag{25}$$

$$\sum_{j \in I} z_{ji}^T = x_i, \quad \forall i \in I \tag{26}$$

$$z_{ij}^T \in \mathbb{Z}_+, \quad \forall i,\ j \in I \tag{27}$$

In the benchmark policy with rebalancing frequency Δ, the fleet is rebalanced to fit future demand concentration every Δ periods. Rebalancing will take place at time $m\Delta$, $m = 0, 1, \ldots, M$, with $M\Delta = T$. Let D_i^m be the average total demand that will originate from zone i from period $m\Delta + 1$ to period T, with $D_i^M = D_i^0$ as a special case. Let r_i^m be the number of current available vehicles in period m after all demand has been fulfilled, with $\sum_{i \in I} r_i^0 = S$ as a special case. We assume that all rentals or relocations must end by period T. So $\sum_{i \in I} r_i^M = S$. This is to guarantee that the system can be restored to initial status. Let v_i^m be the number of vehicles that are currently being rented or relocated but will be returned to zone i, with $v_i^0 = 0$ as a special case. The decision is the number of vehicles relocated from zone i to zone j, denoted by z_{ij}^m. The objective is to make the proportion of $v_i^m + \sum_{j \in I} z_{ji}^m$ as close to the proportion of D_i^m as possible. We solve the following constrained least-squares problem:

$$\min_{\mathbf{z}^m} \quad \sum_{i \in I} \left(\frac{v_i^m + \sum_{j \in J} z_{ji}^m}{S} - \frac{D_i^m}{\sum_{i \in I} D_i^m} \right)^2 \tag{28}$$

$$\text{s.t.} \quad \sum_{j \in I} z_{ij}^m = r_i^m, \quad \forall i \in I \tag{29}$$

$$\sum_{i\in I}\sum_{j\neq i} z_{ij}^m \leq R^m \tag{30}$$

$$z_{ij}^m \in \mathbb{Z}_+ \quad \forall i,\ j\in I \tag{31}$$

Constraint (29) guarantees that the total number of relocated vehicles equals to the number of available vehicles (after fulfilling demand) in the current period for each zone. Constraint (30) is the relocation capacity constraint. In the last period, R^M is assumed to be infinity to guarantee that the initial state can be restored. The benchmark policy is implemented in a similar way as Algorithm 1, except that vehicles are only relocated in periods $m\Delta$, $m=0,1,\ldots,M$, and the relocation decisions are obtained using Model (28)–(31).

Appendix C: Proofs of Theorem 1

Proof of Theorem 1. Let $(q^k,\mathbf{w},\mathbf{x}^1)$ be an arbitrary solution satisfying the set of cuts $L^k(q^k,\mathbf{w},\mathbf{x}^1)\geq \mathbf{0}$ and define

$$q' := q^k - \sum_{i\in I}\left(\sum_{a=(n_{it},n_{i,t+1})\in A^{\text{idle}}} \widehat{\lambda}_a^1\right) w_i - \sum_{i\in I}(\widehat{\pi}_{i0}^1-\widehat{\pi}_{iT}^1)x_i^1 - \sum_{a\in A^{\text{one}}\cup A^{\text{two}}} \widehat{\lambda}_a^1 u_a^k.$$

Then $q'\geq 0$ and

$$\begin{aligned} q' &\geq q' - \left[q^k - \sum_{i\in I}\left(\sum_{a=(n_{it},n_{i,t+1})\in A^{\text{idle}}} \widehat{\lambda}_a^2\right) w_i - \sum_{i\in I}(\widehat{\pi}_{i0}^2-\widehat{\pi}_{iT}^2)x_i^1 - \sum_{a\in A^{\text{one}}\cup A^{\text{two}}} \widehat{\lambda}_a^2 u_a^k\right] \\ &= \delta - \sum_{i\in I}\alpha_i w_i - \sum_{i\in I}\beta_i x_i^1. \end{aligned}$$

That is, $(q',\mathbf{w},\mathbf{x}^1)\in U := \left\{(q',\mathbf{w},\mathbf{x}^1)\in \mathbb{R}_+\times\mathbb{Z}_+^{|I|}\times\mathbb{Z}_+^{|I|} : q' + \sum_{i\in I}\alpha_i w_i + \sum_{i\in I}\beta_i x_i^1 - \delta \geq 0\right\}$. Applying Proposition 1, we obtain the cut

$$\begin{aligned} q' &+ \sum_{i\in I}\frac{\min\{\lceil\Delta\alpha_i\rceil\,\mathbf{frac}(\Delta\delta),\mathbf{frac}(\Delta\alpha_i)+\lfloor\Delta\alpha_i\rfloor\,\mathbf{frac}(\Delta\delta)\}}{\Delta} w_i \\ &+ \sum_{i\in I}\frac{\min\{\lceil\Delta\beta_i\rceil\,\mathbf{frac}(\Delta\delta),\mathbf{frac}(\Delta\beta_i)+\lfloor\Delta\beta_i\rfloor\,\mathbf{frac}(\Delta\delta)\}}{\Delta} x_i^1 \\ &- \frac{\lceil\Delta\delta\rceil\,\mathbf{frac}(\Delta\delta)}{\Delta} \geq 0 \end{aligned} \tag{32}$$

valid for U. Finally, we substitute the expression q' to obtain (19), valid for$L^k(q^k,\mathbf{w},\mathbf{x}^1)\geq\mathbf{0}$. □

Appendix D: Branch-and-Cut Algorithm with MIR Procedure

We outline the branch-and-cut algorithm with MIR below in pseudo-code. For ease of referencing the coefficients in the cuts, we define α and β as the coefficients of variables $\mathbf{w}$ and $\mathbf{x}^1$, respectively, and $-\delta$ as the constant in the cut. Therefore, the cuts for the risk-neutral model are of the form

$$q^k + \sum_{i\in I}\alpha_i w_i + \sum_{i\in I}\beta_i x_i - \delta \geq 0.$$

There are two main parts to the algorithm. The outer algorithm is a regular branch-and-cut algorithm that branches on fractional x_i's and adds cuts generated from the subproblems. The inner algorithm (lines 14–37 in both algorithms) is a Benders decomposition algorithm with an additional MIR procedure that pairs Benders cuts to generate additional valid cuts. The most violated cut is added to the relaxed master problem, while the remaining cuts are stored for subsequent pairings with the MIR procedure.

Algorithm 2 Branch-and-cut algorithm with MIR for carsharing system design model

Initialize **MP** with no cuts
Initialize `sol`:=`null`
Initialize `optval`:= 0
for $k \in K$ **do**
    Initialize $\texttt{cutlist}^k$
end for
Initialize $S := \{0\}$
Initialize $s := 0$
Define $\texttt{Problem}_0$ as **MP**
while $S \neq \emptyset$ **do**
    Define $\overline{s} := \min\{s : s \in S\}$
    Initialize `repeat`:=`true`
    while `repeat`=`true` **do**
        `repeat`←`false`
        Solve $\texttt{Problem}_{\overline{s}}$ to obtain optimal solution $(\widetilde{\mathbf{w}}, \widetilde{\mathbf{x}}^1, \widetilde{\mathbf{q}})$ and objective $\widetilde{\texttt{obj}}$
        for $k \in K$ **do**
            Solve $\mathbf{SP}_k(\widetilde{\mathbf{w}}, \widetilde{\mathbf{x}}^1)$ to obtain optimal dual solution $(\widehat{\pi}, \widehat{\lambda})$ and optimal objective $\widehat{q}^k$
            if $\widehat{q}^k > \widetilde{q}^k$ **then**
                `repeat`←`true`
                Define `cutA` as cut (18)
                Add `cutA` to $\texttt{cutlist}^k$
                for $\texttt{cutB} \in \texttt{cutlist}^k$ **do**
                    $\Delta \leftarrow 1$
                    Apply Theorem 1 to `cutA` and `cutB` to generate $\texttt{cutC}_0$
                    Add $\texttt{cutC}_0$ to $\texttt{cutlist}^k$
                    for $i \in I$ **do**
                        $\Delta \leftarrow \frac{1}{|\alpha_i^{\mathtt{B}} - \alpha_i^{\mathtt{A}}|}$, where $\alpha^{\mathtt{A}}$ and $\alpha^{\mathtt{B}}$ are coefficients of $\mathbf{w}$ in `cutA` and `cutB` respectively
                        Apply Theorem 1 to `cutA` and `cutB` to generate $\texttt{cutC}_i$
                        Add $\texttt{cutC}_i$ to $\texttt{cutlist}^k$
                        $\Delta \leftarrow \frac{1}{|\beta_i^{\mathtt{B}} - \beta_i^{\mathtt{A}}|}$, where $\beta^{\mathtt{A}}$ and $\beta^{\mathtt{B}}$ are coefficients of $\mathbf{x}^1$ in `cutA` and `cutB` respectively
                        Apply Theorem 1 to `cutA` and `cutB` to generate $\texttt{cutD}_i$
                        Add $\texttt{cutD}_i$ to $\texttt{cutlist}^k$
                    end for
                end for
                Among `cutA` and $\texttt{cutC}_i$ and $\texttt{cutD}_i$ $\forall i \in I$, add to $L^k \geq \mathbf{0}$ the cut with the smallest

$$\frac{\delta - (\widetilde{q}^k + \sum_{i \in I} \alpha_i \widetilde{w}_i + \sum_{i \in I} \beta_i \widetilde{x}_i^1)}{1 + \sum_{i \in I} (\alpha_i)^2 + \sum_{i \in I} (\beta_i)^2}$$

            end if
        end for
    end while
    if $\widetilde{\texttt{obj}} < \texttt{optval}$ **then**
        if $\exists i : \widetilde{x}_i^1 - \lfloor \widetilde{x}_i^1 \rfloor \neq 0$ **then**
            Define $\overline{i} := \min\{i : \widetilde{x}_i^1 - \lfloor \widetilde{x}_i^1 \rfloor \neq 0\}$
            Define $\texttt{Problem}_{s+1}$ as $\texttt{Problem}_{\overline{s}}$ with additional constraint $\mathbf{x}_{\overline{i}}^1 \leq \lfloor \widetilde{x}_{\overline{i}}^1 \rfloor$
            Define $\texttt{Problem}_{s+2}$ as $\texttt{Problem}_{\overline{s}}$ with additional constraint $\mathbf{x}_{\overline{i}}^1 \geq \lceil \widetilde{x}_{\overline{i}}^1 \rceil$
            $S \leftarrow S \cup \{s+1, s+2\}$
            $s \leftarrow s + 2$
        else if $\exists i : \widetilde{w}_i - \lfloor \widetilde{w}_i \rfloor \neq 0$ **then**
            Define $\overline{i} := \min\{i : \widetilde{w}_i - \lfloor \widetilde{w}_i \rfloor \neq 0\}$
            Define $\texttt{Problem}_{s+1}$ as $\texttt{Problem}_{\overline{s}}$ with additional constraint $\mathbf{w}_{\overline{i}} \leq \lfloor \widetilde{w}_{\overline{i}} \rfloor$
            Define $\texttt{Problem}_{s+2}$ as $\texttt{Problem}_{\overline{s}}$ with additional constraint $\mathbf{w}_{\overline{i}} \geq \lceil \widetilde{w}_{\overline{i}} \rceil$
            $S \leftarrow S \cup \{s+1, s+2\}$
            $s \leftarrow s + 2$
        else
            `optval`←$\widetilde{\texttt{obj}}$
            `sol`←$(\widetilde{\mathbf{q}}, \widetilde{\mathbf{x}}^1, \widetilde{\mathbf{w}})$
        end if
    end if
    $S \leftarrow S \setminus \{\overline{s}\}$
end while
`optval` is optimal objective function value and `sol` is optimal solution