Online Appendix

The online appendix contains the proofs of all results in the paper.

A. Proofs of Results in Section 4 and Appendix A

Proof of Proposition 1. From Parlar et al. (2011) the limiting distribution of W has density

$$f(x) = \begin{cases} f(0)e^{-(\lambda-\mu)x}, & 0 < x < \gamma, \\ f(0)e^{\mu\gamma-\lambda x}, & x \geq \gamma, \end{cases} \quad \text{where,} \quad f(0) = \begin{cases} \frac{\lambda(\lambda-\mu)}{\lambda-\mu e^{-(\lambda-\mu)\gamma}}, & \lambda \neq \mu, \\ \frac{\lambda}{1+\lambda\gamma}, & \lambda = \mu. \end{cases} \tag{34}$$

The outdates occur at instances when W hits zero, the long-run average rate of which is $f(0)$. Therefore, we have $q^F = f(0)/\lambda$ which yields (4). To obtain $A^F(\cdot)$, note that the remaining age of units at the time of transfusion is embedded at epochs right before jumps on a W sample path when $0 < W < \gamma$. Thus, from (34) and using the PASTA property of the demand process we have for $x < \gamma$, $A^F(x) = \lim_{t\to\infty} P(\gamma - x \leq W(t) < \gamma)/P(0 < W(t) < \gamma) = \int_{\gamma-x}^{\gamma} f(y)dy \,/ \int_0^{\gamma} f(y)dy$, which after simplification gives (5). Using (5) the derivation of (6) is straightforward. For the case where $\lambda = \mu$, the results are obtained by letting $\lambda \to \mu$ and applying L'Hôpital's rule. □

Proof of Proposition 2. We only give a proof for the case with $\lambda/\mu > 1$. The case with $\lambda/\mu < 1$ can be shown similarly and the case with $\lambda = \mu$ is trivial noting that $\lambda q^F = \mu\ell^F = \mu/(1+\mu\gamma)$ and $E[A^F]$ is a constant. Fix $\lambda/\mu > 1$. Then using the expression for q^F in (4) and the identity in (3) relating it to ℓ^F we have

$$\mu\ell^F = \lambda q^F - (\lambda - \mu) = \frac{\mu(\lambda/\mu - 1)e^{-\mu(\lambda/\mu-1)\gamma}}{\lambda/\mu - e^{-\mu(\lambda/\mu-1)\gamma}} \leq \frac{\mu(\lambda/\mu - 1)e^{-\mu(\lambda/\mu-1)\gamma}}{\lambda/\mu - 1} = \mu e^{-\mu(\lambda/\mu-1)\gamma},$$

which goes to zero exponentially fast as $\mu \to \infty$. To see this, note that $\mu e^{-\mu(\lambda/\mu-1)\gamma} \leq C_1 e^{-C_2\mu}$ for constants $C_1, C_2 > 0$ as long as $C_1 C_2 \leq \mu(\lambda/\mu - 1)\gamma$, which clearly holds for sufficiently large μ. The results for $E[A^F]$ can be verified easily using (6). □

Proof of Lemma 1. We first derive $\hat{h}(\theta)$. Note that with probability $e^{-\lambda T}$, we have $W(0) > T$, i.e., the first jump is greater than T and hence $\tau = 0$. Conditioning on $W(0)$ we have

$$\hat{h}(\theta) \equiv E[e^{-\theta\tau} | W(\tau) > T] = \frac{E[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)>T\}}]}{P(W(\tau) > T)} = \frac{\int_0^T \hat{h}_x(\theta)\lambda e^{-\lambda x}dx + e^{-\lambda T}}{\int_0^T \hat{h}_x(0)\lambda e^{-\lambda x}dx + e^{-\lambda T}}.$$

Next, we consider $\hat{g}(\theta)$. Note that given $W(\tau) = 0$, the first jump must be some $x \leq T$. Again by conditioning on $W(0)$ we get

$$\hat{g}(\theta) \equiv E[e^{-\theta\tau} | W(\tau) = 0] = \frac{E[e^{-\theta\tau}\mathbf{1}_{\{W(\tau)=0\}}]}{P(W(\tau) = 0)} = \frac{\int_0^T \hat{g}_x(\theta)\lambda e^{-\lambda x}dx}{\int_0^T \hat{g}_x(0)\lambda e^{-\lambda x}dx}.$$

Computing the integrals and noting that $P(W(\tau) = 0) + P(W(\tau) > T) = 1$, gives the results. □

Proof of Proposition 3. Let $t = 0$ be the time the outdate occurs. Then W has just hit zero, causing a jump with size $W(0) \in (0, \infty)$ independent of the history of the process. If $W(0) > T$, then the residual busy period is over with $R = 0$. Otherwise, the residual busy period ends the first time W upcrosses T (see Figure 1 for a realization of R). However, before this happens, W may hit zero as new outdates may occur. Therefore, the probability that there are $M = m \geq 0$ outdates during the residual busy period is equal to the probability that W hits zero m times before upcrossing T. Note that using the strong Markov property of W, every time it hits zero the process regenerates and a new i.i.d. cycle starts. Also, the probability of hitting zero before crossing over T for each cycle is $p \equiv P(W(\tau) > T)$. Therefore, we have $P(M = m) = (1-p)^m p$. From Lemma 1, it then follows for the length of the residual busy period given the number of outdates, that $\hat{r}_m(\theta) \equiv E[e^{-sR} | M = m] = \hat{h}(\theta)\left(\hat{g}(\theta)\right)^m$. Removing the condition on M, we get the LT of the length of the busy period:

$$\hat{r}(\theta) \equiv E[e^{-\theta R}] = \sum_{m=0}^{\infty} P(M = m) E[e^{-\theta R} | M = m] = \sum_{m=0}^{\infty} (1-p)^m p \hat{h}(\theta)\left(\hat{g}(\theta)\right)^m = \frac{p\hat{h}(\theta)}{1 - (1-p)\hat{g}(\theta)},$$

as claimed. $\square$

Proof of Proposition 4. Each busy period starts with a fresh unit arriving at an empty system. Let $t = 0$ be the start of the busy period, then $W(0) = T$. The busy period ends the first time W upcrosses T. Each time W hits zero before this happens, a unit is outdated. Note that starting from level T two cases can occur: either W upcrosses T before hitting zero, with probability $P_T(W(\tau) > T) = \hat{h}_T(0)$, or it first hits zero, with probability $P_T(W(\tau) = 0) = \hat{g}_T(0) = 1 - \hat{h}_T(0)$. In the first case, the busy period ends with no outdates occurring during it, so $P(N = 0) = \hat{h}_T(0)$, and the LT of the conditional length of the busy period is $\hat{z}_0(\theta) \equiv E[e^{-\theta Z} | N = 0] = E_T[e^{-\theta\tau} | W(\tau) > T] = \hat{h}_T(\theta)/\hat{h}_T(0)$. In the second case after W hits zero, by the strong Markov property of W, a new i.i.d. cycle independent of the history of the process starts and the time until W upcrosses T has the same distribution as a residual busy period. Thus, for $n > 0$ we have $P(N = n) = \hat{g}_T(0) P(M = n-1) = \hat{g}_T(0) p (1-p)^{n-1}$. That is, to have n outdates during the busy period, starting from $W(0) = T$, W must first hit zero and then before crossing level T it must hit zero $n - 1$ ($n \geq 1$) additional times. It follows that $\hat{z}_n(\theta) \equiv E[e^{-\theta Z} | N = n] = E_T[e^{-s\tau} | W(\tau) = 0] E[e^{-\theta R} | M = n-1] = \left(\hat{g}_T(\theta)/\hat{g}_T(0)\right)\hat{r}_{n-1}(\theta)$. Substituting from (25) we get the result for the $n > 0$ case in (28). Finally, by the same argument and again due to the strong Markov property of the process W, given $W(0) = T$ we have $Z = \tau + \mathbf{1}_{\{W(\tau)=0\}} R$. It follows that

$$\begin{aligned} \hat{z}(\theta) \equiv E[e^{-\theta Z}] &= E_T[e^{-\theta\tau} \mathbf{1}_{\{W(\tau) > T\}}] + E_T[e^{-\theta(\tau + R)} \mathbf{1}_{\{W(\tau)=0\}}] \\ &= E_T[e^{-\theta\tau} \mathbf{1}_{\{W(\tau) > T\}}] + E[e^{-\theta R}] E_T[e^{-\theta\tau} \mathbf{1}_{\{W(\tau)=0\}}] = \hat{h}_T(\theta) + \hat{r}(\theta)\hat{g}_T(\theta). \qquad (35) \end{aligned}$$

Substituting $\hat{r}(\theta)$ from (26) we get (29) which completes the proof. $\square$

Proof of Proposition 5. The proof is elementary and hence omitted. $\square$

B. Proofs of Results in Section 5

B.1. Proofs of Theorems 1 and 2

Proof of Theorem 1. We first show that the cdf of $\tilde{S}_{2,k}$ converges to that of $\tilde{S}_2$ for all $x \in [0, \infty)$. Then, the second part follows from the continuity theorem for Laplace transforms (see Feller 2008, page 431). Let A_j denote the event that "the unit is allocated during the j^{th} idle period" and let $\bar{A}_k$ denote the event that "the unit is not allocated during any of the first k idle periods". Note that $\cup_{j=1}^{k} A_j = \{\tilde{S}_2 < U_k\}$ and $\bar{A}_k = \{\tilde{S}_2 \geq U_k\}$. Now consider the cdf of $\tilde{S}_{2,k}$. Using (13) we have

$$\begin{aligned} P(\tilde{S}_{2,k} \leq x) &= \sum_{j=1}^{k} P(\tilde{S}_{2,k} \leq x | A_j) P(A_j) + P(\tilde{S}_{2,k} \leq x | \bar{A}_k) P(\bar{A}_k) \\ &= \sum_{j=1}^{k} P(\tilde{S}_2 \leq x | A_j) P(A_j) + P(U_k \leq x | \bar{A}_k) P(\bar{A}_k). \end{aligned} \tag{36}$$

Letting $k \to \infty$ in (36), the last term on the RHS vanishes. To see this note that $\lim_{k\to\infty} P(U_k \leq x|\bar{A}_k)P(\bar{A}_k) = \lim_{k\to\infty} P(U_k \leq x, \bar{A}_k) \leq \lim_{k\to\infty} P(U_k \leq x) = 0$, where the last equality holds because U_k is the k^{th} renewal epoch of a delayed renewal process. Thus, for any $x \in [0, \infty)$ we have $\lim_{k\to\infty} P(\tilde{S}_{2,k} \leq x) = \sum_{j=1}^{\infty} P(\tilde{S}_2 \leq x|A_j)P(A_j) = P(\tilde{S}_2 \leq x)$. Note that since x is finite the last equality follows even if the cdf of $\tilde{S}_2$ is improper. Hence, the proof is complete. $\square$

Proof of Theorem 2. It suffices to show that for each $\epsilon > 0$, we have $\sum_{k=1}^{\infty} P(|S_{2,k} - S_2| \geq \epsilon) < \infty$. Then from the Borel-Cantelli Lemma we have $P(|S_{2,k} - S_2| \geq \epsilon \ i.o.) = 0$ implying that $P(S_{2,k} \to S_2) = 1$ (Billingsley 2008, page 70) as claimed. To this end, we consider the random variable $\tilde{S}_2$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and decompose the sample space Ω for $k \geq 1$ as

$$\begin{aligned} &C_{1,k} \equiv C_{2,k}\{\omega \in \Omega; \tilde{S}_2(\omega) \leq U_k(\omega)\}, \qquad C_{2,k} \equiv \{\omega \in \Omega; \gamma - T \leq U_k(\omega) < \tilde{S}_2(\omega)\}, \\ &C_{3,k} \equiv C_{4,k}\{\omega \in \Omega; U_k(\omega) < \gamma - T \leq \tilde{S}_2(\omega)\}, \qquad C_{4,k} \equiv \{\omega \in \Omega; U_k(\omega) < \tilde{S}_2(\omega) < \gamma - T\}, \end{aligned}$$

such that $\cup_{i=1}^{4} C_{i,k} = \Omega$ for any $k \geq 1$. Note that since $S_{2,k} = \min(\tilde{S}_{2,k}, \gamma - T)$, for any $\omega \in C_{1,k} \cup C_{2,k}$ we have $S_2(\omega) = S_{2,k}(\omega)$. Thus, for each $\epsilon > 0$, $\{|S_{2,k} - S_2| \geq \epsilon\} \subseteq C_{3,k} \cup C_{4,k}$, implying that for all $k \geq 1$, $P(|S_{2,k} - S_2| \geq \epsilon) \leq P(C_{3,k} \cup C_{4,k})$. Therefore,

$$\sum_{k=1}^{\infty} P(|S_{2,k} - S_2| \geq \epsilon) \leq \sum_{k=1}^{\infty} P(C_{3,k} \cup C_{4,k}) \leq \sum_{k=1}^{\infty} P(U_k \leq \gamma - T).$$

It remains for us to show that $\sum_{k=1}^{\infty} P(U_k \leq \gamma - T) < \infty$. Indeed, defining the stopping time $\sigma = \inf\{n \geq 1; U_n > \gamma - T\}$ we have $\sum_{k=1}^{\infty} P(U_k \leq \gamma - T) = \sum_{k=1}^{\infty} P(\sigma > k) = E[\sigma] < \infty$, where the inequality follows from the fact that U_k is the k^{th} renewal epoch of a delayed renewal process and hence the expected time for it to pass any constant threshold is finite. $\square$

B.2. Proof of Theorem 3 Consider the k^{th} modified system. Note that given the number of units that are in front of the tagged unit at the beginning of any idle period, its *remaining* sojourn time is independent of the past. For the k^{th} modified system, let $\hat{\varphi}^k_{\nu,i}(\theta), 1 \le i \le k+1$ denote the LT of the remaining sojourn time of the tagged unit at the beginning of the i^{th} idle period, given that it has ν units in front of it. Then, $\hat{\varphi}^k_{\nu,1}(\theta)$ is the LT of the remaining sojourn time of the unit at the beginning of the first idle period, given that the number of units moving to Stage 2 during the residual busy period is $\nu \ge 0$. Recall that M denotes the number of outdates during the residual busy period and $\hat{r}_\nu(\theta)$ denotes the LT of the length of the residual busy period given $M = \nu$. Thus, we can write the LT of $\tilde{S}_{2,k}$ as

$$E[e^{-\theta \tilde{S}_{2,k}}] = \sum_{\nu=0}^{\infty} P(M=\nu)\hat{r}_\nu(\theta)\hat{\varphi}^k_{\nu,1}(\theta). \tag{37}$$

We first express $\hat{\varphi}^k_{\nu,1}(\theta)$ for $k \ge 1$ (Proposition 6) by obtaining a recursive relation for $\hat{\varphi}^k_{\nu,i}(\theta), 1 \le i \le k+1$ (Lemma 2). With $\hat{\varphi}^k_{\nu,1}(\theta)$ in hand and using (37) we then find the LT of $\tilde{S}_{2,k}$, which after taking $k \to \infty$ and applying Theorem 1 gives us the LT of $\tilde{S}_2$ presented in Theorem 3.

The first result is a recursive relation for $\hat{\varphi}^k_{\nu,i}(\theta)$, which we then use to obtain $\hat{\varphi}^k_{\nu,1}(\theta)$.

LEMMA 2. *For $1 \le i \le k$ and $\nu \ge 0$ we have*

$$\hat{\varphi}^k_{\nu,i}(\theta) = \sum_{l=0}^{\nu}\sum_{n=0}^{\infty} P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\hat{\varphi}^k_{\nu+n-l,i+1}(\theta) + \left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \tag{38}$$

Proof of Lemma 2. Consider the tagged unit at the beginning of the i^{th} idle period with ν units in front of it. We condition on the number of demands arriving during the i^{th} idle period and consider two cases: (i) there are $L = l \le \nu$ and (ii) there are $L \ge \nu+1$ demands during the i^{th} idle period. In case (i), the unit is not allocated during the i^{th} idle period and hence will be in the system at the beginning of the $(i+1)^{st}$ idle period. Conditioning on the number of outdates during the $(i+1)^{st}$ busy period $N = n \ge 0$, the time interval between the start of the i^{th} and $(i+1)^{st}$ idle periods has LT $\hat{i}_l(\theta)\hat{z}_n(\theta)$. Also, the unit will have $\nu+n-l$ units in front of it at the beginning of the $(i+1)^{st}$ idle period and hence the LT of its remaining sojourn time at the beginning of the $(i+1)^{st}$ idle period is $\hat{\varphi}^k_{\nu+n-l,i+1}(\theta)$. It follows that the remaining sojourn time of the unit at the beginning of the i^{th} idle period given $N = n$ and $L = l \le \nu$ has LT $\hat{i}_l(\theta)\hat{z}_n(\theta)\hat{\varphi}^k_{\nu+n-l,i+1}(\theta)$. Removing the conditions on L and N, the first term on the right-hand side (RHS) follows. In case

(ii), the unit is allocated during the i^{th} idle period. Thus, its remaining sojourn is equal to the time it takes until the $(\nu+1)^{st}$ demand arrival. Note that given the idle period has not ended, the time between demand arrivals are exponentially distributed with rate $\lambda+\mu$. Thus, the time until the arrival of the $(\nu+1)^{st}$ demand is Erlang distributed with parameter $(\lambda+\mu)$ and $(\nu+1)$ phases, and hence its LT is given by

$$\left(\frac{\lambda+\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \tag{39}$$

Also, from Proposition 5 the event $\{L \geq \nu+1\}$ has probability

$$\sum_{l=\nu+1}^{\infty} P(L=l) = 1 - \sum_{l=0}^{\nu} \left(\frac{\mu}{\mu+\lambda}\right)^{l} \left(\frac{\lambda}{\mu+\lambda}\right) = \left(\frac{\mu}{\mu+\lambda}\right)^{\nu+1},$$

which after being multiplied by (39) gives the second term on the RHS. $\square$

Note that by definition of $\hat{\varphi}^k_{\nu,i}(\theta)$ we have $\hat{\varphi}^k_{\nu,k+1}(\theta)=1$ for all $\nu \geq 0$. Thus, for a given k and starting from $i=k$ one can use Lemma 2 to recursively solve for $\hat{\varphi}^k_{\nu,1}(\theta)$. Define the nested sum $Y_\nu(d)$ for non-negative integers ν and d as

$$Y_\nu(d) \equiv \sum_{j_0=0}^{0} \xi_{j_0}(\theta) \sum_{j_1=0}^{1} \xi_{j_1}(\theta) \sum_{j_2=0}^{2-j_1} \xi_{j_2}(\theta) \sum_{j_3=0}^{3-j_2-j_1} \xi_{j_3}(\theta) \cdots \sum_{j_{d-1}=0}^{(d-1)-j_{d-2}-\cdots-j_1} \xi_{j_{d-1}}(\theta) \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1} \binom{\nu+1}{d-j_{d-1}-\cdots-j_1}, \tag{40}$$

where we adopt the convention that an empty sum is equal to 0 and an empty product equal to 1. Note that d is the number of sums in $Y_\nu(d)$. The next proposition expresses $\hat{\varphi}^k_{\nu,1}(\theta)$ as a function of k.

Proposition 6. *The LT of the remaining sojourn of a unit at the beginning of the first idle period in the k^{th} modified system, given it has $\nu \geq 0$ units in front of it, is given by*

$$\hat{\varphi}^k_{\nu,1}(\theta) = \sum_{i=0}^{k-1} Y_\nu(i) \left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right) \left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i} + \left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k}. \tag{41}$$

Before proving the result we need the following lemma.

Lemma 3. *For $\nu, i \geq 0$ the following identity holds:*

$$\sum_{l=0}^{\nu} \sum_{n=0}^{\infty} P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)Y_{\nu+n-l}(i) = \left(\frac{\lambda}{\lambda+\mu+\theta}\right) Y_\nu(i+1). \tag{42}$$

Proof. Substituting for $P(L=l), \hat{i}_l(\theta)$ and $Y_{\nu+n-l}(i)$ into the LHS from (31), (32) and (40), respectively and changing the order of sums, after some simplifications we can rewrite the LHS as

$$\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\sum_{j_2=0}^{2-j_1}\xi_{j_2}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)$$
$$\sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{n}P(N=n)\hat{z}_n(\theta)\sum_{l=0}^{\nu}\binom{\nu+n-l+1}{i-j_{i-1}-\cdots-j_1}. \tag{43}$$

Using Lemma 4 the last sum is

$$\sum_{l=0}^{\nu}\binom{\nu+n-l+1}{i-j_{i-1}-\cdots-j_1}=\sum_{j_i=0}^{i-j_{i-1}-\cdots-j_1}\binom{\nu+1}{(i+1)-j_i-j_{i-1}-\cdots-j_1}\binom{n+1}{j_i},$$

which allows us to rewrite (43) as

$$\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\sum_{j_2=0}^{2-j_1}\xi_{j_2}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)$$
$$\sum_{j_i=0}^{i-j_{i-1}-\cdots-j_1}\sum_{n=0}^{\infty}\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{n}P(N=n)\hat{z}_n(\theta)\binom{n+1}{j_i}\binom{\nu+1}{(i+1)-j_i-\cdots-j_1}. \tag{44}$$

Rearranging the terms in (44) and using (59) we arrive at the RHS:

$$\left(\frac{\lambda}{\lambda+\mu+\theta}\right)\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\sum_{j_2=0}^{2-j_1}\xi_{j_2}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)$$
$$\sum_{j_i=0}^{i-j_{i-1}-\cdots-j_1}\xi_{j_i}(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\binom{\nu+1}{(i+1)-j_i-\cdots-j_1}=\left(\frac{\lambda}{\lambda+\mu+\theta}\right)Y_\nu(i+1).$$

This completes the proof. ☐

Proof of Proposition 6. The proof is by induction on k. For $k=1$, noting that $\hat{\varphi}^{k}_{\nu,k+1}(\theta)=1$ and using (38) we have

$$\hat{\varphi}^{1}_{\nu,1}(\theta)=\sum_{l=0}^{\nu}P(L=l)\hat{i}_l(\theta)\sum_{n=0}^{\infty}P(N=n)\hat{z}_n(\theta)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \tag{45}$$

Noting that

$$\sum_{l=0}^{\nu}P(L=l)\hat{i}_l(\theta)=\left(\frac{\lambda}{\mu+\lambda}\right)\left(1-\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\right), \tag{46}$$

and using (30), we can simplify (45) to obtain

$$\begin{aligned}\hat{\varphi}^{1}_{\nu,1}(\theta)&=\left(1-\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\right)\hat{i}(\theta)\hat{z}(\theta)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\\&=\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}(1-\hat{i}(\theta)\hat{z}(\theta))+\hat{i}(\theta)\hat{z}(\theta)=Y_\nu(0)(1-\hat{i}(\theta)\hat{z}(\theta))+\hat{i}(\theta)\hat{z}(\theta),\end{aligned}$$

which establishes (41) for $k=1$. No assume that (41) holds for some $k\geq 1$. From (38) we have

$$\hat{\varphi}_{\nu,1}^{k+1}(\theta)=\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\hat{\varphi}_{\nu+n-l,2}^{k+1}(\theta)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}. \tag{47}$$

Note that by construction of the k^{th} modified system we have $\hat{\varphi}_{\nu+n-l,2}^{k+1}(\theta)=\hat{\varphi}_{\nu+n-l,1}^{k}(\theta)$. Substituting $\hat{\varphi}_{\nu+n-l,1}^{k}(\theta)$ for $\hat{\varphi}_{\nu+n-l,2}^{k+1}(\theta)$ in (47) and using (41) we get

$$\begin{aligned}\hat{\varphi}_{\nu,1}^{k+1}(\theta)&=\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\\&\quad\times\left(\sum_{i=0}^{k-1}Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}+(\hat{i}(\theta)\hat{z}(\theta))^{k}\right)+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\\&=\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\sum_{i=0}^{k-1}Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}\\&\quad+\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)(\hat{i}(\theta)\hat{z}(\theta))^{k}+\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}.\end{aligned} \tag{48}$$

We start with the first term in (48). Rearranging the terms and applying Lemma 3 we have

$$\begin{aligned}&\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)\sum_{i=0}^{k-1}Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}\\&=\sum_{i=0}^{k-1}\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)Y_{\nu+n-l}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}\\&=\sum_{i=0}^{k-1}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)Y_{\nu}(i+1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}\\&=\sum_{i=1}^{(k+1)-1}Y_{\nu}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{(k+1)-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}.\end{aligned} \tag{49}$$

The second term in (48) can be evaluated by rearranging the sums and using (46):

$$\sum_{l=0}^{\nu}\sum_{n=0}^{\infty}P(L=l)P(N=n)\hat{i}_l(\theta)\hat{z}_n(\theta)(\hat{i}(\theta)\hat{z}(\theta))^{k}=\left(1-\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\right)\left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k+1}. \tag{50}$$

Finally, substituting (49) and (50) into (48) we get

$$\begin{aligned}\hat{\varphi}_{\nu,1}^{k+1}(\theta)&=\left(\frac{\mu}{\mu+\lambda+\theta}\right)^{\nu+1}\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k+1}\right)\\&\quad+\sum_{i=1}^{(k+1)-1}Y_{\nu}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{(k+1)-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}+\left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k+1}\\&=\sum_{i=0}^{(k+1)-1}Y_{\nu}(i)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{(k+1)-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^{i}+\left(\hat{i}(\theta)\hat{z}(\theta)\right)^{k+1}.\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 3. Substituting $\hat{\varphi}^k_{\nu,1}(\theta)$ from (41) into (37) we have

$$E[e^{-\theta\tilde{S}_{2,k}}]=\sum_{i=0}^{k-1}\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)Y_\nu(i)\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i+\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)(\hat{\imath}(\theta)\hat{z}(\theta))^k. \tag{51}$$

We deal with the two terms separately. First, substituting for $Y_\nu(i)$ in the first term and using Lemma 5, we have

$$\begin{aligned}
&\sum_{\nu=0}^{\infty}\sum_{i=0}^{k-1}P(M=\nu)\hat{r}_\nu(\theta)Y_\nu(i)\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\
&=\sum_{i=0}^{k-1}\Bigg[\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\binom{\nu+1}{i-j_{i-1}-\cdots-j_1}\\
&\quad\times\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\Bigg]=\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k}\right)\\
&\quad+\sum_{i=1}^{k-1}\Bigg[\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)\left(\frac{\mu}{\lambda+\mu+\theta}\right)^{\nu+1}\binom{\nu+1}{i-j_{i-1}-\cdots-j_1}\\
&\quad\times\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\Bigg]=\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)\\
&\quad+\sum_{i=1}^{k-1}\Bigg[\xi_0(\theta)\sum_{j_1=0}^{1}\xi_{j_1}(\theta)\cdots\sum_{j_{i-1}=0}^{(i-1)-j_{i-2}-\cdots-j_1}\xi_{j_{i-1}}(\theta)\beta_{i-j_{i-1}-\cdots-j_1}(\theta)\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\Bigg].
\end{aligned}\tag{52}$$

Using the definition of $X(d,w)$ in (18) for $w=1$, (52) becomes

$$\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)+\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i. \tag{53}$$

Next, the second term in (51) is simply

$$\sum_{\nu=0}^{\infty}P(M=\nu)\hat{r}_\nu(\theta)(\hat{\imath}(\theta)\hat{z}(\theta))^k=(\hat{\imath}(\theta)\hat{z}(\theta))^k\hat{r}(\theta). \tag{54}$$

Substituting (53) and (54) back into (51) we have

$$\begin{aligned}
E[e^{-\theta\tilde{S}_{2,k}}]&=(\hat{\imath}(\theta)\hat{z}(\theta))^k\hat{r}(\theta)+\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)\\
&\quad+\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i
\end{aligned}\tag{55}$$

$$\begin{aligned}
&=(\hat{\imath}(\theta)\hat{z}(\theta))^k\hat{r}(\theta)+\left(1-(\hat{\imath}(\theta)\hat{z}(\theta))^{k}\right)\beta_0(\theta)\\
&\quad+\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i-\xi_0(\theta)\sum_{i=1}^{k-1}X(i,1)(\hat{\imath}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i.
\end{aligned}\tag{56}$$

Letting $k\to\infty$ in (56), the first term goes to 0 and the second term converges to $\beta_0(\theta)$. Therefore, to get the final result it remains to show that for all $\theta>0$ the last term converges to 0 as $k\to\infty$. To this end, consider $E[e^{-\theta\tilde{S}_{2,k}}]$ in (55) and note that it converges if and only if the sequence $\{F_k\}_{k\geq 2}$ with $F_k \equiv \sum_{i=1}^{k-1} X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i$, is convergent. However, from Theorem 1 we know that $E[e^{-\theta\tilde{S}_{2,k}}]$ converges and therefore $\{F_k\}$ is indeed convergent. Thus, we have $|F_{k+1}-F_k|\to 0$ as $k\to\infty$. Now observe that

$$\begin{aligned}
F_{k+1}-F_k &= \sum_{i=1}^{k} X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k+1-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i - \sum_{i=1}^{k-1} X(i,1)\left(1-(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\right)\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\
&= X(k,1)(1-\hat{i}(\theta)\hat{z}(\theta))\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^k + (1-\hat{i}(\theta)\hat{z}(\theta))\sum_{i=1}^{k-1} X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i\\
&= (1-\hat{i}(\theta)\hat{z}(\theta))\sum_{i=1}^{k} X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i,
\end{aligned}$$

and hence, $\lim_{k\to\infty}\sum_{i=1}^{k} X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i = 0$, for all $\theta>0$. Finally, noting that for $k\geq 2$, $\sum_{i=1}^{k} X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i \geq \sum_{i=1}^{k-1} X(i,1)(\hat{i}(\theta)\hat{z}(\theta))^{k-i}\left(\frac{\lambda}{\lambda+\mu+\theta}\right)^i$, we can conclude that the last term in (56) vanishes as $k\to\infty$, which completes the proof. □

C. Side Lemmas

LEMMA 4. *For $\omega\geq 0$ we have*

$$\sum_{l=0}^{v}\binom{v+n-l+1}{\omega} = \sum_{\kappa=0}^{\omega}\binom{v+1}{\omega+1-\kappa}\binom{n+1}{\kappa}. \tag{57}$$

Proof. To save space in this proof we use $\mathcal{C}(a,b)$ to denote the Binomial coefficient $\binom{a}{b}$. Starting from the left-hand side (LHS) of (57) we first claim that

$$\sum_{l=0}^{v}\mathcal{C}(v+n-l+1,\omega) = \mathcal{C}(v+n+2,\omega+1)-\mathcal{C}(n+1,\omega+1), \tag{58}$$

which can be proved by induction on v. For $v=0$ (58) becomes $\mathcal{C}(n+1,\omega)=\mathcal{C}(n+1,\omega+1)-\mathcal{C}(n+1,\omega+1)$, which is the Pascal's recurrence (see, e.g., Gross 2007, page 218). Now assume for some $v\geq 1$ that $\sum_{l=0}^{v-1}\mathcal{C}(v+n-l,\omega)=\mathcal{C}(v+n+1,\omega+1)-\mathcal{C}(n+1,\omega+1)$, then

$$\begin{aligned}
\sum_{l=0}^{v}\mathcal{C}(v+n-l+1,\omega) &= \mathcal{C}(v+n+1,\omega)+\sum_{l=1}^{v}\mathcal{C}(v+n-l+1,\omega) = \mathcal{C}(v+n+1,\omega)+\sum_{l=0}^{v-1}\mathcal{C}(v+n-l,\omega)\\
&= \mathcal{C}(v+n+1,\omega)+\mathcal{C}(v+n+1,\omega+1)-\mathcal{C}(n+1,\omega+1) \quad \text{(induction hypothesis)}\\
&= \mathcal{C}(v+n+2,\omega+1)-\mathcal{C}(n+1,\omega+1), \quad \text{(Pascal's recurrence)}
\end{aligned}$$

as claimed. Next, applying Vandermonde's convolution (see, e.g., Gross 2007, page 226) to the first term we get $\mathcal{C}(v+n+2,\omega+1)=\sum_{\kappa=0}^{\omega+1}\mathcal{C}(v+1,\omega+1-\kappa)\mathcal{C}(n+1,\kappa)=\sum_{\kappa=0}^{\omega}\mathcal{C}(v+1,\omega+1-\kappa)\mathcal{C}(n+1,\kappa)+\mathcal{C}(n+1,\omega+1)$, which after substituting in (58) gives (57). □

Lemma 5. *For $i \geq 0$ we have*

$$\sum_{n=0}^{\infty} \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{n} P(N=n)\hat{z}_n(\theta) \binom{n+1}{i} = \xi_i(\theta), \tag{59}$$

and

$$\sum_{m=0}^{\infty} \left(\frac{\mu}{\lambda+\mu+\theta}\right)^{m+1} P(M=m)\hat{r}_m(\theta) \binom{m+1}{i} = \beta_i(\theta) \tag{60}$$

with $\xi_i(\theta)$ and $\beta_i(\theta)$ given in (16) and (17), respectively.

Proof. We give a proof for (60); (59) can be obtained similarly. Substituting for $P(M=l)$ and $\hat{r}_m(\theta)$ from (24) and (25) into (60), and using the definitions in (15) the LHS becomes

$$c_1(\theta) \sum_{m=0}^{\infty} (c_2(\theta))^m \binom{m+1}{i},$$

establishing (60) for $i=0$. To obtain the formula for $i \geq 1$, note that

$$\begin{aligned}\frac{i!}{(1-c_2(\theta))^{i+1}} = \frac{d^i}{d\,(c_2(\theta))^i} \sum_{m=0}^{\infty} (c_2(\theta))^m &= \sum_{m=0}^{\infty} \frac{d^i}{d\,(c_2(\theta))^i} (c_2(\theta))^m \\ &= \sum_{m=0}^{\infty} m(m-1)\cdots(m-i+1)\,(c_2(\theta))^{m-i} \\ &= \sum_{m=i}^{\infty} m(m-1)\cdots(m-i+1)\,(c_2(\theta))^{m-i}.\end{aligned}$$

Multiplying both sides by $c_2(\theta)^{i-1}/i!$ yields:

$$\frac{(c_2(\theta))^{i-1}}{(1-c_2(\theta))^{i+1}} = \sum_{m=i}^{\infty} (c_2(\theta))^{m-1} \binom{m}{i} = \sum_{m=i-1}^{\infty} (c_2(\theta))^m \binom{m+1}{i} = \sum_{m=0}^{\infty} (c_2(\theta))^m \binom{m+1}{i},$$

from which (60) follows. □