Appendix
Technical Details and Proofs

Proof of Lemma 1: In this proof, we consider a cluster C_j of size s and its retailers $1, 2, \dots, s$ in a market with conditions c, v, and t. We find the general formula for Retailer i's profit Π_i for $i = 1, 2, \dots, s$, show that there exists a unique symmetric price equilibrium under profit maximization, and obtain an implicit function for the equilibrium price through the first order condition.

The General Formula for a Retailer i's Profit Π_i. To simplify notation, we define $p_{s+1} = v$ then $p_i \leq p_{s+1} = v$ for $i = 1, 2, \dots, s$. To handle possible repeats in the multiset $[p_1, p_2, \dots, p_s]$ we define, for any $1 \leq \gamma \leq s$, the number $\alpha(\gamma) = \min\{1 \leq m \leq s+1 : p_m > p_\gamma\}$. Fix $i \leq r \leq s$ then for $1 - \frac{p_{\alpha(\gamma)} - p_i}{v} < \beta_i < 1 - \frac{p_\gamma - p_i}{v}$ we have: $\min\left\{1, \beta_i + \frac{p_m - p_i}{v}\right\} = \begin{cases} \beta_i + \frac{p_m - p_i}{v} & \text{for } m \in \{1, 2, \dots, \gamma\} \setminus i \\ 1 & \text{for } m = \gamma+1, \gamma+2, \dots, s \end{cases}$

$$Prob\,[\,\beta_i v - p_i \geq, \quad \beta_m v - p_m \leq \beta_i v - p_i \text{ for } m \in \{1, 2, \cdots s\} \setminus i]$$

$$= \sum_{[p_\gamma : p_\gamma \geq p_i, 1 \leq \gamma \leq s]} \int_{\beta_i = 1 - \frac{p_{\alpha(\gamma)} - p_i}{v}}^{1 - \frac{p_\gamma - p_i}{v}} \prod_{[p_m : p_m \leq p_\gamma] \setminus p_i} \left(\beta_i + \tfrac{p_m - p_i}{v}\right) d\beta_i$$

So the profit for retailer i is $\Pi_i = (p_i - c) \cdot D_j \cdot \sum_{[p_\gamma : p_\gamma \geq p_i, 1 \leq \gamma \leq s]} \int_{\beta_i = 1 - \frac{p_{\alpha(\gamma)} - p_i}{v}}^{1 - \frac{p_\gamma - p_i}{v}} \prod_{[p_m : p_m \leq p_\gamma] \setminus p_i} \left(\beta_i + \tfrac{p_m - p_i}{v}\right) d\beta_i$

Note that the profit function is dependent only on p_i and the values of $p_m - p_i$ for $m = 1, 2, ..., \widehat{i}, ..., s$. This is consistent with the fact that the profit of retailer is only dependent on the size of its pricing relative to the other retailer in the same cluster. In other words, reordering the prices amongst the retailers will still yield the same set of profit functions $\{\Pi_i; i = 1, \dots, s\}$. Thus we may, without loss of generality, assume that $p_1 \leq \cdots \leq p_s < p_{s+1} = v$. Next we characterize the optimal prices for the retailers. First we compute $\frac{\partial \Pi_i}{\partial p_i}$. Let $e_m(x_1, ..., x_i)$ denote the elementary symmetric polynomial in $x_1, ..., x_i$ of order m.

$$\Pi_i = (p_i - c) \cdot D_j \cdot \textstyle\sum_{m=1}^{i} \frac{1}{i-m+1} \left[1 - \left(1 - \frac{p_{\alpha(i)} - p_i}{v}\right)^{i-m+1}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_{i-1} - p_i}{v}\right)$$
$$+ (p_i - c) \cdot D_j \cdot \textstyle\sum_{m=1}^{s} \frac{1}{s-m+1} \left[\left(1 - \frac{p_s - p_i}{v}\right)^{s-m+1} - \left(\frac{p_i}{v}\right)^{s-m+1}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_s - p_i}{v}\right)$$
$$+ (p_i - c) \cdot D_j \cdot \textstyle\sum_{r=i+1}^{s-1} \sum_{m=1}^{r} \frac{1}{\gamma-m+1} \left[\left(1 - \frac{p_\gamma - p_i}{v}\right)^{\gamma-m+1} - \left(1 - \frac{p_{\alpha(\gamma)} - p_i}{v}\right)^{\gamma-m+1}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_\gamma - p_i}{v}\right)$$

$$\frac{\partial \Pi_i}{\partial p_i} = D_j \cdot \textstyle\sum_{m=1}^{i} \frac{1}{i-m+1} \left[1 - \left(1 - \frac{p_{\alpha(i)} - p_i}{v}\right)^{i-m+1}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_{i-1} - p_i}{v}\right)$$
$$- (p_i - c) \cdot D_j \cdot \textstyle\sum_{m=1}^{i} \frac{1}{v} \left(1 - \frac{p_{\alpha(i)} - p_i}{v}\right)^{i-m} e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_{i-1} - p_i}{v}\right)$$
$$+ (p_i - c) \cdot D_j \cdot \textstyle\sum_{m=1}^{i} \frac{1}{i-m+1} \left[1 - \left(1 - \frac{p_{\alpha(i)} - p_i}{v}\right)^{i-m+1}\right] \frac{\partial}{\partial p_i} e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_{i-1} - p_i}{v}\right)$$
$$+ D_j \cdot \textstyle\sum_{m=1}^{s} \frac{1}{s-m+1} \left[\left(1 - \frac{p_s - p_i}{v}\right)^{s-m+1} - \left(\frac{p_i}{v}\right)^{s-m+1}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_s - p_i}{v}\right)$$
$$+ (p_i - c) \cdot D_j \cdot \textstyle\sum_{m=1}^{s} \frac{1}{v} \left[\left(1 - \frac{p_s - p_i}{v}\right)^{s-m} - \left(\frac{p_i}{v}\right)^{s-m}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_s - p_i}{v}\right)$$
$$+ (p_i - c) \cdot D_j \cdot \textstyle\sum_{m=1}^{s} \frac{1}{s-m+1} \left[\left(1 - \frac{p_s - p_i}{v}\right)^{s-m+1} - \left(\frac{p_i}{v}\right)^{s-m+1}\right] \frac{\partial}{\partial p_i} e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_s - p_i}{v}\right)$$
$$+ D_j \cdot \textstyle\sum_{r=i+1}^{s-1} \sum_{m=1}^{r} \frac{1}{\gamma-m+1} \left[\left(1 - \frac{p_\gamma - p_i}{v}\right)^{\gamma-m+1} - \left(1 - \frac{p_{\alpha(\gamma)} - p_i}{v}\right)^{\gamma-m+1}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_\gamma - p_i}{v}\right)$$
$$+ (p_i - c) \cdot D_j \cdot \textstyle\sum_{r=i+1}^{s-1} \sum_{m=1}^{r} \frac{1}{v} \left[\left(1 - \frac{p_\gamma - p_i}{v}\right)^{r-m} - \left(1 - \frac{p_{\alpha(\gamma)} - p_i}{v}\right)^{r-m}\right] e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_\gamma - p_i}{v}\right)$$
$$+ (p_i - c) \cdot D_j \cdot \textstyle\sum_{r=i+1}^{s-1} \sum_{m=1}^{r} \frac{1}{\gamma-m+1} \left[\left(1 - \frac{p_\gamma - p_i}{v}\right)^{\gamma-m+1} - \left(1 - \frac{p_{\alpha(\gamma)} - p_i}{v}\right)^{\gamma-m+1}\right] \frac{\partial}{\partial p_i} e_{m-1}\left(\frac{p_1 - p_i}{v}, \dots, \frac{p_\gamma - p_i}{v}\right)$$

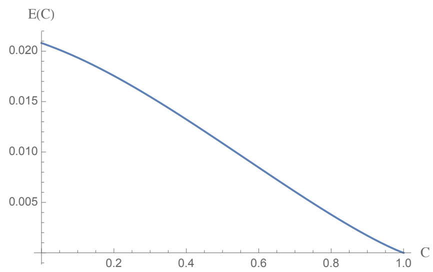
Existence of Unique Price Equilibrium That is Symmetric. We note that the profit Π_i depends on the values of p_i and $Q_{iw} = p_w - p_i$ for $w = 1, 2, \cdots, \widehat{i}, \cdots, s$. Direct computation gives $\frac{\partial \Pi_i}{\partial Q_{iw}} = \frac{1}{v}\Pi_i(p_1, p_2, \cdots, \widehat{p_w}, \cdots, p_s)$ for each fixed $i \in \{2, \cdots, s\}$ and $1 \leq w < i$. Here $\Pi_i(p_1, p_2, \cdots, \widehat{p_w}, \cdots, p_s)$ denotes the profit of Retailer i for the cluster $C_j \setminus w$. Thus $\frac{\partial \Pi_s}{\partial Q_{sw}} > 0$ for all $1 \leq w \leq s-1$ whenever retailer s has the largest price p_s in the cluster. So Π_s cannot maximize at any points $(p_1, p_2, \cdots, p_s)$ such that $p_s > p_w$ for some $1 \leq w < s$. This implies that the profit functions Π_i could only simultaneously maximize if $p_1 = p_2 = \cdots = p_s$. Next, we find the equation $\frac{\partial \Pi_i}{\partial p_i}(p, \ldots, p) = 0$. Using $e_0 \equiv 1$ and for $m \neq 1$, $e_{m-1}(0, \ldots, 0) = 0 = \frac{\partial}{\partial p_i} e_m(0, \ldots, 0)$ and $\frac{\partial}{\partial p_i} e_1(0, \ldots, 0)\Big|_{p_1 = \ldots = p_\gamma} = \frac{1-\gamma}{v}$, we obtain $\frac{\partial \Pi_i}{\partial p_i}(p, \ldots, p) = -\frac{D_j(p-c)}{v} + \frac{D_j}{s}\left[1 - \frac{p^s}{v^s}\right] = 0$ which gives $p^s = v^{s-1}[v - sp + sc]$. Dividing by v^s, we have $\left(\frac{p}{v}\right)^s = -s\left(\frac{p}{v}\right) + \frac{sc}{v} + 1$. The right hand side of the equation is a linear function in $\frac{p}{v}$ while the left hand side is the monomial $\left(\frac{p}{v}\right)^s$. Thus by graphical considerations there can only be one positive unique p given $s \geq 1$ and c. Moreover, the profit for retailer i at equal pricing p reduces to $\Pi_i(p, \ldots, p) = \frac{D_j(p-c)}{s}\left[1 - \frac{p^s}{v^s}\right] = \Pi(p)$. The second derivative $\frac{d^2\Pi}{dp^2} = -\frac{p^{s-1}}{v^s} - \frac{(s-1)D_j(p-c)p^{s-2}}{v^s} + \frac{D_j}{s}\left(-\frac{sp^{s-1}}{v^s}\right) < 0$. Thus the critical point $p = p^*$ is a unique local maximum for Π and hence also a global maximum. Using $(p^*)^s = v^{s-1}[v - sp^* + sc]$, we obtain $\Pi(p^*) = \frac{D_j}{v}(p^* - c)^2$.

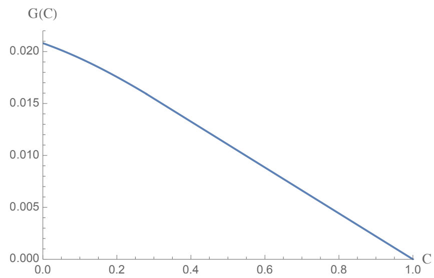
Monotonicity of Optimal Price vs s, c, and v. Set $P = p/v$ and $C = c/v$ then $p^s = v^s - sv^{s-1}p + sv^{s-1}c$ reduces to $P^s = 1 - sP + sC$. Here we treat $P(s) = p(c, v, s)/v$ as function of s. Apply Logarithmic differentiation and $P^s = 1 - sP + sC$ to obtain: $\frac{dP}{ds} = \frac{-1 + P^s(1 - \ln(P^s))}{s^2(P^{s-1} + 1)}$. Note that $0 < x(1 - \ln(x)) < 1$ for $0 < x < 1$. Thus for $0 < P < 1$ and $s > 0$ we have $0 < P^s(1 - \ln(P^s)) < 1$. So $\frac{dP}{ds} = \frac{1}{v}\frac{dp}{ds} < 0$. Therefore, the equilibrium price decreases in s. From $p^s = v^s - sv^{s-1}p + sv^{s-1}c$, $\frac{\partial p}{\partial c} = \frac{v^{s-1}}{p^{s-1} + v^{s-1}} = \frac{1}{(p/v)^{s-1} + 1} > 0$ and $\frac{\partial p}{\partial v} = \frac{v^{s-1} - (s-1)v^{s-2}p + (s-1)v^{s-2}c}{p^{s-1} + v^{s-1}}$. Thus the equilibrium price increases in c. We also have $p(v, c, s)^s = v^s - sv^{s-1}p(v, c, s) + sv^{s-1}c$ and $p(v, c, s) < p(v, c, s-1)$. Thus, for each $v > 0$ we have $v^{s-1} - (s-1)v^{s-2}p(v, c, s) + (s-1)v^{s-2}c > v^{s-1} - (s-1)v^{s-2}p(v, c, s-1) + (s-1)v^{s-2}c = p(v, c, s-1)^{s-1} > 0$. Thus $\frac{\partial p}{\partial v} > 0$. Thus the equilibrium price increases in v.

Proof of Lemma 2: We will first derive the expression for r_n in the non-overlapping case and then show that r_n increases in cluster size s and is independent of n. Let C_j be a cluster of size s. Backward substitution from Stage 3 fixes prices p_i for $i \in C_j$ at optimal price p. For $i \in C_j$, let $Y_i = \beta_i v - p$ where β_i are iid and uniformly distributed over $[0, 1]$. Thus Y_i are iid and uniformly distributed over $[-p, v-p]$. Let $X = \max\left\{0, \max_{i \in C_j}(Y_i)\right\}$. Then X is a random variable over $[0, v-p]$. For any $0 \leq x \leq v-p$, we compute and obtain $F_X(x) = \frac{(x+p)^s}{s!(v^s)}$. Therefore, $f_X(x) = \frac{d}{dx}[F_X(x)] = \frac{s(x+p)^{s-1}}{v^s}$. Hence, $E[X] = \int_0^{v-p} x f_X(x)\, dx = \frac{sv}{s+1} - p + \frac{p^{s+1}}{(s+1)v^s}$. But p satisfies $p^s = v^{s-1}[v - sp + sc]$, so $E[X] = \frac{sv}{(1+s)}\left[1 - \frac{p}{v} - \frac{p}{v}\left(\frac{p-c}{v}\right)\right]$. At optimal r_n, $\|x - L_j\| = r_n$ and $E[U(x, C_j)] = 0$. This gives $r_n = \frac{sv}{t(1+s)}\left[1 - \frac{p}{v} - \frac{p}{v}\left(\frac{p-c}{v}\right)\right]$ which is independent of n. We compute $\frac{dr_n}{ds} = \frac{r_n}{s(1+s)} - \frac{sv}{t(1+s)}\left[\frac{1}{v} + \frac{p}{v^2} + \frac{(p-c)}{v^2}\right]\frac{dp}{ds}$. Since $p > c$ and $\frac{dp}{ds} < 0$, we have $\frac{dr_n}{ds} > 0$. So r_n is increases in s. We compare and contrast that $r_o = \frac{s}{2n}$ which is independent of t and increasing in s.

Proof of Proposition 1: In this proof, we first estimate the constraint curve $P^s + s(P-C) - 1 = 0$ where $P = p/v$ and $C = c/v$. Then show that there exists a unique optimal s^* that solves Problem (7).

Estimating the Constraint Curve. By Galois Theory, we cannot solve for P in the polynomial $P^s + s(P-C) - 1 = 0$ for given C and any $s \geq 5$. Thus we apply Newton's Method to find estimates for the positive real root P that satisfies $P^s + s(P-C) - 1 = 0$. Note that the feasible root P is between $C < P \leq \frac{1+C}{2} < 1$ for all $s \geq 1$. Here $P = \frac{1+C}{2}$ is the solution of the constraint equation when $s = 1$. $P > C$ is observed from the graphs of P^s and $1 - s(P-C)$. If we apply Newton's Method starting at $P = 0$ to find the positive root of $P^s + s(P-C) = 0$, then the first iterate is $P \approx C + \frac{1}{s}$. This is a good estimate for large s. For smaller values of s, we use the tangent line to the curve $P^s + s(P-C) - 1 = 0$ at the point $(P,s) = \left(\frac{1+C}{2}, 1\right)$. The equation of this tangent is given by $P = \frac{C+1}{2} + \left(\frac{C-1}{4} - \frac{C+1}{4}\ln\left(\frac{C+1}{2}\right)\right)(s-1)$. We may also write $s \approx 1 + \frac{2(1+C-2P)}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}$ for smaller s. We solve simultaneously the equations $P = \frac{C+1}{2} + \left(\frac{C-1}{4} - \frac{C+1}{4}\ln\left(\frac{C+1}{2}\right)\right)(s-1)$ and $P = C + \frac{1}{s}$ to find the intersection point: $P = F(C) = \frac{1}{8}\left(3 + 5C + (C+1)\ln\left(\frac{C+1}{2}\right) - \sqrt{A(C)}\right)$ and $s = \frac{1}{(F(C)-C)}$ where $A(C) = (C-1)(9C+7) + (C+1)\ln\left(\frac{C+1}{2}\right)\left((C+1)\ln\left(\frac{C+1}{2}\right) - 2(3C+5)\right)$. Then the estimate for P given s and C is: $P \approx C + \frac{1}{s}$ for $\frac{1}{F(C)-C} \leq s < +\infty$ and $P \approx \frac{C+1}{2} + \left(\frac{C-1}{4} - \frac{C+1}{4}\ln\left(\frac{C+1}{2}\right)\right)(s-1)$ for $1 \leq s \leq \frac{1}{F(C)-C}$. Making s the subject we get: $s \approx \frac{1}{P-C}$ for $C < P \leq F(C)$ and $s \approx 1 + \frac{2(1+C-2P)}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}$ for $F(C) \leq P \leq \frac{C+1}{2}$. We numerical compute the maximum error in using our estimator for P. Below are the graphs of the maximal error $E(C)$ for using the estimate $P = C + \frac{1}{s}$ and the maximal error $G(C)$ for using $P = \frac{C+1}{2} + \left(\frac{C-1}{4} - \frac{C+1}{4}\ln\left(\frac{C+1}{2}\right)\right)(s-1)$. We also numerically verify that $|P - (C + \frac{1}{s})| \leq E(C)e^{-(1-C)(s - \frac{1}{F(C)-C})}$ for any $0 \leq C \leq 1$ and $\frac{1}{F(C)-C} \leq s < \infty$ and $|P - (\frac{C+1}{2} + \left(\frac{C-1}{4} - \frac{C+1}{4}\ln\left(\frac{C+1}{2}\right)\right)| \leq G(C) - \frac{G(C)(F(C)-C)^2}{(F(C)-C-1)^2}\left(s - \frac{1}{F(C)-C}\right)^2$ for any $0 \leq C \leq 1$ and $1 \leq s \leq \frac{1}{F(C)-C}$.





Finally we discuss the existence of unique solution. Using the estimate for s above, we obtain an estimate for the profit function. Specifically, for the non-overlap case we have: $\Pi_n \approx \Pi_{1,n}(P,C) = \frac{2(P-C)^2(P(C-P-1)+1)}{P-C+1}$ for $C \leq P \leq F(C)$ and $\Pi_n \approx \Pi_{2,n}(P,C) = \frac{(P-C)^2(P(C-P-1)+1)\left(C+(C+1)\log\left(\frac{C+1}{2}\right)-4P+3\right)}{(C+1)\log\left(\frac{C+1}{2}\right)-2P+2}$ for $F(C) \leq P \leq \frac{C+1}{2}$. For the overlap case we have: $\Pi_o \approx \Pi_{1,o}(P,C) = P - C$ for $C \leq P \leq F(C)$ and $\Pi_o \approx \Pi_{2,o}(P,C) = (P-C)^2\left(\frac{2(C-2P+1)}{1-C+(C+1)\ln\left(\frac{C+1}{2}\right)} + 1\right)$ for $F(C) \leq P \leq \frac{C+1}{2}$.

Let's work on Π_n first. The equation $\frac{d\Pi_{1,n}}{dP} = 0$ gives only one possible critical point $P_{1,n}(C)$ for $0 < C < 1$. However, $P_{1,n}(C) - F(C) > 0$. Thus $P_{1,n}(C)$ is not in the domain of $\Pi_{1,n}$ and so does not have a critical point. Since $\Pi_{1,n} \geq 0$ and $\Pi_{1,n}(C) = 0$, $\Pi_{1,n}$ is increasing on $C < P < F(C)$. The equation $\frac{d\Pi_{2,n}}{dP}(P,C) = 0$

gives only one possible critical point $P_{2,n}(C)$ for $0 < C < 1$. We check that $F(C) < P_{2,n}(C) < \frac{1+C}{2}$. First derivative test shows that $P = P_{2,n}(C)$ is a local maximum. Indeed, $\frac{d\Pi_{2,n}}{dP}((P_{2,n}(C) + \frac{1+C}{2})/2, C) < 0$ and $\frac{d\Pi_{2,n}}{dP}((P_{2,n}(C) + F(C))/2, C) > 0$ for all $0 \leq C \leq 1$. Thus there is only one unique local maximum for Π_n and it must also be its global maximum. By Lemma 1, $P^s + s(P - C) - 1 = 0$ defines a monotone decreasing function s in P so $P_n^* = P_{2,n}(C)$ corresponds to an optimal cluster size s_n^*. We optimize Π_o using the same method for Π_n. It clear that $\Pi_{1,o} = P - C$ is increasing for $C \leq P \leq F(C)$. We obtain a unique local maximum at $P^* = P_{2,o}(C)$ that solve $\frac{d\Pi_{2,o}}{dP}(P, C) = 0$ and $F(C) < P_{2,o}(C) < \frac{1+C}{2}$. As above, $P_o^* = P_{2,o}(C)$ also gives an optimal s_o^* that maximizes profit Π_o.

Proof of Proposition 2: Equilibrium Cluster Size

When the cluster size is s_d^*, a retailer can achieve maximum profit. Therefore, if n is divisible by s_d^*, then the pareto optimal equilibrium is: every cluster has the same cluster size $\frac{n}{s_d^*}$.

If n is indivisible by s_d^*, i.e., $(n \mod s_d^*) > 0$, then $(n \mod s_d^*)$ clusters have size $s_d^* + 1$ and the rest clusters have size s_d^*. Below we show no retailers have incentives to deviate. Retailers in clusters with size s_d^* already have maximum profit and thus they do not want to deviate. Suppose that a retailer i is in a cluster with size $s_d^* + 1$. If he deviates and joins a cluster with size s_d^*, then the new cluster size is $s_d^* + 1$ after he joins it. So his profit stay the same. Thus he will not deviate to a cluster with size s_d^*. If he joins a cluster with size $s_d^* + 1$, then the new cluster size is $s_d^* + 2$ after he joins it. Thus, he will not deviate to a cluster with size $s_d^* + 1$ since $\Pi_i(s_d^* + 1) > \Pi_i(s_d^* + 2)$. Therefore, no retailers deviate.

Next, we verify if it is a pareto optimal equilibrium. Retailers in clusters with size s_d^* already have maximum profits. Retailer in clusters wtih size $s_d^* + 1$ can further improve profits if the cluster size decreases by 1. This can only be done by putting residue retailers in new clusters, in order not to hurt other retailers' profits by increasing their cluster sizes. It can be easily shown that in an equilibrium, there cannot exist more than one cluster with size less than s_d^*. Therefore, there exists only one more candidate for pareto optimal equilibrium. In this alternative candidate, there exists only one cluster with size $(n \mod s_d^*)$, which is less than s_d^*, and $\left\lfloor \frac{n}{s_d^*} \right\rfloor$ clusters with size s_d^*.

If $\Pi_i(s_d^* + 1) < \Pi_i(n \mod s_d^*)$, then retailers in the residue cluster with size $(n \mod s_d^*)$ do not want to deviate. Also no cluster can further improve profit without hurting other clusters' profits. So the pareto optimal equilibrium is: one cluster with size $(n \mod s_d^*)$, which is less than s_d^*, and $\left\lfloor \frac{n}{s_d^*} \right\rfloor$ clusters with size s_d^*. Otherwise, if $\Pi_i(s_d^* + 1) \geq \Pi_i(n \mod s_d^*)$, then retailers in the residue cluster have incentive to deviate. Thus there does not exist an equilibrium that can further improve a retailer's profit without hurting another retailer's profit. Therefore, the pareto optimal equilibrium is: $(n \mod s_d^*)$ clusters with size $s_d^* + 1$ and the rest with size s_d^*.

Proof of Proposition 3: Impact of Cost-Valuation Ratio on Equilibrium Cluster Size

From the optimal price $P^*(C)$ (P_n^* and P_o^*) found in Proposition 1, there exist a corresponding optimal size s^* (s_n^* and s_o^*). Since $F(C) < P^* < \frac{1+C}{2}$, the constraint estimator gives us $s^*(C) \approx 1 +$

$\frac{2(1+C-2P^*(C))}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}$. Since $P^*(C)>C$, $s^*(C)\approx 1+\frac{2(1+C-2P^*(C))}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}\geq 1+\frac{2(1-C)}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}$. By L'Hopital's rule $\lim\limits_{C\to 1^-}\frac{2(1-C)}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}=+\infty$. Thus $\lim\limits_{C\to 1^-}s^*(C)=+\infty$. The estimated P^* found are good estimates since its values are estimated with a tangent over a closed interval between C and $\frac{1+C}{2}$. But the values s^* are large values so the estimate above is not the best. WE improve this estimate by applying Newton's Method to the curve $(P^*(C))^s+s(P^*(C)-C)-1=0$. Specifically, we take $s_0=1+\frac{2(1+C-2P^*(C))}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}$, and define $s_{k+1}=s_k-\frac{(P^*(C))^{s_k}+s_k(P^*(C)-C)-1}{(P^*(C))^{s_k}\ln(P^*(C))+P^*(C)}$ for $k\geq 0$. The second iterate s_2 is already a very good estimate of $s^*(C)$ for $0<C<0.9$; we check using Mathematica that s_3 does not significantly improve s_2. For $0.9<C<1$, we recommend using the third or more iterates. We give the graph of s_2 for the overlapping case in Figure **??** illustrating $\frac{ds^*}{dC}>0$ and $s^*(C)\to\infty$ as $C\to 1^-$.

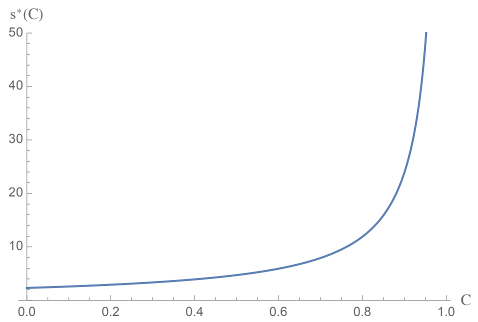


Figure 1 **Cluster Size Increases in C**

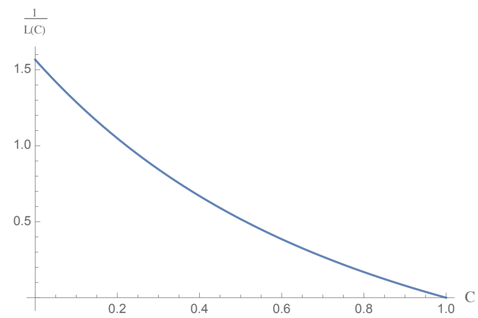


Figure 2 **The Gap $L(C)$ increases in C**

Proof of Proposition 4: Impact of Retailer Density on Clustering Equilibrium

In Problem (7), one of the two conditions $r\leq r_n$ and $r\leq r_o$ must be binding at optimality. If $r_n<r_o$, then $r\leq r_n$ binds, $r=r_n$ and retailers form non-overlapping clusters. On the contrary, if $r_o<r_n$, then $r\leq r_o$ binds, $r=r_o$ and retailers form overlapping clusters. Therefore, we compare r_o and r_n to see which case is the equilibrium. Since $r_o=s/(2n)$ and $r_n=\frac{(sv)}{(s+1)t}\left(-\frac{p(p-c)}{vv}-\frac{p}{v}+1\right)$, we have $\frac{r_n}{r_o}=\frac{n}{t}\frac{2\left(p(c-p)-pv+v^2\right)}{(s+1)v}$. Thus, if $\frac{n}{t}>\frac{(s+1)v}{2\left(p(c-p)-pv+v^2\right)}$, $r_n>r_o$ and thus the clustering equilibrium is the overlapping case. Otherwise, $r_n<r_o$ and thus the clustering equilibrium is the non-overlapping case.

Proof of Proposition 5: Cluster Size Comparison

In the proof of Proposition 1, we found estimates for the optimal prices $P_n^*(C)$ and $P_o^*(C)$. We check using Mathematica that there exists a threshold $0<C_h<1$ such that $P_n^*(C_h)=P_o^*(C_h)$. Moreover $P_n^*(C)<P_o^*(C)$ for $0<C<C_h$ and $P_n^*(C)>P_o^*(C)$ for $C_h<C<1$. By Lemma 1, for each fixed C, $\frac{dP}{ds}<0$. Thus $s_n^*(C)>s_o^*(C)$ for $0<C<C_h$ and $s_n^*(C)<s_o^*(C)$ for $C_h<C<1$.

Proof of Proposition 6: Socially Optimal Cluster Size

Using the estimate for s in the above discussion, we obtain an estimate for the social welfare function as continuous piecewise defined functions. $SW_n\approx SW_{1,n}(P,C)=\frac{(P-C)(P(C-P-1)+1)\left(-3CP+2(C-1)C+P^2+P+1\right)}{(-C+P+1)^2}$ for $C\leq P\leq F(C)$ and $SW_n\approx SW_{2,n}(P,C)=\frac{G(P,C)}{H(P,C)}$ for $F(C)\leq P\leq\frac{C+1}{2}$ where $G(P,C)=(-CP+P^2+$

$P-1)(-C-(C+1)\ln\left(\frac{C+1}{2}\right)+4P-3)\times(-(C+1)(4C^2-7CP+P(3P-1)+1)\ln\left(\frac{C+1}{2}\right)\ -8C^2-(17C+7)P^2+C(9C+14)P+C+8P^3+P-1)$, and $H(P,C)=4\left(-C\ln\left(\frac{C+1}{2}\right)-\ln\left(\frac{C+1}{2}\right)+2P-2\right)^2$.

We compute the critical points of $SW_{2,n}$. The equation $\frac{dSW_{2,n}}{dP}=0$ gives two possible critical points $P_{2,SW}(C)$ such that $0<P_{2,SW}(C)<1$ for $0<C<1$ and $P_{2,SW}(0)<0.5$. However we check using Mathematica $F(C)-P_{2,SW}(C)>0$. Thus both critical points are not in the domain of $SW_{2,n}$ so there are no critical point. Since $SW_{2,n}\geq 0$ and $SW_{2,n}(F(C))>SW_{2,n}\left(\frac{1+C}{2}\right)$, $SW_{2,n}$ can only be decreasing for $F(C)<P<\frac{1+C}{2}$. Next we compute critical points of $SW_{1,n}$. The equation $\frac{dSW_{1,n}}{dP}(P,C)=0$ gives only one possible critical point $P_{1,SW}(C)$ such that $0<P_{1,SW}(C)<1$ for $0<C<1$ and $P_{1,SW}(0)<0.5$. We also directly check that $C<P_{1,SW}(C)<F(C)$. Next we show that $P=P_{1,SW}(C)$ is a global maximum. Indeed, using Mathematica we check $\frac{dSW_{1,n}}{dP}((P_{1,SW}(C)+C)/2,C)>0$ and $\frac{dSW_{1,n}}{dP}((P_{1,SW}(C)+F(C))/2,C)<0$ for all $0\leq C\leq 1$. Thus by first derivative test, there is only one unique local maximum for SW_n so must also be the global maximum. Moreover, the local maximum occur at $P^*_{n,SW}=P_{1,SW}(C)$ in the interval $C<P<F(C)$. Also by Lemma 1, s is decreasing in P if $P^s+s(P-C)-1=0$. Thus $P^*_{n,SW}=P_{1,SW}(C)$ corresponds to a socially optimal cluster size $s^*_{n,SW}$ in the non-overlapping case.

Proof of Proposition 7: Socially Optimal Cluster Size V.S. Equilibrium Cluster Size

Here we compare the optimal prices for profit and social welfare in non-overlapping markets. Specifically, we have $C<P^*_{n,SW}<F(C)<P^*_n<(1+C)/2$ for all $0\leq C\leq 1$. By Lemma 1, for each fixed C, $\frac{dP}{ds}<0$. Thus $s^*_n(C)<s^*_{n,SW}(C)$ for all $0\leq C\leq 1$. Note that $s^*_n(C)\approx 1+\frac{2(1+C-2P^*_n(C))}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}$ and $s^*_{n,SW}(C)\approx\frac{1}{P^*_{n,SW}(C)-C}$. Let $L(C)=\frac{1}{P^*_{n,SW}(C)-C}-\left(1+\frac{2(1+C-2P^*_n(C))}{1-C+(1+C)\ln\left(\frac{C+1}{2}\right)}\right)$. Then $s^*_{n,SW}(C)-s^*_n(C)\approx L(C)$. Using Mathematica, we check that $\frac{1}{L(C)}$ is strictly decreasing and $\frac{1}{L(1)}=0$ (See Figure **??**). Thus $L(C)$ is increasing and $\lim_{C\to 1^-}L(C)=+\infty$. Therefore $s^*_{n,SW}(C)-s^*_n(C)$ is increasing and the difference gets larger and larger.

Proof of Proposition 8: A Retailer's Profit As a Function of n

Here we consider markets with conditions c, v, and t. Let n be the total number of retailers in the market with k clusters. In a cluster of size s, let $\Pi_N(s)$ and $\Pi_O(s)$ be the profit of a retailer in the cluster when the market is non-overlapping and overlapping respectively. Let $\Pi(n)$ be the maximum profit of a retailer in a market with n retailers. In this discussion, we assume that all clusters have the same average cluster size $s=\frac{n}{k}$. Thus we view $\Pi_N(s)$ and $\Pi_O(s)$ as continuous functions of s, and $\Pi(n)$ as a continuous function of n. Let c_e be the entry cost for a retailer to join this market. Let $\widehat{n}$ be the number of retailers such that $r_N=r_O$. Then the market is non-overlapping if $n<\widehat{n}$ and overlapping if $n>\widehat{n}$. Let s^*_N be the unique cluster size that maximizes $\Pi_N(s)$ subjected to $p^s=v^{s-1}(v-sp+sc)$. Note that $s^*_N<\widehat{n}$.

First, if $0<n<s^*_N$ then $\Pi(n)=\Pi_N(n)$. Indeed, $\Pi_N(s)$ is increasing for $0<s<s^*_N$ since it maximizes at $s=s^*_N$. Thus this market will have only have one cluster, and the profit of a retailer is $\Pi(n)=\Pi_N(n)$.

Second, if $s^*_N<n<\widehat{n}$ then $\Pi(n)$ is section of a damped oscillating function with horizontal asymptote $\Pi=\Pi_N(s^*_N)$. Indeed, if $h>0$ retailer join a market with k clusters of size s they have two options. Option

A: the new retailers decide to join existing clusters. Then the average cluster size is $\frac{ks+h}{k} = s + \frac{h}{k} > s$ and the profit for a retailer is $\Pi_N\left(s+\frac{h}{k}\right)$. Option B: the new retailers decide to start a new cluster. Then the average cluster size is $\frac{ks+h}{k+1} = s - \frac{s-h}{k+1} < s$ and the profit for a retailer is $\Pi_N\left(s - \frac{s-h}{k+1}\right)$. New retailers decide on Option A if $\Pi_N\left(s+\frac{h}{k}\right) > \Pi_N\left(s-\frac{s-h}{k+1}\right)$. But decide on Option B if $\Pi_N\left(s+\frac{h}{k}\right) < \Pi_N\left(s-\frac{s-h}{k+1}\right)$. For convenience, we define $L_k(h) = s_N^* - \frac{s_N^*-h}{k+1}$ and $R_k(h) = s_N^* + \frac{h}{k}$. We note that $L_k(h) \le s_N^* \le R_k(h)$ for $0 \le h \le s_N^*$. Now consider a non-overlapping market with k clusters each of size s_N^* (so $s_N^* < ks_N^* < \widehat{n}$). A new retailer joining this market at this point, it will not start a new cluster as none of the existing retailers will join the new cluster since they are making optimal profit. So the new retailer will decide on Option A and join the existing k clusters. In fact for any $0 < h < s_N^*$, h incoming retailers will also decide on Option A as long as $\Pi_N(R_k(h)) > \Pi_N(L_k(h))$. In this scenario, the average cluster size $s = R_k(h)$ is larger than s_N^* and s increases as the number of incoming retailers h increases. Moreover $\Pi(n)$ decreases as the total number of retailers $n = ks_N^* + h$ increases. The incoming h retailers decide on Option A till h reaches some threshold h_k (dependent on number of clusters k) such that $\Pi_N(R_k(h_k)) = \Pi_N(L_k(h_k))$. An incoming retailer at this critical scenario (where $n = ks_N^* + h_k$) has the incentive to start a new cluster (Option B) so that the new average cluster size $s = L_k(h_k)$ drops below s_N^* while the number of clusters increases to $k+1$. Moreover, for $h_k < h < s_N^*$, h incoming retailers will join the existing $k+1$ clusters (Option A) since the average cluster size $s = L_k(h) < s_N^*$ the optimal cluster size. Thus the cluster size s and $\Pi(n)$ increase as h increases till the number of retailers $n = ks_N^* + h$ reaches $(k+1)s_N^*$ (the next integral multiple of s_N^*) and the profit $\Pi((k+1)s_N^*)$ is maximal. This gives one oscillation between $ks_N^* \le n \le (k+1)s_N^*$. In other words, the condition $\Pi_N(R_k(h_k)) = \Pi_N(L_k(h_k))$ gives $n = ks_N^* + h_k$ that minimizes $\Pi(n)$ for $ks_N^* < n < (k+1)s_N^*$. Moreover this minimum value is given by $\Pi_N(L_k(h_k))$. We illustrate the decisions of incoming retailers based on profits in the figures **??**, **??**, and **??**.

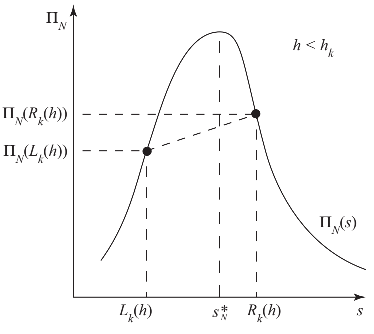


Figure 3 **Decide on Option A** $(h < h_k)$

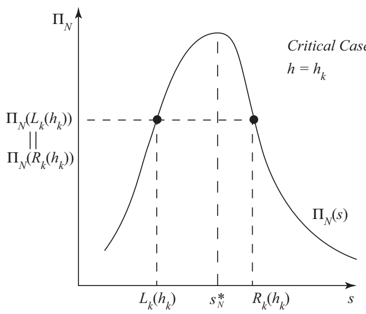


Figure 4 **Decide on Option B** $(h = h_k)$

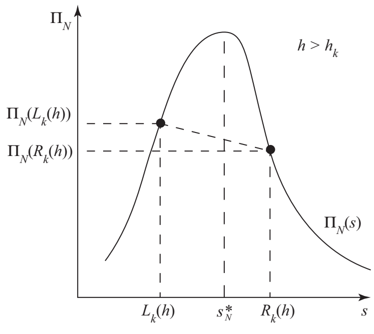


Figure 5 **Decide on Option A** $(h > h_k)$

Next, we show that the minimum values of $\Pi(n)$ is monotonic increasing and approaches the maximal possible profit $c_{e2} = \Pi_N(s_N^*)$ as n gets larger. Indeed, as discussed above there exists h_k between 0 and s_N^* such that $\Pi_N(R_k(h_k)) = \Pi_N(L_k(h_k))$. Let $\Delta_k(h) = R_k(h) - L_k(h) = \frac{s_N^*}{k+1} + h\left(\frac{1}{k} - \frac{1}{k+1}\right)$. Since $\frac{1}{k} > \frac{1}{k+1}$, we have $\Delta_k(0) = \frac{s_N^*}{k+1} < \Delta_k(h) \le \frac{s_N^*}{k} = \Delta_k(s_N^*)$ for $0 < h \le s_N^*$. We emphasize that $L_k(h_k) < s_N^* < R_k(h_k)$

and $\Delta_k(h_k)$ measures the distance between the values $L_k(h_k)$ and $R_k(h_k)$. In particular we have $\frac{s_N^*}{k+2} < \Delta_{k+1}(h_{k+1}) \leq \frac{s_N^*}{k+1} < \Delta_k(h_k) \leq \frac{s_N^*}{k}$. This implies that $\Delta_k(h_k)$ strictly decreases as k increases. Moreover, since Π_N has a unique maximum and $\Pi_N(L_k(h_k)) = \Pi_N(R_k(h_k))$, we must have $L_k(h_k) < L_{k+1}(h_{k+1}) < R_{k+1}(h_{k+1}) < R_k(h_k)$ as illustrated in figure **??**. This implies that $L_k(h_k)$ strictly increases while $R_k(h_k)$ strictly decreases as k increases.

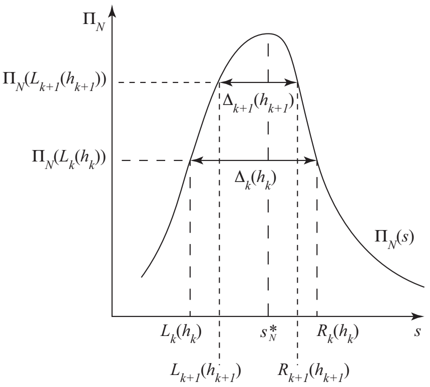


Figure 6 **A retailer's profit**

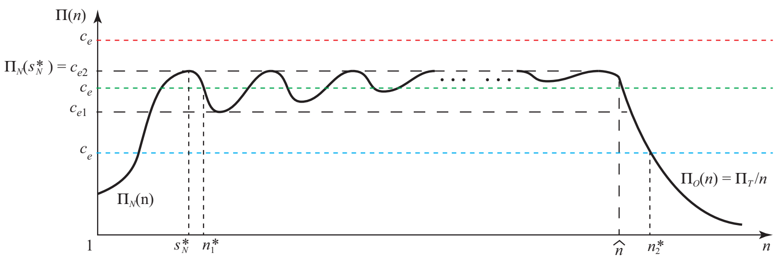


Figure 7 **The equilibrium number of retailers**

For $ks_N^* \leq n \leq (k+1)s_N^*$, Π minimizes at $n = ks_N^* + h_k$ and has value $\Pi_N(L_k(h_k))$. Since $L_k(h_k) < L_{k+1}(h_{k+1}) < s_N^*$, $\Pi_N(L_k(h_k)) < \Pi_N(L_{k+1}(h_{k+1}))$. Thus the sequence of minimums of Π increases as n increases. We also observe that as $\widehat{n}$ increases, the largest K such that $Ks_N^* < \widehat{n}$ also increases. As K increases $\Delta_K(h_K)$ approaches zero. Indeed, we have $\frac{s_N^*}{K+1} < \Delta_K(h_K) \leq \frac{s_N^*}{K}$. Since $\Delta_K(h_K) = R_K(h_K) - L_K(h_K)$ approaches zero, $R_K(h_K)$ and $L_K(h_K)$ approaches s_N^*. Thus the sequence of minimums $\Pi_N(L_k(h_k))$ approaches $\Pi_N(s_N^*)$ as K gets larger or equivalently as $\widehat{n}$ increases.

Third, if $\widehat{n} < n$ then $\Pi(n) = \Pi_T/n$ where Π_T is a constant. In an overlapping market, we optimize the profit function $\Pi_O = \dfrac{2r}{v}(p - c + bs)^2$ subjected to $r = \dfrac{s}{2n}$ and $p^s = v^{s-1}[v - sp + s(c - bs)]$. Fixing $n > \widehat{n}$ and substituting $r = \frac{s}{2n}$, the problem reduces to optimizing $n\Pi_O = \dfrac{s}{v}(p - c + bs)^2$ subjected to $p^s = v^{s-1}[v - sp + s(c - bs)]$. Let Π_T be the maximum of $\frac{s}{v}(p - c + bs)^2$ subjected to $p^s = v^{s-1}[v - sp + s(c - bs)]$. Then the maximum of $n\Pi_O^*(n)$ is Π_T. Thus $\Pi(n) = \Pi_O^*(n) = \Pi_T/n$ for all $n > \widehat{n}$.

Proof of Proposition 9: The Impact of An Entry Cost

We can read off the results for the entry cost c_e of joining a market from the graph of Π (Figure **??**). We have: (1) When $c_e \leq c_{e1}$, as illustrated by the blue dotted line, the equilibrium number of retailers is n_2^* where $c_e = \Pi_O(n_2^*)$. Indeed, $\Pi_O(n_2^* - 1) > c_e$ and $\Pi_O(n_2^* + 1) < c_e$. So no existing retailer wants to leave the market and no new retailer wants to join the market. Therefore, the equilbrium retailer number is n_2^* and $n_2^* > \widehat{n}$. (2) When $c_{e1} < c_e \leq c_{e2}$, as illustrated by the green dotted line, the equilibrium number of retailers is n_1^* where $c_e = \Pi_N(n_1^*)$. Indeed, $\Pi_N(n_1^* - 1) > c_e$ and $\Pi_N(n_1^* + 1) < c_e$. So no existing retailer wants to leave the market and no new retailer wants to join the market. Therefore, the equilibrium retailer number is n_1^* and $s_N^* < n_1^* < \widehat{n}$. (3) When $c_e > c_{e2}$, $\Pi_N(S_N^*) < c_e$. Therefore, the market is not profitable and no retailers join the market.