Online Supplements for “The Cash Flow Advantages of 3PLs as Supply Chain Orchestrators”

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Appendix A: Exogenous Service Rate

This appendix assumes that the service rate (w_l) is exogenous. This corresponds to the case in Lemma 2 where the manufacturer decides the optimal solution given the 3PL’s service rate.

As Lemma 2 shows, if all firms’ opportunity costs are the same, neither P nor T can strictly outperform the other. Compared with results in Section 3, this observation indicates that fixing the 3PL’s service rate minimizes the 3PL’s capability to facilitate a mutually beneficial outcome for all firms in P.

With different cash opportunity costs, we find that P can reemerge as a preferred choice for all firms. From Lemma 2, we have ${q_P^*}' = \frac{dq_P^*}{d\ell_g} < 0$. Define $A(\ell_g) = (\frac{p\bar{F}(q_P^*)}{1+\ell_r a_l} - w_l)q_P^* + [(w_l - c_l)\ell_r + \frac{c_m(1+\ell_o a_m)(\ell_g - \ell_s)}{1+(\ell_o-\ell_g)a_m}]{q_P^*}'$. Comparing firms’ profits between T and P yields the following property.

Corollary 1 *Suppose w_l is exogenous and the firms have different cash opportunity costs. Let $\tilde{\ell} = \frac{(1+\ell_o a_m a_b)(\ell_o - \ell_r)}{a_m(1+\ell_o a_b)}$ and $\check{\ell}$ be the solution to $\pi_{lT}(q_T^*) = \pi_{lP}(q_P^*, \ell_g)$.*

1. *If $a_l \geq \frac{(c_l - w_l){q_P^*}'}{A(\ell_g)}$ and $\frac{d\pi_{lP}(q_P^*)}{d\ell_g} \geq 0$ then $\check{\ell} < \tilde{\ell}$, and $T \prec P$ for all $\ell_g \in [\check{\ell}, \tilde{\ell}]$;*
2. *If $\frac{d\pi_{lP}(q_P^*)}{d\ell_g} < 0$, then $T \prec P$ when $\ell_g \leq \min[\check{\ell}, \tilde{\ell}]$.*

Thus, the qualitative findings with endogenous w_l and different cash opportunity costs still hold for exogenous w_l.

Appendix B

Proof of Lemma 1: Taking first derivative of Eq. (3) with respect to (w.r.t.) q_{bi}, the first-order-condition (FOC) yields

$$q_{bi}^*(w_{li}, w_{mi}) = \begin{cases} \bar{F}^{-1}\left(\frac{w_{mi}(1+\ell_o a) + w_{li}(1+\ell_r a)}{p}\right), & \text{if } i = T, \\ \bar{F}^{-1}\left(\frac{(w_{mi}+w_{li})(1+\ell_r a)}{p}\right), & \text{if } i = P. \end{cases}$$

We have $q_{bT}^*(w_m, 0) = w_{bP}^*(w_m, 0) = q_b^*(w_m) = \bar{F}^{-1}(w_m/P)$. If $w_{li} \not\equiv 0$ and $a \not\equiv 0$, we have $w_{mT}(1+\ell_o a) + w_{lT}(1+\ell_r a) > w_m$, and then $q_{bT}^*(w_m, w_l) < q_b^*(w_m)$.

Let $w_{mT} = w_{mP} = w_m$ and $w_{lT} = w_{lP} = w_l$. Since $\ell_o > \ell_r$, we have $w_{mT}(1+\ell_o a) + w_{lT}(1+\ell_r a) > (w_{mP} + w_{lP})(1+\ell_r a)$, and then obtain $q_{bP}^*(w_m, w_l) > q_{bT}^*(w_m, w_l)$. Q.E.D.

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Proof of Lemma 2: We first prove that $\pi_{mi}(w_{mi})$ is unimodal w.r.t. $w_{mi} \geq 0$. The first derivative of $\pi_{mi}(w_{mi})$ w.r.t. w_{mi} is $\frac{d\pi_{mi}(w_{mi})}{dw_{mi}} = \frac{\partial \pi_{mi}}{\partial q^*_{bi}} \frac{dq^*_{bi}}{dw_{mi}} + \frac{\partial \pi_{mi}}{\partial w_{mi}}$, and we have,

$$\frac{d\pi_{mi}(w_{mi})}{dw_{mi}} = \begin{cases} \frac{1+\ell_o a}{ph(q^*_{bi})}\left[pH(q^*_{bi}) + \frac{c_m(1+\ell_o a)}{\bar{F}(q^*_{bi})} - \frac{w_{mi}(1+\ell_o a)}{\bar{F}(q^*_{bi})}\right] & \text{if } i = T, \\ \frac{1+(\ell_o-\ell_g)a}{ph(q^*_{bi})}\left[pH(q^*_{bi}) + \frac{(1+\ell_o a)(1+\ell_r a)}{1+(\ell_o-\ell_g)a}\frac{c_m}{\bar{F}(q^*_{bi})} - \frac{w_{mi}(1+\ell_r a)}{\bar{F}(q^*_{bi})}\right] & \text{if } i = P. \end{cases}$$

It is straightforward that $pH(q^*_{bi})$ and $\frac{1}{\bar{F}(q^*_{bi})}$ decrease with w_{mi} since $\frac{dq^*_{bi}}{dw_{mi}} < 0$. And $\frac{w_{mi}}{\bar{F}(q^*_{bi})}$ increases with w_{mi} since $\left[\frac{w_{mi}}{\bar{F}(q^*_{bi})}\right]' = \frac{\bar{F}(q^*_{bi})-w_{mi}(1+\ell_r a)}{\bar{F}^2(q^*_{bi})} > 0$. Hence, $\frac{d\pi_{mi}(w_{mi})}{dw_{mi}}$ decreases with w_{mi}. Then, if $w_{mi} = 0$, we have $\frac{d\pi_{mi}(w_{mi})}{dw_{mi}} > 0$; if $w_{mi} \to \propto$, $\frac{d\pi_{mi}(w_{mi})}{dw_{mi}} < 0$. Therefore, $\pi_{mi}(w_{mi})$ is unimodal w.r.t. w_{mi}. Solving the first order condition $\frac{d\pi_{mi}(w_{mi})}{dw_{mi}} = 0$ results in

$$w^*_{mi}(w_{li}) = \begin{cases} c_m + \frac{pq^*_{bT}(w_{lT}, w^*_{mT}) f(q^*_{bT}(w_{lT}, w^*_{mT}))}{1+\ell_o a}, & \text{if } i = T, \\ \eta_m c_m + \frac{pq^*_{bP}(w_{lP}, w^*_{mP}) f(q^*_{bP}(w_{lP}, w^*_{mP}))}{1+\ell_r a}, & \text{if } i = P, \end{cases}$$

where $\eta_m = \frac{1+\ell_o a}{1+(\ell_o-\ell_g)a}$. Given $\ell_g \geq 0$, we get $\eta_m \geq 1$.

For a fixed $w_l = w_{lT} = w_{lP}$, from Lemma 1, we have, $p\bar{F}(q^*_{bT}) = w_{mT}(1+\ell_o a) + w_l(1+\ell_r a)$ and $p\bar{F}(q^*_{bP}) = (w_{mP} + w_l)(1+\ell_r a)$. Submitting w^*_{mi} into these two equations, we can obtain the optimal order quantities $q_{bi}(w^*_{mi}, w_l)$ solving the following equations

$$\begin{aligned} p\bar{F}(q^*_{bT}) - pq^*_{bT} f(q^*_{bT}) - w_l(1+\ell_r a) &= c_m(1+\ell_o a), \\ p\bar{F}(q^*_{bP}) - pq^*_{bP} f(q^*_{bP}) - w_l(1+\ell_r a) &= c_m \frac{1+\ell_o a}{\eta_l}. \end{aligned}$$

Note that $qf(q) = H(q)\bar{F}(q)$ increases with q, since $[H(q)\bar{F}(q)]' = \bar{F}(q)[h(q)[1-H(q)]+qh'(q)] > 0$. Then, we have $p\bar{F}(q) - pqf(q)$ decreases in q. Let $\ell_g = \ell_s$, we have $q^*_{bT}(w^*_{mT}, w_l) = q^*_{bP}(w^*_{mP}, w_l)$, since $\eta_l = 1$. Since $\eta_m > 1$ and $\ell_o > \ell_r$, we have $w^*_{mP} \geq w^*_{mT}$. If $\ell_g < \ell_s$, we have $\eta_l > 1$ and $q^*_{bT}(w^*_{mT}, w_l) < q^*_{bP}(w^*_{mP}, w_l)$; otherwise, $q^*_{bT}(w^*_{mT}, w_l) \geq q^*_{bP}(w^*_{mP}, w_l)$. Q.E.D.

Proof of Lemma 3: From Eq. (10),

$$\frac{d\pi_{li}(q^*_{bi}(w_{li}))}{dq^*_{bi}(w_{li})} = \begin{cases} p\bar{F}(q^*_{bT})[(1-H(q^*_{bT}))^2 - q^*_{bT}H'(q^*_{bT})] - c_m(1+\ell_o a) - c_l(1+\ell_r a), \\ p\bar{F}(q^*_{bP})\left[(1-H(q^*_{bP}))(1-\eta_l H(q^*_{bP})) - \eta_l q^*_{bP}H'(q^*_{bP})\right] - c_m(1+\ell_o a) - c_l(1+\ell_r a). \end{cases}$$

Define $G(p, q_{li}, 1) = p\bar{F}(q_{li})[(1-H(q_{li}))^2 - q_{li}H'(q_{li})] > 0$. Since $H(q_{li})$ and $q_{li}H'(q_{li})$ increase with q_{li}, and $\bar{F}(q_{li})$ decreases with q_{li}, $G(p, q_{li}, 1)$ decreases with q_{li}. Similarly, define $G(p, q_{li}, \eta_l) \equiv p\bar{F}(q_{li})[(1-H(q_{li}))(1-\eta_l H(q_{li})) - \eta_l q_{li}H'(q_{li})]$, which decreases with q_{li}. In addition, we have $\frac{d\pi_{li}(q_{li})}{dq_{li}}|_{q_{li}=0} = p - c_m(1+\ell_o a) - c_l(1+\ell_r a) > 0$, and $\frac{d\pi_{li}(q_{li})}{dq_{li}}|_{q_{li}\to\propto} = -c_m(1+\ell_o a) - c_l(1+\ell_r a) < 0$. Thus, $\pi_{li}(q_{li})$ is unimodal in $q_{li} \geq 0$. Solving the FOC leads to the result in Lemma 3. Q.E.D.

Proof of Proposition 1: **Part (i):** From Lemma 3, we get $G(p, q^*_{lP}, \eta_l) = G(p, q^*_{lT}, 1) \leq G(p, q^*_{lT}, \eta_l)$ since $\ell_g \geq \ell_s$ and $\eta_l \leq 1$. Since $G(p, q, \eta_l)$ decreases in q, we obtain $q^*_{lT} \leq q^*_{lP}$.
Part (ii): Based on the proof of Part (iv), we have $w^*_{lT}(1+\ell_r a) = p\bar{F}(q^*_{lT})[1-H(q^*_{lT})] - c_m(1+\ell_o a)$ and $w^*_{lP}(1+\ell_r a) = p\bar{F}(q^*_{lP})[1-H(q^*_{lP})] - c_m\frac{(1+\ell_o a)}{\eta_l}$. Since $\bar{F}(q)[1-H(q)]$ decreases with q and $q^*_{lT} \leq q^*_{lP}$, we have $p\bar{F}(q^*_{lP})[1-H(q^*_{lP}) \leq p\bar{F}(q^*_{lT})[1-H(q^*_{lT})]$. And since $\ell_g \geq \ell_s$, we have $\frac{c_m(1+\ell_o a)}{\eta_l} \geq c_m(1+\ell_o a)$. Thus, we obtain $w^*_{lP} \leq w^*_{lT}$.
Part (iii): According to Lemma 3, q^*_{lP} (i.e., q^*_P) solves the following equation.

$$G(p, q^*_P, \eta_l) = c_m(1+\ell_o a) + c_l(1+\ell_r a).$$

It is easy to see that the left hand side (LHS) of the above equation increases with ℓ_g, but decreases with q. Since the value of LHS decreases with q, we should increase q to make the equation satisfied when ℓ_g increases. Thus, given that the right hand side is fixed, we have q^*_{lP} increases with ℓ_g.

We can rewrite the buyer's profit as $\pi^*_{bP} = p\mathbb{E}[D \wedge q^*_{lP}] - q\bar{F}(q^*_{lP})]$. Taking first order and second order derivatives with respect to q^*_{lP}, we have $\frac{d\pi_{bP}(q^*_{lP})}{dq^*_{lP}} = p\bar{F}(q^*_{lP})H(q^*_{lP}) > 0$ and $\frac{d^2\pi_{lP}(q^*_{lP})}{d{q^*_{lP}}^2} =$

$p\bar{F}(q_{lP}^*)[h(q_{lP}^*)(1-H(q_{lP}^*)) + q_{lP}^* h'(q_{lP}^*)] > 0$. Therefore, $\pi_{bP}(q_{lP}^*)$ convexly increases with q_{lP}^*. Since q_{lP}^* increases with ℓ_g, $\pi_{bP}(q_{lP}^*)$ convexly increases with ℓ_g.

The profit of 3PL is: $\pi_{lP}(q_{lP}^*) = \left[p\bar{F}(q_{lP}^*)\left(1-\eta_l H(q_{lP}^*)\right) - c_m(1+\ell_o) - c_l(1+\ell_r a)\right] q_{lP}^*$. We then have $\frac{d\pi_{lP}(q_{lP}^*)}{d\ell_g} = \frac{p\ a}{1+\ell_r a} q_{lP}^* \bar{F}(q_{lP}^*) H(q_{lP}^*) > 0$ and $\frac{d^2\pi_{lP}(q_{lP}^*)}{d\ell_g{}^2} = \frac{pa}{1+\ell_r a}\bar{F}(q_{lP}^*)[H(q_{lP}^*)(1-H(q_{lP}^*)) + q_{lP}^* H'(q_{lP}^*)]\frac{dq_{lP}^*}{d\ell_g} > 0$. Thus we can show that $\pi_{lP}(q_{lP}^*)$ convexly increases with ℓ_g.

Part (iv): Given q_{li}^*, we have

$$w_{mi}^* = \begin{cases} \frac{pq_{lT}^* f(q_{lT}^*) + c_m(1+\ell_o a)}{1+\ell_o a}, & \text{if } i = T, \\ \frac{pq_{lP}^* f(q_{lP}^*) + c_m(1+\ell_o a)/\eta_l}{1+\ell_r a}, & \text{if } i = P, \end{cases}$$

and

$$w_{li}^*(1+\ell_r a) = \begin{cases} p\bar{F}(q_{lT}^*)[1-H(q_{lT}^*)] - c_m(1+\ell_o a), & \text{if } i = T, \\ p\bar{F}(q_{lP}^*)[1-H(q_{lP}^*)] - c_m(1+\ell_o a)/\eta_l, & \text{if } i = P. \end{cases}$$

Submitting $w_{li}^*(q_{li}^*)$ and $w_{mi}^*(q_{li}^*)$ into Eq. (3), we have

$$\pi_{bi}(q_{li}^*) = \begin{cases} p\ \mathbb{E}[D \wedge q_{lT}^*] - q_{lT}^* \bar{F}(q_{lT}^*), & \text{if } i = T, \\ p\ \mathbb{E}[D \wedge q_{lP}^*] - q_{lP}^* \bar{F}(q_{lP}^*), & \text{if } i = P. \end{cases}$$

Since $\frac{d\pi_{bi}(q_{li}^*)}{dq_{li}^*} = q_{li}^* f(q_{li}^*) \geq 0$, we have $\pi_{bT}(q_{lT}^*) \leq \pi_{bP}(q_{lP}^*)$ because $q_{lT}^* \leq q_{lP}^*$ conditional on $\ell_g \geq \ell_s$ according to Proposition 1. Similarly, we have $\pi_{lT}(q_{lT}^*) \leq \pi_{lP}(q_{lP}^*)$.

Since the buyer and the 3PL's profits in P increase with ℓ_g but those in T are independent of ℓ_g, the buyer and the 3PL firm's preference of P to T grows stronger as ℓ_g increases. Q.E.D.

Proof of Proposition 2 Part (i): Note that $\pi_{mP}(q_{lP}^*, \ell_g) = p\eta_l q_{lP}^* \bar{F}(q_{lP}^*) H(q_{lP}^*)$. $\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g} = p\eta_l \bar{F}(q_{lP}^*)[H(q_{lP}^*)(1-H(q_{lP}^*)) + q_{lP}^* H'(q_{lP}^*)]\frac{dq_{lP}^*}{d\ell_g} - p\frac{a}{1+\ell_r a} q_{lP}^* \bar{F}(q_{lP}^*) H(q_{lP}^*)$. And

$$\frac{dq_{lP}^*}{d\ell_g} = \frac{a}{1+\ell_r a} \frac{H(q_{lP}^*) - H^2(q_{lP}^*) + q_{lP}^* H'(q_{lP}^*)}{\frac{h(q_{lP}^*)(c_m'+c_l')}{p\bar{F}(q_{lP}^*)} + H'(q_{lP}^*) + \eta_l[2H'(q_{lP}^*)(1-H(q_{lP}^*)) + q_{lP}^* H''(q_{lP}^*)]} \geq 0.$$

According to Lemma 3, q_{lP}^* solves the following equation $p\bar{F}(q_{lP}^*)[(1-H(q_{lP}^*))(1-\eta_l H(q_{lP}^*)) - \eta_l q_{lP}^* H'(q_{lP}^*)] = c_m' + c_l'$, where $c_m' = c_m(1+\ell_o a)$ and $c_l' = c_l(1+\ell_r a)$.

Furthermore, we have, $p\eta_l \bar{F}(q_{lP}^*)[H(q_{lP}^*)(1-H(q_{lP}^*)) + q_{lP}^* H'(q_{lP}^*)] = p\bar{F}(q_{lP}^*)(1-H(q_{lP}^*)) - c_m' - c_l'$. Then, we have,

$$\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g} = [p\bar{F}(q_{lP}^*)(1-H(q_{lP}^*)) - c_m' - c_l']\left[-\frac{p\ a}{1+\ell_r a}\frac{p\bar{F}(q_{lP}^*) q_{lP}^* H(q_{lP}^*)}{p\bar{F}(q_{lP}^*)(1-H(q_{lP}^*)) - c_m' - c_l'} + \frac{dq_{lP}^*}{d\ell_g}\right].$$

Define $M1(\ell_g) = \frac{p\bar{F}(q_{lP}^*) q_{lP}^* H(q_{lP}^*)}{p\bar{F}(q_{lP}^*)(1-H(q_{lP}^*)) - c_m' - c_l'}$. As ℓ_g increases, $q(\ell_g)$ increases, $p\bar{F}(q_{lP}^*) q_{lP}^* H(q_{lP}^*)$ increases, but $p\bar{F}(q_{lP}^*)(1-H(q_{lP}^*)) - c_m' - c_l'$ decreases. Thus, $M1(\ell_g)$ increases with ℓ_g.

Let $M2(\ell_g) = \frac{dq_{lP}^*}{d\ell_g}$.

Case I: $\frac{dM2(\ell_g)}{d\ell_g} \geq 0$

Therefore, both $M1(\ell_g)$ and $M2(\ell_g)$ are monotonic increasing with ℓ_g.

If we can show that $M2(\ell_g) - M1(\ell_g) > 0$ when ℓ_g is a very small quantity making $q_{bP} \to 0$. $M2(\ell_g) - M1(\ell_g) < 0$ when ℓ_g is a very large value making $q_{lP}^* \to \check{q}$, which solves $p\bar{F}(1-H(q)) = c_m' + c_l'$. Let $q_{lP}^*(\ell_g) \to 0$, we have, $\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g}|_{q_{lP}^*(\ell_g)=0} = \frac{a}{[1+(\ell_o-\ell_g)a]p\bar{F}(q_{lP}^*)}\frac{[p\bar{F}(q_{lP}^*) - c_m' - c_l']^2}{\frac{h(q_{lP}^*)(c_m'+c_l')}{p\bar{F}(q_{lP}^*)}} > 0$.

Let $q_{lP}^*(\ell_g) \to \check{q}$, we have, $\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g}|_{q_{lP}^*(\ell_g)=\check{q}} = -p\frac{a}{1+\ell_r a} q_{lP}^* \bar{F}(q_{lP}^*) H(q_{lP}^*) < 0$. Let $\check{q}$ solves $p\bar{F}(q_{lP}^*)(1-H(q_{lP}^*)) - c_m' - c_l' = 0$, we get $\frac{\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g}|_{q_{lP}^*(\ell_g)\to\check{q}} < 0$. Thus, $\pi_{mP}(q_{lP}^*(\ell_g))$ is a unimodal function ℓ_g, and $q_{lP}^*(\ell_g^*)$ satisfying $\frac{d\pi_{mP}^*(q_{lP}^*(\ell_g))}{d\ell_g} = 0$.

Case II: $\frac{dM2(\ell_g)}{d\ell_g} < 0$

$\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g} = [p\bar{F}(q_{lP}^*)(1-H(q_{lP}^*)) - c_m' - c_l'][-M1(\ell_g) + M2(\ell_g)]$. According to Case I, we show

that $M1(\ell_g)$ increases with ℓ_g, and $-M1(\ell_g) + M2(\ell_g)$ decreases with ℓ_g. Thus, $\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g}$ decreases in ℓ_g. As shown in Case I, $\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g}|_{q_{lP}^*(\ell_g)=0} > 0$ and $\frac{d\pi_{mP}(q_{lP}^*,\ell_g)}{d\ell_g}|_{q_{lP}^*(\ell_g)=\check{q}} < 0$. Therefore, $\pi_{mP}(q_{lP}^*, \ell_g)$ is a unimodal function in ℓ_g.
Part (ii): Part (ii) is a direct result of Part (i). Since $\pi_{mP}(q_{lP}^*(\ell_g))$ is a unimodal function in ℓ_g, there must exist a unique $\overline{\ell} > \ell_o - \ell_r$ such that $\pi_{mP}(q_{lP}^*(\ell_g)) = \pi_{mT}(q_{lT}^*(\ell_g))$ and $\pi_{mP}(q_{lP}^*(\ell_g)) < \pi_{mT}(q_{lT}^*(\ell_g))$ if $\ell_g > \overline{\ell}$. Q.E.D.

Proof of Corollary 1: **Part (i):** We have $G(p, q_{lP}^*, \eta_l) = c_m(1+\ell_o a) + c_l(1+\ell_r a) = p\bar{F}(q^C(a)) > G(p, q^C(a), \eta_l)$ since $\eta_l \leq 1$ and $\ell_g \geq \ell_s$. Since $G(p, q, \eta_l)$ decreases with q, we have $q_{lP}^* < q^C(a)$.
Part (ii): We have $\pi_T^{SC}(q_{lT}^*) \leq \pi_P^{SC}(q_{lP}^*) < \pi_T^{SC}(q^C(a)) = \pi_P^{SC}(q^C(a))$ because of $q_{lT}^* \leq q_{lP}^* < q^C(a)$. Similarly, we can show that $q_{lT}^* > q_{lP}^*$ and $\pi_T^{SC}(q_{lT}^*) > \pi_P^{SC}(q_{lP}^*)$ when $\ell_g < \ell_s$. Q.E.D.

Proof of Proposition 3: We now prove that, when the manufacturer is the Stackelberg leader, $\pi_{mP}(q_{mP}^*) = \pi_{mT}(q_{mT}^*)$, $\pi_{lP}(q_{mP}^*) = \pi_{lT}(q_{mT}^*)$, and $\pi_{bP}(q_{mP}^*) = \pi_{bT}(q_{mP}^*)$. We solve the game backward. Given the manufacturer and the 3PL's decisions, the buyer problem is the same as in Lemma 1 where the 3PL is the Stackelberg leadership. From Lemma 1, we have $w_{lT} = \frac{p\bar{F}(q_{bT}^*)-w_{mT}(1+\ell_o a)}{1+\ell_r a}$, and $w_{lP} = \frac{p\bar{F}(q_{bP}^*)-w_{mP}(1+\ell_r a)}{1+\ell_r a}$. Submitting $w_{li}(q_{bi}^*)$ into Eq. (5), rewriting the 3PL firm's problem, and taking the first order derivative w.r.t. q_{li}, we have,

$$\frac{d\pi_{li}(q_{li})}{dq_{li}} = \begin{cases} p\bar{F}(q_{lT})[1 - H(q_{lT})] - w_{mT}(1 + \ell_o a) - c_l(1 + \ell_r a), & \text{if } i = T, \\ p\bar{F}(q_{lP})[1 - H(q_{lP})] - w_{mP}(1 + (\ell_o - \ell_g)a) - c_l(1 + \ell_r a), & \text{if } i = P. \end{cases}$$

Denoting q_{li}^* that solves $\frac{d\pi_{li}(q_{li})}{dq_{li}} = 0$, we then obtain

$$w_{mi}(q_{li}^*) = \begin{cases} \frac{p\bar{F}(q_{lT}^*)[1-H(q_{lT}^*)]-c_l(1+\ell_r a)}{1+\ell_o a}, & \text{if } i = T, \\ \frac{p\bar{F}(q_{lP}^*)[1-H(q_{lP}^*)]-c_l(1+\ell_r a)}{1+(\ell_o-\ell_g)a}, & \text{if } i = P. \end{cases}$$

Submitting $w_{mi}(q_{li}^*)$ into Eq. (4), we can rewrite the manufacturer's profit function as below:

$$\pi_{mi}(w_{mi}(q_{li}^*)) = \pi_{mP}(q_{mi}) = \begin{cases} (p\bar{F}(q_{mT})[1 - H(q_{mT})] - c_l(1 + \ell_r a) - c_m(1 + \ell_o a))q_{mT}, & \text{if } i = T, \\ (p\bar{F}(q_{mP})[1 - H(q_{mP})] - c_l(1 + \ell_r a) - c_m(1 + \ell_o a))q_{mP}, & \text{if } i = P. \end{cases}$$

Therefore, we have $q_{mT}^* = q_{mP}^*$, which solves $\frac{d\pi_{mP}(q_{mP})}{dq_{mP}} = 0$. As a result, $\pi_{mP}(q_{mP}^*) = \pi_{mT}(q_{mT}^*)$. The buyer's profit is: $\pi_{bi}(q_{mi}^*) = p\mathbb{E}[D \wedge q_{mi}^*] - p\ q_{mi}^* \bar{F}(q_{mi}^*)$. Since $q_{mT}^* = q_{mP}^*$, we have $\pi_{bT}(q_{mT}^*) = \pi_{bP}(q_{mP}^*)$. Similarly, we can show that $\pi_{lT}(q_{mT}^*) = \pi_{lP}(q_{mP}^*)$. Q.E.D.

Proof of Proposition 4: The buyer, the manufacturer, and the 3PL's profit functions under Case T and Case B can be written as follows.

$$\pi_{bi}(q_{bi}) = \begin{cases} p\mathbb{E}[D \wedge q_{bi}] - [w_{mi}(1 + \ell_o a_b) + w_{li}(1 + \ell_r a_b)]q_{bi}, & \text{if } i = T, \\ p\mathbb{E}[D \wedge q_{bi}] - w_{mi}(1 + (\ell_o - \ell_g)a_b)q_{bi} - w_{li}q_{bi}(1 + \ell_r a_b), & \text{if } i = B, \end{cases} \tag{1}$$

$$\pi_{mi}(w_{mi}) = \begin{cases} (w_{mi} - c_m)(1 + \ell_o a_m)q_{bi}, & \text{if } i = T, \\ [w_{mi}(1 + (\ell_o - \ell_g)a_m) - c_m(1 + \ell_o a_m)]\, q_{bi}, & \text{if } i = B. \end{cases} \tag{2}$$

$$\pi_{li}(w_{li}) = \begin{cases} (w_{li} - c_l)(1 + \ell_r a_l)q_{bi}, & \text{if } i = T, \\ (w_{li} - c_l)\,(1 + \ell_r a_l)q_{bi}, & \text{if } i = B. \end{cases} \tag{3}$$

Case one: the 3PL leadership. We solve the game backward. First, we solve the optimal ordering level q_{bT}^* and q_{bB}^*, and get $w_{mT}(q_{bT}^*, w_{lT})$ and $w_{mB}(q_{bB}^*, w_{lB})$; Next, submitting $w_{mT}(q_{bT}^*, w_{lT})$ and $w_{mB}(q_{bB}^*, w_{lB})$ into manufacturer's profit function, we change $\pi_{mT}(w_{mT})$ and $\pi_{mB}(w_{mB})$ into $\pi_{mT}(q_{mT}, w_{lT})$ and $\pi_{mB}(q_{mB}, w_{lB})$, solve the optimal problems of q_{mT}^* and q_{mB}^* instead of w_{mT}^* and w_{mB}^*, and obtain $w_{lT}(q_{mT}^*)$ and $w_{lB}(q_{mB}^*)$; Finally, submitting $w_{lT}(q_{mT}^*)$ and $w_{lB}(q_{mB}^*)$ into 3PL's profit functions, we change the optimal problems of $\pi_{lT}(w_{lT})$ and $\pi_{lB}(w_{lB})$ into $\pi_{lT}(q_{lT})$ and $\pi_{lB}(q_{lB})$, and obtain q_{lT}^*(i.e., q_T^*) and q_{lB}^*(i.e., q_B^*) solving the following equations, respectively.

$$\begin{cases} G(p, q_T^*, 1) - c_l(1 + \ell_r a_b) - c_m(1 + \ell_o a_b) = 0, \\ G(p, q_B^*, 1) - c_l(1 + \ell_r a_b) - c_m(1 + \ell_o a_b)\frac{1+(\ell_o-\ell_g)a_b}{1+(\ell_o-\ell_g)a_m}\frac{1+\ell_o a_m}{1+\ell_o a_b} = 0. \end{cases}$$

As a result, the firms' profits can be written as:

$$\begin{cases} \pi_{bT}(q_T^*) = p\mathbb{E}[D \wedge q_T^*] - p\bar{F}(q_T^*)q_T^*, \\ \pi_{bB}(q_B^*) = p\mathbb{E}[D \wedge q_B^*] - p\bar{F}(q_B^*)q_P^*. \end{cases}$$

$$\begin{cases} \pi_{mT}(q_T^*) = p\frac{1+\ell_o a_m}{1+\ell_o a_b} q_T^* \bar{F}(q_T^*) H(q_T^*), \\ \pi_{mB}(q_B^*) = p\frac{1+(\ell_o-\ell_g)a_m}{1+(\ell_o-\ell_g)a_b} q_B^* \bar{F}(q_B^*) H(q_B^*). \end{cases}$$

$$\begin{cases} \pi_{lT}(q_T^*) = pq_T^* \frac{1+\ell_r a_l}{1+\ell_r a_b} \bar{F}(q_T^*)[1-H(q_T^*)] - \frac{(1+\ell_o a_b)(1+\ell_r a_l)}{1+\ell_r a_b} c_m q_T^* - (1+\ell_r a_l) c_l q_T^*, \\ \pi_{lB}(q_B^*) = pq_B^* \frac{1+\ell_r a_l}{1+\ell_r a_b} \bar{F}(q_B^*)[1-H(q_B^*)] - \frac{(1+\ell_o a_m)(1+\ell_r a_l)(1+(\ell_o-\ell_g)a_b)}{(1+\ell_r a_b)(1+(\ell_o-\ell_g)a_m)} c_m q_B^* - (1+\ell_r a_l) c_l q_B^*. \end{cases}$$

With the same cash opportunity cost, we have $q_T^* = q_B^*$, and then obtain that $\pi_{bT}(q_T^*) = \pi_{bB}(q_B^*)$, $\pi_{mT}(q_T^*) = \pi_{mB}(q_B^*)$, and $\pi_{lT}(q_T^*) = \pi_{lB}(q_B^*)$.

Case two: the manufacturer leadership. The profit functions of players are the same as those in Case one. Similar to Case one, we have $q_T^* = q_B^*$ if the cash opportunity cost is equal, and get $\pi_{bT}(q_T^*) = \pi_{bB}(q_B^*)$, $\pi_{lT}(q_T^*) = \pi_{lB}(q_B^*)$, and $\pi_{mT}(q_T^*) = \pi_{mB}(q_B^*)$. Q.E.D.

Proof of Lemma 4: We solve the game in its general form backward. For the buyer, solving $\frac{d\pi_{bi}(q_{bi})}{dq_{bi}} = 0$ in Eq. (3), we have the buyer's optimal ordering solution q_{bi}^*. Since q_{bi}^* monotonically decreases with w_{mi}, there exists a one-by-one mapping between q_{bi}^* and w_{mi}, that is,

$$w_{mi}(q_{bi}^*) = \begin{cases} \frac{p\bar{F}(q_{bi}^*) - w_{li}(1+\ell_r a_b)}{1+\ell_o a_b}, & i = T, \\ \frac{p\bar{F}(q_{bi}^*) - w_{li}(1+\ell_r a_b)}{1+\ell_r a_b}, & i = P. \end{cases}$$

Submitting $w_{mi}(q_{bi}^*)$ into the manufacturer's profit function, and solving $\frac{d\pi_{mi}(q_{mi})}{dq_{mi}} = 0$, we then have

$$w_{li}(q_{mi}^*) = \begin{cases} \frac{p\bar{F}(q_{mi}^*)[1-H(q_{mi}^*)] - c_m(1+\ell_o a_b)}{1+\ell_r a_b}, & i = T, \\ \frac{p\bar{F}(q_{mi}^*)[1-H(q_{mi}^*)]}{1+\ell_r a_b} - \frac{c_m(1+\ell_o a_m)}{1+(\ell_o-\ell_g)a_m}, & i = P. \end{cases}$$

For P, submitting $w_{lP}(q_{mi}^*)$ into $w_{mP}(q_{mi}^*)$, we have, $w_{mP}(q_{mP}^*) = \frac{p\bar{F}(q_{mP}^*)H(q_{mP}^*)}{1+\ell_r a_b} + \frac{c_m(1+\ell_o a_m)}{1+(\ell_o-\ell_g)a_m}$. Then, submitting $w_{li}(q_{mi}^*)$ and $w_{mP}(q_{mi}^*)$ into 3PL's profit function, we have

$$\pi_{li}^l(q_{li}) = \begin{cases} p\ q_{li} \frac{1+\ell_r a_l}{1+\ell_r a_b} \bar{F}(q_{li})[1-H(q_{li})] - \frac{(1+\ell_o a_b)(1+\ell_r a_l)}{1+\ell_r a_b} c_m q_i - c_l(1+\ell_r a_l) q_{li}, & i = T, \\ p\ q_{li} \frac{1+\ell_r a_l}{1+\ell_r a_b} \bar{F}(q_{li})[1 - \frac{1+(\ell_o-\ell_g)a_l}{1+\ell_r a_l} H(q_{li})] - \frac{(1+\ell_o a_m)[1+(\ell_o-\ell_g)a_l]}{1+(\ell_o-\ell_g)a_m} c_m q_i - c_l(1+\ell_r a_l) q_{li}, & i = P. \end{cases}$$

Define $\eta_m = \frac{1+\ell_o a_m}{1+(\ell_o-\ell_g)a_m}$, $\eta_l = \frac{1+(\ell_o-\ell_g)a_l}{1+\ell_r a_l}$ and $\eta_b = \frac{1+\ell_r a_b}{1+\ell_o a_b}$. Taking the first derivative of $\pi_{li}(q_{li})$ w.r.t. q_{li}, and letting $\frac{d\pi_{li}(q_{li})}{dq_{li}} = 0$, we can solve the 3PL's optimal solution q_{lT}^*(i.e., q_T^*) and q_{lP}^* (i.e., q_P^*) by the following equations.

$$\begin{cases} G(p, q_T^*, 1) - c_l(1+\ell_r a_b) - c_m(1+\ell_o a_b) = 0, \\ G(p, q_P^*, \eta_l) - c_l(1+\ell_r a_b) - c_m(1+\ell_o a_b)\eta_m\eta_l\eta_b = 0. \end{cases} \tag{4}$$

Q.E.D.

Proof of Proposition 5: We first show that, if $a_m \geq a_l$, $\eta_m\eta_l$ increases in ℓ_g and vise versa. We obtain that $\frac{d\eta_m\eta_l}{d\ell_g} = \frac{1+\ell_o a_m}{(1+\ell_r a_l)[1+(\ell_o-\ell_g)a_m]^2}(a_m - a_l)$. Therefore, if $a_m \geq a_l$, we have $\frac{d\eta_m\eta_l}{d\ell_g} \geq 0$, and vise versa.

Part (i): *First:* $\pi_{lP} \geq \pi_{lT}$ *iff* $a_b \geq a_m$. Given $\ell_g = \ell_o - \ell_r$, $\eta_l = 1$. In Eq. (4), we observe $\eta_m\eta_b \leq 1$ when $a_b \geq a_m$,, which suggests $q_P^* \geq q_T^*$ because $p\bar{F}(q)[(1-H(q))^2 - qH'(q)]$ decreases with q. Consequently,

$$\begin{cases} \pi_{lT}(q_T^*) = pq_T^* \frac{1+\ell_r a_l}{1+\ell_r a_b} \bar{F}(q_T^*)[1-H(q_T^*)] - \frac{(1+\ell_o a_b)(1+\ell_r a_l)}{1+\ell_r a_b} c_m q_T^* - (1+\ell_r a_l) c_l q_T^*, \\ \pi_{lP}(q_P^*) = pq_P^* \frac{1+\ell_r a_l}{1+\ell_r a_b} \bar{F}(q_P^*)[1-H(q_P^*)] - \frac{(1+\ell_g a_m)(1+\ell_r a_l)}{1+\ell_r a_m} c_m q_P^* - (1+\ell_r a_l) c_l q_P^*. \end{cases} \tag{5}$$

Since $a_b \geq a_m$, we have $\frac{(1+\ell_r a_l)(1+\ell_o a_b)}{1+\ell_r a_b} \geq \frac{(1+\ell_r a_l)(1+\ell_o a_m)}{1+\ell_r a_m}$, then we show $\pi_{lT}(q_T^*) \leq \pi_{lP}(q_T^*)$.

Second: $\pi_{mP} \geq \pi_{mT}$ *if* $a_b \geq a_m$.

Submitting q_{lT}^* and q_{lP}^* into the manufacture's profit, we have,

$$\begin{cases} \pi_{mT}(q_T^*) = p\frac{1+\ell_o a_m}{1+\ell_o a_b} q_T^* \bar{F}(q_T^*) H(q_T^*), \\ \pi_{mP}(q_P^*) = p\frac{1+\ell_r a_m}{1+\ell_r a_b} q_P^* \bar{F}(q_P^*) H(q_P^*). \end{cases} \tag{6}$$

Following our proof in Proposition 1, in which we show that $\bar{F}(q)H(q)$ increases in q, we have $q_T^*\bar{F}(q_T^*)H(q_T^*) \leq q_P^*\bar{F}(q_P^*)H(q_P^*)$ since $q_T^* \leq q_P^*$. Because $a_b \geq a_m$ and $\ell_o \geq \ell_r$, we have $\frac{1+\ell_o a_m}{1+\ell_o a_b} \leq$

$\frac{1+\ell_r a_m}{1+\ell_r a_b}$. Consequently, if $a_b \geq a_m$, $\pi_{mT}(q_T^*) \leq \pi_{mP}(q_P^*)$.

Third: $\pi_{bP} \geq \pi_{bT}$ *if* $a_b \geq a_m$.

Similarly, the buyer's profit can be rewritten as

$$\begin{cases} \pi_{bT}(q_T^*) = p\mathbb{E}[D \wedge q_T^*] - p\bar{F}(q_T^*)q_T^*, \\ \pi_{bP}(q_P^*) = p\mathbb{E}[D \wedge q_P^*] - p\bar{F}(q_P^*)q_P^*. \end{cases} \tag{7}$$

We can show $\frac{d\pi_{bi}(q_{li}^*)}{dq_{li}^*} = p\bar{F}(q_{li}^*)H(q_{li}^*) \geq 0$. Therefore, $\pi_{bT}(q_T^*) \leq \pi_{bP}(q_P^*)$ given $q_T^* \leq q_P^*$ if $a_b \geq a_m$.

Based on the above analysis, we can infer that $q_{lT}^* = q_{lP}^*$ if $a_b = a_m$ regardless of the value of a_l. Thus, all firms are indifferent of P and T if $a_b = a_m$. For tie-breaking, our analysis assumes P is taken when firms are indifferent of both cases.

Part (ii):

If $\ell_g > \ell_s$, we have $\eta_l < 1$. Furthermore, when $a_l > a_m$, $\eta_m\eta_l$ decreases in ℓ_g. As we show in Part (i), $\ell_g = \ell_s$, $\eta_m\eta_b < 1$ if $a_b \geq a_m$. Then, we can show that if $a_m \leq a_l, a_b$, $\eta_m\eta_l\eta_b < 1$, we have $q_P^* > q_T^*$. In Eq. (5) and (6), if $a_b \geq a_m$ and $q_P^* > q_T^*$, we have $\pi_{lP}(q_P^*) > \pi_{lT}(q_T^*)$ and $\pi_{mP}(q_P^*) > \pi_{mT}(q_T^*)$. In Eq. (7), if $q_P^* > q_T^*$, we have $\pi_{bP}(q_P^*) > \pi_{bT}(q_T^*)$. As a result, the proof of Part (ii) is well done.

Part (iii):

Since $a_m > a_l$, we have $\eta_m\eta_l$ increases in ℓ_g. If $\ell_g < \ell_s$, we have $\eta_m\eta_l < 1$. And we already have $\eta_b < 1$. Then, we obtain $\eta_m\eta_l\eta_b < 1$.

According to Eq. (4), we have q_T^* and q_P^*, further q_P^* decreases in η_l. If $\ell_g < \ell_s$, we obtain η_l increases in a_l. Let $\hat{a}_l$ solve $q_T^*(a_l) = q_P^*(a_l)$. Then, we have $q_P^* \geq q_T^*$ if $a_l \leq \hat{a}_l$.

Similar to the proof in Part (ii), we can show that all firms prefer P to T if $a_b > a_m > a_l$ and even $a_b \geq \hat{a}_b(\ell_g)$. Q.E.D.

Proof of Lemma 5: Solving Eq. (3), we have $w_{lT} = \frac{p\bar{F}(q)-w_{mT}(1+\ell_o a_b)}{1+\ell_r a_b}$, and $w_{lP} = \frac{p\bar{F}(q)-w_{mP}(1+\ell_r a_b)}{1+\ell_r a_b}$. Submitting w_{lT} and w_{lP} into Eq. (5), we have $w_{mT} = \frac{p\bar{F}(q)[1-H(q)]-c_l(1+\ell_r a_b)}{1+\ell_o a_b}$ and

$$w_{mP} = \frac{p\bar{F}(q)[1-H(q)](1+\ell_r a_l)}{(1+\ell_r a_b)[1+(\ell_o-\ell_g)a_l]} - c_l\frac{1+\ell_r a_l}{1+(\ell_o-\ell_g)a_l}.$$

Submitting w_{mT} and w_{mP} into Eq. (4), we can solve q_{mT}^* and q_{mP}^* as described in Lemma 5. Q.E.D.

Proof of Proposition 6: Part (i): From Lemma 5, we know q_{mT}^* (i.e., q_T^*) and q_{mP}^* (i.e., q_P^*) solving the following equations, respectively,

$$\begin{cases} G(p, q_T^*, 1) - c_l(1+\ell_r a_b) - c_m(1+\ell_o a_b) = 0 \\ G(p, q_P^*, 1) - c_l(1+\ell_r a_b) - \eta_m\eta_l\eta_b c_m(1+\ell_o a_b) = 0. \end{cases} \tag{8}$$

As we know, $G(p,q,1)$ decreases with q. Thus, if $\eta_m\eta_l\eta_b \leq 1$, we immediately have $q_P^* \geq q_T^*$.

Part (ii): We first prove that $\pi_{bT}(q_T^*) \leq \pi_{bP}(q_P^*)$, $\pi_{lT}(q_T^*) \leq \pi_{lP}(q_P^*)$, and $\pi_{mT}(q_T^*) \leq \pi_{mP}(q_P^*)$, when $q_T^* \leq q_P^*$. Similar to the proof of Lemma 5, we submit $w_{lT}(q_T^*)$, $w_{mT}(q_T^*)$, $w_{lP}(q_P^*)$, and $w_{mP}(q_P^*)$ into all firms' profit functions. And we obtain $\pi_{bT}(q_T^*) = p\mathbb{E}[D \wedge q_T^*] - p\bar{F}(q_T^*)q_T^*$ and $\pi_{bP}(q_P^*) = p\mathbb{E}[D \wedge q_P^*] - p\bar{F}(q_P^*)q_P^*$; $\pi_{lT}(q_T^*) = p\frac{1+\ell_r a_l}{1+\ell_r a_b}\bar{F}(q_T^*)H(q_T^*)q_T^*$ and $\pi_{lP}(q_P^*) = p\frac{1+\ell_r a_l}{1+\ell_r a_b}\bar{F}(q_P^*)H(q_P^*)q_P^*$; $\pi_{mT}(q_T^*) = \frac{1+\ell_o a_m}{1+\ell_o a_b}[p\bar{F}(q_T^*)(1-H(q_T^*)) - c_l(1+\ell_r a_b)] - c_m(1+\ell_o a_m)q_T^*$ and

$$\pi_{mP}(q_P^*) = \frac{1+\ell_r a_l}{1+\ell_r a_b}\frac{1+(\ell_o-\ell_g)a_m}{1+(\ell_o-\ell_g)a_l}[p\bar{F}(q_P^*)(1-H(q_P^*)) - c_l(1+\ell_r a_b)] - c_m q_P^*(1+\ell_o a_m).$$

We can show that $\frac{d\pi_{bi}(q)}{dq} > 0$ and $\frac{d\pi_{li}(q)}{dq} > 0$. Therefore, $\pi_{bT}(q_{mT}^*) \leq \pi_{bP}(q_{mP}^*)$ and $\pi_{lT}(q_{mT}^*) \leq \pi_{lP}(q_{mP}^*)$ when $q_{mT}^* \leq q_{mP}^*$.

Consider the manufacturer. The inequality of $1 + \ell_o a_b \geq \frac{(1+\ell_o a_m)(1+\ell_r a_b)[1+(\ell_o-\ell_g)a_l]}{(1+\ell_r a_l)[1+(\ell_o-\ell_g)a_m]}$ (i.e., $\eta_m\eta_l\eta_b \leq 1$) is equivalent to $\frac{1+\ell_o a_m}{1+\ell_o a_b} \leq \frac{1+\ell_r a_l}{1+\ell_r a_b}\frac{1+(\ell_o-\ell_g)a_m}{1+(\ell_o-\ell_g)a_l}$. Therefore, for any given q, we have $\pi_{mP}(q) \geq \pi_{mT}(q)$. Let $q = q_T^*$, we obtain $\pi_{mP}(q_T^*) \geq \pi_{mT}(q_T^*)$. Given $\eta_m\eta_l\eta_b \leq 1$, we also have $q_P^* \geq q_T^*$. Since q_P^* is the optimal solution for the manufacturer in P, we must have

$\pi_{mP}(q_P^*) \geq \pi_{mP}(q_T^*)$. Therefore, the manufacturer is better off as long as $\eta_m \eta_l \eta_b \leq 1$, because $\pi_{mP}(q_P^*) \geq \pi_{mP}(q_T^*) \geq \pi_{mT}(q_T^*)$.

If $\eta_b \geq \frac{a_m}{a_l}\delta$ and $a_l \geq a_b$, we can rewrite $\eta_m \eta_l \eta_b \leq 1$ as $\ell_g \geq \frac{(1+\ell_o a_m)(a_l - a_b)(\ell_o - \ell_r)}{(1+\ell_o a_m)(a_l - a_b) + (1+\ell_r a_l)(a_b - a_m)} = \frac{1}{1+\beta}\ell_s = \xi \ell_s$. Therefore, all firms prefer P to T, as long as $\ell_g \in [\xi \ell_s, \ell_o]$. Here, without loss of generality, we assume P is preferred if there is a tie between P and T.

Otherwise if $\eta_b < \frac{a_m}{a_l}\delta$ and $a_l < a_b$, we can rewrite $\eta_m \eta_l \eta_b \leq 1$ as $\ell_g \leq \frac{(1+\ell_o a_m)(a_b - a_l)(\ell_o - \ell_r)}{(1+\ell_o a_m)(a_b - a_l) + (1+\ell_r a_l)(a_m - a_l)} = \frac{1}{1+\beta}\ell_s = \xi \ell_s$. $\ell_g \leq \xi \ell_s$. That is, all firms prefer P to T, as long as $\ell_g \in [0, \xi \ell_s]$. Q.E.D.

Proof of Corollary 2: Corollary 2 is a special case of Proposition 6 and can be obtained immediately by plugging the corresponding conditions. Thus, due to limited space, the proof is omitted. Q.E.D.

Proof of Proposition 7: The proof can be obtained from comparing Eq. (4) with Eq. (8), then thus omitted because of limited space. Q.E.D.

Proof of Proposition 8: Part (1): When the buyer borrows capital from the bank, we have

$$\pi_{bi}(q_{bi}) = \begin{cases} p\mathbb{E}[D \wedge q_{bi}] - [w_{mi}(1+\ell_o a_b) + w_{li}(1+\ell_r a_b)]q_{bi}(1+r_f), & \text{if } i = T, \\ p\mathbb{E}[D \wedge q_{bi}] - [(w_{mi} + w_{li})(1+\ell_r a_b)]q_{bi}(1+r_f), & \text{if } i = P. \end{cases} \tag{9}$$

Taking derivative of q_{bi} in Eq.(9), we have $w_{mT} = \frac{p\bar{F}(q) - w_{lT}(1+\ell_r a_b)(1+r_f)}{(1+\ell_o a_b)(1+r_f)}$ and $w_{mP} = \frac{p\bar{F}(q) - w_{lP}(1+\ell_r a_b)(1+r_f)}{(1+\ell_r a_b)(1+r_f)}$. Submitting w_{mT} and w_{mP} into manufacturer's profit function, and taking derivative of q_{mi}, we have $w_{lT} = \frac{\frac{p}{1+r_f}\bar{F}(q)[1-H(q)] - c_m(1+\ell_o a_m)}{1+\ell_r a_b}$, $w_{lP} = \frac{\frac{p}{1+r_f}\frac{1+(\ell_o-\ell_g)a_m}{1+\ell_r a_b}\bar{F}(q)[1-H(q)] - c_m(1+\ell_o a_m)}{1+(\ell_o-\ell_g)a_m}$, and $w_{mP} = \frac{p\bar{F}(q)H(q)}{(1+r_f)(1+\ell_r a_b)} + \frac{c_m(1+\ell_o a_m)}{1+(\ell_o-\ell_g)a_m}$. Let $a_i = a$, where $i = b, m, l$. Then, we submit w_{lT}, w_{lP} and w_{mP} into 3PL's profit function, and rewrite it as the following. $\pi_{lT} = \frac{p}{1+r_f}\bar{F}(q_{lT})[1 - H(q_{lT})]q_{lT} - c_m(1 + \ell_o a)q_{lT} - c_l(1+\ell_r a)q_{lT}$, and $\pi_{lP} = \frac{p}{1+r_f}\bar{F}(q_{lP})[1 - \frac{1+(\ell_o-\ell_g)a}{1+\ell_r a}H(q_{lP})]q_{lP} - c_m(1+\ell_o a)q_{lP} - c_l(1 + \ell_r a)q_{lP}$. Solving $\frac{d\pi_{lT}}{dq_{lT}} = 0$ and $\frac{d\pi_{lP}}{dq_{lP}} = 0$ results in Proposition 8.

Part (2): For any given r_f, q_i^* solves the following equations:

$$\begin{cases} G(\frac{p}{1+r_f}, q_i^*, 1) = c_m(1+\ell_o a) + c_l(1+\ell_r a), & \text{if } i = T, \\ G(\frac{p}{1+r_f}, q_i^*, \eta_l) = c_m(1+\ell_o a) + c_l(1+\ell_r a), & \text{if } i = P. \end{cases}$$

The RHS in the above equations is fixed for any given ℓ_s and ℓ_g. The LHSs decrease with q_i^*. If r_f increases, $\frac{p}{1+r_f}$ decreases. To keep the equations hold, both $G(\frac{p}{1+r_f}, q_i^*, 1)$ and $G(\frac{p}{1+r_f}, q_i^*, \eta_l)$ must increase. Consequently q_T^* and q_P^* must decrease with r_f.

Next, we prove that $\frac{d\pi_{li}(q_i^*)}{dr_f} < 0$, $\frac{d\pi_{mi}(q_i^*)}{dr_f} < 0$, and $\frac{d\pi_{bi}(q_i^*)}{dr_f} < 0$. The buyer's profit is $\pi_{bi}(q_i^*) = p[\mathbb{E}[D \wedge q_i^*] - \bar{F}(q_i^*)q_i^*]$. Since $\frac{d\pi_{bi}(q_i^*)}{dq_i^*} > 0$ and $\frac{dq_i^*}{dr_f} < 0$, we have $\frac{d\pi_{bi}(q_i^*)}{dr_f} < 0$. The manufacturer's profit functions are $\pi_{mT}(q_T^*) = \frac{p}{1+r_f}q_T^*\bar{F}(q_T^*)H(q_T^*)$ and $\pi_{mP}(q_P^*) = \frac{p}{1+r_f}\eta_l q_P^*\bar{F}(q_P^*)H(q_P^*)$, in T and P, respectively. We have $q\bar{F}(q)H(q)$ increases with q, and then $q_i^*\bar{F}(q_i^*)H(q_i^*)$ decreases with r_f, because $\frac{dq_i^*}{dr_f} < 0$. Because $\frac{p}{1+r_f}$ decreases with r_f, both $\pi_{mT}(q_T^*)$ and $\pi_{mP}(q_P^*)$ decrease with r_f. For the 3PL, we have $\frac{d\pi_{li}(q_i^*)}{dr_f} = \frac{\partial \pi_{li}(q_i^*)}{\partial q_i^*}\frac{dq_i^*}{dr_f} + \frac{\partial \pi_{li}(q_i^*)}{\partial r_f} = \frac{\partial \pi_{li}(q_i^*)}{\partial r_f}$. We then obtain $\frac{d\pi_{lT}(q_T^*)}{dr_f} = -\frac{p}{(1+r_f)^2}\bar{F}(q_T^*)[1 - H(q_T^*)]q_T^* < 0$ and $\frac{d\pi_{lP}(q_P^*)}{dr_f} = -\frac{p}{(1+r_f)^2}\bar{F}(q_P^*)[1 - \frac{1+(\ell_o-\ell_g)a}{1+\ell_r a}H(q_P^*)]q_P^* < 0$. Q.E.D.

Proof of Proposition 9: Part (i): Similar to the proof in Lemma 3, the first order conditions of Eq. (19) in T and P, respectively, give us

$$\frac{p}{n}\bar{F}(q_{lT})[(1 - H(q_{lT}))(n - H(q_{lT})) - q_{lT}H'(q_{lT})] = c_l' + c_m', \tag{10}$$

$$\frac{p}{n}\bar{F}(q_{lP})[(1 - \eta_l H(q_{lP}))(n - H(q_{lP})) - \eta_l q_{lP}H'(q_{lP})] = c_l' + c_m'. \tag{11}$$

For a fixed q_{lT}, as n increases, $(p - \frac{p}{n}H(q_{lT}))$ and $-\frac{p}{n}q_{lT}H'(q_{bT})$ increases, and then $\bar{F}(q_{lT})[(1 -$

$H(q_{lT}))(p - \frac{p}{n}H(q_{lT})) - \frac{p}{n}q_{lT}H'(q_{lT})]$ increases. Given that the right hand sides are constant, as n increases, q_{li} must increase to satisfy the first equation. The same logic applies to the second equation. Then, we can show that $\frac{\partial q_T^*}{\partial n} > 0$ and $\frac{\partial q_P^*}{\partial n} > 0$.

The proof is similar to that of Proposition 1. Given any n, the firms' preference is independent of n but hinges on the values of ℓ_g and ℓ_s. We have $q_{lP}^* \geq q_{lT}^*$ iff $\ell_g \geq \ell_s$; otherwise $q_{lP}^* < q_{lT}^*$.

Part (ii): We use the contradiction approach to prove this. Assume that $\frac{\partial q_P^*}{\partial n} \leq \frac{\partial q_T^*}{\partial n}$ if $\ell_g \geq \ell_s$. Let $G1(n) = \frac{p}{n}\bar{F}(q_{lT})[(1 - H(q_{lT}))(n - H(q_{lT})) - q_{lT}H'(q_{lT})]$ in Eq. (10), and $G2(n) = \frac{p}{n}\bar{F}(q_{lP})[(1 - \eta_l H(q_{lP}))(n - H(q_{lP})) - \eta_l q_{lP}H'(q_{lP})]$ in Eq. (11). And $G1(n)$ and $G2(n)$ increase in n. Since $\frac{\partial q_P^*}{\partial n} \leq \frac{\partial q_T^*}{\partial n}$, we should find an n equal to $\tilde{n}$ satisfying $q_T^*(\tilde{n}) > q_P^*(\tilde{n})$. If $\ell_g \geq \ell_s$, we have $\eta_l \leq 1$. Then for any given n, we have $G1(n) \leq G2(n)$. Consequently, from Eq. (10) and (11), we have $q_T^* \leq q_P^*$, which contradicts the previous result. Therefore, we have $\frac{\partial q_P^*}{\partial n} \geq \frac{\partial q_T^*}{\partial n}$. Q.E.D.

Proof of Proposition 10: For limited space, we focus only on the 3PL leadership game. The result is the same for manufacturer leadership. According to Lemma 3, the following FOC conditions must hold.

$$\begin{cases} G(p, q_T^*, 1) = c_m(1 + \ell_o a) + c_l(1 + \ell_r a), \\ G(p, q_P^*, \eta_l) = c_m(1 + \ell_o a) + c_l(1 + \ell_r a). \end{cases}$$

As proved in Lemma 3, for any given ℓ_g, LHS decreases with q in the above equations. Note that $\ell_o = L_{si} + \ell_r$. When L_{si} increases, RHS increases in both Models T and P. To keep the equations hold, LHS must increase and, therefore, q_i^* must decrease with L_{si}. Meanwhile, for the buyer, its profit is $\pi_{bi}(q_i^*) = p[\mathbb{E}[D \wedge q_i^*] - q_i^* \bar{F}(q_i^*)]$. It can be proved that $\frac{d\pi_{bi}(q_i^*)}{dq_i^*} > 0$. Since $\frac{dq_i^*}{dL_{si}} < 0$, we have $\frac{d\pi_{bi}(q_i^*)}{dL_{si}} < 0$. For the 3PL, its profit in Model T is $\pi_{lT}(q_T) = p\bar{F}(q_T)[1 - H(q_T)]q_T - c_m(1 + \ell_o a)q_T - c_l(1 + \ell_r)q_T$. $\frac{d\pi_{lT}(q_T^*)}{dL_{sT}} = \frac{\partial \pi_{lT}(q_T^*)}{\partial q_T^*}\frac{dq_T^*}{dL_{sT}} + \frac{\partial \pi_{lT}(q_T^*)}{\partial L_{sT}}$. From Lemma 3, we have $\frac{\partial \pi_{lT}(q_T^*)}{\partial q_T^*} = 0$, and $\frac{d\pi_{lT}(q_T^*)}{dL_{sT}} = \frac{\partial \pi_{lT}(q_T^*)}{\partial L_{sT}} = -c_m q_T^* a < 0$. The 3PL's profit in Model P is $\pi_{lP}(q_P) = p\bar{F}(q_P)[1 - \eta_l H(q_P)]q_P - c_m(1 + \ell_o a)q_P - c_l(1 + \ell_r)q_P$. Similarly, we have $\frac{d\pi_{lT}(q_P^*)}{dL_{sP}} = -\frac{pa}{1+\ell_r a}q_P^*\bar{F}(q_P^*)H(q_P^*) - c_m q_P^* a < 0$.

Because both Π_{bi} and Π_{li}, $i = T, P$, decrease with L_{si}, the Nash bargaining product for any given $\theta_b \in [0, 1]$ decreases with L_{si}. Therefore, the optimal Nash bargaining solution is achieved at the lower bound $L_{si}^* = \ell_s$, $i = T, P$. Q.E.D.

Proof of Proposition 11: The proof is provided conceptually right after the proposition. Q.E.D.

Proof of Corollary 3: Let $\tilde{\ell} = \frac{(1+\ell_o a_m a_b)(\ell_o - \ell_r)}{a_m(1+\ell_o a_b)}$ and $\check{\ell}$ solve $\pi_{lT}(q_T^*) = \pi_{lP}(q_P^*, \ell_g)$. If $\ell_g \leq \tilde{\ell}$, we have $\frac{1+(\ell_o-\ell_g)a_m}{1+\ell_r a_b} \geq \frac{1+\ell_o a_m}{1+\ell_o a_m}$. From Lemma 2, we obtain $q_P^* \geq q_T^*$. To prove $T \prec P$, we need to prove $\pi_{bT}(q_T^*) \leq \pi_{bP}(q_P^*)$, $\pi_{mT}(q_T^*) \leq \pi_{mP}(q_P^*)$, and $\pi_{lT}(q_T^*) \leq \pi_{lP}(q_P^*)$. Submitting q_T^* and q_P^* into buyer's profit function, we have $\pi_{bT}(q_T^*) = p\min[D \wedge q_T^*] - p\bar{F}(q_T^*)q_T^*$ and $\pi_{bP}(q_P^*) = p\min[D \wedge q_P^*] - p\bar{F}(q_P^*)q_P^*$. Then we have $\pi_{bT}(q_T^*) \leq \pi_{bP}(q_P^*)$ if $q_T^* \leq q_P^*$ or $\ell_g \leq \tilde{\ell}$. Similarly, $\pi_{mT}(q_T^*) \leq \pi_{mP}(q_P^*)$ when $\ell_g \leq \tilde{\ell}$.

Since $\frac{d\pi_{lP}(q_P^*)}{d\ell_g} = \frac{\partial \pi_{lP}(q_P^*)}{\partial q_P^*}\frac{dq_P^*}{d\ell_g} + \frac{\partial \pi_{lP}(q_P^*)}{d\ell_g}$, we have $\frac{d\pi_{lP}(q_P^*)}{d\ell_g} = \{(\frac{p\bar{F}(q_P^*)}{1+\ell_r a_l} - w_l)q_P^* + [(w_l - c_l)\ell_r + \frac{c_m(1+\ell_o a_m)(\ell_g-\ell_s)}{1+(\ell_o-\ell_g)a_m}]q_P^{*\prime}\}a_l + (w_l - c_l)q_P^{*\prime}$, where $q_P^{*\prime} = \frac{dq_P^*}{d\ell_g} < 0$. Let $A(\ell_g) = (\frac{p\bar{F}(q_P^*)}{1+\ell_r a_l} - w_l)q_P^* + [(w_l - c_l)\ell_r + \frac{c_m(1+\ell_o a_m)(\ell_g-\ell_s)}{1+(\ell_o-\ell_g)a_m}]q_P^{*\prime}$, we get $\frac{d\pi_{lP}(q_P^*)}{d\ell_g} = A(\ell_g)a_l + (w_l - c_l)q_P^{*\prime}$, then $\frac{d\pi_{lP}(q_P^*)}{d\ell_g}$ is a linear function of a_l. If $a_l < \frac{(c_l-w_l)q_P^{*\prime}}{A(\ell_g)}$, $\pi_{lP}(q_P^*)$ decreases in ℓ_g. Then $\ell_g \leq [\check{\ell}, \tilde{\ell}]$, $T \prec P$. Otherwise, $\pi_{lP}(q_P^*)$ increases in ℓ_g. If $\check{\ell} < \tilde{\ell}$, we have $T \prec P$. Q.E.D.