Technical Appendix - Financing Capacity Investment Under Demand Uncertainty: An Optimal Contracting Approach

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Appendix A: Preliminary Lemmas

Lemma 1. *$\forall q \in \mathbb{R}_+$, $g_{1,q}/g_{0,q}$ is strictly increasing over $[sq, rq]$, where $g_{e,q}$ is the probability distribution of P_e, q.*

Proof: The distributions of payoff are $g_{e,q}(x) = f_e\left(P_{e,q}^{-1}(x)\right)$ for $x \in [sq, rq)$ and $g_{e,q}(rq) = \bar{F}_e(q)$. Hence $g_{1,q}/g_{0,q}$ is increasing over $[sq, rq)$ because f_0 and f_1 satisfy MLRP. Moreover, we have

$$\begin{aligned}
\frac{g_{1,q}(rq)}{g_{0,q}(rq)} &= \frac{\bar{F}_1(q)}{\bar{F}_0(q)} = \frac{\int_q^{+\infty} f_1(x)\,dx}{\int_q^{+\infty} f_0(x)\,dx} = \frac{\int_q^{+\infty} \frac{f_1(x)}{f_0(x)} f_0(x)\,dx}{\int_q^{+\infty} f_0(x)\,dx} > \frac{\int_q^{+\infty} \frac{f_1(q)}{f_0(q)} f_0(x)\,dx}{\int_q^{+\infty} f_0(x)\,dx} \text{ by MLRP} \\
&> \frac{f_1(q)}{f_0(q)} > \frac{f_1(x)}{f_0(x)} \quad \text{for all } x \in [0, q) \text{ by MLRP} \\
&> \frac{g_{1,q}(p)}{g_{0,q}(p)} \quad \text{for all } p \in [sq, rq).
\end{aligned}$$

■

Lemma 2. *For $q \in \mathbb{R}_+$, consider two feasible claims A and B with repayment functions $R_A(\cdot)$ and $R_B(\cdot)$ satisfying $\mathbb{E}[R_A(P_{1,q})] = \mathbb{E}[R_B(P_{1,q})]$ and $\exists p^* \in (sq, rq)$ such that $\forall p \in [sq, p^*)$ $R_A(p) \geq R_B(p)$ and $\forall p \in (p^*, rq]$ $R_A(p) \leq R_B(p)$ with strict inequalities for a set of payoffs. Constraint (19)'s left-hand side is strictly larger for A than for B.*

Proof: We want to show that the left-hand side of (19) is weakly larger for A than for B, i.e.,

$$\mathbb{E}[P_{1,q} - R_A(P_{1,q})] - \mathbb{E}[P_{0,q} - R_A(P_{0,q})] - \Delta\kappa > \mathbb{E}[P_{1,q} - R_B(P_{1,q})] - \mathbb{E}[P_{0,q} - R_B(P_{0,q})] - \Delta\kappa.$$

After simplification and using condition $\mathbb{E}[R_A(P_{1,q})] = \mathbb{E}[R_B(P_{1,q})]$, this amounts to showing that $\mathbb{E}[\Delta(P_{0,q})] > 0$, where $\Delta(\cdot) \equiv R_A(\cdot) - R_B(\cdot)$. Therefore, we have

$$\mathbb{E}[\Delta(P_{0,q})] = \int_{sq}^{p^*} \Delta(p)\, g_{0,q}(p)\, dp - \int_{p^*}^{rq} -\Delta(p)\, g_{0,q}(p)\, dp = \int_{sq}^{p^*} \Delta(p) \frac{g_{0,q}(p)}{g_{1,q}(p)} g_{1,q}(p)\, dp - \int_{p^*}^{rq} -\Delta(p) \frac{g_{0,q}(p)}{g_{1,q}(p)} g_{1,q}(p)\, dp$$

$$> \frac{g_{0,q}(p^*)}{g_{1,q}(p^*)} \int_{sq}^{p^*} \Delta(p)\, g_{1,q}(p)\, dp - \frac{g_{0,q}(p^*)}{g_{1,q}(p^*)} \int_{p^*}^{rq} -\Delta(p)\, g_{1,q}(p)\, dp > \frac{g_{0,q}(p^*)}{g_{1,q}(p^*)} \mathbb{E}[\Delta(P_{1,q})] > 0.$$

■

LEMMA 3. *If $W \geq (c-s)q$ then $\partial \hat{d}(q)/\partial q = 0$ and $\partial K(q)/\partial q = c$. If $W < (c-s)q$ then*

$$\frac{\partial \hat{d}(q)}{\partial q} = \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)} \quad \text{and} \quad \frac{\partial K(q)}{\partial q} = \frac{c-s}{\bar{F}_1\left(\hat{d}(q)\right)} + s.$$

Proof: If $W \geq (c-s)q$, (22) implies $K(q) \leq sq$ and $\hat{d}(q) = 0$. Therefore $\partial \hat{d}(q)/\partial q = 0$, and taking the first-order derivative of (22) with respect to q we deduce yields $\partial K/(q)\,\partial q = c$.

If $W < (c-s)q$, (22) implies $K(q) > sq$ and $\hat{d}(q) = (K(q) - sq)/(r-s)$. Hence,

$$\frac{\partial K(q)}{\partial q} - (r-s)\frac{\partial \hat{d}(q)}{\partial q} F_1\left(\frac{K-sq}{r-s}\right) = c. \tag{1}$$

Since $\partial \hat{d}(q)/\partial q = (\partial K(q)/\partial q - s)/(r-s)$ this expression can be rewritten as

$$s + (r-s)\frac{\partial \hat{d}(q)}{\partial q} - (r-s)\frac{\partial \hat{d}(q)}{\partial q} F_1\left(\hat{d}(q)\right) = c$$

$$\text{or,} \quad \frac{\partial \hat{d}(q)}{\partial q} = \frac{c-s}{r-s}\frac{1}{1-F_1\left(\hat{d}(q)\right)} = \left(1 - \frac{r-c}{r-s}\right)\frac{1}{1-F_1\left(\hat{d}(q)\right)} = \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)}$$

and the corresponding derivative of $K(q)$ is obtained from (1). ■

Appendix B: Proofs

B.1. Proof of Proposition 2

Step 1. We adapt Innes (1990) to show we can focus on debt claims. For a given capacity $q \in [\underline{q}_1, \bar{q}_1]$, the firm's problem is to find a claim $R(\cdot)$ satisfying (17), (18) and (19). Consider such a claim, $R(\cdot)$, assuming one exists, and consider the debt claim with face value $K \in (sq, rq)$ defined as the unique solution to

$$\mathbb{E}[P_{1,q} \wedge K] = \mathbb{E}[R(P_{1,q})]. \tag{2}$$

Note that the debt claim satisfies (17), because contract $R(\cdot)$ does. Also, like all debt claims, it satisfies (18).

Notice that (18) implies that $\exists p^* \in (sq, rq)$ such that $\forall p \leq p^*$, $R(p) \leq p \wedge K$ and $\forall p \geq p^*$, $R(p) \geq p \wedge K$. Hence, by Lemma 2, (19) is strictly tighter for contract $R(\cdot)$ than for the debt claim. In that sense, $R(\cdot)$ is dominated by the debt claim.

Step 2. Define function h as $h(K) \equiv K - (r-s)\int_0^{\left(\frac{K-sq}{r-s}\right)^+} F_1(x)\,dx - (cq-W)^+$. Expression (22) is written as $h(K) = 0$. Also, $h(\cdot)$ is strictly increasing as $h'(K) = 1$ for $K \leq sq$ and $h'(K) = 1 - F_1\left(\frac{K-sq}{r-s}\right)$ for $K \geq sq$.

If $W \geq (c-s)\,q$ then $(cq-W)^{+} \leq sq$ and $h\left((cq-W)^{+}\right) = (cq-W)^{+} - (r-s)\int_0^0 F_1(x)\,dx - (cq-W)^{+} = 0$. Therefore $K(q) = (cq-W)^{+} \in [0, sq]$. Debt is riskfree because $K(q) \leq sq$ while $P_{1,q} \geq sq$. Conversely, if $W < (c-s)\,q$ then $(cq-W)^{+} > sq$ and so $h(sq) = sq - (r-s)\int_0^0 F_1(x)\,dx - (cq-W)^{+} < 0$. Moreover

$$h(rq) = rq - (r-s)\int_0^q F_1(x)\,dx - (cq-W)^{+} = [\pi_1(q) - \kappa_1] + \left[\kappa(1) + cq - (cq-W)^{+}\right] > 0.$$

Indeed, the first bracket term non-negative over $[0, \bar{q}_1]$ and the second one is strictly positive because $\kappa_1 > 0$.

Therefore $K(q) \in (sq, rq)$. Moreover, debt is risky because $K(q) > sq$ and $P_{1,q} \geq sq$.

B.2. Proof of Lemma 2

We have

$$\frac{\partial L(q, \Delta\kappa)}{\partial q} = (F_0(q) - F_1(q)) - \frac{\partial \hat{d}(q)}{\partial q}\left(F_0\left(\hat{d}(q)\right) - F_1\left(\hat{d}(q)\right)\right). \tag{3}$$

If $W \geq (c-s)\,q$, $\hat{d}(q) = 0$ from Lemma 3 (see Appendix A), which implies $\partial L(q, \Delta\kappa)/\partial q = (F_0(q) - F_1(q)) > 0$. This means that $L(q, \Delta\kappa)$ is increasing in q and that $q_1^{\max} = \bar{q}_1 > q_1^{FB}$.

If $W < (c-s)\,q$, $\partial \hat{d}(q)/\partial q = \bar{F}_1(q_1^{FB})/\bar{F}_1\left(\hat{d}(q)\right)$ from Lemma 3. Hence, for $W < (c-s)\,q_1^{FB}$, we can rewrite (3) as

$$\begin{aligned}\frac{\partial L(q, \Delta\kappa)}{\partial q} &= (F_0(q) - F_1(q)) - \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)}\left(F_0\left(\hat{d}(q)\right) - F_1\left(\hat{d}(q)\right)\right)\\ &= \left(\bar{F}_1(q) - \bar{F}_0(q)\right) - \frac{\bar{F}_1(q_1^{FB})}{\bar{F}_1\left(\hat{d}(q)\right)}\left(\bar{F}_1\left(\hat{d}(q)\right) - \bar{F}_0\left(\hat{d}(q)\right)\right).\end{aligned}$$

For $q \leq q_1^{FB}$, we have $\bar{F}_1(q) \geq \bar{F}_1(q_1^{FB})$ and so

$$\begin{aligned}\frac{\partial L(q, \Delta\kappa)}{\partial q} &\geq \left(\bar{F}_1(q) - \bar{F}_0(q)\right) - \frac{\bar{F}_1(q)}{\bar{F}_1\left(\hat{d}(q)\right)}\left(\bar{F}_1\left(\hat{d}(q)\right) - \bar{F}_0\left(\hat{d}(q)\right)\right)\\ &\geq \bar{F}_1(q)\left[\left(1 - \frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right) - \left(1 - \frac{\bar{F}_0\left(\hat{d}(q)\right)}{\bar{F}_1\left(\hat{d}(q)\right)}\right)\right] = \bar{F}_1(q)\left[\frac{\bar{F}_0\left(\hat{d}(q)\right)}{\bar{F}_1\left(\hat{d}(q)\right)} - \frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right].\end{aligned}$$

Note now that $\bar{F}_0/\bar{F}_1$ is decreasing. Indeed,

$$\frac{d}{dx}\frac{\bar{F}_0(x)}{\bar{F}_1(x)} = \frac{-f_0(x)\,\bar{F}_1(x) + f_1(x)\,\bar{F}_0(x)}{\left(\bar{F}_1(x)\right)^2} < 0$$

since MLRP implies that D_1 stochastically dominates D_0 according to the hazard rate order.[1]

Note that for all $q < \bar{q}_1$, $\hat{d}(q) < q$. Hence for all $q \leq q_1^{FB}$, $\partial L(q, \Delta\kappa)/\partial q > 0$, which implies $q_1^{\max} > q_1^{FB}$.

B.3. Proof of Theorem 1

Recall that (4) implies an upper bound on $\Delta\kappa$, i.e., $\pi_1(q_1^{FB}) - \pi_0(q_0^{FB}) > \Delta\kappa$. If $\pi_1(q_1^{FB}) - \pi_0(q_0^{FB}) \leq (r-s)\,L(q_1^{FB}, 0)$, then for all values of $\Delta\kappa$ such that (4) holds, we have $L(q_1^{FB}, \Delta\kappa) \geq 0$ so that $q^* = q_1^{FB}$ for all W and we have thresholds $\Delta\underline{\kappa} = \Delta\bar{\kappa} = 0$. Assume now that $\pi_1(q_1^{FB}) - \pi_0(q_0^{FB}) < (r-s)\,L(q_1^{FB}, 0)$. If $W \geq (c-s)\,q_1^{FB}$ then $\hat{d}(q_1^{FB}) = 0$. In that case, (4) implies $L(q_1^{FB}, \Delta\kappa) > 0$ and so $q^* = \inf S(\Delta\kappa) = q_1^{FB}$. For all q, $L(q, 0) > 0$ and for $\Delta\kappa$ large enough $L(q, \Delta\kappa) < 0$. Moreover, $L(q, \Delta\kappa)$ is continuous and strictly decreasing with $\Delta\kappa$. Therefore, we can define $\Delta\underline{\kappa}$ and $\Delta\bar{\kappa}$ by $L(q_1^{FB}, \Delta\underline{\kappa}) = 0$ and $L(\hat{q}_1, \Delta\bar{\kappa}) = 0$.

For $\Delta\kappa \leq \Delta\underline{\kappa}$, $L(q_1^{FB}, \Delta\kappa) \geq 0$ and so $q^* = \inf S(\Delta\kappa) = q_1^{FB}$. For $\Delta\kappa > \Delta\bar{\kappa}$, $S(\Delta\kappa) = \varnothing$ and $q^* = 0$.

For $\Delta\underline{\kappa} < \Delta\kappa \leq \Delta\bar{\kappa}$, $L(q_1^{FB}, \Delta\kappa) < 0$ and $L(\hat{q}_1, \Delta\kappa) \geq 0$ so $q^* = \inf S(\Delta\kappa) > q_1^{FB}$ and increases with $\Delta\kappa$.

[1] To see this, note that $f_1(x)\,\bar{F}_0(x) = f_0(x)\int_x^{+\infty}(f_1(x)/f_0(x))\,/\,(f_1(t)/f_0(t))\,f_1(t)\,dt \leq f_0(x)\int_x^{+\infty} f_1(t)\,dt = f_0(x)\,\bar{F}_1(x)$ where the inequality holds since under MLRP, for all $t > x$, we have $f_1(x)/f_0(x) < f_1(t)/f_0(t)$.

B.4. Formalization of the Bonus and Demand Differentiation Effects

Consider a capacity increase from q_1^{FB} to $q > q_1^{FB}$. For $e \in \{0,1\}$, the project's payoff (1) increases by $\Delta P_{e,q} = P_{e,q} - P_{e,q_1^{FB}}$. Focusing for now on demand realizations above q_1^{FB}, we decompose the expected payoff increase into two elements:

$$\mathbb{E}\left[\Delta P_{e,q}|D_e \geq q_1^{FB}\right] = b\left(q\right) + a_e\left(q\right) \tag{4}$$

where the payoff increase's mean for $e=1$ and its expected deviation from that mean are denoted $b\left(q\right) \equiv \mathbb{E}\left[\Delta P_{1,q}|D_1 \geq q_1^{FB}\right]$ and $a_e\left(q\right) \equiv \mathbb{E}\left[\Delta P_{e,q} - b(q)|D_e \geq q_1^{FB}\right]$.

The first term is akin to a fixed bonus $b\left(q\right)$ paid to the firm if demand exceeds q_1^{FB}. We refer to its impact on constraint (25) as the *bonus effect*. Note that this bonus being the same for all demand realizations above q_1^{FB}, implementing it would not require being able to tell these apart. The second term reflects deviations from $b\left(q\right)$. These differ across demand realizations which is only allowed by the fact that $q > q_1^{FB}$ allows the firm to meet otherwise unmet demand realizations in $(q_1^{FB}, q]$. We thus call this term's impact on constraint (25) the *demand differentiation effect*.

Proposition 1. *For capacity $q \geq q_1^{FB}$, the bonus and demand differentiation effects are captured by $\beta\left(q\right) \equiv b\left(q\right) \cdot \left(\bar{F}_1\left(q_1^{FB}\right) - \bar{F}_0\left(q_1^{FB}\right)\right)$ and $\alpha\left(q\right) \equiv a_1\left(q\right)\bar{F}_1\left(q_1^{FB}\right) - a_0\left(q\right)\bar{F}_0\left(q_1^{FB}\right)$. We have $L\left(q,\Delta\kappa\right) - L\left(q_1^{FB},\Delta\kappa\right) = \left[\beta\left(q\right) + \alpha\left(q\right) - \varphi\left(q\right)\right] / \left(r-s\right)$ with $\beta\left(q\right)$, $\alpha\left(q\right)$ and $\varphi\left(q\right)$ equal to zero at q_1^{FB} and strictly increasing over $(q_1^{FB}, +\infty)$. Further, when $q = q_1^{FB}$, the bonus effect is first-order (i.e. $\partial\beta\left(q_1^{FB}\right)/\partial q > 0$), and the demand differentiation effect second-order (i.e. $\partial\alpha\left(q_1^{FB}\right)/\partial q = 0$).*

Proof: We have

$$\begin{aligned}
\mathbb{E}\left[\Delta P_{e,q}\right] &= \bar{F}_e\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{e,q}|D_e \geq q_1^{FB}\right] + F_e\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{e,q}|D_e \leq q_1^{FB}\right] \\
&= \bar{F}_e\left(q_1^{FB}\right)\left(b\left(q\right) + a_e\left(q\right)\right) + F_e\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{e,q}|D_e \leq q_1^{FB}\right]
\end{aligned}$$

Hence given $\left(r-s\right)L\left(q,\Delta\kappa\right) = \left(\mathbb{E}\left[P_{1,q}\right] - \mathbb{E}\left[P_{0,q}\right]\right) - \left(\mathbb{E}\left[K\left(q\right) \wedge P_{1,q}\right] - \mathbb{E}\left[K\left(q\right) \wedge P_{0,q}\right]\right) - \Delta\kappa$, we have

$$\begin{aligned}
\left(r-s\right)\left(L\left(q,\Delta\kappa\right) - L\left(q_1^{FB},\Delta\kappa\right)\right) &= \left(\mathbb{E}\left[\Delta P_{1,q}\right] - \mathbb{E}\left[\Delta P_{0,q}\right]\right) - \left(\mathbb{E}\left[K\left(q\right) \wedge P_{1,q}\right] - \mathbb{E}\left[K\left(q\right) \wedge P_{0,q}\right]\right) \\
&\quad + \left(\mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{1,q_1^{FB}}\right] - \mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{0,q_1^{FB}}\right]\right) \\
&= \begin{pmatrix} \bar{F}_1\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{1,q}|D_1 \geq q_1^{FB}\right] \\ +F_1\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{1,q}|D_1 \leq q_1^{FB}\right] \end{pmatrix} - \begin{pmatrix} \bar{F}_0\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{1,q}|D_0 \geq q_1^{FB}\right] \\ +F_0\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{0,q}|D_0 \leq q_1^{FB}\right] \end{pmatrix} \\
&\quad - \left(\mathbb{E}\left[K\left(q\right) \wedge P_{1,q}\right] - \mathbb{E}\left[K\left(q\right) \wedge P_{0,q}\right]\right) + \mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{1,q_1^{FB}}\right] - \mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{0,q_1^{FB}}\right] \\
&= \beta\left(q\right) + \alpha\left(q\right) - \varphi\left(q\right)
\end{aligned}$$

where denoting $\Delta\bar{F} \equiv \bar{F}_1 - \bar{F}_0$ and $\Delta f \equiv f_1 - f_0$ we have

$$\begin{aligned}
\varphi\left(q\right) &\equiv F_0\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{0,q}|D_0 \leq q_1^{FB}\right] - F_1\left(q_1^{FB}\right) \cdot \mathbb{E}\left[\Delta P_{1,q}|D_1 \leq q_1^{FB}\right] \\
&\quad + \left(\mathbb{E}\left[K\left(q\right) \wedge P_{1,q}\right] - \mathbb{E}\left[K\left(q\right) \wedge P_{0,q}\right]\right) - \left(\mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{1,q_1^{FB}}\right] - \mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{0,q_1^{FB}}\right]\right) \\
&= \left(r-s\right)\int_0^{\hat{d}(q)} \Delta\bar{F}\left(x\right)dx - \int_0^{q_1^{FB}} s\left(q - q_1^{FB}\right)\Delta f\left(x\right)dx - \left(\mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{1,q_1^{FB}}\right] - \mathbb{E}\left[K\left(q_1^{FB}\right) \wedge P_{0,q_1^{FB}}\right]\right).
\end{aligned}$$

Hence $\partial L(q,\Delta\kappa)/\partial q = (\partial\beta(q)/\partial q + \partial\alpha(q)/\partial q - \partial\varphi(q)/\partial q)/(r-s)$. Since $\partial\hat{d}(q)/\partial q > 0$, we have

$$\frac{\partial\varphi(q)}{\partial q} = (r-s)\left(\frac{\partial\hat{d}(q)}{\partial q}\right)\Delta\bar{F}\left(\hat{d}(q)\right) + s\Delta\bar{F}\left(q_1^{FB}\right) > 0.$$

Besides, we have

$$\begin{aligned}
\beta(q) &= \left(\int_{q_1^{FB}}^{q}(sq+(r-s)x)f_1(x)\,dx + rq\bar{F}_1(q) - rq_1^{FB}\bar{F}_1\left(q_1^{FB}\right)\right)\Delta\bar{F}\left(q_1^{FB}\right)/\bar{F}_1\left(q_1^{FB}\right)\\
&= \left(sq\left(F_1(q)-F_1\left(q_1^{FB}\right)\right) + \int_{q_1^{FB}}^{q}(r-s)xf_1(x)\,dx + rq\bar{F}_1(q) - rq_1^{FB}\bar{F}_1\left(q_1^{FB}\right)\right)\frac{\Delta\bar{F}\left(q_1^{FB}\right)}{\bar{F}_1\left(q_1^{FB}\right)}\\
\frac{\partial\beta(q)}{\partial q} &= \frac{\Delta\bar{F}\left(q_1^{FB}\right)}{\bar{F}_1\left(q_1^{FB}\right)}\left(s\left(F_1(q)-F_1\left(q_1^{FB}\right)\right) + sqf_1(q) + (r-s)qf_1(q) + r\bar{F}_1(q) - rqf_1(q)\right)\\
&= \left[s\left(F_1(q)-F_1\left(q_1^{FB}\right)\right) + r\bar{F}_1(q)\right]\Delta\bar{F}\left(q_1^{FB}\right)/\bar{F}_1\left(q_1^{FB}\right) > 0.
\end{aligned}$$

Further, we have

$$\begin{aligned}
\alpha(q) &= \mathbb{E}\left[\Delta P_{1,q} - b(q)|D_1 \geq q_1^{FB}\right]\bar{F}_1\left(q_1^{FB}\right) - \mathbb{E}\left[\Delta P_{0,q} - b(q)|D_0 \geq q_1^{FB}\right]\bar{F}_0\left(q_1^{FB}\right)\\
&= \mathbb{E}\left[\Delta P_{1,q}|D_1 \geq q_1^{FB}\right]\bar{F}_1\left(q_1^{FB}\right) - \mathbb{E}\left[\Delta P_{0,q}|D_0 \geq q_1^{FB}\right]\bar{F}_0\left(q_1^{FB}\right) - b(q)\Delta\bar{F}\left(q_1^{FB}\right)\\
&= \left(\mathbb{E}\left[P_{1,q}|D_1 \geq q_1^{FB}\right] - rq_1^{FB}\right)\bar{F}_1\left(q_1^{FB}\right) - \left(\mathbb{E}\left[P_{0,q}|D_0 \geq q_1^{FB}\right] - rq_1^{FB}\right)\bar{F}_0\left(q_1^{FB}\right) - \beta(q)\\
&= \int_{q_1^{FB}}^{q}(sq+(r-s)x)\Delta f(x)\,dx + rq\Delta\bar{F}(q) - rq_1^{FB}\Delta\bar{F}\left(q_1^{FB}\right) - \beta(q),\\
\frac{\partial\alpha(q)}{\partial q} &= s\left(\Delta F(q) - \Delta F\left(q_1^{FB}\right)\right) + rq\Delta f(q) + r\Delta\bar{F}(q) - rq\Delta f(q) - \frac{\partial\beta(q)}{\partial q}\\
&= s\left(\Delta F(q) - \Delta F\left(q_1^{FB}\right)\right) + r\Delta\bar{F}(q) - \frac{1}{\bar{F}_1\left(q_1^{FB}\right)}\left(s\left(F_1(q)-F_1\left(q_1^{FB}\right)\right) + r\bar{F}_1(q)\right)\Delta\bar{F}\left(q_1^{FB}\right)\\
&= (r-s)\bar{F}_1(q)\left(\frac{\bar{F}_0\left(q_1^{FB}\right)}{\bar{F}_1\left(q_1^{FB}\right)} - \frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right)
\end{aligned}$$

which is zero for $q = q_1^{FB}$ and strictly positive for $q > q_1^{FB}$ due to MLRP. Indeed,

$$\begin{aligned}
\frac{\partial}{\partial q}\left(\frac{\bar{F}_0(q)}{\bar{F}_1(q)}\right) &= \frac{-f_0(q)\bar{F}_1(q) + f_1(q)\bar{F}_0(q)}{\left(\bar{F}_1(q)\right)^2} = \frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\bar{F}_0(q)}{\bar{F}_1(q)} - \frac{f_0(q)}{f_1(q)}\right)\\
&= \frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\int_q^{+\infty}f_0(x)\,dx}{\bar{F}_1(q)} - \frac{f_0(q)}{f_1(q)}\right) = \frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\int_q^{+\infty}\frac{f_0(x)}{f_1(x)}f_1(x)\,dx}{\bar{F}_1(q)} - \frac{f_0(q)}{f_1(q)}\right)\\
&> \frac{f_1(q)\bar{F}_1(q)}{\left(\bar{F}_1(q)\right)^2}\left(\frac{\int_q^{+\infty}\frac{f_0(q)}{f_1(q)}f_1(x)\,dx}{\bar{F}_1(q)} - \frac{f_0(q)}{f_1(q)}\right) \quad \text{by MLRP}\\
&> 0.
\end{aligned}$$

■

The *bonus effect* is captured by $\beta(q)$. Indeed, the incentive impact of bonus $b(q)$ is via the higher likelihood $\bar{F}_1\left(q_1^{FB}\right) - \bar{F}_0\left(q_1^{FB}\right)$ of getting it by working rather than shirking. The *demand differentiation effect* is captured by $\alpha(q)$ as the incentive impact of deviations $\Delta P_{e,q} - b(q)$ from $b(q)$ is again via the difference between $e = 1$ and $e = 0$. Term $\varphi(q)$ captures the adverse incentive effect of increasing capacity, namely the higher salvage value for low demands (i.e., below q_1^{FB}), and the higher debt needed to fund the extra capacity. Hence, raising q above q_1^{FB} boost incentives (i.e., $L(q,\Delta\kappa) > L\left(q_1^{FB},\Delta\kappa\right)$) only if $\beta(q) > 0$ or $\alpha(q) > 0$.

B.5. Proof of Proposition 3

Consider (25)'s left-hand side as a function $L(W,q,\Delta\kappa)$. We have $L(0,q_1^{FB},0)=\int_{\hat{d}\left(q_1^{FB}\right)}^{q_1^{FB}}(F_0(x)-F_1(x))\,dx>0$. Recall that (4) implies an upper bound on $\Delta\kappa$, i.e., $\pi_1(q_1^{FB})-\pi_0(q_0^{FB})>\Delta\kappa$.

Case 1: $\pi_1(q_1^{FB})-\pi_0(q_0^{FB})<(r-s)\,L(0,q_1^{FB},0)$. In that case, for all values of $\Delta\kappa$ such that (4) holds we have $L(0,q_1^{FB},\Delta\kappa)>0$. Since $\partial L(W,q,\Delta\kappa)/\partial W\geq 0$, this implies $L(W,q_1^{FB},\Delta\kappa)\geq 0$ and so $q^*=q_1^{FB}$ for all W, i.e. $\underline{W}=\overline{W}=0$.

Case 2: $\pi_1(q_1^{FB})-\pi_0(q_0^{FB})>(r-s)\,L(0,q_1^{FB},0)$. In that case, values of $\Delta\kappa$ such that condition (4) holds exist such that $L(0,q_1^{FB},\Delta\kappa)=\int_{\hat{d}\left(q_1^{FB}\right)}^{q_1^{FB}}(F_0(x)-F_1(x))\,dx-\frac{\Delta\kappa}{(r-s)}<0$. Since L is continuous and $\partial L(0,q,\Delta\kappa)/\partial\Delta\kappa<0$, we can define $\Delta\kappa_0$ as the sole solution to $L(0,q_1^{FB},\Delta\kappa)=0$.

For all $\Delta\kappa\leq\Delta\kappa_0$, $L(0,q_1^{FB},\Delta\kappa)\geq 0$ which given $\partial L(W,q,\Delta\kappa)/\partial W\geq 0$ implies $L(W,q_1^{FB},\Delta\kappa)\geq 0$ for all W so that $\underline{W}=\overline{W}=0$. Now consider $\Delta\kappa>\Delta\kappa_0$, which implies $L(0,q_1^{FB},\Delta\kappa)<0$. Note that for $W\geq(c-s)\,q_1^{FB}$ we have $\hat{d}(q_1^{FB})=0$ which implies

$$\begin{aligned}
L\left((c-s)\,q_1^{FB},q_1^{FB},\Delta\kappa\right) &= \int_0^{q_1^{FB}}(F_0(x)-F_1(x))\,dx-\frac{\Delta\kappa}{(r-s)}=\frac{\pi_1(q_1^{FB})-\pi_0(q_1^{FB})-\Delta\kappa}{(r-s)}\\
&> \frac{\pi_1(q_1^{FB})-\pi_0(q_0^{FB})-\Delta\kappa}{(r-s)}>0 \text{ due to (4).}
\end{aligned}$$

Note that $q_1^{\max}(W)$ is a function of W. If $L(0,q_1^{\max}(0),\Delta\kappa)\geq 0$ then let $\underline{W}=0$. Else define $\underline{W}$ as the smallest solution to $L(W,q_1^{\max}(W),\Delta\kappa)=0$. We have for all $W<\underline{W}$ (if any), $L(W,q_1^{\max}(W),\Delta\kappa)<0$ and so $L(W,q,\Delta\kappa)<0$ for all $q\leq\bar{q}_1$. Hence the project is abandoned. Define $\bar{W}$ as the unique solution to $L(W,q_1^{FB},\Delta\kappa)=0$. Since $L(W,q_1^{FB},\Delta\kappa)$ is non-decreasing in W, we have for all $W\geq\bar{W}$, $L(W,q_1^{FB},\Delta\kappa)\geq 0$ and so $q^*=q_1^{FB}$. For $W=\underline{W}$, $q^*=q_1^{\max}(\underline{W})$ and for all $W\in[\underline{W},\bar{W})$, $q^*(W)=\inf\{q\in[q_1^{FB},q_1^{\max}(W)]\text{ s.t. }L(W,q,\Delta\kappa)\geq 0\}$. By definition, $q_1^{\max}(W)\leq\bar{q}_1$ for all W. Hence, the previous condition can be rewritten as$q^*(W)=\inf\{q\in[q_1^{FB},\bar{q}_1]\text{ s.t. }L(W,q,\Delta\kappa)\geq 0\}$. Since for all q, $\partial L(W,q,\Delta\kappa)/\partial W\geq 0$, $q^*(W)$ decreases with W over $[\underline{W},\bar{W})$.

B.6. Proof of Proposition 4

All else equal, (4) defines upper and lower bounds for r: $r_{min}\equiv\min\{r\in\mathbb{R}_+\text{s.t. }\max_q\pi_1(q)\geq\kappa_1\}$ and $r_{max}\equiv\max\{r\in\mathbb{R}_+\text{s.t. }\max_q\pi_0(q)\leq\kappa_0\}$. The proof consists in showing that an increase in unit revenue r relaxes all constraints in problem (5).

Constraint (16): Denote $\underline{q}_1(r)$, $q_1^{FB}(r)$, $\bar{q}_1(r)$ and $q^*(r)$ the values of $\underline{q}_1$, q_1^{FB}, q^* and q^* for a given r, respectively. Note that $\underline{q}_1(r)$ decreases with r while $q_1^{FB}(r)$ and $\bar{q}_1(r)$ increase with r. Therefore in particular, an increase in r relaxes (16).

Constraint (17): An increase in r leads to an increase in payoff $P_{1,q}$ which $R(\cdot)$ being monotonic (as per (18)), implies an increase is (17)'s right-hand side.

Constraint (18) is independent of r.

Constraint (19): Denote $L(r,q)$ the value of $L(q,\Delta\kappa)$ for a given r where we have dropped the reference to $\Delta\kappa$ as it is not useful here. We show that (25), is relaxed by an increase by showing that for all q, $L(r,q)$ is non-decreasing with r. To do so, we start by showing that $K(q)$ is non-increasing with r.

Case 1: $K(q)<sq$. In that case, expression (22) writes as $K=(cq-W)^+$and therefore $\partial K/\partial r=0$.

Case 2. $K(q) > sq$. In that case, differentiating expression (22) with respect to r yields:

$$\frac{\partial K}{\partial r} = \int_0^{\hat{d}(q)} F_1(x)\,dx + (r-s)\frac{\partial \hat{d}(q)}{\partial r} F_1 \hat{d}(q) = \int_0^{\hat{d}(q)} F_1(x)\,dx + \left(\frac{\partial K}{\partial r} - \hat{d}(q)\right) F_1\left(\hat{d}(q)\right)$$

$$\frac{\partial K}{\partial r} = -\frac{1}{1 - F_1\left(\hat{d}(q)\right)} \int_0^{\hat{d}(q)} \left(F_1\left(\hat{d}(q)\right) - F_1(x)\right) dx < 0$$

Hence, $K(q)$ is non-increasing in r, which implies that $\hat{d}(q)$ and thus $L(r,q)$ are non-decreasing in r.

We can now complete the proof. First, we show that if $r' > r$, $q^*(r') = 0$ implies $q^*(r) = 0$. Since all constraints being laxer for higher values of r, if the problem is feasible for r (i.e., $q^*(r) > 0$) then it is feasible for all $r' > r$ (i.e., $q^*(r') > 0$). Thus we can define $\underline{r} \equiv \min\{r \in [r_{\min}, r_{\max}] \text{ s.t. } q^*(r) \geq 0\}$.

Second, we show that if $r' > r$, $q^*(r) = q_1^{FB}(r)$ implies $q^*(r') = q_1^{FB}(r')$. We know that $q^*(r) = q_1^{FB}(r) > 0$ implies $q^*(r') > 0$ which in turn implies $q^*(r') \geq q_1^{FB}(r')$. We only need to show that $L(r', q_1^{FB}(r')) \geq 0$. Note that $q_1^{FB}(r) \in \left[\underline{q}_1(r'), q_1^{FB}(r')\right]$. Indeed, $q_1^{FB}(r) \leq q_1^{FB}(r')$ (as per standard newsvendor arguments), and $q_1^{FB}(r) \geq \underline{q}_1(r) \geq \underline{q}_1(r')$. Since $L(r', \cdot)$ is increasing over $\left[\underline{q}_1(r'), q_1^{FB}(r')\right]$ we have:

$$L\left(r', q_1^{FB}(r')\right) \geq L\left(r', q_1^{FB}(r)\right) \geq L\left(r, q_1^{FB}(r)\right) \geq 0$$

Thus we can define $\bar{r} \equiv \min\{r \in [r_{\min}, r_{\max}] \text{ s.t. } q^*(r) = q_1^{FB}(r)\}$.

Third, assume $(r, r') \in (\underline{r}, \bar{r})^2$. We show that if $r' > r$ then $q^*(r') \leq q^*(r)$. By definition of $\underline{r}$ and $\bar{r}$, we have $q^*(r) > q_1^{FB}(r)$ and $q^*(r') > q_1^{FB}(r')$. We know that since $q^*(r)$ satisfies all constraints for r it does so too for r'. Moreover, $q_1^{FB}(r') < q^*(r)$ otherwise q_1^{FB} would also satisfy all constraints for r' and we would have $q^*(r') = q_1^{FB}(r')$. Hence, $q^*(r) \in [q_1^{FB}(r'), \bar{q}_1(r'))$ and satisfies all constraints. We know that $q^*(r')$ is the smallest $q \in [q_1^{FB}(r'), \bar{q}_1(r'))$ satisfying all constraints. Hence $q^*(r') \leq q^*(r)$.

B.7. Proof of Theorem 2

Step 1. As a first step, we consider the problem as if funding amount I and capacity q were chosen directly by the investor, not by the firm. We relax that temporary assumption in Step 5 below. Denote $(I^{\star\star}, q^{\star\star}, e^{\star\star})$ the equilibrium outcome under the temporary assumption.

The project being viable only for effort $e = 1$, the investor's optimal choice of a contract must induce the firm to work, or lead to project abandonment. Given this, the investor's problem can be written as:

$$\max_{q,I,R(\cdot)} \mathbb{E}[R(P_{1,q})] - I \tag{5}$$

$$s.t.$$

$$\mathbb{E}[R(P_{1,q})] \geq I \tag{6}$$

$$\mathbb{E}[P_{1,q} - R(P_{1,q})] + I - cq - \kappa_1 \geq 0 \tag{7}$$

$$I \geq cq \tag{8}$$

$$\forall (p, p') \in [sq, rq]^2 \text{ with } p > p', \quad R(p) \leq p \quad \text{and} \quad R(p') \leq R(p) \tag{9}$$

$$\mathbb{E}[P_{1,q} - R(P_{1,q})] - \mathbb{E}[P_{0,q} - R(P_{0,q})] - \Delta\kappa \geq 0 \tag{10}$$

and if the previous problem is not feasible then the firm abandons the project $(q = 0)$.

Recall that to satisfy our model's assumption, κ_0 and κ_1 must be such that $\pi_0(q_0^{FB}) < \kappa_0 < \kappa_1 < \pi_1(q_1^{FB})$ or equivalently $\pi_0(q_0^{FB}) < \kappa_0$ and $0 < \Delta\kappa < \Delta\kappa^{\max}$.

Step 2. Next, we show that for a given funding amount and capacity, debt financing is optimal.

Lemma 4. *For all $\Delta\kappa$, a unique threshold $\hat{q} < q_1^{FB}$ exists such that:*

- *No capacity $q \in (0, \hat{q})$ can be optimal.*
- *For any $q \in [\hat{q}, \bar{q}_1]$, a unique debt claim with face value $\bar{K}(q) \in [sq, rq]$ exists such that incentive compatibility constraint (10) is binding, i.e., the unique solution to*

$$(r-s)\int_{(K-sq)/(r-s)}^{q} \left(\bar{F}_1 - \bar{F}_0\right)(u)\,du - \Delta\kappa = 0 \tag{11}$$

Moreover, financing capacity q with that debt claim is (weakly) optimal.

Proof. First, we prove the optimality of debt financing for all q. In (5), claim $R(\cdot)$ appears only as $\mathbb{E}[R(P_{1,q})]$ except in (10). From Lemma 2, we know that given capacity q, for all claims $R(\cdot)$, a debt claim with face value K such that with $\mathbb{E}[K \wedge P_{1,q}] = \mathbb{E}[R(P_{1,q})]$ exists, is unique and relaxes (10). Hence given (q, I) that debt claim weakly dominates R.

Next, we show the existence of $\hat{q} < q_1^{FB}$ and the existence and uniqueness of $\bar{K}(q)$ for all $q \geq \hat{q}$. For all $q > 0$ and a given debt claim with face value K, (10)'s LHS can be written as

$$\begin{aligned}
L(q,K) &= \int_{\bar{d}(K)}^{q} (sq + (r-s)u - K)(f_1 - f_0)(u)\,du + (rq - K)\left(\bar{F}_1 - \bar{F}_0\right)(q) - \Delta\kappa \\
&= (rq - K)(F_1 - F_0)(q) - (r-s)\int_{\bar{d}(K)}^{q} (F_1 - F_0)(u)\,du + (rq - K)\left(\bar{F}_1 - \bar{F}_0\right)(q) - \Delta\kappa \\
&= (r-s)\int_{\bar{d}(K)}^{q} \left(\bar{F}_1 - \bar{F}_0\right)(u)\,du - \Delta\kappa
\end{aligned}$$

where $\bar{d}(K) \equiv (K - sq)^+ / (r-s)$. We have $L(0,0) = -\Delta\kappa < 0$ and

$$\frac{\partial L(q,0)}{\partial q} = \frac{\partial}{\partial q}\left((r-s)\int_0^q \left(\bar{F}_1 - \bar{F}_0\right)(u)\,du - \Delta\kappa\right)(r-s)\left(\bar{F}_1 - \bar{F}_0\right)(q) > 0.$$

Hence, $L(q_1^{FB}, 0) = \left(\mathbb{E}\left[P_{1,q_1^{FB}}\right] - cq_1^{FB} - \kappa_1\right) - \left(\mathbb{E}\left[P_{0,q_1^{FB}}\right] - cq_1^{FB} - \kappa_0\right) > (\pi_1(q_1^{FB}) - \kappa_1) - (\pi_0(q_0^{FB}) - \kappa_0)$, which is positive as by assumption, the first term in brackets is positive and the second one is negative. Hence we can define a unique threshold $\hat{q} < q_1^{FB}$ such that $L(\hat{q}, 0) = 0$. Note also that for all $q > 0$

$$\frac{\partial L(q,K)}{\partial K} = -(r-s)\frac{\partial \bar{d}(K)}{\partial K}\left(\bar{F}_1 - \bar{F}_0\right)\left(\bar{d}(K)\right)$$

Given that $\bar{d}(K) \equiv (K - sq)^+ / (r-s)$, we have $\partial L(q,K)/\partial K = 0$ for $K \in [0, sq]$ and $\partial L(q,K)/\partial K < 0$ for $K \in [sq, rq]$. Hence, for all $q \in (0, \hat{q})$ and all $K \geq 0$, we have $L(q,K) \leq L(q,0) < 0$ and (10) is violated. Hence $q \in (0, \hat{q})$ cannot be optimal and we can focus on $q \in [\hat{q}, \bar{q}_1]$. For all $q \in [\hat{q}, \bar{q}_1]$, given that $\bar{d}(rq) = q$, we have $L(q, rq) = 0 - 0 - \Delta\kappa < 0 < L(q, 0)$. Finally, for a given $q \geq \hat{q}$, $L(q,K)$ is non-negative and constant in K over $[0, sq]$ and strictly decreasing in K over $[sq, rq]$. Hence, a unique K exists such that $L(q,K) = 0$. Note that $K \in (sq, rq)$. This proves the existence and uniqueness of $\bar{K}(q)$ for all $q \geq \hat{q}$.

Last, we show the optimality of $\bar{K}(q)$ for a given $q \geq \hat{q}$. We can minimize $\mathbb{E}[R(P_{1,q})]$ so that (10) is binding. Indeed, the objective and all conditions other than (10) depend on $R(\cdot)$ only via $\mathbb{E}[R(P_{1,q})] - I$, while (8), the only other constraint, only puts a lower bound on I. ■

Step 3. We can rewrite the problem as follows. For a given capacity q, define the firm's minimum rent, i.e., the smallest expected payoff needed to induce effort. It is (10)'s LHS under debt financing with face value

$\bar{K}(q)$, i.e., $\Gamma(q) \equiv \mathbb{E}\left[P_{1,q} - \bar{K}(q) \wedge P_{1,q}\right] - (\kappa_0 + \Delta\kappa)$. The investor's expected payoff equals total surplus, i.e., the project's value $\pi_1(q) - \kappa_1$, net of the investor's rent $\Gamma(q)$ and of any transfer in excess of funding needs $I - cq$. The investor's problem is thus

$$\max_{q \in [\hat{q}, \bar{q}_1], I \geq 0} (\pi_1(q) - \kappa_1) - \Gamma(q) - (I - cq) \tag{12}$$

$$s.t.$$

$$(\pi_1(q) - \kappa_1) - \Gamma(q) - (I - cq) \geq 0 \tag{13}$$

$$\Gamma(q) + (I - cq) \geq 0 \tag{14}$$

$$I - cq \geq 0 \tag{15}$$

$$R(\cdot) \text{ is the debt claim with face value } \bar{K}(q). \tag{16}$$

Since the objective decreases with I which is only bounded from below by constraints (14) and (15), one of them is binding. The problem can thus be rewritten as

$$\max_{q \in [\hat{q}, \bar{q}_1]} (\pi_1(q) - \kappa_1) - [\Gamma(q)]^+ \tag{17}$$

$$s.t.$$

$$(\pi_1(q) - \kappa_1) - [\Gamma(q)]^+ \geq 0 \tag{18}$$

$$I = cq - (\Gamma(q))^- \quad \text{and} \quad R(p) = p \wedge \bar{K}(q) \tag{19}$$

where $(x)^+ = \max(0, x)$, $(x)^- = \min(0, x)$, and a solution exists if and only if

$$\max_{q \in [\hat{q}, \bar{q}_1]} (\pi_1(q) - \kappa_1) - [\Gamma(q)]^+ \geq 0 \tag{20}$$

in which case the solution is

$$q^{**} = \arg \max_{q \in [\hat{q}, \bar{q}_1]} (\pi_1(q) - \kappa_1) - (\Gamma(q))^+ \tag{21}$$

and otherwise the project is abandoned ($q^{**} = 0$).

Step 4. We can now prove the results in Theorem 2 still under the temporary assumption.

Step 4.1 First, we show that $\Gamma(q, \kappa_0, \Delta\kappa)$ is strictly decreasing in q and strictly increasing in $\Delta\kappa$. By definition of $\bar{K}(q)$, we have

$$(r - s) \int_{\bar{d}(\bar{K}(q))}^{q} \left(\bar{F}_1 - \bar{F}_0\right)(u)\, du = \Delta\kappa. \tag{22}$$

Taking the first derivative with respect to q gives

$$\left(\bar{F}_1 - \bar{F}_0\right)(q) - \frac{\partial \bar{d}\left(\bar{K}(q)\right)}{\partial q} \left(\bar{F}_1 - \bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right) = 0 \quad \text{or} \quad \frac{\partial \bar{d}\left(\bar{K}(q)\right)}{\partial q} = \frac{\left(\bar{F}_1 - \bar{F}_0\right)(q)}{\left(\bar{F}_1 - \bar{F}_0\right)\left(\bar{d}\left(\bar{K}(q)\right)\right)}.$$

Given that $\bar{K}(q) > sq$, $\bar{d}\left(\bar{K}(q)\right) = \left(\bar{K}(q) - sq\right) / (r - s)$ and $\partial\bar{K}/\partial q\,(q) = (r - s)\,\partial\bar{d}/\partial q\left(\bar{K}(q)\right) + s$. Now since $\Gamma(q, \kappa_0, \Delta\kappa) = \left(rq - \bar{K}(q)\right) - (r - s)\int_{\bar{d}(\bar{K}(q))}^{q} F_1(u)\, du - \kappa_1$, we have

$$\begin{aligned}
\frac{\partial \Gamma(q, \kappa_0, \Delta\kappa)}{\partial q} &= r - \frac{\partial \bar{K}(q)}{\partial q} - (r - s) F_1(q) + (r - s) \frac{\partial \bar{d}\left(\bar{K}(q)\right)}{\partial q} F_1\left(\bar{d}\left(\bar{K}(q)\right)\right) \\
&= r - \left((r - s) \frac{\partial \bar{d}\left(\bar{K}(q)\right)}{\partial q} + s\right) - (r - s) F_1(q) + (r - s) \frac{\partial \bar{d}\left(\bar{K}(q)\right)}{\partial q} F_1\left(\bar{d}\left(\bar{K}(q)\right)\right)
\end{aligned}$$

$$= (r-s)\left(\bar F_1(q) - \frac{\partial \bar d\left(\bar K(q)\right)}{\partial q}\bar F_1\left(\bar d\left(\bar K(q)\right)\right)\right) = (r-s)\left(\bar F_1(q) - \frac{\left(\bar F_1 - \bar F_0\right)(q)}{\left(\bar F_1 - \bar F_0\right)\left(\bar d\left(\bar K(q)\right)\right)}\bar F_1\left(\bar d\left(\bar K(q)\right)\right)\right)$$
$$= \frac{(r-s)}{\left(\bar F_1 - \bar F_0\right)\left(\bar d\left(\bar K(q)\right)\right)}\left(\bar F_1(q)\left(\bar F_1 - \bar F_0\right)\left(\bar d\left(\bar K(q)\right)\right) - \left(\bar F_1 - \bar F_0\right)(q)\,\bar F_1\left(\bar d\left(\bar K(q)\right)\right)\right)$$
$$= \frac{(r-s)}{\left(\bar F_1 - \bar F_0\right)\left(\bar d\left(\bar K(q)\right)\right)}\left(\bar F_0(q)\,\bar F_1\left(\bar d\left(\bar K(q)\right)\right) - \bar F_1(q)\,\bar F_0\left(\bar d\left(\bar K(q)\right)\right)\right)$$
$$= \frac{(r-s)\,\bar F_0(q)\,\bar F_0\left(\bar d\left(\bar K(q)\right)\right)}{\left(\bar F_1 - \bar F_0\right)\left(\bar d\left(\bar K(q)\right)\right)}\left(\frac{\bar F_1\left(\bar d\left(\bar K(q)\right)\right)}{\bar F_0\left(\bar d\left(\bar K(q)\right)\right)} - \frac{\bar F_1(q)}{\bar F_0(q)}\right)$$

which is strictly negative given that $\bar F_1/\bar F_0$ is strictly increasing (from MLRP) and $\bar d\left(\bar K(q)\right) < q$ since $K < rq$.

Moreover, (22) implies that as $\Delta\kappa$ increases, $\bar d(K)$ and hence $\bar K(q)$ must decrease strictly, and so $\mathbb{E}\left[\bar K(q) \wedge P_{0,q}\right]$ must also decrease strictly. Further, by definition of $\Gamma(q,\kappa_0,\Delta\kappa)$ and $\bar K(q)$, we have $\Gamma(q,\kappa_0,\Delta\kappa) = \mathbb{E}\left[P_{1,q} - \bar K(q) \wedge P_{1,q}\right] - \kappa_1 = \mathbb{E}\left[P_{0,q} - \bar K(q) \wedge P_{0,q}\right] - \kappa_0$. Hence $\partial\Gamma(q,\kappa_0,\Delta\kappa)/\partial\Delta\kappa = -\partial\mathbb{E}\left[\bar K(q) \wedge P_{0,q}\right]/\partial\Delta\kappa > 0$.

Step 4.2. Next, assume $\Gamma(q_1^{FB},\kappa_0,\Delta\kappa^{\max}) \le 0$. We have$(\pi_1(q_1^{FB}) - \kappa_1) - (\Gamma(q_1^{FB},\kappa_0,\Delta\kappa))^+ = (\pi_1(q_1^{FB}) - \kappa_1) > 0$. Hence, (20) holds. Moreover, for all $q \in [\hat q, \bar q_1]$ with $q \neq q_1^{FB}$ we have $(\pi_1(q) - \kappa_1) - (\Gamma(q,\kappa_0,\Delta\kappa))^+ \le (\pi_1(q) - \kappa_1) < (\pi_1(q_1^{FB}) - \kappa_1)$. Hence by (21), optimality implies $q^* = q_1^{FB}$.

Step 4.3. Now assume $\Gamma(q_1^{FB},\kappa_0,\Delta\kappa^{\max}) > 0$. Because $\lim_{\Delta\kappa\to 0}\Gamma(q_1^{FB},\kappa_0,\Delta\kappa) = -\kappa_0 < 0$and $\Gamma(q_1^{FB},\kappa_0,\Delta\kappa)$ is strictly increasing in $\Delta\kappa$, we can define a unique threshold $\Delta\underline{\kappa}$ by$\Gamma(q_1^{FB},\kappa_0,\Delta\underline{\kappa}) = 0$. For all $\Delta\kappa \in (0,\Delta\underline{\kappa}]$, $\Gamma(q_1^{FB},\kappa_0,\Delta\kappa) \le 0$ and as per the previous reasoning, the first-best obtains: $q^* = q_1^{FB}$.

Consider now $\kappa_1 \in (\Delta\underline{\kappa},\Delta\kappa^{\max})$. Note that $\lim_{\Delta\kappa\to\Delta\kappa^{\max}}(\pi_1(q) - (\kappa_0+\Delta\kappa)) - (\Gamma(q,\kappa_0,\Delta\kappa))^+ < 0$. Indeed, $\forall q > q_1^{FB}$ the first term is strictly negative for $\Delta\kappa \to \Delta\kappa^{\max}$ by definition of q_1^{FB} and for $q = q_1^{FB}$, the first term equals zero but $\Gamma(q_1^{FB},\kappa_0,\Delta\kappa) > 0$ as by definition of $\Delta\underline{\kappa}$. Besides, the first term decreases strictly with $\Delta\kappa$ and $\Gamma(q,\kappa_0,\Delta\kappa)$ increases strictly with $\Delta\kappa$. Hence we can define a unique $\Delta\bar\kappa < \Delta\kappa^{\max}$ as the minimum $\Delta\kappa$ such that $\forall q \ge q_1^{FB}$, $(\pi_1(q) - (\kappa_0+\Delta\kappa)) - (\Gamma(q,\kappa_0,\Delta\kappa))^+ \le 0$. For $\Delta\kappa \in (\Delta\bar\kappa,\Delta\kappa^{\max})$, the problem has no solution and the project is abandoned ($q^* = 0$).

Note that $\Delta\underline{\kappa} < \Delta\bar\kappa$, i.e., $\exists q$ exists such that $(\pi_1(q) - (\kappa_0+\Delta\underline{\kappa})) - (\Gamma(q,\kappa_0,\Delta\underline{\kappa}))^+ > 0$. Indeed, the condition holds for $q = q_1^{FB}$: the first term is $\pi_1(q_1^{FB}) - \kappa_1 > 0$ and the second term is zero.

Last, we show that for $\Delta\kappa \in (\Delta\underline{\kappa},\Delta\bar\kappa)$, the optimal capacity exceeds q_1^{FB}. For $\Delta\kappa$ such that $\pi_1(q_1^{FB} \;- (\kappa_0+\Delta\kappa)) - \Gamma(q_1^{FB},\kappa_0,\Delta\kappa) < 0$, the optimum cannot equal q_1^{FB} and must therefore exceed q_1^{FB}. Indeed, the first term in objective (20) is maximum at $q = q_1^{FB}$ and Γ is strictly decreasing in q. Hence, the optimum cannot be below (or equal) to q_1^{FB} and since we are in the case in which a non-zero optimum exists, that optimum must be strictly above q_1^{FB}. Consider now $\Delta\kappa$ such that $\pi_1(q_1^{FB} - (\kappa_0+\Delta\kappa)) - \Gamma(q_1^{FB},\kappa_0,\Delta\kappa) \ge 0$. Since $\Gamma(q_1^{FB},\kappa_0,\Delta\kappa) > 0$, it must be that the objective is also $(\pi_1(q) - (\kappa_0+\Delta\kappa)) - \Gamma(q,\kappa_0,\Delta\kappa)$ and is strictly positive in a right-neighborhood of q_1^{FB}. Hence, in that neighborhood,

$$\frac{\partial}{\partial q}(\pi_1(q) - (\kappa_0+\Delta\underline{\kappa})) - \frac{\partial}{\partial q}\Gamma(q,\kappa_0,\Delta\underline{\kappa}).$$

The first term is zero at $q = q_1^{FB}$ by definition of q_1^{FB}. The second term is strictly negative. Hence the derivative is strictly positive at $q = q_1^{FB}$. Therefore the objective is strictly positive in a right-neighborhood of q_1^{FB} and strictly increasing at q_1^{FB}. Hence $q^{**} > q_1^{FB}$.

Step 5. We now relax the temporary assumption and revert to the model assumption that the firm chooses I, q and e. The following lemma exhibits a debt contract $R(\cdot,\cdot,\cdot)$ such that the firm's best response is to choose the optimal $I^\star$, $q^\star$ and $e^\star$ derived under the temporary assumption. Define a loan schedule as a financial contract setting an interest rate i contingent on loan size I and capacity q, i.e, where $i(I,q)$.

LEMMA 5. *There exists a non-linear loan schedule such that the firm's best response is to choose I^{**}, q^{**} and e^{**}. Moreover, one such loan schedule is such that $(1+i(I^{**},q^{**}))I = \bar{K}(q^{**})$.*

Proof: To prove the result, it suffices to show that the investor can set a loan schedule such that the firm chooses $q = q^{**}$ and that if $q = q^{**} > 0$, it also chooses $I = I^{**}$. Indeed, if $q^{**} > 0$, set $i(I^{**},q^{**}) = (\bar{K}(q^{**}) - I^{**}/I^{**}$. Given this, if the firm chooses $I = I^{**}$ and $q = q^{**}$, it must repay $(1+i)I = \bar{K}(q^{**})$. Given the definition of $\bar{K}(q^{**})$, the firm finds it optimal to exert effort $e^* = e^{**} = 1$. In that case, the firm's expected payoff, denoted as X^{**}, is non-negative. Therefore the outcome is (I^{**},q^{**},e^{**}) and Theorem 2 would hold.

For given I and q, define $X_{(I,q)}(i)$ as the firm's expected payoff given its best response if its only choice were to either abandon the project (in which case its payoff is zero) or borrow I and set up capacity q and choose its effort $e \in \{0,1\}$ accordingly. We have $X_{(I,q)}(0) \geq 0$ because the firm can always opt to abandon the project which ensure a payoff of zero. We also have that for $i = (r-c)q/I$, $X_{(I,q)}(i) = 0$. Indeed, for that interest rate, for any effort level $e \in \{0,1\}$ and any demand realization D_e, the firm's revenue $(I - cq) + r(D_e \wedge q)$ is not sufficient to meet the debt repayment $(1+i)I$. This is because $(I-cq) + r(D_e \wedge q) = (I - cq) + rq \leq (1 + (r-c)q/I)I \leq (1+i)I$. Last, $X_{(I,q)}(i)$ is weakly decreasing with the interest rate i. Indeed, all else equal, a higher interest rate decreases the firm's net payoff. Given this, we can define uniquely $i^*(I,q)$ as the smallest (non-negative) interest rate i such that $X_{(I,q)}(i) \leq X^*$.

Now suppose the investor offers the following interest rate schedule. If $q^{**} = 0$, then $i(I,q) = i^*(I,q)$ for all (I,q), else $i(I^{**},q^{**}) = (\bar{K}(q^{**}) - I^{**}/I^{**}$ and $i(I,q) = i^*(I,q)$ for all $(I,q) \neq (I^{**},q^{**})$. The firm's optimal optimal response will be as follows. If $q^{**} = 0$ then the firm chooses $q = 0$ as, by definition of $i^*(I,q)$, it is (weakly) better off abandoning the project than borrowing. Else, it will set $q = q^{**}$ as it ensures itself a positive payoff $X^{**} > 0$. ■

Because the investor's problem is less constraint when he chooses I and q than when the firm's chooses these quantities, the lemma implies that quantities I^{**}, q^{**} are also optimal for the original problem, in which the investor only chooses the financial terms.

B.8. Proof of Proposition 5

The proof consists in showing that the constrains on the monopolist's problem (5) are stricter than those of problem (5) in the competitive case. Note that condition (16) is implied by (6) and (7). Condition (17) is implied by (6) and (8) (recall that for brevity, $W = 0$). Finally, conditions (18) and (19) are the same as (9) and (10), respectively. Thus if problem (5) is feasible, so is problem (5), i.e., $q_S^* > 0$ implies $q_C^* > 0$. Moreover, we have established that if (5) is feasible, q_C^* is the smallest $q \geq q_1^{FB}$ satisfying all constraints. Thus if (5) is feasible then $q_S^* \geq q_C^*$.

References

Innes, Robert D. 1990. Limited liability and incentive contracting with ex-ante action choices. *Journal of Economic Theory* **52**(1) 45–67.