Selling Off-Grid Light to Liquidity Constrained Consumers: Online Appendix

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Appendix A: Proofs

Proof of Lemma 1. The result follows by noting that the corresponding partial derivatives have the desired sign. □

Proof of Proposition 1. Let $\rho_k = \mu/(\lambda p_k)$.
(i) Let $\Psi_k = Q_k/\lambda + \mathbb{E}[p_k Q_k \tilde{\varepsilon}/\mu - Q_k/\lambda]^+ = Q_k(1/\lambda + \mathbb{E}[p_k\tilde{\varepsilon}/\mu - 1/\lambda]^+) = Q_k\hat{\Psi}_k$, where $\hat{\Psi}_k$ is independent of Q_k. Then

$$C_k = \frac{I + p_k Q_k + \beta B_k}{\Psi_k} = \frac{p_k}{\hat{\Psi}_k} + \frac{I + \beta B_k}{Q_k \hat{\Psi}_k}.$$

The long-run monetary cost is independent of Q_k. Note that

$$\frac{B_k}{Q_k} = \frac{1}{Q_k}\int_{\rho_k}^{\infty} b\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right) dF(\varepsilon) \quad \text{and} \quad \frac{\partial(B_k/Q_k)}{\partial Q_k} = \frac{H_k}{Q_k^2};$$

here

$$H_k = \int_{\rho_k}^{\infty} \left[\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right) b'\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right) - b\left(\frac{p_k Q_k \varepsilon}{\mu} - \frac{Q_k}{\lambda}\right)\right] dF(\varepsilon).$$

The integrand (in brackets) is of the form $zb'(z) - b(z)$, where $z \geq 0$; it is positive and increasing because it equals zero when $z = 0$ and its derivative is $zb''(z) \geq 0$ (as follows from the convexity of b). Therefore, H_k is also positive and increasing in Q_k. The derivative of C_k w.r.t. Q_k is then given by

$$\frac{\partial C_k}{\partial Q_k} = \frac{1}{Q_k^2 \hat{\Psi}_k}[-I + \beta H_k].$$

Since $\lim_{Q_k \to 0} H_k = 0$ and since $\lim_{Q_k \to \infty} H_k = \infty$ (by our assumption that $\lim_{z\to\infty} zb'(z) - b(z) = \infty$), it follows that the term in brackets crosses the x-axis exactly once from below, which yields the unique minimum Q_k^*.
(ii) Observe that

$$\frac{\partial^2 C_k}{\partial I \partial Q_k} = -\frac{1}{Q_k^2 \hat{\Psi}_k} \leq 0 \quad \text{and} \quad \frac{\partial^2 C_k}{\partial \beta \partial Q_k} = \frac{H_k}{Q_k^2 \hat{\Psi}_k} \geq 0.$$

The desired result now follows from Topkis (1998). □

Proof of Lemma 2. The derivative of Δ w.r.t. I is

$$\frac{\partial \Delta}{\partial I} = \frac{\partial C}{\partial I} - \frac{\partial C_k^*}{\partial I} = \frac{1}{\Psi} - \frac{1}{Q_k^* \hat{\Psi}_k}. \quad (1)$$

According to Proposition 1(ii), Q_k^* is increasing in I. Hence the derivative in (1) is also increasing in I and so Δ is convex in I. It is easy to see that $\lim_{I\to 0} Q_k^* = 0$ and $\lim_{I\to\infty} Q_k^* = \infty$, from which it follows that $\lim_{I\to 0} \partial\Delta/\partial I = -\infty$ and $\lim_{I\to\infty} \partial\Delta/\partial I = 1/\Psi > 0$. So here the derivative of Δ crosses the x-axis exactly once from below, which leads to a unique minimum. □

Proof of Proposition 2. First note that the preference region for bulbs exists if and only if (iff) the minimum value of Δ is less than zero. We shall demonstrate that this minimum is negative iff $P/Q \le p_k$. Let Δ_m denote the minimum value of Δ, and let this minimum be achieved at I_m. Then

$$\Delta_m = \left(\frac{I}{\Psi} - \frac{I}{Q_k^*\hat{\Psi}_k}\right)\bigg|_{I_m} + \left(\frac{P}{\Psi} - \frac{p_k}{\hat{\Psi}_k}\right)\bigg|_{I_m} + \left(\frac{\beta B}{\Psi} - \frac{\beta B_k^*}{Q_k^*\hat{\Psi}_k}\right)\bigg|_{I_m}. \tag{2}$$

By (1), the first term in parentheses is equal to zero at I_m. The second term in (2) is independent of I_m. To show that it is negative iff $P/Q \le p_k$, we rewrite P/Ψ as $p_l/\hat{\Psi}_l$; here $p_l = P/Q$ is the unit price of bulbs and $\hat{\Psi}_l = \Psi/Q$. Now note that the function $p/(1/\lambda + \mathbb{E}[p\tilde{\varepsilon}/\mu - 1/\lambda]^+)$ is increasing in p because its derivative,

$$\frac{F(\mu/(\lambda p))/\lambda}{(1/\lambda + \mathbb{E}[p\tilde{\varepsilon}/\mu - 1/\lambda]^+)^2},$$

is positive. Therefore, if $p_l \le (>)\, p_k$ then $p_l/\hat{\Psi}_l \le (>)\, p_k/\hat{\Psi}_k$.

We now show that the third term of (2) is also negative iff $P/Q \le p_k$. By (1), the sign of this term depends only on the sign of $B - B_k^*$ evaluated at I_m, which is given by

$$\mathbb{E}b\left[\frac{P\tilde{\varepsilon}}{\mu} - \frac{Q}{\lambda}\right]^+ - \mathbb{E}b\left[\frac{p_k Q_k^*\tilde{\varepsilon}}{\mu} - \frac{Q_k^*}{\lambda}\right]^+\bigg|_{I_m} = \mathbb{E}b\left[\frac{P\tilde{\varepsilon}}{\mu} - \frac{Q}{\lambda}\right]^+ - \mathbb{E}b\left[\frac{\Psi}{\hat{\Psi}_k}\left(\frac{p_k\tilde{\varepsilon}}{\mu} - \frac{1}{\lambda}\right)\right]^+.$$

Now we can see that

$$\left(\frac{P\tilde{\varepsilon}}{\mu} - \frac{Q}{\lambda}\right) - \frac{\Psi}{\hat{\Psi}_k}\left(\frac{p_k\tilde{\varepsilon}}{\mu} - \frac{1}{\lambda}\right) = \frac{\Psi\tilde{\varepsilon}}{\mu}\left(\frac{p_l}{\hat{\Psi}_l} - \frac{p_k}{\hat{\Psi}_k}\right) + \frac{Q}{\lambda\hat{\Psi}_k}(\hat{\Psi}_l - \hat{\Psi}_k)$$

is negative iff $p_l \le p_k$. The desired result follows by noting that the positive part, the monotonically increasing b, and the expectation preserve this relationship.

Therefore, if (i) $p_k < P/Q$, then Δ_m is positive and hence Δ is positive (as in Figure 2(a)), (ii) $p_k \ge P/Q$ and $\beta \le \hat{\beta}$, then Δ_m is negative but $\lim_{I\to 0}\Delta \le 0$; Δ starts negative and crosses the horizontal axis only once at I_1 (as in Figure 2(b)), and (iii) $p_k \ge P/Q$ and $\beta > \hat{\beta}$, then Δ_m is negative and $\lim_{I\to 0}\Delta > 0$; Δ starts positive and crosses the horizontal axis twice at I_0 and I_1 (as in Figure 2(c)). □

Proof of Lemma 3. Because the I_j ($j \in \{0,1\}$) are zeros of Δ, we can use the implicit function theorem to write their derivatives w.r.t. β as

$$\frac{\partial I_j}{\partial \beta} = -\frac{\partial\Delta/\partial\beta}{\partial\Delta/\partial I}\bigg|_{I_j}.$$

By definition, $\partial\Delta/\partial I$ is negative (resp. positive) at I_0 (resp. I_1). Using the technique employed for proving Lemma 2 and Proposition 2, we can show that, for any given I, there are two zeros of Δ w.r.t. β; we label them β_0 and β_1 (with $\beta_0 < \beta_1$). At any zero of Δ, by definition we have

$$I\left(\frac{1}{\Psi} - \frac{1}{\Psi_k^*}\right) + \beta\left(\frac{B}{\Psi} - \frac{B_k^*}{\Psi_k^*}\right) = \frac{p_k}{\hat{\Psi}_k} - \frac{P}{\Psi} \implies I\frac{\partial\Delta}{\partial I} + \beta\frac{\partial\Delta}{\partial\beta} > 0. \tag{3}$$

Consider a point on the curve $I_0(\beta)$. By definition, $\partial\Delta/\partial I < 0$. We can use (3) to show that $\partial\Delta/\partial\beta > 0$ at this zero; hence $I_0(\beta)$ is increasing in β. Now consider a point on the curve $\beta_0(I)$. By definition, $\partial\Delta/\partial\beta < 0$, and again by (3) we have $\partial\Delta/\partial I > 0$ at this zero. Hence the $\beta_0(I)$ curve is increasing and so its inverse, the $I_1(\beta)$ curve, must also be increasing. □

Proof of Lemma 4. First, we characterize the long-run cost C_{bk} of using bulbs with kerosene to avoid blackouts. As evident from Figure 4(c), the consumption process renews after every bulb recharge. Therefore, a cycle constitutes one sub-cycle where bulb is used, and $\hat{n}$ sub-cycles where kerosene is used. Here, $\hat{n}$ is the smallest n satisfying $M_n \geq P$. To simplify the analysis, we approximate integral $\hat{n}$ as a real number. Then

$$\hat{n} = \frac{\log(P/M_0)}{\log(1-\alpha+\alpha\gamma)},$$

where $\gamma = \mu/(\lambda p_k)$. Recall that $\gamma > 1$. The length of a single cycle, given by the sum of its sub-cycle lengths, is

$$\frac{Q}{\lambda} + \frac{\alpha M_0}{\lambda p_k} + \frac{\alpha M_1}{\lambda p_k} + \cdots + \frac{\alpha M_{\hat{n}-1}}{\lambda p_k} = \frac{Q}{\lambda} + \frac{P/\mu - Q/\lambda}{1-1/\gamma}.$$

Since blackouts are completely avoided in this case, the blackout cost in a cycle is zero. The total inconvenience incurred in a cycle is the sum of inconvenience I in recharging the bulb once, and inconvenience I_k in purchasing kerosene $\hat{n}$ times, i.e., $I + \hat{n} I_k$. The monetary cost incurred in a cycle is the sum of amounts paid to recharge the bulb once and to purchase kerosene in the subsequent $\hat{n}$ sub-cycles. It is given by

$$P + \alpha M_0 + \alpha M_1 + \cdots + \alpha M_{\hat{n}-1} = P + \frac{P - \mu Q/\lambda}{\gamma - 1}.$$

Then the long-run cost, which is the total cost incurred in a cycle divided by the cycle length, is given by

$$C_{bk} = \frac{I + \hat{n} I_k + P + \frac{P-\mu Q/\lambda}{\gamma-1}}{\frac{Q}{\lambda} + \frac{P/\mu - Q/\lambda}{1-1/\gamma}}.$$

If consumer uses only kerosene, then the resultant long-run cost is $C_k = (I_k + p_k Q_k)/(Q_k/\lambda) = \mu/\gamma + I_k\lambda/Q_k$, where Q_k is the quantity of kerosene that consumer chooses to purchase in every cycle. Now, note that $C_{bk} > \mu/\gamma$, which follows from

$$\left(I + \hat{n} I_k + P + \frac{P - \mu Q/\lambda}{\gamma - 1}\right)\gamma - \left(\frac{Q}{\lambda} + \frac{P/\mu - Q/\lambda}{1-1/\gamma}\right)\mu$$
$$= (I + \hat{n} I_k)\gamma + P\gamma - \mu Q/\lambda \; > \; (I + \hat{n} I_k)\gamma + P - \mu Q/\lambda \; > \; 0.$$

Since $C_{bk} - C_k = (C_{bk} - \mu/\gamma) - I_k\lambda/Q_k$, by setting $Q_k > I_k\lambda/(C_b - \mu/\gamma) > 0$, we obtain $C_{bk} > C_k$. □

Proof of Lemma 5. The result follows if we show that C_k^* is increasing in p_k. Here, we use the notation from Proposition 1. By envelope theorem, we have

$$\frac{\partial C_k^*}{\partial p_k} = \left.\frac{\partial C_k}{\partial p_k}\right|_{Q_k^*} = \frac{\partial\big(p_k/\hat{\Psi}_k\big)}{\partial p_k} + \frac{1}{Q_k^*}\frac{\partial}{\partial p_k}\left.\left\{\frac{I + \beta B_k}{\hat{\Psi}_k}\right\}\right|_{Q_k^*},$$

where the first term is positive because $p_k/\hat{\Psi}_k$ is increasing in p_k. The sign of the second term depends on

$$\hat{\Psi}_k \left.\frac{\partial \beta B_k}{\partial p_k}\right|_{Q_k^*} - \frac{\partial \hat{\Psi}_k}{\partial p_k}\,(I + \beta B_k)|_{Q_k^*} \;=\; \left[\frac{1}{\lambda} + \int_{\rho_k}^{\infty}\left(\frac{p_k\varepsilon}{\mu} - \frac{1}{\lambda}\right) dF(\varepsilon)\right] \left.\frac{\partial \beta B_k}{\partial p_k}\right|_{Q_k^*} - \frac{\partial \hat{\Psi}_k}{\partial p_k}\,\beta(H_k + B_k)|_{Q_k^*}$$
$$\geq \beta \int_{\rho_k}^{\infty} \frac{p_k\varepsilon}{\mu}\, dF(\varepsilon) \int_{\rho_k}^{\infty} \frac{Q_k^*\varepsilon}{\mu} b'(Q_k^* z)\, dF(\varepsilon) - \beta \int_{\rho_k}^{\infty} \frac{\varepsilon}{\mu}\, dF(\varepsilon) \int_{\rho_k}^{\infty} Q_k^* z b'(Q_k^* z)\, dF(\varepsilon)$$
$$= \beta \int_{\rho_k}^{\infty} \frac{\varepsilon}{\mu}\, dF(\varepsilon) \int_{\rho_k}^{\infty} \frac{Q_k^*}{\lambda} b'(Q_k^* z)\, dF(\varepsilon) \;\geq\; 0,$$

where the first step follows because $I = \beta H_k$ at the optimum Q_k^* (see Proposition 1), the second step follows because B_k is increasing in p_k and $H_k + B_k = \int_{\rho_k}^{\infty} Q_k z b'(Q_k z)\, dF(\varepsilon)$ where $z = p_k\varepsilon/\mu - 1/\lambda$, and the last step follows because b is increasing. □

The four lemmas that follow will be used to prove Propositions 3 and 4. Lemmas A.1 and A.2 characterize the properties of I_0 and I_1 when $b(z) = z^2$; Lemmas A.3 and A.4 characterize the properties of the distribution of I. The proofs of these lemmas can be obtained from the authors upon request.

Lemma A.1. *If $b(z) = z^2$, then the zeros of Δ are given by*

$$\sqrt{I_0} = x\psi\left(\frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k} - \sqrt{\delta}\right) \quad \text{and} \quad \sqrt{I_1} = x\psi\left(\frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k} + \sqrt{\delta}\right), \tag{4}$$

where

$$\delta = \beta\left(\frac{\hat{B}_k}{\hat{\Psi}_k^2} - \frac{\xi}{\psi^2}\right) + \frac{1}{x\psi}\left(\frac{p_k}{\hat{\Psi}_k} - \frac{1}{\psi}\right), \quad \xi = \int_{\frac{\mu q}{\lambda}}^{\infty}\left(\frac{\varepsilon}{\mu} - \frac{q}{\lambda}\right)^2 dF(\varepsilon), \quad \text{and} \quad \hat{B}_k = \int_{\frac{\mu}{\lambda p_k}}^{\infty}\left(\frac{p_k\varepsilon}{\mu} - \frac{1}{\lambda}\right)^2 dF(\varepsilon).$$

Lemma A.2. *Let $i_0 = \sqrt{I_0}$ and $i_1 = \sqrt{I_1}$. If $b(z) = z^2$, then:*

(i) *ψ/i_1 is U-shaped in q;*

(ii) *$\frac{\partial i_1/\partial q}{\partial\psi/\partial q}$ is decreasing in q; and*

(iii) *$-\psi\frac{\partial i_0/\partial q}{\partial\psi/\partial q}$ is decreasing in q.*

Lemma A.3. *Let g and G be the PDF and the CDF, respectively, of a positive random variable Z such that $zg'(z)/g(z)$ and $zg(z)/G(z)$ are decreasing in z. Call this property* (P).

(i) *Random variable Y such that $Z = uY^v$ also satisfies property* (P) *for all $u > 0$ and $v > 0$.*

(ii) *The gamma, log-normal, Erlang, chi-squared, chi, Weibull, exponential, power-law, and uniform distributions satisfy property* (P).

Lemma A.4. *Let g and G be the PDF and CDF (respectively) of a positive random variable such that hazard rate $g(z)/(1-G(z))$ is increasing in z and the function $zg(z)/G(z)$ is decreasing in z. Then:*

(i) *$G(z)$ is log-concave in z;*

(ii) *$(\alpha - G(z))g'(z) + g(z)^2 \geq 0$ for $0 \leq \alpha \leq 1$; and*

(iii) *$(\alpha - G(z))(zg'(z) + g(z)) + zg(z)^2 \geq 0$ for $0 \leq \alpha \leq 1$.*

Proof of Proposition 3. It is easy to verify that δ defined in Lemma A.1 is decreasing in x, and $x^2\delta$ is increasing in x.

(i) This result follows once we note that the terms in parentheses in the expressions for $\sqrt{I_0}$ and $\sqrt{I_1}$ in (4) are positive and increasing in x. (Note that if $p_k \geq P/Q$ then $\hat{B}_k/\hat{\Psi}^2 \geq \xi/\psi^2$.)

(ii) First, observe that

$$I_1 - I_0 = 4\psi^2 x\frac{\sqrt{\beta\hat{B}_k}}{\hat{\Psi}_k}\sqrt{x^2\delta} \quad \text{and} \quad \frac{I_0}{I_1} = \left(\frac{\sqrt{\beta\hat{B}_k}\Big/\hat{\Psi}_k - \sqrt{\delta}}{\sqrt{\beta\hat{B}_k}\Big/\hat{\Psi}_k + \sqrt{\delta}}\right)^2.$$

The first function above is increasing in x because $x^2\delta$ is increasing in x, and the second is increasing in x because δ is decreasing in x. So now we have that

$$\frac{\partial I_0/\partial x}{\partial I_1/\partial x} = \sqrt{\frac{I_0}{I_1}}\left[\frac{\sqrt{\beta\hat{B}_k}\Big/\hat{\Psi}_k - \partial\sqrt{x^2\delta}/\partial x}{\sqrt{\beta\hat{B}_k}\Big/\hat{\Psi}_k + \partial\sqrt{x^2\delta}/\partial x}\right]$$

is increasing in x because:

$$\frac{\partial^2\sqrt{x^2\delta}}{\partial x^2} = \frac{-(p_k/\hat{\Psi}_k - 1/\psi)^2}{4\psi^2(x^2\delta)^{3/2}} < 0,$$

the term in brackets is positive (from part (i)), and I_0/I_1 is increasing in x. Also, $g(I_0)/g(I_1)$ is increasing in x because

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{g(I_0)}{g(I_1)}\right) &= \frac{g(I_0)}{g(I_1)}\left\{\left(\frac{g'(I_0)}{g(I_0)} - \frac{g'(I_1)}{g(I_1)}\right)\frac{\partial I_0}{\partial x} - \frac{g'(I_1)}{g(I_1)}\left(\frac{\partial I_1}{\partial x} - \frac{\partial I_0}{\partial x}\right)\right\}\\ &\geq \frac{g(I_0)}{g(I_1)}\left(\frac{\partial(I_1 - I_0)/\partial x}{I_1 - I_0}\right)\left\{\frac{g'(I_0)I_0}{g(I_0)} - \frac{g'(I_1)I_1}{g(I_1)}\right\} \geq 0.\end{aligned}$$

The first inequality follows because g'/g is decreasing and I_0/I_1 is increasing in x, from which it follows that $(I_1 - I_0)\partial I_0/\partial x \geq I_0\partial(I_1 - I_0)/\partial x$. The second inequality follows because $zg'(z)/G(z)$ is decreasing and $I_1 - I_0$ is increasing in x.

Next we use the preceding results to show that demand D is unimodal in x. Note that $I_0 > 0$ is equivalent to $x > x_\beta$ for $x_\beta = \psi(p_k/\hat{\Psi}_k - 1/\psi)/(\beta\xi)$. Then, for $x \leq x_\beta$, we have that $D = G(I_1)$ is increasing in x. Otherwise, $D = G(I_1) - G(I_0)$ and its derivative w.r.t. x is

$$\frac{\partial D}{\partial x} = g(I_1)\frac{\partial I_1}{\partial x}\left[1 - \frac{g(I_0)}{g(I_1)}\frac{\partial I_0/\partial x}{\partial I_1/\partial x}\right] \equiv g(I_1)\frac{\partial I_1}{\partial x}\left[1 - h(x)\right].$$

We can see that $h(x)$ increases with x. Also, $\lim_{x\to x_\beta} h(x) = 0$ and $\lim_{x\to\infty} h(x) = \infty$. Therefore, the function $1 - h(x)$ is decreasing in x and crosses the x-axis exactly once from above, which yields a unimodal D. □

Proof of Lemma 6.

(i) First note the following properties w.r.t. bulb capacity Q: Long-run inconvenience and monetary costs decrease with Q because, by Lemma 1(ii), cycle length increases with Q. Since B is decreasing in Q, Ψ is increasing in Q, and both B and Ψ are positive, it follows that their ratio (and hence the long-run blackout cost) is decreasing in Q. Now recall that varying capacity-price ratio q at fixed scale x is equivalent varying Q at fixed recharge price P. Therefore, all the above properties also hold w.r.t. q.

(ii) Because I_j ($j \in \{0,1\}$) are zeros of Δ, we can use the implicit function theorem and write

$$\frac{\partial I_j}{\partial q} = -\left.\frac{\partial\Delta/\partial q}{\partial\Delta/\partial I}\right|_{I_j}.$$

The result then follows by noting that Δ decreases with q (since, by part (i), C decreases with q) and that Δ is downward sloping (resp., upward sloping) at I_0 (resp., at I_1).

(iii) The derivative of demand w.r.t. q is

$$\frac{\partial D}{\partial q} = g(I_1)\frac{\partial I_1}{\partial q} - g(I_0)\frac{\partial I_0}{\partial q}.$$

The result now follows easily from part (ii). □

Proof of Proposition 4. The derivative of revenue w.r.t. q is

$$\frac{\partial R}{\partial q} = \frac{1}{\psi^2}\left(\psi\frac{\partial D}{\partial q} - D\frac{\partial \psi}{\partial q}\right) = \frac{D}{\psi^2}\frac{\partial \psi}{\partial q}\left(\frac{\psi}{D}\frac{\partial D/\partial q}{\partial \psi/\partial q} - 1\right) \equiv \frac{D}{\psi^2}\frac{\partial \psi}{\partial q}\big(h(q)-1\big).$$

We can use (8) in the paper to translate the condition $\beta > (\leq)\,\hat{\beta}$ into $q < (\geq)\, q_\beta$ for some unique q_β. Then, for $q < q_\beta$, we have $D = G(I_1) - G(I_0)$. Let $i_0 = \sqrt{I_0}$ and $i_1 = \sqrt{I_1}$. We can now rewrite $h(q)$ as

$$h(q) = 2\left[\frac{I_1 g(I_1)}{G(I_1)-G(I_0)}\right]\left[\frac{\partial i_1/\partial q}{\partial \psi/\partial q}\right]\left[\frac{\psi}{i_1}\right] + 2\left[\frac{g(I_0)}{G(I_1)-G(I_0)}\right]\left[-\psi\frac{\partial i_0/\partial q}{\partial \psi/\partial q}\right][i_0].$$

It now follows from Lemmas A.2 and A.4 that each term in brackets is positive and also decreasing in q. Therefore, $h(q)$ is also decreasing in q for $q < q_\beta$.

If $q \geq q_\beta$ then $D = G(I_1)$. By Lemma A.2(i), ψ/i_1 is U-shaped in q; we use $\overline{q}$ to denote the minimum of this function. (One can easily verify that $\overline{q} > q_\beta$.) Then, for $q \geq q_\beta$, we have

$$h(q) = 2\left[\frac{I_1 g(I_1)}{G(I_1)}\right]\left[\frac{\partial i_1/\partial q}{\partial \psi/\partial q}\right]\left[\frac{\psi}{i_1}\right].$$

Since the domain extends only up to $\overline{q}$, it follows that all the terms in brackets are both positive and decreasing in q; therefore, $h(q)$ is decreasing in $\left[\underline{q}, \overline{q}\right]$. Moreover, $\lim_{q\to\underline{q}} h(q) = \infty$ and so $h(q)-1$ crosses the x-axis at most once (from above); hence R is unimodal in this domain. □

Proof of Proposition 5. Put $\rho = Q\mu/(\lambda P)$.

(i) This claim follows from Lemma 1(ii).

(ii) The results w.r.t. μ follow from Lemma 1(ii). The term P/Ψ is increasing in P because

$$\frac{\partial (P/\Psi)}{\partial P} = \frac{1}{\Psi^2}\left\{\Psi - P\frac{\partial \Psi}{\partial P}\right\} = \frac{Q}{\lambda \Psi^2}F(\rho) \geq 0.$$

(iii) First note that the shape of $\beta B/\Psi$ is same as that of B/Ψ. The derivative of latter w.r.t. P is given by

$$\frac{\partial (B/\Psi)}{\partial P} = \frac{1}{\Psi^2}\left\{\Psi\frac{\partial B}{\partial P} - B\frac{\partial \Psi}{\partial P}\right\}.$$

The term in braces can alternatively be written as

$$\begin{aligned}
&\left[\frac{Q}{\lambda} + \int_\rho^\infty \left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right) dF(\varepsilon)\right]\frac{\partial B}{\partial P} - B\frac{\partial \Psi}{\partial P} \\
&\quad = \frac{Q}{\lambda}F(\rho)\frac{\partial B}{\partial P} + \int_\rho^\infty \frac{P\varepsilon}{\mu}\, dF(\varepsilon)\frac{\partial B}{\partial P} - B\frac{\partial \Psi}{\partial P} \\
&\quad \geq \int_\rho^\infty \frac{P\varepsilon}{\mu}\, dF(\varepsilon) \int_\rho^\infty b'\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\frac{\varepsilon}{\mu}\, dF(\varepsilon) - \int_\rho^\infty b\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right) dF(\varepsilon)\int_\rho^\infty \frac{\varepsilon}{\mu}\, dF(\varepsilon) \\
&\quad = \int_\rho^\infty \frac{\varepsilon}{\mu}\, dF(\varepsilon)\left\{\int_\rho^\infty \left[b'\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\frac{P\varepsilon}{\mu} - b\left(\frac{P\varepsilon}{\mu} - \frac{Q}{\lambda}\right)\right] dF(\varepsilon)\right\} \geq 0.
\end{aligned}$$

The first inequality follows from Lemma 1(i). The last inequality follows by noting that the term in brackets is always positive for any given ε because its derivative w.r.t. P is $b''(P\varepsilon/\mu - Q/\lambda)P(\varepsilon/\mu)^2 \geq 0$ and its value at the lowest feasible $P = Q\mu/(\lambda\varepsilon)$ is $b'(0)Q/\lambda \geq 0$.

The result w.r.t. μ follows immediately from the result w.r.t. P because P and μ co-occur in the expression for B/Ψ, with P in the numerator and μ in the denominator. □

Proof of Lemma 7. After setting $\rho = Q\mu/(\lambda P)$, we can rewrite L_n as follows:

$$L_n = \frac{P}{\mu}\mathbb{E}[\tilde{e}_n - \rho]^+ = \frac{P}{\mu}\left(\mathbb{E}\tilde{e}_n - \rho + \mathbb{E}[\rho - \tilde{e}_n]^+\right).$$

Because $\tilde{e}_n$ is a mean-preserving spread of $\tilde{e}_{n+1}$, we can use Definition 1.5.1 of Müller and Stoyan (2002) to write $\tilde{e}_n \leq_{icv} \tilde{e}_{n+1}$ or, equivalently, $-\tilde{e}_{n+1} \leq_{icx} -\tilde{e}_n$. Using their Theorem 1.5.7(ii), we now deduce that $\mathbb{E}[\rho - \tilde{e}_{n+1}]^+ \leq \mathbb{E}[\rho - \tilde{e}_n]^+$. Since $\mathbb{E}\tilde{e}_n = \mathbb{E}\tilde{\varepsilon}$ for all n, it follows that $L_{n+1} \leq L_n$. Given this inequality, from Theorem 1.5.7(i) of Müller and Stoyan (2002) we obtain $\tilde{e}_{n+1} \leq_{icx} \tilde{e}_n$. Since $b(P[z-\rho]^+/\mu)$ is an increasing convex function in z, it follows that $B_{n+1} \leq B_n$. □

Lemma A.5. *In (9) from the paper, Δ is U-shaped in I. Let Δ_m be the minimum value of Δ. Then – for any given μ, λ, P, Q, and p_k – there exists a threshold $\zeta \geq 0$ such that $\Delta_m \leq (>)\, 0$ if $P/Q \leq (>)\, p_k(1+\zeta)$.*

Proof of Lemma A.5. We first characterize the optimal cost with bulbs and kerosene. Then we describe the shape and minimum value of Δ.

Optimal cost with bulbs. The unconstrained minimization of $C(T) = (I + P + \beta(T - Q/\lambda)^2)/T$ w.r.t. T yields the optimal cycle length T^* and optimal cost $C(T^*)$:

$$T^* = \sqrt{\left(\frac{Q}{\lambda}\right)^2 + \frac{I+P}{\beta}}; \qquad C(T^*) = 2\beta\left(T^* - \frac{Q}{\lambda}\right).$$

We require that the cycle length be greater than P/μ, so if $T^* > P/\mu$ then the optimal cost is $C(T^*)$; otherwise, it is $C(P/\mu)$. We can rewrite the condition $T^* \leq (>)\, P/\mu$ as $I \leq (>)\, \hat{I}$, where the threshold $\hat{I} = \beta[(P/\mu)^2 - (Q/\lambda)^2] - P$.

Optimal cost with kerosene. The Lagrangian for the constrained optimization of $C_k(Q_k, T_k)$ in (9) from the paper is given by

$$\mathcal{L}(Q_k, T_k, \chi_1, \chi_2) = C_k(Q_k, T_k) - \chi_1(T_k - p_k Q_k/\lambda) - \chi_2(Q_k M - T_k).$$

Any local minimum satisfies the following Karush–Kuhn–Tucker conditions:

$$\frac{\partial C_k}{\partial Q_k} + \frac{\chi_1 p_k}{\lambda} - \chi_2 M = 0, \quad \frac{\partial C_k}{\partial T_k} - \chi_1 + \chi_2 = 0, \quad \chi_1\left(T_k - \frac{p_k Q_k}{\lambda}\right) = 0, \quad \chi_2(Q_k M - T_k) = 0,$$

$$T_k \geq \frac{p_k Q_k}{\lambda}, \quad Q_k M \geq T_k, \quad Q_k \geq 0, \quad T_k \geq 0, \quad \chi_1 \geq 0, \quad \chi_2 \geq 0.$$

We consider three cases as follows.

1. $\chi_1 = \chi_2 = 0$. This case is not possible because the equations $\partial C_k/\partial Q_k = 0$ and $\partial C_k/\partial T_k = 0$ are inconsistent.

2. $\chi_1 = 0$ and $\chi_2 > 0$. This case results in the optimal values $Q_k = 0$ and $T_k = 0$, which lead to infinite cost.

3. $\chi_1 > 0$ and $\chi_2 = 0$. This case simply reduces to optimizing C_k w.r.t. Q_k, with $T_k = p_k Q_k/\mu$; it is the deterministic version of the problem considered in Section 3.2. Since the corresponding optimal cost is finite, it follows that this is the only feasible solution. The optimal solution is given by

$$Q_k^* = \frac{1}{L_k}\sqrt{\frac{I}{\beta}}, \quad T_k^* = \frac{p_k Q_k^*}{\mu}, \quad \text{and} \quad C_k^* = \mu + \frac{2\mu L_k}{p_k}\sqrt{I\beta} \quad \text{for } L_k = \frac{p_k}{\mu} - \frac{1}{\lambda}.$$

Shape of Δ. First we suppose that $\hat{I} > 0$, in which case $P/\mu > Q/\lambda$. It follows that Δ is equal to $\Delta_<$ for $I \le \hat{I}$ or is equal to $\Delta_>$ for $I > \hat{I}$; here

$$\Delta_< = \frac{I + P + \beta(P/\mu - Q/\lambda)^2}{P/\mu} - \mu - \frac{2\mu L_k}{p_k}\sqrt{I\beta} \quad \text{and} \quad \Delta_> = 2\beta(T^* - Q/\lambda) - \mu - \frac{2\mu L_k}{p_k}\sqrt{I\beta}. \tag{5}$$

Note that Δ is continuous at $\hat{I}$. The corresponding derivatives w.r.t. I are given by

$$\frac{\partial \Delta_<}{\partial I} = \frac{\mu}{P} - \frac{\mu L_k}{p_k}\sqrt{\frac{\beta}{I}} \quad \text{and} \quad \frac{\partial \Delta_>}{\partial I} = \frac{1}{\sqrt{I}}\left\{\sqrt{\frac{I}{(Q/\lambda)^2 + (I+P)/\beta}} - \frac{\mu L_k}{p_k}\sqrt{\beta}\right\}. \tag{6}$$

Since $\partial \Delta_</\partial I$ is increasing in I, $\partial \Delta_>/\partial I$ crosses the x-axis at most once from below (because the term in braces is increasing in I), and $\lim_{I\to\hat{I}} \partial\Delta_</\partial I = \lim_{I\to\hat{I}} \partial\Delta_>/\partial I$, it follows that $\partial\Delta/\partial I$ also crosses the x-axis at most once from below. Finally, since $\lim_{I\to 0} \partial\Delta_</\partial I < 0$ and $\lim_{I\to\infty} \partial\Delta_>/\partial I = 0$ (from above, since the term in braces in (6) is positive as $I \to \infty$), $\partial\Delta/\partial I$ crosses the x-axis exactly once and so Δ is U-shaped in I.

Now we consider the case when $\hat{I} \le 0$; then $\Delta = \Delta_>$ for all I. Because $\lim_{I\to 0} \partial\Delta_>/\partial I < 0$ and $\lim_{I\to\infty} \partial\Delta_>/\partial I = 0$ (from above), Δ is again U-shaped in I.

Minimum value of Δ. As before, we first consider the case $\hat{I} > 0$. Let Δ_m denote the minimum value of Δ, which is achieved at I_m. Since Δ has only one minimum, it is either from $\Delta_<$ or $\Delta_>$ depending on the sign of $\lim_{I\to\hat{I}} \partial\Delta/\partial I$.

On the one hand, if $\lim_{I\to\hat{I}} \partial\Delta/\partial I \ge 0$, then I_m is obtained by setting $\partial\Delta_</\partial I = 0$. It follows from (5) and (6) that Δ_m is given by

$$\Delta_m = I_m\left(\frac{\mu}{P} - \frac{\mu L_k}{p_k}\sqrt{\frac{\beta}{I_m}}\right) + \beta\mu P\left(\frac{(P/\mu - Q/\lambda)^2}{P^2} - \frac{L_k^2}{p_k^2}\right).$$

By (6), the first term is equal to zero – and the second term is less than zero – iff $P/Q \le p_k$.

On the other hand, if $\lim_{I\to\hat{I}} \partial\Delta/\partial I < 0$ then I_m is obtained by setting $\partial\Delta_>/\partial I = 0$. We can now use (5) and (6) to obtain

$$\Delta_m = 2\beta\sqrt{\left[1 - \left(\frac{\mu L_k}{p_k}\right)^2\right]\left[\left(\frac{Q}{\lambda}\right)^2 + \frac{P}{\beta}\right]} - \frac{2\beta Q}{\lambda} - \mu.$$

Then $\Delta_m \le 0$ iff

$$\frac{P}{Q} < \frac{\frac{\beta Q}{\lambda^2}\left(\frac{\mu L_k}{p_k}\right)^2 + \frac{\mu^2}{4\beta Q} + \frac{\mu}{\lambda}}{1 - \left(\frac{\mu L_k}{p_k}\right)^2} = p_k\left\{\frac{\frac{\beta Q}{\mu\lambda}\left(\frac{\mu L_k}{p_k}\right)^2 + \frac{\mu\lambda}{4\beta Q} + 1}{2 - \frac{\mu}{\lambda p_k}}\right\}. \tag{7}$$

The term in the braces is greater than one because

$$\frac{\beta Q}{\mu\lambda}\left(\frac{\mu L_k}{p_k}\right)^2 + \frac{\mu\lambda}{4\beta Q} - 1 + \frac{\mu}{\lambda p_k} \ge 2\sqrt{\frac{\beta}{\mu\lambda}\left(\frac{\mu L_k}{p_k}\right)^2 \frac{\mu\lambda}{4\beta}} - 1 + \frac{\mu}{\lambda p_k} = 0.$$

We can therefore rewrite the term in braces in (7) as $1 + \zeta$ for some $\zeta \ge 0$, so the condition in (7) reduces to $P/Q \le p_k(1+\zeta)$.

Finally, we consider the case $\hat{I} \le 0$; then $\Delta = \Delta_>$ for all I. It follows that $\Delta_m \le 0$ iff $P/Q \le p_k(1+\zeta)$. So in all possible cases (as just described), $\Delta_m \le 0$ if and only if $P/Q \le p_k(1+\zeta)$ for some $\zeta \ge 0$. ☐

References

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