Product-line Pricing under Discrete Mixed Multinomial Logit Demand

Hongmin Li • Scott Webster • Nicholas Mason • Karl Kempf

A Online Appendix

A.1 Proof of Non-equal Markup

Lemma 2. *Assume $b_{ik} = b$ for all i, k. Let p_i^* be the optimal price for product i. Then in general, $p_i^* - c_i \neq p_j^* - c_j$ for $i \neq j$.*

Proof. Since $b_{ik} = b$, the first-order optimality condition becomes

$$p_i - c_i = \frac{1}{b} + \sum_k \frac{w_k q_{ik}}{q_i} r_k \ . \tag{10}$$

Note that $\sum_k \frac{w_k q_{ik}}{q_i} r_k = \frac{\sum_k w_k q_{0k} A_{ik} e^{-bp_i} r_k}{\sum_{k'} w_{k'} q_{0k'} A_{ik'} e^{-bp_i}} = \sum_k \frac{w_k q_{0k} A_{ik}}{\sum_{k'} w_{k'} q_{0k'} A_{ik'}} r_k$ is a weighted average of r_k with the weights given by $\frac{w_k q_{0k} A_{ik}}{\sum_{k'} w_{k'} q_{0k'} A_{ik'}}$. Since $A_{ik} \neq A_{ik'}$ for $k \neq k'$ (otherwise, segments k and k' become degenerate and are considered the same segment), the weights depend on the product index i.

Assume for contradiction that $p_i^* - c_i = \theta$ for all i. Then

$$\begin{aligned} r_k(\mathbf{p}^*) &= \sum_{i'} (p_{i'}^* - c_{i'}) q_{i'k}(\mathbf{p}^*) = \theta \sum_{i'} q_{i'k}(\mathbf{p}^*) = \theta(1 - q_{0k}(\mathbf{p}^*)) \\ &= \theta\Big(1 - \frac{1}{1 + \sum_{j=1}^{n} A_{jk} e^{-bp_j^*}}\Big) = \theta\Big(1 - \frac{1}{1 + e^{-b\theta} \sum_{j=1}^{n} A_{jk} e^{-bc_j}}\Big) \end{aligned}$$

whose value depends on the segment index k, thus in general r_k's are not equal across segments. Since the right side of equation (10) is a weighted average of the vector $(r_1, r_2, \ldots, r_m)$ with nonequal values and the weights depend on the product index i, this weighted average is a value that depends on i. This contradicts the assumption that $p_i^* - c_i = \theta$ for all i. □

A.2 Proof of Lemma 1

Proof. Since $b_{ik} = b$, the first-order optimality condition becomes

$$p_i - c_i = \frac{1}{b} + \frac{w_A q_{iA}}{q_i} r_A + \frac{w_B q_{iB}}{q_i} r_B = \frac{1}{b} + r_B + \frac{w_A q_{iA}}{q_i}(r_A - r_B) \ .$$

Thus $p_i^* - c_i \geq p_j^* - c_j$ if and only if $\left(\frac{w_A q_{iA}}{q_i} - \frac{w_A q_{jA}}{q_j}\right)(r_A - r_B) \geq 0$. It is easy to verify that $\left(\frac{w_A q_{iA}}{q_i} - \frac{w_A q_{jA}}{q_j}\right)$ has the same sign as $[(a_{iA} - a_{iB}) - (a_{jA} - a_{jB})]$.

$$\Big(\text{Note that } \frac{w_A q_{iA}}{q_i} \geq \frac{w_A q_{jA}}{q_j} \Leftrightarrow \frac{q_i}{w_A q_{iA}} \leq \frac{q_j}{w_A q_{jA}} \Leftrightarrow \frac{w_A q_{iA} + w_B q_{iB}}{w_A q_{iA}} \leq \frac{w_A q_{jA} + w_B q_{jB}}{w_A q_{jA}}$$
$$\Leftrightarrow \frac{q_{iB}}{q_{iA}} \leq \frac{q_{jB}}{q_{jA}} \Leftrightarrow \frac{q_{iB}}{q_{jB}} \leq \frac{q_{iA}}{q_{jA}} \Leftrightarrow \frac{e^{a_{iB} - b_i p_i}}{e^{a_{jB} - b_j p_j}} \leq \frac{e^{a_{iA} - b_i p_i}}{e^{a_{jA} - b_j p_j}} \Leftrightarrow a_{iA} - a_{iB} \geq a_{jA} - a_{jB} \ .\Big)$$

Therefore, $p_i^* - c_i \geq p_j^* - c_j$ if and only if $[(a_{iA} - a_{iB}) - (a_{jA} - a_{jB})](r_A - r_B) \geq 0 \quad$ for $i \neq j$. □

A.3 An Example with Preference-value-inconsistent Optimal Prices

Consider a two-product (products 1 and 2) two-segment (segments A and B) example with $w_A = w_B = 0.5, b_{ik} = 1, c_i = 0, a_{1A} = 6, a_{2A} = 5, a_{1B} = 3, a_{2B} = 1$. Product 1 has higher price-independent utility values than product 2 for both segments of customers. The optimal prices for product 1 and product 2 are $p_1 = 4.011, p_2 = 4.387$, which is a sequence that is the opposite of the preference value sequence.

A.4 Proof of Proposition 1

Proof. To simplify presentation, we suppress the product subscript in our notation (e.g., q_k in place of q_{1k}). Note that the purchase probability of the product by a segment k customer is $\mathbf{q}^k = q_k$. Accordingly, $f_k(q_1) = \frac{A_k\left(\frac{q_1}{A_1(1-q_1)}\right)}{1+A_k\left(\frac{q_1}{A_1(1-q_1)}\right)}$, $g_k(q_k) = \frac{1}{b}\log\left(\frac{A_k(1-q_k)}{q_k}\right)$, and the profit contribution from a segment k customer as a function of q_1 simplifies to

$$\hat{R}_k(q_1) = (g_k(f_k(q_1)) - c)\, f_k(q_1) = \left(\frac{\log A_k - \log(\lambda_k q_1) + \log(1-q_1) - bc}{b}\right)\frac{\lambda_k q_1}{1 - q_1 + \lambda_k q_1}$$

where $\lambda_k = A_k/A_1$. We can derive that

$$-bz\frac{\partial^2 \hat{R}_k}{\partial q_1^2} = \frac{\lambda_k^2}{x} + \frac{2\lambda_k}{y} + \frac{x}{y^2} + 2L(\lambda_k - 1)^2\left(\frac{1}{z} - \frac{x}{z^2}\right) + \frac{2}{z}(\lambda_k - 1)(L - \lambda_k) - \frac{2x}{yz}(\lambda_k - 1)\ ,$$

where $x := \lambda_k q_1$, $y := q_{01}$, $z := \lambda_k q_1 + q_{01}$, and $L := \log A_k - \log(\lambda_k q_1) + \log q_{01} - bc$. Assume without loss of generality that $\lambda_k \geq 1$. From $\frac{\max_k A_k}{\min_k A_k} \leq 2$, we have $\lambda_k \leq 2$, equivalently, $\lambda_k \geq 2(\lambda_k - 1)$. Therefore,

$$\begin{aligned}
-bz\frac{\partial^2 R_k}{\partial {q_1}^2} &\geq \frac{\lambda_k^2}{x} + \frac{2\lambda_k}{y} + \frac{x}{y^2} + \frac{2}{z}L(\lambda_k-1)^2 - \frac{2x}{z^2}L(\lambda_k-1)^2 - \frac{2}{z}\lambda_k(\lambda_k-1) - \frac{2x}{yz}(\lambda_k-1)\\
&\geq \frac{\lambda_k^2}{x} + \frac{2\lambda_k}{y} + \frac{x}{y^2} - \frac{2}{z}\lambda_k(\lambda_k-1) - \frac{2x}{yz}(\lambda_k-1)\\
&= \lambda_k\left[\frac{\lambda_k}{x} - \frac{2}{z}(\lambda_k-1)\right] + \frac{2}{y}\left[\lambda_k - \frac{x}{z}(\lambda_k-1)\right] + \frac{x}{y^2} \geq \frac{x}{y^2} \geq 0
\end{aligned}$$

where the first inequality holds due to $L > 0$ (i.e., $L = b(p-c) > 0$), the second inequality holds because $\frac{x}{z} \leq 1$, and the third inequality holds due to $\lambda_k \geq 2(\lambda_k - 1)$ and $x \leq z$. Therefore, $\hat{R}_k$ is concave on Ω_1. The weighted sum of concave functions is concave, and thus $\hat{\Pi}$ is concave on Ω_1. □

A.5 Proof of Proposition 2

Proof. From (7), to establish the concavity of $\hat{\Pi}(\mathbf{q}^1)$, it suffices to show concavity of $\hat{R}_k(\mathbf{q}^1)$. Recall that $f_k(\mathbf{q}^1)$ is the vector of product purchase probabilities for segment k as a function of vector $\mathbf{q}^1$. Let $f_{ik}(\mathbf{q}^1)$ denote the ith element in $f_k(\mathbf{q}^1)$. Then we can write profit contribution of product i in segment k as $\hat{R}_{ik}(\mathbf{q}^1) = (g_{ik}(f_k(\mathbf{q}^1)) - c_i) f_{ik}(\mathbf{q}^1)$ and the segment-k profit as $\hat{R}_k(\mathbf{q}^1) = \sum_{i=1}^{n} \hat{R}_{ik}(\mathbf{q}^1)$.

From (4) and (6),

$$\begin{aligned}
b_i\hat{R}_{ik}(\mathbf{q}^1) &= \frac{A_{ik}\left(\frac{q_{i1}}{A_{i1}q_{01}}\right)}{1 + \sum_{j=1}^{n} A_{jk}\left(\frac{q_{j1}}{A_{j1}q_{01}}\right)}\left[\log\left(\frac{A_{i1}q_{01}}{q_{i1}}\right) - b_i c_i\right] = \frac{\frac{A_{ik}q_{i1}}{A_{i1}q_{01}}}{1 + \sum_{j=1}^{n}\left(\frac{A_{jk}q_{j1}}{A_{j1}q_{01}}\right)}\left[\log A_{ik} - \log\left(\frac{A_{ik}q_{i1}}{A_{i1}q_{01}}\right) - b_i c_i\right]\\
&= \frac{\lambda_{ik}q_{i1}}{q_{01} + \sum_{j=1}^{n}(\lambda_{jk}q_{j1})}\left[\log A_{ik} - \log(\lambda_{ik}q_{i1}) + \log q_{01} - b_i c_i\right]
\end{aligned}$$

where $\lambda_{ik} = A_{ik}/A_{i1}$ and $q_{01} = 1 - \sum_{i'=1}^{n} q_{i'1}$. For a given segment k, define $Z_i\left(\mathbf{q}^1\right) := b_i \hat{R}_{ik}\left(\mathbf{q}^1\right)$. We can derive that

$$\begin{aligned}
\frac{\partial^2 Z_i}{\partial q_{\ell 1}^2} &= -\frac{x_i}{y^2 z} + \frac{2x_i}{z^3}(\lambda_{\ell k}-1)^2 L_i + \frac{2x_i}{yz^2}(\lambda_{\ell k}-1) \quad \ell \neq i \\
\frac{\partial^2 Z_i}{\partial q_{i 1}^2} &= -\frac{x_i}{y^2 z} + \frac{2x_i}{z^3}(\lambda_{ik}-1)^2 L_i + \frac{2x_i}{yz^2}(\lambda_{ik}-1) - \frac{\lambda_{ik}^2}{x_i z} - \frac{2\lambda_{ik}}{yz} - \frac{2\lambda_{ik}(\lambda_{ik}-1)}{z^2}(L_i - 1) \\
\frac{\partial^2 Z_i}{\partial q_{\ell 1}\partial q_{j1}} &= -\frac{x_i}{y^2 z} + \frac{2x_i}{z^3}(\lambda_{\ell k}-1)(\lambda_{jk}-1) L_i + \frac{x_i}{yz^2}(\lambda_{\ell k}-1+\lambda_{jk}-1) \quad \ell, j \neq i \\
\frac{\partial^2 Z_i}{\partial q_{i 1}\partial q_{\ell 1}} &= -\frac{x_i}{y^2 z} + \frac{2x_i}{z^3}(\lambda_{ik}-1)(\lambda_{\ell k}-1) L_i + \frac{x_i}{yz^2}(\lambda_{ik}-1+\lambda_{\ell k}-1) \\
&\quad -\frac{\lambda_{ik}}{yz} - \frac{1}{z^2}\lambda_{ik}(\lambda_{\ell k}-1)(L_i-1) \quad \ell \neq i\ ,
\end{aligned}$$

where $x_i := \lambda_{ik} q_{i1}$, $y := q_{01}$, $z := q_{01} + \sum_j \lambda_{jk} q_{j1}$, and $L_i := \log A_{ik} - \log x_i + \log y - b_i c_i = b_i(p_i - c_i)$. Let the Hessian of $\hat{R}_k\left(\mathbf{q}^1\right) = \sum_{i=1}^{n} \hat{R}_{ik}(\mathbf{q}^1) = \sum_{i=1}^{n} Z_i(\mathbf{q}^1)/b_i$ be H. The function $\hat{R}_k\left(\mathbf{q}^1\right)$ is concave on Ω_1 if and only if $\boldsymbol{\theta}^{\mathrm{T}} H \boldsymbol{\theta} < 0$ for any nonzero vector $\boldsymbol{\theta}^{\mathrm{T}} = (\theta_1, \theta_2, \ldots, \theta_n)$ and any $\mathbf{q}^1 \in \Omega_1$. Note that

$$\begin{aligned}
\boldsymbol{\theta}^{\mathrm{T}} H \boldsymbol{\theta} &= \sum_{\ell=1}^{n} \theta_\ell^2 \frac{\partial^2 \sum_{i=1}^{n} Z_i/b_i}{\partial q_{\ell 1}^2} + \sum_{\ell=1}^{n}\sum_{j\neq \ell}^{n} \theta_\ell \theta_j \frac{\partial^2 \sum_{i=1}^{n} Z_i/b_i}{\partial q_{\ell 1}\partial q_{j1}} \\
&= -\sum_i \frac{1}{b_i x_i z}\left(\theta_i \lambda_{ik} + \frac{x_i}{y}\sum_\ell \theta_\ell\right)^2 - \sum_i \frac{L_i - 1}{b_i z^2} 2\theta_i \lambda_{ik}\left[\sum_\ell \theta_\ell(\lambda_{\ell k}-1)\right] \\
&\quad + \sum_i \frac{2x_i L_i}{b_i z^3}\left[\sum_\ell \theta_\ell(\lambda_{\ell k}-1)\right]^2 + \sum_i \frac{2x_i}{b_i y z^2}\left[\sum_\ell \theta_\ell^2(\lambda_{\ell k}-1) + \sum_\ell \sum_{j\neq\ell} \theta_\ell\theta_j(\lambda_{\ell k}-1+\lambda_{jk}-1)\right]\ .
\end{aligned}$$

Define $G(\boldsymbol{\lambda}) := \boldsymbol{\theta}^{\mathrm{T}} H \boldsymbol{\theta}$ where $\boldsymbol{\lambda} = [\lambda_{ik}]$ for $i = 1, ..., n$ and $k = 1, ..., m$ (i.e., $\boldsymbol{\lambda}$ is an $n \times m$ matrix). The function G has a strict negative value at $\boldsymbol{\lambda} = \mathbf{1}$ (because it is not possible to find a nonzero vector $\boldsymbol{\theta}$ such that $\theta_i \lambda_{ik} + \frac{x_i}{y}\sum_\ell \theta_\ell = 0$ for all i). G is continuous in $\boldsymbol{\lambda}$. So there must exist a rectangle region near $\boldsymbol{\lambda} = \mathbf{1}$ in which the values of G stay negative. □

A.6 Proof of Proposition 3

Proof. The concavity of $\overline{\Pi}(\mathbf{q}^1)$ and $\underline{\Pi}(\mathbf{q}^m)$ follows from Lemma 2 in Li and Huh (2011). From (2) and $A_{i1} \leq A_{ik} \leq A_{im}$, $q_{01} \geq q_{0k} \geq q_{0m}$. From (4),

$$p_i = \frac{1}{b_i}\Big[\log A_{i1} - \log q_{i1} + \log\big(1 - \sum_j q_{j1}\big)\Big]\ . \tag{11}$$

Since $q_{ik} = A_{ik} q_{0k} \frac{q_{i1}}{A_{i1} q_{01}}$,

$$\begin{aligned}
\hat{\Pi}(\mathbf{q}^1) &= \sum_i (p_i(\mathbf{q}^1) - c_i)\sum_k w_k q_{ik} = \sum_i (g_{i1}(\mathbf{q}^1) - c_i)\sum_k w_k A_{ik} q_{0k}\left(\frac{q_{i1}}{A_{i1}q_{01}}\right) \\
&\leq \sum_i \frac{q_{i1}}{b_i}\Big[\log A_{i1} - \log q_{i1} + \log\big(1 - \sum_j q_{j1}\big) - b_i c_i\Big]\sum_k w_k\left(\frac{A_{ik}}{A_{i1}}\right) = \overline{\Pi}(\mathbf{q}^1)\ .
\end{aligned}$$

We can similarly show that $\tilde{\Pi}(\mathbf{q}^m) \geq \underline{\Pi}(\mathbf{q}^m)$. □

A.7 Corollary 1 (Upper Bounds of $\overline{\Pi}(\mathbf{q}^1)$ and $\underline{\Pi}(\mathbf{q}^m)$)

Corollary 1. *Let* $\overline{\mathbf{q}^1} = argmax_{\mathbf{q}^1}\, \overline{\Pi}(\mathbf{q}^1)$ *and* $\overline{\mathbf{p}^1}$ *be the corresponding price vector. In addition, let* $\overline{\mathbf{q}^m} = argmax_{\mathbf{q}^m}\, \underline{\Pi}(\mathbf{q}^m)$ *and* $\overline{\mathbf{p}^m}$ *be the corresponding price vector.*
(i) The maximum of $\overline{\Pi}(\mathbf{q}^1)$ *is given by* $\overline{\theta}$ *where* $\overline{\theta}$ *is the unique solution to the single-variable equation*

$$\overline{\theta} = \sum_i \left(\frac{e^{a_{i1} - b_i c_i - 1 - \frac{b_i \overline{\theta}}{\sum_k w_k A_{ik}/A_{i1}}}}{b_i} \sum_k w_k A_{ik}/A_{i1} \right) .$$

(ii) The maximum of $\underline{\Pi}(\mathbf{q}^m)$ *is given by* $\underline{\theta}$ *where* $\underline{\theta}$ *is the unique solution to the single-variable equation*

$$\underline{\theta} = \sum_i \left(\frac{e^{a_{im} - b_i c_i - 1 - \frac{b_i \underline{\theta}}{\sum_k w_k A_{ik}/A_{im}}}}{b_i} \sum_k w_k A_{ik}/A_{im} \right) .$$

Proof. Since $\overline{\Pi}(\mathbf{q}^1)$ is concave in $\mathbf{q}^1$, we take the first order derivative with respect to q_{j1}, and set it to zero to obtain the first-order condition

$$p_j - c_j = \frac{1}{b_j} + \frac{\overline{\theta}}{\sum_k w_k A_{jk}/A_{j1}} , \tag{12}$$

$$\text{where } \overline{\theta} = \sum_i \frac{q_{i1}/q_{01}}{b_i} \sum_k w_k A_{ik}/A_{i1} . \tag{13}$$

Thus

$$q_{i1}/q_{01} = e^{a_{i1} - b_i p_i} = e^{a_{i1} - b_i c_i - 1 - b_i \frac{\overline{\theta}}{\sum_k w_k A_{ik}/A_{i1}}} . \tag{14}$$

From (13) and (14), we have $\overline{\theta} = \sum_i \frac{e^{a_{i1} - b_i c_i - 1 - \frac{b_i \overline{\theta}}{\sum_k w_k A_{ik}/A_{i1}}}}{b_i} \sum_k w_k A_{ik}/A_{i1}$. Therefore, the profit $\overline{\Pi}$ can be rewritten as

$$\begin{aligned} \overline{\Pi} &= \sum_i (p_i - c_i) q_{i1} \sum_k w_k A_{ik}/A_{i1} = \sum_i \left(\frac{1}{b_i} + \frac{\overline{\theta}}{\sum_k w_k A_{ik}/A_{i1}} \right) q_{i1} \sum_k w_k A_{ik}/A_{i1} \\ &= \left(\sum_i \frac{q_{i1}/q_{01}}{b_i} \sum_k w_k A_{ik}/A_{i1} \right) q_{01} + \overline{\theta} \sum_i q_{i1} = \overline{\theta} q_{01} + \overline{\theta} \sum_i q_{i1} = \overline{\theta} , \end{aligned}$$

where the second equality follows from (12) and the fourth equality follows from (13). This proves (i). The proof of (ii) follows a similar argument. □

A.8 Proof of Proposition 4

Proof. Note that $f_1\left(\mathbf{q}^1\right) = \mathbf{q}^1$. Thus $\hat{R}_1\left(\mathbf{q}^1\right) = R_1\left(f_1\left(\mathbf{q}^1\right)\right) = R_1\left(\mathbf{q}^1\right)$ is a profit function based on MNL demand which, as noted above, is concave.

What remains is to show that $\hat{R}_k\left(\mathbf{q}^1\right)$ is quasiconcave for $k \geq 2$. Without loss of generality, we set $k = 2$. Let us outline the main steps of our proof. We first show that $\Omega_4 := \left\{ f_2\left(\mathbf{q}^1\right) | \mathbf{q}^1 \in \Omega_1 \right\}$ is a convex set by decomposing function f_2 into a composition of more elementary functions, and show that each of these functions preserves convexity. We then explain why the convexity of Ω_4 implies that superlevel set $S_\alpha\left(R_2, \Omega_4\right) = \left\{ \mathbf{q}^2 \in \Omega_4 | R_2\left(\mathbf{q}^2\right) \geq \alpha \right\}$ is convex. Finally, we show that the inverse image $S_\alpha\left(R_2, \Omega_4\right)$ under function f_2 is convex (using a similar decomposition approach), which implies that superlevel set $S_\alpha\left(\hat{R}_2, \Omega_1\right) = \left\{ \mathbf{q}^1 \in \Omega_1 | \hat{R}_2\left(\mathbf{q}^1\right) \geq \alpha \right\}$ is convex, thereby proving that $\hat{R}_2$ is quasiconcave. Our proof will rely on the following definitions, remark, and property.

Definition 1. Function $f: R^n \to R$ is *quasiconcave* if its domain is convex and its superlevel sets $S_\alpha(f, \mathbf{dom}\ f) = \{\mathbf{x} \in \mathbf{dom}\ f | f(\mathbf{x}) \geq \alpha\}$ are convex for all $\alpha \in R$ (Boyd and Vandenberghe 2004, p. 95).

Definition 2. Let $A \in R^{n\times m}, \mathbf{b} \in R^n, \mathbf{c} \in R^m, d \in R$. Function $f: R^m \to R^n$ with $f(\mathbf{x}) = (A\mathbf{x}+\mathbf{b})/(\mathbf{c}^{\mathrm{T}}\mathbf{x}+d)$ defined on $\mathbf{dom}\ f = \left\{\mathbf{x}|\mathbf{c}^{\mathrm{T}}\mathbf{x}+d > 0\right\}$ is a *linear-fractional function* (Boyd and Vandenberghe 2004, p. 41).

Let $C \in \mathbf{dom}\ f$ be a convex set. Note that $S_\alpha(f, C) = \{\mathbf{x} \in C | f(\mathbf{x}) \geq \alpha\} = C \cap S_\alpha(f, \mathbf{dom}\ f)$. The following remark follows from the fact that the intersection of two convex sets is a convex set.

Remark 1. Let $C \in \mathbf{dom}\ f$ be a convex set. If f is quasiconcave on $\mathbf{dom}\ f$, then f is quasiconcave on C.

Property 1. Let f be a linear-fractional function and let $C \in \mathbf{dom}\ f$ be a convex set. Then image $D = \{f(\mathbf{x}) | \mathbf{x} \in C\}$ is a convex set. Furthermore, the inverse image of a convex set under a linear-fractional function is also convex, i.e., $\left\{f^{-1}(\mathbf{y}) | \mathbf{y} \in D\right\}$ is convex if D is convex (Boyd and Vandenberghe 2004, p. 42).

Note that $\Omega_1 = \left\{\mathbf{q}^1 | \sum_{i=1}^{n} q_{i1} \leq 1,\ q_{i1} \geq 0,\ g_{i1}(\mathbf{q}^1) \geq c_i\ \forall i\right\} = \mathbf{dom}\ \hat{R}_k$ is a convex set. Consider the following function F_1 that maps $\mathbf{q}^1 \in \Omega_1$ to $\mathbf{x} \in R^n$: $\mathbf{x} = F_1(\mathbf{q}^1) = \left(\frac{q_{11}/A_{11}}{1-\sum_{l=1}^{n} q_{l1}}, ..., \frac{q_{n1}/A_{n1}}{1-\sum_{l=1}^{n} q_{l1}}\right)$. Function F_1 is a linear-fractional function (see Definition 2), and thus it follows from Property 1 that the image of Ω_1 under F_1, $\Omega_2 = \left\{F_1(\mathbf{q}^1) | \mathbf{q}^1 \in \Omega_1\right\}$, is a convex set. Next consider the following function F_2 that maps $\mathbf{x} \in \Omega_2$ to $\mathbf{y} \in R^n$: $\mathbf{y} = F_2(\mathbf{x}) = \left(A_{12}x_1^{b_{12}/b_{11}}, ..., A_{n2}x_n^{b_{n2}/b_{n1}}\right)$. The image of Ω_2 under F_2 is $\Omega_3 = \{F_2(\mathbf{x}) | \mathbf{x} \in \Omega_2\} = \left\{A_{12}x_1^{b_{12}/b_{11}}, ..., A_{n2}x_n^{b_{n2}/b_{n1}} | \mathbf{x} \in \Omega_2\right\}$. We next show that Ω_3 is a convex set. Let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ denote two distinct points in Ω_2, so that $F_2\left(\mathbf{x}^{(1)}\right)$ and $F_2\left(\mathbf{x}^{(2)}\right)$ are two points in Ω_3. Note that Ω_3 is convex if and only if $\alpha F_2\left(\mathbf{x}^{(1)}\right) + (1-\alpha) F_2\left(\mathbf{x}^{(2)}\right) \in \Omega_3$ for all $\alpha \in [0,1]$ and all $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ in Ω_2. We use a subscript on function F_2 to denote the functional element in vector $F_2(\mathbf{x})$, i.e., $F_{2i}(x_i) = A_{i2}x_i^{b_{i2}/b_{i1}}$. Thus,

$$\alpha F_{2i}\left(x_i^{(1)}\right) + (1-\alpha) F_{2i}\left(x_i^{(2)}\right) = \alpha A_{i2}\left(x_i^{(1)}\right)^{b_{i2}/b_{i1}} + (1-\alpha) A_{i2}\left(x_i^{(2)}\right)^{b_{i2}/b_{i1}}.$$

Assume without loss of generality that $x_i^{(1)} \leq x_i^{(2)}$. Then, because $F_{2i}(x_i)$ is strictly increasing in x_i,

$$A_{i2}\left(x_i^{(1)}\right)^{b_{i2}/b_{i1}} \leq \alpha A_{i2}\left(x_i^{(1)}\right)^{b_{i2}/b_{i1}} + (1-\alpha) A_{i2}\left(x_i^{(2)}\right)^{b_{i2}/b_{i1}} \leq A_{i2}\left(x_i^{(2)}\right)^{b_{i2}/b_{i1}}$$

and there exists $x_i^{(3)} \in \left[x_i^{(1)}, x_i^{(2)}\right]$ such that $\alpha A_{i2}\left(x_i^{(1)}\right)^{b_{i2}/b_{i1}} + (1-\alpha) A_{i2}\left(x_i^{(2)}\right)^{b_{i2}/b_{i1}} = A_{i2}\left(x_i^{(3)}\right)^{b_{i2}/b_{i1}}$, and equivalently, there exists $\theta_i \in [0,1]$ that satisfies

$$\alpha A_{i2}\left(x_i^{(1)}\right)^{b_{i2}/b_{i1}} + (1-\alpha) A_{i2}\left(x_i^{(2)}\right)^{b_{i2}/b_{i1}} = A_{i2}\left(\theta_i x_i^{(1)} + (1-\theta_i) x_i^{(2)}\right)^{b_{i2}/b_{i1}} = A_{i2}\left(x_i^{(3)}\right)^{b_{i2}/b_{i1}}.$$

Of course, if $b_{i2}/b_{i1} = 1$, then $\theta_i = \alpha$. Combining the above, we have the following identity:

$$\alpha F_{2i}\left(x_i^{(1)}\right) + (1-\alpha) F_{2i}\left(x_i^{(2)}\right) = F_{2i}\left(\theta_i x_i^{(1)} + (1-\theta_i) x_i^{(2)}\right)$$

for some $\theta_i \in [0,1]$ and all i. Therefore, $\alpha F_2\left(\mathbf{x}^{(1)}\right) + (1-\alpha) F_2\left(\mathbf{x}^{(2)}\right) \in \Omega_3$ if and only if

$$\mathbf{x}^{(3)} := \left(\theta_1 x_1^{(1)} + (1-\theta_1) x_1^{(2)}, ..., \theta_n x_n^{(1)} + (1-\theta_n) x_n^{(2)}\right) \in \Omega_2. \tag{15}$$

To determine whether (15) holds, we need to characterize set Ω_2. Note that $\mathbf{x} = F_1\left(\mathbf{q}^1\right) = \left(\frac{q_{11}}{A_{11}q_{01}}, ..., \frac{q_{n1}}{A_{n1}q_{01}}\right)$. For pair $i, j \in \{1, ..., n\}$ with $i \neq j$, let $\Delta = \sum\limits_{l=1}^{n} q_{l1} - q_{i1} - q_{j1}$. We keep q_{01} and Δ fixed, and examine the (x_i, x_j) curve as q_{i1} varies over its feasible range $[0, 1 - q_{01} - \Delta]$. Note that $q_{j1} = 1 - q_{01} - \Delta - q_{i1}$, and thus

$$x_j = \frac{1 - q_{01} - \Delta - q_{i1}}{A_{j1}q_{01}} = \frac{1 - q_{01} - \Delta}{A_{j1}q_{01}} - \frac{A_{i1}}{A_{j1}}\left(\frac{q_{i1}}{A_{i1}q_{01}}\right) = \frac{1 - q_{01} - \Delta}{A_{j1}q_{01}} - \frac{A_{i1}}{A_{j1}}x_i$$

for $x_i \in \left[0, \frac{1-q_{01}-\Delta}{A_{i1}q_{01}}\right]$. It is apparent that the function $x_j(x_i)$ is a line with slope $-A_{i1}/A_{j1}$ connecting points $\left(0, \frac{1-q_{01}-\Delta}{A_{j1}q_{01}}\right)$ and $\left(\frac{1-q_{01}-\Delta}{A_{i1}q_{01}}, 0\right)$. By letting $\delta := q_{01} + \Delta$ vary over interval $[0, 1]$ and q_{01} vary over interval $[0, \delta - \Delta]$, we see that our $x_j(x_i)$ curves cover the entire positive orthant in two dimensions. This holds for all $i, j \in \{1, ..., n\}$ with $i \neq j$, and thus Ω_2 is the positive orthant in n dimensions. Therefore, (15) holds if $\mathbf{x}^{(3)}$ is in the positive orthant. This is clearly the case because $\theta_i \in [0, 1]$ for all i and both $\mathbf{x}^{(1)} \in \Omega_2$ and $\mathbf{x}^{(2)} \in \Omega_2$ are in the positive orthant. By the above arguments, we have shown that $\alpha F_2\left(\mathbf{x}^{(1)}\right) + (1-\alpha)F_2\left(\mathbf{x}^{(2)}\right) \in \Omega_3$, and thus Ω_3 is a convex set.

Finally, consider the following function F_3 that maps $\mathbf{y} \in \Omega_3$ to $\mathbf{z} \in R_n$:

$$\mathbf{z} = F_3(\mathbf{y}) = \left(\frac{y_1}{1 + \sum\limits_{j=1}^{n} y_j}, ..., \frac{y_n}{1 + \sum\limits_{j=1}^{n} y_j}\right).$$

F_3 is a linear-fractional function (see Definition 2), and thus it follows from Property 1 that the image of Ω_3 under F_3, $\Omega_4 = \{F_3(\mathbf{y})\,|\mathbf{y} \in \Omega_3\}$, is a convex set.

Now, to conclude that $\hat{R}_2$ is quasiconcave, we need to show that all of its superlevel sets are convex. Note that $\hat{R}_2\left(\mathbf{q}^1\right) = R_2\left(F_3\left(F_2\left(F_1\left(\mathbf{q}^1\right)\right)\right)\right) = R_2\left(f_2\left(\mathbf{q}^1\right)\right)$. We see that $\hat{R}_2\left(\mathbf{q}^1\right)$ is obtained by evaluating R_2 at a point in convex set Ω_4. Because the segment profit function $R_2\left(\mathbf{q}^2\right)$ is concave (and quasiconcave) on $\textbf{dom}\ R_2 = \left\{q_{i2}|\sum\limits_{i=1}^{n} q_{i2} \leq 1,\ q_{i2} \geq 0\ \forall i\right\}$ and $\Omega_4 \subset \textbf{dom}\ R_2$ is convex set, we know from Remark 1 that R_2 is quasiconcave on Ω_4, and thus $S_\alpha(R_2, \Omega_4) = \left\{\mathbf{q}^2 \in \Omega_4 | R_2\left(\mathbf{q}^2\right) \geq \alpha\right\}$ is a convex set for any α (follows from Definition 1). To establish that

$$S_\alpha\left(\hat{R}_2, \Omega_1\right) = \left\{\mathbf{q}^1 | \mathbf{q}^1 \in \Omega_1, \hat{R}_2\left(\mathbf{q}^1\right) = R_2\left(F_3\left(F_2\left(F_1\left(\mathbf{q}^1\right)\right)\right)\right) \geq \alpha\right\}$$

is a convex set, we need to show that the inverse image of convex set $S_\alpha(R_2, \Omega_4)$ under $f_2 = F_3 \circ F_2 \circ F_1$ is convex. Now F_1 and F_3 are linear-fractional functions, and from Property 1, we know that an inverse image of a convex set under F_1 and F_3 is a convex set. What remains is to show that the inverse image of a convex set under F_2 is a convex set.

Let $D \in \Omega_3$ be a convex set. Its inverse image under F_2 is $C = \left\{F_2^{-1}(\mathbf{y})\,|\mathbf{y} \in D\right\}$. Recall that $F_2(\mathbf{x}) = \left(A_{12}x_1^{b_{12}/b_{11}}, ..., A_{n2}x_n^{b_{n2}/b_{n1}}\right)$, and thus

$$F_2^{-1}(\mathbf{y}) = \left(F_{21}^{-1}(y_1), ..., F_{2n}^{-1}(y_n)\right) = \left(\left(\frac{y_1}{A_{12}}\right)^{b_{11}/b_{12}}, ..., \left(\frac{y_n}{A_{n2}}\right)^{b_{n1}/b_{n2}}\right).$$

Suppose that $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are in D. Then $\mathbf{x}^{(1)} = F_2^{-1}\left(\mathbf{y}^{(1)}\right)$ and $\mathbf{x}^{(2)} = F_2^{-1}\left(\mathbf{y}^{(2)}\right)$. Inverse image C is convex if and only if $\alpha\mathbf{x}^{(1)} + (1 - \alpha\mathbf{x}^{(2)}) \in C$ for all $\alpha \in [0, 1]$ and for all $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ in C. Because the elements of $\mathbf{y}$ are independent, if the above condition holds for the i^{th} element in $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, then it holds for all elements. Note that

$$\alpha x_i^{(1)} + (1-\alpha)\,x_i^{(2)} = \alpha\left(\frac{y_i^{(1)}}{A_{i2}}\right)^{b_{i1}/b_{i2}} + (1-\alpha)\left(\frac{y_i^{(2)}}{A_{i2}}\right)^{b_{i1}/b_{i2}}$$

and that $\left(\frac{y_i}{A_{i2}}\right)^{b_{i1}/b_{i2}}$ is a strictly increasing function. Thus, there exists $\theta_i \in [0,1]$ that satisfies

$$\alpha\left(\frac{y_i^{(1)}}{A_{i2}}\right)^{b_{i1}/b_{i2}} + (1-\alpha)\left(\frac{y_i^{(2)}}{A_{i2}}\right)^{b_{i1}/b_{i2}} = \left(\frac{\theta_i y_i^{(1)} + (1-\theta_i)\, y_i^{(2)}}{A_{i2}}\right)^{b_{i1}/b_{i2}} = F_{2i}^{-1}\left(\theta_i y_i^{(1)} + (1-\theta_i)\, y_i^{(2)}\right).$$

Because D is convex, we know that $\theta_i y_i^{(1)} + (1-\theta_i)\, y_i^{(2)} \in D_i := \{y_i | \mathbf{y} \in D\}$, which implies $\alpha x_i^{(1)} + (1-\alpha)\, x_i^{(2)} \in C_i$. Therefore, C is a convex set.

Let us summarize the implications of the above. We now know that inverse image of a convex set under F_1, under F_2, and under F_3 is a convex set. Therefore, beginning with convex set $S_\alpha(R_2, \Omega_4)$, we obtain its convex inverse image under F_3. From this convex set, we obtain its convex inverse image under F_2, then repeat to obtain the convex inverse image under F_1. This process results in convex set $S_\alpha\left(\hat{R}_2, \Omega_1\right)$, which proves that $\hat{R}_2$ is quasiconcave on Ω_1. Therefore $\hat{R}_k$ is quasiconcave for any segment k. ☐

A.9 Proof of Proposition 5

Proof. We first show that the sequence generated by Algorithm 2 has at least one limit point. From equation (3), the optimal price $p_i, i = 1, 2, \ldots, n$ must be bounded in the interval

$$\left[c_i + \frac{1}{\max_k b_{ik}},\; c_i + \frac{1}{\min_k b_{ik}} + \max_k \rho_k\right] \tag{16}$$

where ρ_k is the optimal profit from a segment k customer if prices of all products are set to maximize segment k profit only. Specifically, ρ_k solves the single-variable equation (Li and Huh 2011, Theorem 2) $\rho_k = \sum_{j=1}^{n} \frac{e^{a_{jk} - b_{jk} c_{jk} - 1} e^{-b_{jk}\rho_k}}{b_{jk}}$ and is finite. Thus the optimal price $p_i, i = 1, 2, \ldots, n$ must be finite. Hence we assume that one always starts with a finite price vector in Algorithm 2.
Note that given any bounded margin vector at the t^{th} iteration,

$$\hat{p}_i^{t+1} = \hat{p}_i^t + \alpha^t d_i^t = \hat{p}_i^t + \alpha^t\left(\frac{1}{\sum\limits_{k=1}^{m}\left(\frac{w_k q_{ik}}{q_i}\right) b_{ik}} + \sum_{k=1}^{m}\left(\frac{w_k b_{ik} q_{ik}}{\sum\limits_{l=1}^{m} w_l b_{il} q_{il}}\right) r_k^t - \hat{p}_i^t\right)$$

where $r_k^t = \sum\limits_{i=1}^{n} \hat{p}_i^t q_{ik}\left(\hat{\mathbf{p}}^t\right)$. Since $\alpha^t \in [0,1]$, $\hat{p}_i^{t+1}$ is bounded in the interval $\left[\min\left(\hat{p}_i^t, M^t\right), \max\left(\hat{p}_i^t, M^t\right)\right]$ where $M^t = \frac{1}{\sum\limits_{k=1}^{m}\left(\frac{w_k q_{ik}}{q_i}\right) b_{ik}} + \sum\limits_{k=1}^{m}\left(\frac{w_k b_{ik} q_{ik}}{\sum\limits_{l=1}^{m} w_l b_{il} q_{il}}\right) r_k^t$. M^t is the sum of the multiplicative inverse of a weighted average of b_{ik} values and a weighted average of the segment profits r_k^t. Since $r_k^t \le \rho_k$ and ρ_k is finite, M^t is bounded by a finite constant $\frac{1}{\min_{ik} b_{ik}} + \rho_k$. As a result, $\hat{p}_i^{t+1} \le \max\{\hat{p}_i^t, \frac{1}{\min_{ik} b_{ik}} + \rho_k\}$. Hence, the sequence $\{\hat{\mathbf{p}}^t\}$ is bounded and consequently has at least one limit point (see Bertsekas 2003, Proposition A.5, p. 666). Furthermore,

$$\nabla h\left(\hat{\mathbf{p}}^t\right)^{\mathrm{T}} \mathbf{d}^t = -\sum_{i=1}^{n}\sum_{k=1}^{m} w_k b_{ik} q_{ik}\left(\hat{p}_i^t - \sum_{k=1}^{m}\left(\frac{w_k b_{ik} q_{ik}}{\sum\limits_{l=1}^{m} w_l b_{il} q_{il}}\right) r_k^t - \frac{1}{\sum\limits_{k=1}^{m}\left(\frac{w_k q_{ik}}{q_i}\right) b_{ik}}\right)^2 < 0$$

unless $\hat{\mathbf{p}}^t$ is already a stationary point. Hence, $\{\mathbf{d}^t\}$ is gradient-related to $\{\hat{\mathbf{p}}^t\}$ and every limit point of the sequence $\{\hat{\mathbf{p}}^t\}$ is a stationary point of h (See Bertsekas 2003, Proposition 1.2.1, p. 43). ☐

A.10 Proof of Proposition 6

Proof. The proof follows the same argument as in the proof of Proposition 5 and is omitted here. ☐

A.11 Data Fitting Details

In the following, we provide the details of data fitting and testing. Because not all product attributes are relevant for all customer segments, Intel suggested that segment-specific subset of regressors should be used to prevent problems stemming from over-fitting or over-simplifying. To that end, we test a variety of models for each segment where a model refers to a particular subset of the regressors.

We use the first three generations of products (13 SKUs) to parameterize the demand model and the fourth generation of products (3 SKUs) to test the model, mimicking the practical context at Intel. For any given customer segment, the market share prediction is computed for each product; we select the model using the mean absolute error (MAE) for the market share of each product.

Table 6 presents a summary of goodness-of-fit and test measures for the selected model for each segment including the Estrella index (which is a value between 0 and 1, larger number corresponding to a better fit), the training MAE, and the test MAE. The model for each segment is chosen by focusing primarily on the test MAE and secondly on the training MAE, and by balancing model parsimony and the test errors.

Table 6: Fit and Forecast Accuracy.

Segment	Chosen Model	Estrella Index	Training MAE	Test MAE
1	TDP, Performance, Price/performance	89%	10%	17%
2	Frequency, TDP, Price, Price/performance	62%	13%	1%
3	Performance, Price/performance	78%	12%	2%
4	Performance, Price/performance	53%	13%	9%
5	Frequency, Price, Price/performance	71%	10%	6%
6	Performance, Price/performance	50%	14%	10%
7	Performance, Price/performance	54%	13%	13%

The coefficients and the corresponding standard errors (in parenthesis) of the selected regression model for each segment are given in Table 7.

Table 7: Linear Utility Coefficients for Each Customer Segment.

Segment	Frequency	TDP	Performance	Price	Price/performance
1	–	-0.2791 (0.0095)	0.02885 (0.00066)	–	-0.786 (0.170)
2	2.097 (0.162)	-0.0244 (0.0086)	–	0.00105 (0.00051)	-3.677 (0.131)
3	–	–	0.00936 (0.00031)	–	-0.993 (0.106)
4	–	–	0.00267 (0.00027)	–	-2.201 (0.099)
5	2.512 (0.135)	–	–	0.00490 (0.00032)	-2.846 (0.109)
6	–	–	0.00729 (0.00030)	–	-0.615 (0.084)
7	–	–	0.00777 (0.00030)	–	-0.625 (0.087)

A.12 Segment-specific Sales Distribution among Products

Table 8: Sales Distribution under Current Practice (Each Number Represents Segment-specific Choice Probability for the Corresponding Product).

Product \ Segment	S1	S2	S3	S4	S5	S6	S7
1	0.0008	0.0982	0.0464	0.1505	0.0451	0.0726	0.0661
2	0.0044	0.3359	0.0764	0.1457	0.3169	0.1087	0.1018
3	0.9897	0.3938	0.5274	0.4003	0.3954	0.4716	0.4829

Table 9: Sales Distribution under Profit-improving Solution (Each Number Represents Segment-specific Choice Probability for the Corresponding Product).

Product \ Segment	S1	S2	S3	S4	S5	S6	S7
1	0.0017	0.2063	0.0862	0.3339	0.0848	0.1043	0.0962
2	0.0097	0.6924	0.1425	0.3256	0.5198	0.1565	0.1486
3	0.9807	0.0282	0.3590	0.0957	0.2065	0.3634	0.3736