Online Appendix

The online appendix contains proofs of the theorems in the paper.

Proofs for the Remaining Theoretical Results in Section 4

Proof of Theorem 4.2: Consider $\tau_1 < \tau_2$. For $i = 1, 2$, let $((p_t^i), q^i, (r_t^i)|\xi)$ be the firm's optimal price and quantity decisions at tax rate τ_i, and let $((s_t^i)|\xi)$ be the resulting sales quantity. Given the optimality of these decisions,

$$\begin{aligned} &\mathbb{E}_\xi\left[(1-\tau_1)(p_1^1 s_1^1 + p_2^1 s_2^1) + \tau_1(h(p_1^1)r_1^1 + h(p_2^1)r_2^1) - (1-\tau_1)cq^1 \mid \xi\right] \\ \geq\ &\mathbb{E}_\xi\left[(1-\tau_1)(p_1^2 s_1^2 + p_2^2 s_2^2) + \tau_1(h(p_1^2)r_1^2 + h(p_2^2)r_2^2) - (1-\tau_1)cq^2 \mid \xi\right] \end{aligned}$$

$$\begin{aligned} &\mathbb{E}_\xi\left[(1-\tau_2)(p_1^2 s_1^2 + p_2^2 s_2^2) + \tau_2(h(p_1^2)r_1^2 + h(p_2^2)r_2^2) - (1-\tau_1)cq^2 \mid \xi\right] \\ \geq\ &\mathbb{E}_\xi\left[(1-\tau_2)(p_1^1 s_1^1 + p_2^1 s_2^1) + \tau_2(h(p_1^1)r_1^1 + h(p_2^1)r_2^1) - (1-\tau_1)cq^1 \mid \xi\right] \end{aligned}$$

Summing up $(1-\tau_2)$ times the first inequality and $(1-\tau_1)$ times the second inequality, we have

$$(\tau_1 - \tau_1\tau_2)\mathsf{EATD}(\tau_1) + (\tau_2 - \tau_1\tau_2)\mathsf{EATD}(\tau_2) \geq (\tau_2 - \tau_1\tau_2)\mathsf{EATD}(\tau_1) + (\tau_1 - \tau_1\tau_2)\mathsf{EATD}(\tau_2)\ ,$$

which implies that $\mathsf{EATD}(\tau_1) \leq \mathsf{EATD}(\tau_2)$. □

Proof of Theorem 4.3: We consider a specific strategy under which the firm procures $q = \frac{M/(1+a) - F_c}{v_c}$ units of the product and donates all of them in the regular period with sufficiently high regular price (i.e., $p_1 \geq \frac{a+b}{b}c = \frac{(a+b)M}{(1+a)bq}$).

Under this strategy, the firm incurs a cost of $F_c + v_c q = M/(1+a)$ and claims an enhanced deduction of $(c + h(p_1))q = (c + \min\{ac, b(p_1 - c)\})q = (1+a)cq = M$. Therefore, the after-tax profit $U^*(\tau) \geq \tau M - M/(1+a) = \left(\tau - \frac{1}{1+a}\right)M$. □

Proof of Theorem 4.5: $s_i D_2(s_i, \xi) = s_i(\mathsf{WTP} - s_i)$, and $\frac{\partial(s_i D_2(s_i,\xi))}{\partial s_i} = \mathsf{WTP} - 2s_i$ for $i = 1, 2$. When $F_c \leq \hat{F}_c$, Theorem 4.4(I) shows that $\frac{\partial s_i D_2(s_i,\xi)}{\partial s_i}|_{s_i = s_i^*} = v_c$ for $i = 1, 2$. Therefore, $p_1^* = p_2^* = \frac{\mathsf{WTP} + v_c}{2}$, $s_1^* = s_2^* = \frac{\mathsf{WTP} - v_c}{2}$, $r_1^* = r_2^* = 0$, $q^* = s_1^* + s_2^*$, and the firm's optimal after-tax profit U^* is $(1-\tau)\left(\frac{(\mathsf{WTP} - v_c)^2}{2} - F_c\right)$.

By definition, $\bar{p}_1 = \mathsf{WTP}$. When $\mathsf{WTP} < \frac{a+b}{b}v_c$, $\hat{F} = \breve{F} = \bar{F} = \infty$ by Theorem 4.4. Now we focus on the case $\mathsf{WTP} \geq \frac{a+b}{b}v_c$ and derive the optimal solution when $F_c > \hat{F}$. First, we relax the markdown constraint. It is readily to verify that all the solutions identified satisfy the markdown constraint and this relaxation can be applied without loss of generality.

We derive the optimal solution for a given q (i.e., equivalent to a given c). By Theorem 4.4, when $F_c > \hat{F}$, $p_1^* \geq \frac{a+b}{b}c^*$ and $h(p_1^*) = ac^*$, and the firm's problem can be rewritten as the following convex optimization:

$$\begin{aligned} \max_{s_1, s_2} \quad & (1-\tau)\left(s_1(\mathsf{WTP} - s_1) + s_2(\mathsf{WTP} - s_2) - \frac{cF_c}{c - v_c}\right) + \tau ac\left(\frac{F_c}{c - v_c} - s_1 - s_2\right), \\ \text{s.t.} \quad & \frac{a+b}{b}c \leq p_1(s_1, \xi) \leq \bar{p}_1\ . \end{aligned}$$

Notice that $\frac{a+b}{b} \geq 1 > \frac{\tau}{1-\tau}a$, thus $\frac{2(a+b)}{b} - \frac{\tau}{1-\tau}a > \frac{a+b}{b}$. Denote $\breve{c} \equiv \max\left\{v_c, \frac{\mathsf{WTP}}{\frac{2(a+b)}{b} - \frac{\tau a}{1-\tau}}\right\}$. We consider the potential c value when it belongs to $(v_c, \breve{c}]$, $\left(\breve{c}, \frac{b\mathsf{WTP}}{a+b}\right)$, or equals to $\frac{b\mathsf{WTP}}{a+b}$.

When $c \in (v_c, \breve{c}]$, the solution to the relaxed problem is $s_1 = s_2 = \frac{\mathsf{WTP} - \frac{\tau}{1-\tau}ac}{2}$, and $p_1 = p_2 = \frac{\mathsf{WTP} + \frac{\tau}{1-\tau}ac}{2} \geq \frac{a+b}{b}c$. Therefore, both $\frac{a+b}{b}c \leq p_1(s_1, \xi) \leq \bar{p}_1$ and the markdown constraint $p_1 \geq p_2$ are satisfied. The firm's after-tax profit is $(1-\tau)\frac{\left(\mathsf{WTP} - \frac{\tau}{1-\tau}ac\right)^2}{2} - (1 - \tau - \tau a)F_c\frac{c}{c - v_c}$.

When $c\in\left(\breve{c},\frac{b\mathsf{WTP}}{a+b}\right]$, the solution to the relaxed problem violates the constraint $\frac{a+b}{b}c\le p_1(s_1,\xi)$. Therefore, at optimal, $p_1=\frac{a+b}{b}c$, $s_1=\mathsf{WTP}-\frac{a+b}{b}c$, $p_2=\frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac}{2}$, and $s_2=\frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac}{2}$. It is easy to verify that both $p_1(s_1,\xi)\le\bar{p}_1$ and the markdown constraint $p_1\ge p_2$ are satisfied. The firm's after-tax profit is $\left((1-\tau)\frac{a+b}{b}-\tau a\right)c\left(\mathsf{WTP}-\frac{a+b}{b}c\right)+(1-\tau)\frac{\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)^2}{4}-(1-\tau-\tau a)F_c\frac{c}{c-v_c}$.

Therefore, to find the optimal solution for a given F_c, we can search for the optimal c. When $F_c>\hat{F}$, Theorem 4.4 shows that $r_1^*>0$. As a result, when the optimal $c\in\left(v_c,\frac{b\mathsf{WTP}}{a+b}\right)$, we can find the optimal solution using the first-order condition over the profit function.

When $c\in(v_c,\breve{c}]$, $-\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)+\frac{(1-\tau-\tau a)F_cv_c}{(c-v_c)^2}=0$ by the first-order condition (that is, $F_c=\frac{\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)(c-v_c)^2}{(1-\tau-\tau a)v_c}$). Denote $F_1(x)\equiv\frac{\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ax\right)(x-v_c)^2}{(1-\tau-\tau a)v_c}$. Now we show that $F_1(x)$ is a monotone increasing function on $(v_c,\breve{c}]$. Therefore, when $F_c\in(0,F_1(\breve{c})]$, there is a unique solution c^* to $F_c=F_1(c)$ on $(v_c,\breve{c}]$.

When $c>v_c$, the derivative $F_1'(c)$ has the same sign as $2\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)-\frac{\tau}{1-\tau}a(c-v_c)$. The term would be non-negative if and only if $c\le\frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a}+\frac{1}{3}v_c$. It suffices to show that $\frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}}\le\frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a}<\frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a}+\frac{1}{3}v_c$. Notice that $\frac{2\mathsf{WTP}}{3\frac{\tau}{1-\tau}a}\ge\frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}}\Leftrightarrow\frac{4(a+b)}{b}-\frac{2\tau a}{1-\tau}\ge\frac{3\tau a}{1-\tau}\Leftarrow 4+4\frac{\tau}{1-\tau}a\ge 5\frac{\tau}{1-\tau}a$, which is true.

It is easy to verify that the solution c^* to $F_c=F_1(c)$ on $(v_c,\breve{c}]$ is a maximizer for the original profit maximization problem by the second-order condition. Furthermore, when $c\in(\breve{c},\infty)$ solves $F_c=F_1(c)$ for $F_c\le(\breve{c})$, c is a minimizer.

When $c\in\left(\breve{c},\frac{b\mathsf{WTP}}{a+b}\right)$, $\left((1-\tau)\frac{a+b}{b}-\tau a\right)\left(\mathsf{WTP}-\frac{2(a+b)}{b}c\right)-\frac{1}{2}\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)+\frac{(1-\tau-\tau a)F_cv_c}{(c-v_c)^2}=0$ by the first-order condition (that is, $F_c=\frac{\left(\frac{1}{2}\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)-\left((1-\tau)\frac{a+b}{b}-\tau a\right)\left(\mathsf{WTP}-\frac{2(a+b)}{b}c\right)\right)(c-v_c)^2}{(1-\tau-\tau a)v_c}$). Denote $F_2(x)\equiv\frac{\left(\frac{1}{2}\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ax\right)-\left((1-\tau)\frac{a+b}{b}-\tau a\right)\left(\mathsf{WTP}-\frac{2(a+b)}{b}x\right)\right)(x-v_c)^2}{(1-\tau-\tau a)v_c}$. It is easy to verify that $F_1(\breve{c})=F_2(\breve{c})$. Now we show that $F_2(x)$ is a monotone increasing function on $(\breve{c},\infty)$. Therefore, when $F_c\in\left(F_2(\breve{c}),F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)\right)$, there is a unique solution c^* to $F_c=F_2(c)$ on $\left(\breve{c},\frac{b\mathsf{WTP}}{a+b}\right)$.

When $c>\breve{c}\ge v_c$, the derivative $F_2'(c)$ has the same sign as $\left(-\frac{1}{2}\tau a\frac{\tau a}{1-\tau}+\left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)(c-v_c)+\left(\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)-2\left((1-\tau)\frac{a+b}{b}-\tau a\right)\left(\mathsf{WTP}-\frac{2(a+b)}{b}c\right)\right)=\left(-\frac{1}{2}\tau a\frac{\tau a}{1-\tau}+\left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)(3c-v_c)-\left(2(1-\tau)\frac{a+b}{b}-3\tau a\right)\mathsf{WTP}$.

The term would be non-negative if and only if $c\ge\frac{\left(2(1-\tau)\frac{a+b}{b}-3\tau a\right)\mathsf{WTP}}{3\left(-\frac{1}{2}\tau a\frac{\tau}{1-\tau}a+\left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)}+\frac{1}{3}v_c$. Notice that $\breve{c}\ge\frac{2}{3}\frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}}+\frac{1}{3}v_c$ and $\frac{2}{3}\frac{\mathsf{WTP}}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}}+\frac{1}{3}v_c\ge\frac{\left(2(1-\tau)\frac{a+b}{b}-3\tau a\right)\mathsf{WTP}}{3\left(-\frac{1}{2}\tau a\frac{\tau}{1-\tau}a+\left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)}+\frac{1}{3}v_c\Leftrightarrow\frac{2}{\frac{2(a+b)}{b}-\frac{\tau a}{1-\tau}}\ge\frac{\left(2(1-\tau)\frac{a+b}{b}-3\tau a\right)}{\left(-\frac{1}{2}\tau a\frac{\tau}{1-\tau}a+\left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{2(a+b)}{b}\right)}\Leftrightarrow-\tau a\frac{\tau}{1-\tau}a+\left((1-\tau)\frac{a+b}{b}-\tau a\right)\frac{4(a+b)}{b}\ge(1-\tau)\frac{a+b}{b}\frac{4(a+b)}{b}-\tau a\frac{8(a+b)}{b}+3\tau a\frac{\tau}{1-\tau}a\Leftrightarrow\frac{a+b}{b}\ge\frac{\tau}{1-\tau}a$, which is true. It is easy to verify that the solution c^* to $F_c=F_2(c)$ on $\left(\breve{c},\frac{b\mathsf{WTP}}{a+b}\right)$ is a maximizer for the original profit maximization problem by the second-order condition.

The analysis of the first-order conditions further reveals that when $F_c>\hat{F}$, if $F_c\in(0,F_1(\breve{c})]$, it would be suboptimal to choose $c>\breve{c}$; if $F_c\in\left(F_1(\breve{c}),F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)\right)$, it would be suboptimal to choose either $c=\frac{b\mathsf{WTP}}{a+b}$ or $c\in(v_c,\breve{c})$; and if $F_c\ge F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)$, it would be suboptimal to choose $c\in\left(v_c,\max\left\{v_c,\frac{b\mathsf{WTP}}{a+b}\right\}\right)$. Therefore, when $F_c>\hat{F}$, if $F_c\in(0,F_1(\breve{c})]$, $c^*\in(v_c,\breve{c}]$, $p_1^*=p_2^*=\frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^*=s_2^*=\frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^*=\frac{F_c}{c-v_c}$, $r_1^*=q^*-s_1^*-s_2^*$, and $r_2^*=0$, where c^* is the unique solution to $\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)(c-v_c)^2-F_c(1-\tau-\tau a)v_c=0$ on $(v_c,\breve{c}]$; if $F_c\in\left(F_1(\breve{c}),F_2\left(\frac{b\mathsf{WTP}}{a+b}\right)\right)$, $c^*\in\left(\breve{c},\frac{b\mathsf{WTP}}{a+b}\right)$, $p_1^*=\frac{a+b}{b}c^*$, $p_2^*=\frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^*=\mathsf{WTP}-\frac{a+b}{b}c^*$, $s_2^*=\frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^*=\frac{F_c}{c^*-v_c}$, $r_1^*=q^*-s_1^*-s_2^*$, and $r_2^*=0$, where c^* is the unique solution to $\frac{1}{2}\tau a\left(\mathsf{WTP}-\frac{\tau}{1-\tau}ac\right)(c-v_c)^2-(1-$

$\tau)\frac{a+b}{b}-\tau a)\left(\mathsf{WTP}-\frac{2(a+b)}{b}c\right)(c-v_c)^2-F_c(1-\tau-\tau a)v_c=0$ on $\left(\check{c},\frac{b\mathsf{WTP}}{a+b}\right)$; and if $F_c\geq F_2(\frac{b\mathsf{WTP}}{a+b})$, $c^*=\frac{b\mathsf{WTP}}{a+b}$, $p_1^*=\frac{a+b}{b}c^*$, $p_2^*=\frac{\mathsf{WTP}+\frac{\tau}{1-\tau}ac^*}{2}$, $s_1^*=0$, $s_2^*=\frac{\mathsf{WTP}-\frac{\tau}{1-\tau}ac^*}{2}$, $q^*=\frac{F_c}{c^*-v_c}$, $r_1^*=q^*-s_1^*-s_2^*$, and $r_2^*=0$.

Furthermore, the claim that $\hat{F}=\infty$ if and only if $\tau\leq\frac{(a+b)v_c}{(a+b)v_c+ab\mathsf{WTP}}$ directly follows Theorem 4.4 and that $\bar{p}_1=\mathsf{WTP}$ under the linear demand case. The monotone property analyzed above implies that when $F_c>\hat{F}$, c^* is a continuous (weakly) increasing function of F_c. Given that p_1^* and p_2^* are continuous (weakly) increasing functions of c^*, when $F_c>\hat{F}$, p_1^* and p_2^* are also continuous (weakly) increasing functions of F_c. □

Proof of Theorem 4.6: When $\xi=\underline{\xi}=\overline{\xi}$, $s_1^*=D_1(p_1^*,\xi)$, $r_2^*=0$, $s_2^*=q-r_1^*-s_1^*$, and $p_2^*=p_2(s_2^*,\xi)$ by Theorem 4.4; similarly, $s_1^0=D_1(p_1^0,\xi)$, $s_2^0=q-s_1^0$, and $p_2^0=p_2(s_2^0,\xi)$. Therefore,

$$(q^0,p_1^0)=\underset{\substack{q\geq D_1(p_1,\xi)\\ p_1\geq p_2(q-D_1(p_1,\xi),\xi)}}{\arg\max}\; p_1D_1(p_1,\xi)+p_2(q-D_1(p_1,\xi),\xi)(q-D_1(p_1,\xi))-(F_c+v_cq).$$

$$(q^*,p_1^*,r_1^*)=\underset{\substack{q\geq D_1(p_1,\xi)+r_1\\ p_1\geq p_2(q-r_1-D_1(p_1,\xi),\xi)\\ r_1\geq 0}}{\arg\max}\; p_1D_1(p_1,\xi)+p_2(q-r_1-D_1(p_1,\xi),\xi)(q-r_1-D_1(p_1,\xi))-(F_c+v_cq)+\frac{\tau}{1-\tau}h(p_1)r_1.$$

We will prove by contradiction. Suppose that $p_1^*<p_1^0$. Let

$$(q^\#,p_1^\#)=\underset{\substack{q\geq D_1(p_1,\xi)\\ p_1=p_1^*\geq p_2(q-D_1(p_1,\xi),\xi)}}{\arg\max}\; p_1D_1(p_1,\xi)+p_2(q-D_1(p_1,\xi),\xi)(q-D_1(p_1,\xi))-(F_c+v_cq).$$

That is, $(q^\#,p_1^\#)$ maximizes the profit without the enhanced tax deduction when the first period price $p_1^\#$ is fixed at p_1^*. Let $p_2^\#=p_2(q^\#-D_1(p_1^\#,\xi),\xi)$ and $U^\#$ and $(s_1^\#,s_2^\#)$ be the associated profit and sales quantities, respectively. Because the profit maximization problem without the enhanced tax deduction is concave and has a unique solution, $U^0>U^\#$, $p_2^\#\leq p_1^\#=p_1^*<p_1^0$, and $s_1^0=D_1(p_1^0,\xi)<D_1(p_1^*,\xi)=s_1^*=s_1^\#$.

We consider the two possibilities:

$p_2^\#<p_1^\#$: In this case, by Theorem 4.4, $s_2^0=s_2^\#=\arg\max_s(sp_2(s,\xi)-v_cs)$ as the constraints are not binding; thus, $p_2^\#=p_2^0$. Because $U^0>U^\#$, $p_1^0s_1^0-v_cs_1^0>p_1^\#s_1^\#-v_cs_1^\#=p_1^*s_1^*-v_cs_1^*$.

Now we show that (q^*,p_1^0,p_2^*) is feasible and provides a profit higher than U^* with the enhanced tax deduction. (q^*,p_1^0,p_2^*) is feasible because $p_1^0>p_1^*\geq p_2^*$. Under (q^*,p_1^0,p_2^*), the donation quantity in the first period is $q^*-s_1^0-s_2^*=r_1^*+s_1^*-s_1^0>r_1^*$. Furthermore, $p_1^*<p_1^0$, implies that $h(p_1^0)r_1^*\geq h(p_1^*)r_1^*$ and Theorem 4.4 shows that $\frac{\tau}{1-\tau}h(p_1^0)\geq\frac{\tau}{1-\tau}h(p_1^*)>v_c$. Together with $p_1^0s_1^0-v_cs_1^0>p_1^*s_1^*-v_cs_1^*$, the profit under (q^*,p_1^0,p_2^*) is greater than U^*. Thus, we reach a contradiction.

$p_2^\#=p_1^\#$: In this case, by Theorem 4.4, $s_2^\#=D_2(p_1^\#,\xi)=\underset{s\geq D_2(p_1^\#,\xi)}{\arg\max}(sp_2(s,\xi)-v_cs)$ as the constraints is binding. Recall that $(sp_2(s,\xi)-v_cs)$ is concave and $p_1^*=p_1^\#$. Because $D_2(p_1^\#,\xi)=D_2(p_1^*,\xi)$ and $\frac{\tau}{1-\tau}h(p_1^*)>v_c$ by Theorem 4.4, $s_2^*=\underset{s\geq D_2(p_1^*,\xi)}{\arg\max}\left(sp_2(s,\xi)-\frac{\tau}{1-\tau}h(p_1^*)s\right)=s_2^\#$ as the constraint must also be binding. Furthermore, $p_1^0>p_1^*=p_1^\#\Leftrightarrow D_2(p_1^\#,\xi)>D_2(p_1^0,\xi)$, $s_2^0=\underset{s\geq D_2(p_1^0,\xi)}{\arg\max}(sp_2(s,\xi)-v_cs)\leq s_2^\#=s_2^*$ and $p_2^*=p_2^\#\geq p_2^0$. Because $U^0>U^\#$, $p_1^0s_1^0+p_2^0s_2^0-v_c(s_1^0+s_2^0)>p_1^\#s_1^\#+p_2^\#s_2^\#-v_c(s_1^\#+s_2^\#)=p_1^*s_1^*+p_2^*s_2^*-v_c(s_1^*+s_2^*)$.

Now we show that (q^*,p_1^0,p_2^0) is feasible and provides a profit higher than U^* with the enhanced tax deduction. (q^*,p_1^0,p_2^0) is feasible because $p_1^0>p_2^0$. Under (q^*,p_1^0,p_2^0), the donation quantity in the first period is $q^*-s_1^0-s_2^0=r_1^*+s_1^*+s_2^*-(s_1^0+s_2^0)>r_1^*$. The profit under (q^*,p_1^0,p_2^0) is higher than U^* because $p_1^0s_1^0+p_2^0s_2^0-v_c(s_1^0+s_2^0)>p_1^*s_1^*+p_2^*s_2^*-v_c(s_1^*+s_2^*)$, $h(p_1^0)r_1^*\geq h(p_1^*)r_1^*$ due to $p_1^*<p_1^0$, and $\frac{\tau}{1-\tau}h(p_1^0)\geq\frac{\tau}{1-\tau}h(p_1^*)>v_c$ by Theorem 4.4. Thus, we reach a contradiction. □

Proof of Theorem 4.9: Statements I) and II) parts (i) and (ii) directly follow from Theorem 4.8.

When it is optimal for the firm to choose salvaging, the optimal sales quantity $s_2 = \arg\max sp_2(s,\xi)$, which is a concave function and has a unique solution. When either $D_2(p,\xi)=D(p)$ or $D_2(p,\xi)=\xi D(p)$, the optimal price corresponds to $\hat{p}$, the unique price that maximizes $pD(p)$. $\hat{p}\le c$ follows from Theorem 4.8. This proves the first part of the theorem.

To prove the last part, we consider two demand states $\xi'<\xi''$. Let the optimal price, donation quantity, and sales in the second period be (p_2',r_2',s_2') and (p_2'',r_2'',s_2'') for these two states, respectively. Let the available inventory at beginning of second period be I' and I'' for these two states, respectively.

Suppose that $\xi'<\xi''$ and $r_2''>0$, we show that $r_2'>0$ as well. Because $r_2''>0$, donating is preferred over salvage at ξ''. When $D_2(p,\xi)=\xi D(p)$, this implies $(1-\tau)\xi'' p_2'' D(p_2'')+\tau h(p_2'')(I''-\xi'' D(p_2''))>(1-\tau)\xi''\hat{p}D(\hat{p})$.

At ξ', the profit of donating excess inventory at price p_2'' is $(1-\tau)\xi' p_2'' D(p_2'')+\tau h(p_2'')(I'-\xi' D(p_2''))\ge \frac{\xi'}{\xi''}((1-\tau)\xi'' p_2'' D(p_2'')+\tau h(p_2'')(I''-\xi'' D(p_2'')))>(1-\tau)\xi'\hat{p}D(\hat{p})$, which is the profit under salvaging at ξ'. The first inequality holds because $I'\ge I''$ and $\xi'<\xi''$. Therefore, if donating is optimal at ξ'', donating is optimal at $\xi'<\xi''$ when $D_2(p,\xi)=\xi D(p)$. When $D_2(p,\xi)=D(p)$, the same conclusion can be established using a similar argument.

Because the firm's profit is continuous under either donating or salvaging and the tie-breaking favors salvaging, there exists $\check{\xi}$ such that when $\underline{\xi}\le\xi<\check{\xi}$, the firm chooses donation. That is, $r_2^*>0$, $s_2^*=\max\{D_2(p_1^*,\xi),\arg\max_s((1-\tau)sp_2(s,\xi)+\tau h(p_2(s,\xi))(I^*-s)\}$, and $p_2^*=p_2(s_2^*,\xi)>c$ by Theorem 4.8.

Now we show that p_2^* and r_2^* are decreasing in ξ when $\underline{\xi}\le\xi<\check{\xi}$. We will prove by contradiction. Suppose that the optimal solutions $p_2'<p_2''$ under $\xi'<\xi''$. Because of the continuity of the profit function and the uniqueness of the optimal solution under donating, p_2 can be viewed as a continuous function of ξ. Because $I''-D_2(p_2'',\xi)>0$, without loss of generality, we can assume that $I''-D_2(p_2',\xi)>0$. Also, $I'-D_2(p_2'',\xi)>0$ follows from $I'>I''$. Furthermore, when $D_2(p,\xi)=\xi D(p)$, the optimality of p_2' and p_2'' implies that

$$(1-\tau)\xi'' p_2'' D(p_2'')+\tau h(p_2'')(I''-\xi'' D(p_2''))>(1-\tau)\xi'' p_2' D(p_2')+\tau h(p_2')(I''-\xi'' D(p_2')) \tag{EC.1}$$

and

$$(1-\tau)\xi' p_2' D(p_2')+\tau h(p_2')(I'-\xi' D(p_2'))>(1-\tau)\xi' p_2'' D(p_2'')+\tau h(p_2'')(I'-\xi' D(p_2'')). \tag{EC.2}$$

Note that $p_2'<p_2''$ implies that $h(p_2')\le h(p_2'')$. Therefore,

$$\begin{aligned}(1-\tau)(p_2' D(p_2')-p_2'' D(p_2''))&>\tfrac{\tau I'}{\xi'}(h(p_2'')-h(p_2'))+\tau(h(p_2')D(p_2')-h(p_2'')D(p_2''))\\ &\ge\tfrac{\tau I''}{\xi''}(h(p_2'')-h(p_2'))+\tau(h(p_2')D(p_2')-h(p_2'')D(p_2''))>(1-\tau)(p_2' D(p_2')-p_2'' D(p_2'')),\end{aligned}$$

where the first inequality follows from (EC.2), the second inequality follows from $I'>I''$ and $\xi'<\xi''$, and the last inequality follows from (EC.1). Thus we reach a contradiction. When $D_2(p,\xi)=D(p)$, we can reach a contradiction by a similar argument.

Therefore, when $\underline{\xi}\le\xi<\check{\xi}$, p_2^* is (weakly) decreasing in ξ; $s_2^*=D(p_2^*,\xi)$ is (weakly) increasing in ξ; and $r_2^*=I^*-s_2^*$ is (weakly) decreasing in ξ. □

Proof of Lemma 1: We first show that when $p_1'<p_1''$, $U_2(x|p_1',\xi)-U_2(x|p_1',\xi,r_2=0)\le U_2(x|p_1'',\xi)-U_2(x|p_1'',\xi,r_2=0)$. Suppose that the second period price p_2' is part of the optimal solution to $U_2(x|p_1',\xi,r_2=0)$. We consider two possibilities:

$p_2' = p_1'$: In this case, all x units of inventory are sold at $p_2' = p_1'$ under the optimal solution to $U_2(x|p_1', \xi, r_2 = 0)$. Therefore, $U_2(x|p_1, \xi) = p_1 x = U_2(x|p_1', \xi, r_2 = 0)$, and $U_2(x|p_1', \xi) - U_2(x|p_1', \xi, r_2 = 0) = 0 \le U_2(x|p_1'', \xi) - U_2(x|p_1'', \xi, r_2 = 0)$.

$p_2' < p_1'$: In this case, the constraint $p_2' < p_1'$ is not binding, because $U_2(x|p_1', \xi, r_2 = 0)$ can be formulated as a convex optimization problem of the sales quantity. When $p_2' < p_1' < p_1''$, the optimal solution to $U_2(x|p_1', \xi, r_2 = 0)$ is the optimal solution to $U_2(x|p_1'', \xi, r_2 = 0)$. Therefore, $U_2(x|p_1', \xi) - U_2(x|p_1', \xi, r_2 = 0) \le U_2(x|p_1'', \xi) - U_2(x|p_1'', \xi, r_2 = 0)$ because $U_2(x|p_1', \xi) \le U_2(x|p_1'', \xi)$ when $p_1' < p_1''$.

Now we show that when $x' < x''$, $U_2(x'|p_1, \xi) - U_2(x'|p_1, \xi, r_2 = 0) \le U_2(x''|p_1, \xi) - U_2(x''|p_1, \xi, r_2 = 0)$. Suppose that the second period donation r_2' is part of the optimal solution to $U_2(x'|p_1, \xi)$. We consider two possibilities:

$r_2' = 0$: In this case, $U_2(x'|p_1, \xi) - U_2(x'|p_1, \xi, r_2 = 0) = 0 \le U_2(x''|p_1, \xi) - U_2(x''|p_1, \xi, r_2 = 0)$.

$r_2' > 0$: In this case, it suffices to show that $\tau h(p_2') \ge U_2'(x'|p_1, \xi, r_2 = 0)$ because $U_2(x''|p_1, \xi) - U_2(x'|p_1, \xi) \ge h(p_2')(x'' - x')$ and $U_2(x''|p_1, \xi, r_2 = 0) - U_2(x'|p_1, \xi, r_2 = 0) \le U_2'(x'|p_1, \xi, r_2 = 0)(x'' - x')$ due to the concavity of $U_2(x|p_1, \xi, r_2 = 0)$.

Notice that $\tau h(p_2') r_2' > U_2(x'|p_1, \xi, r_2 = 0) - U_2(x' - r_2'|p_1, \xi, r_2 = 0) = U'(x'''|p_1, \xi, r_2 = 0) r_2'$, where $x''' \in (x' - r_2', x')$. Therefore, $\tau h(p_2') > U'(x'''|p_1, \xi, r_2 = 0) > U'(x'|p_1, \xi, r_2 = 0)$ by the concavity of $U_2(x|p_1, \xi, r_2 = 0)$. □

Proof of Theorem 4.10: Notice that $r_1 = 0$ at optimal due to $F_c = 0$ by Theorem 4.7. We prove the result by contradiction. Suppose that $p_1^* < p_1^0$ and $q^* < q^0$, we show that by procuring q^0 and setting the first period price as p_1^0, the firm achieves a profit higher than U^* with the enhanced tax deduction.

$$
\begin{aligned}
U^* &= U_1(q^*) - (1-\tau) v_c q^* = (1-\tau)(-v_c q^* + p_1^* E_\xi[\min\{D_1(p_1^*, \xi), q^*\}]) + E_\xi[U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi)] \\
&= (1-\tau)(-v_c q^* + p_1^* E_\xi[\min\{D_1(p_1^*, \xi), q^*\}]) + E_\xi[U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi, r_2 = 0)] \\
&+ E_\xi[U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi) - U_2((q^* - D_1(p_1^*, \xi))^+ | p_1^*, \xi, r_2 = 0)] \\
&< (1-\tau)(-v_c q^0 + p_1^* E_\xi[\min\{D_1(p_1^0, \xi), q^0\}]) + E_\xi[U_2((q^0 - D_1(p_1^0, \xi))^+ | p_1^0, \xi, r_2 = 0)] \\
&+ E_\xi[U_2((q^0 - D_1(p_1^*, \xi))^+ | p_1^0, \xi) - U_2((q^0 - D_1(p_1^0, \xi))^+ | p_1^0, \xi, r_2 = 0)] \\
&= (1-\tau)(-v_c q^0 + p_1^0 E_\xi[\min\{D_1(p_1^0, \xi), q^0\}]) + E_\xi[U_2((q^0 - D_1(p_1^0, \xi))^+ | p_1^0, \xi)] \\
&\le U_1(q^0) - (1-\tau) v_c q^0
\end{aligned}
$$

The first inequality is due to the optimality of (q^0, p_1^0), Lemma 1, and $(q^0 - D_1(p_1^0, \xi))^+ \ge (q^* - D_1(p_1^*, \xi))^+$ when $p_1^* < p_1^0$ and $q^* < q^0$. The second inequality is due to the definition of U_1. Therefore, we reach a contradiction and we conclude the proof. □