Appendix: Online Supplements

*This appendix include all proofs and additional properties for "**Financing Multiple Heterogeneous Suppliers in Assembly Systems: Buyer Finance versus Bank Finance**" by Shiming Deng, Chaocheng Gu, Gangshu (George) Cai, and Yanhai Li, published by Manufacturing & Service Operations Management.*

Proof of Lemma 2: We prove this lemma by contradiction. Given the equilibrium $\vec{w}$, let $q_m = \min\{\bar{F}^{-1}(\frac{c^j(1+r_s^j)}{w^j}), j = 1, ..., N\}$. Suppose that, in the equilibrium solution, $\bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w^i}) > q_m$ for some supplier i. Note that the assembler's expected profit is $\Pi_n(q) = \mathbf{E_D}\ [(p - \Sigma_{i=1}^N w^i) \cdot \min(D, q)]$. We can improve $\Pi_n(q)$ by keeping q and $(w^1, \cdots, w^{i-1}, w^{i+1}, \cdots, w^N)$ unchanged but lowering w^i so that $\bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w^i}) = q_m$. The assembler incurs a lower purchasing cost for component i and receives a higher profit. This contradicts the assumption that $\vec{w}$ is optimal for the assembler. Therefore, we have $q = \bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w^i}), \forall i$. Similarly, if there exists some supplier i such that $\bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w^i}) > \min_i\{\frac{k^i}{c^i}\}$, then $\Pi_n(q)$ can be improved by lowering w^i but keeping q and $(w^1, \cdots, w^{i-1}, w^{i+1}, \cdots, w^N)$ unchanged. Therefore, in the equilibrium solution, we have $\bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w^i}) \leq \min_i\{\frac{k^i}{c^i}\}, \forall i$. Following the same argument, we can show that the equilibrium q cannot be less than q_m and must be equal to q_m. Q.E.D.

Proof of Lemma 3: The first order derivative of $\Pi_n(q)$ with respect to q is $\frac{\partial \Pi_n(q)}{\partial q} = -\frac{f(q)\sum_{i=1}^N c^i(1+r_s^i)}{[\bar{F}(q)]^2}\int_0^q \bar{F}(x)dx + p\bar{F}(q) - \sum_{i=1}^N c^i(1+r_s^i)$. Following the IFR assumption, $\frac{f(q)}{\bar{F}(q)}$ increases with q. Note that $\int_0^q \bar{F}(x)dx$ increases with q and $\bar{F}(q)$ decreases with q. Therefore, $\frac{\partial \Pi_n(q)}{\partial q}$ is decreasing in q. Thus, $\Pi_n(q)$ is concave in q. Q.E.D.

Proof of Lemma 4: Let $w_{(1)}^i = c^i(1+r_s^i)/\bar{F}(\frac{k^i}{c^i})$ and $w_{(2)}^i = c^i(1+r_f)/\bar{F}(\frac{k^i}{c^i})$. Suppose there exists $w_0^i \leq w_{(2)}^i$, when $w^i = w_0^i$, supplier i borrows $B^i > 0$ and produces $q_0^i > \frac{k^i}{c^i}$. Then the optimal quantity $q_0^i = \bar{F}^{-1}(\frac{c^i(1+r_f)}{w_0^i}) > \frac{k^i}{c^i}$, which implies $w_0^i > c^i(1+r_f)/\bar{F}(\frac{k^i}{c^i}) = w_{(2)}^i$, this contradicts the definition of w_0^i. Therefore, $B^i = 0$ (i.e. $q^i \leq \frac{k^i}{c^i}$) if $w^i \leq w_{(2)}^i$.

(1) When $w^i < w_{(1)}^i$, because $w_{(1)}^i \leq w_{(2)}^i$, it follows the above statement that $B^i = 0$. The supplier i's profit is $\pi_{bk}^i = \mathbf{E}_D[w^i min(q^i, D)] + (k^i - c^i q^i)(1 + r_s^i)$, subject to $c^i q^i \leq k^i$. The optimal quantity $q^i = min(\frac{k^i}{c^i}, \bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w^i})) = \bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w^i})$.

(2) We have shown $q^i \leq \frac{k^i}{c^i}$ if $w^i \leq w_{(2)}^i$. To prove $q^i = \frac{k^i}{c^i}$ when $w_{(1)}^i < w^i \leq w_{(2)}^i$, we next need to show $q^i < \frac{k^i}{c^i}$ is not optimal. Suppose there exists $w_0^i \in [w_{(1)}^i, w_{(2)}^i)$, when $w^i = w_0^i$, supplier i produces $q_0^i < \frac{k^i}{c^i}$. Then the optimal quantity $q_0^i = \bar{F}^{-1}(\frac{c^i(1+r_s^i)}{w_0^i}) < \frac{k^i}{c^i}$, which implies

$w^i_0 < c^i(1+r^i_s)/\bar{F}(\frac{k^i}{c^i}) = w^i_{(1)}$, this contradicts the definition of w^i_0.

(3) When $w^i > w^i_{(2)}$, we next prove $B^i = 0$ (i.e., $q^i \leq \frac{k^i}{c^i}$) is not optimal. Assume $B^i = 0$, the optimal quantity of supplier i is $q^i = \bar{F}^{-1}(\frac{c^i(1+r^i_s)}{w^i}) > \bar{F}^{-1}(\frac{c^i(1+r_f)}{w^i_{(2)}}) = k^i/c^i$, which contradicts the assumption that $B^i = 0$. Q.E.D.

Proof of Lemma 5: The proof follows the same argument for Lemma 2. Q.E.D.

Proof of Theorem 1:

1. For any supplier i, by definition, Π_{bk} is left-continuous at $q = \frac{k^i}{c^i}$. We next show that Π_{bk} is not right-continuous at $q = \frac{k^i}{c^i}$. Note that for each $q = \frac{k^i}{c^i}$, the right-hand limit is $\Pi^+_{bk}(q) = \lim_{q \to (\frac{k^i}{c^i})^+} \Pi_{bk}(q) = \{p - \frac{\sum_{j=i+1}^N c^j + \sum_{j=1}^i (1+r_f)c^j}{\bar{F}(q)}\} \cdot \mathbf{E_D} min(q, D)$, so $\Pi^+_{bk}(q) - \Pi^-_{bk} = -\frac{c^i r_f}{\bar{F}(q)} \cdot \mathbf{E_D} min(q, D) < 0$. Therefore, $q = \frac{k^i}{c^i}$ is a breakpoint for $\forall i \in \{1, 2, \cdots, N\}$.

2. In the subset of $q \in (\frac{k^i}{c^i}, \frac{k^{i+1}}{c^{i+1}}]$, $\Pi_{bk}(q) = \{p - \frac{\sum_{j=i+1}^N c^j + \sum_{j=1}^i (1+r_f)c^j}{\bar{F}(q)}\} \cdot \mathbf{E_D} min(q, D)$ resembles $\Pi_n(q)$. The proof of concavity of $\Pi_{bk}(q)$ follows the same argument in the proof of Lemma 3.

3. The proof follows two steps. First, for any $\frac{k^i}{c^i} < a < b \leq \frac{k^{i+1}}{c^{i+1}}$, $\frac{\partial_- \Pi_{bk}(q)}{\partial q}|_{q=a} > \frac{\partial_- \Pi_{bk}(q)}{\partial q}|_{q=b}$ because Π_{bk} is continuous and concave in each subset $(\frac{k^i}{c^i}, \frac{k^{i+1}}{c^{i+1}}]$, which has been proved in part (2). Second, we then show that if $a = \frac{k^i}{c^i}$, for any b such that $a < b \leq \frac{k^{i+1}}{c^{i+1}}$, then $\frac{\partial_- \Pi_{bk}(q)}{\partial q}|_{q=a} > \frac{\partial_- \Pi_{bk}(q)}{\partial q}|_{q=b}$. This result occurs because,

$$\begin{aligned}
& \frac{\partial_- \Pi_{bk}(q)}{\partial q}|_{q=a} = \frac{\partial_- \Pi_{bk}(q)}{\partial q}|_{q=(\frac{k^i}{c^i})} \\
= \; & \{-\frac{f(q)(\sum_{j=1}^N c^j + \sum_{j=1}^{i-1} r_f c^j)}{[\bar{F}(q)]^2} \int_0^q \bar{F}(x)dx + p\bar{F}(q) - (\sum_{j=1}^N c^j + \sum_{j=1}^{i-1} r_f c^j)\}|_{q=(\frac{k^i}{c^i})} \\
\geq \; & \{-\frac{f(q)(\sum_{j=1}^N c^j + \sum_{j=1}^{i-1} r_f c^j)}{[\bar{F}(q)]^2} \int_0^q \bar{F}(x)dx + p\bar{F}(q) - (\sum_{j=1}^N c^j + \sum_{j=1}^{i-1} r_f c^j)\}|_{q=b} \\
> \; & \{-\frac{f(q)(\sum_{j=1}^N c^j + \sum_{j=1}^{i} r_f c^j)}{[\bar{F}(q)]^2} \int_0^q \bar{F}(x)dx + p\bar{F}(q) - (\sum_{j=1}^N c^j + \sum_{j=1}^{i} r_f c^j)\}|_{q=b} \\
= \; & \frac{\partial_- \Pi_{bk}(q)}{\partial q}|_{q=b}.
\end{aligned}$$

The first inequality follows from the concavity of $\Pi_{bk}(q)$ in each subset $q \in (\frac{k^i}{c^i}, \frac{k^{i+1}}{c^{i+1}}]$. The second inequality holds because the summation upper limit is changed from $i-1$ to i and $r_f c^j \geq 0$. To summarize, $\Pi_{bk}(q)$ is globally left-differentiable and $\frac{\partial_- \Pi_{bk}(q)}{\partial q}$ is globally decreasing in q. Q.E.D.

Proof of Theorem 2: By definition of b_k, $\Pi_{bk}(q)$ decreases in q for $q \in (b_k, \infty)$ since $\frac{\partial_- \Pi_{bk}}{\partial q}$ is decreasing in q (Theorem 1). Hence $q^*_{bk} \leq b_k$. Also, $\frac{\partial_- \Pi_{bk}}{\partial q} > 0$ for $q \in [0, \max_i\{\frac{k^i}{c^i}\}|_{\frac{k^i}{c^i} < b_k}]$, implying that q_{bk}^* can not be any internal point of the interval $(\frac{k^{i-1}}{c^{i-1}}, \frac{k^i}{c^i}]$ $\forall i$ such that $\frac{k^i}{c^i} < b_k$. Therefore, q_{bk}^* can be obtained by comparing $\Pi_{bk}(q)$ at $q = \frac{k^i}{c^i}$ for $\forall i$ such that $\frac{k^i}{c^i} < b_k$ and $\Pi_{bk}(q)$ at $q = \bar{q}$ where $\bar{q}$ lies in the interval that includes b_k and $\frac{\partial \Pi_{bk}(q)}{\partial q}|_{q=\bar{q}} = 0$. Q.E.D.

Proof of the sensitivity results in bank finance:

(1) To prove q_{bk}^* decreases in c^i and in r_f but increases in p, we show that $\Pi_{bk}(q, c^i)$ is a submodular function in (q, c^i), a submodular function in (q, r_f), and a supermodular function in (q, p), respectively. To prove that $\Pi_{bk}(q, c^i)$ is submodular in (q, c^i), we can show that $\Pi_{bk}(q, c^i) - \Pi_{bk}(q, c^{i\prime})$ is decreasing in q for $c^i \geq c^{i\prime}$. For a borrowing supplier i, $\Pi_{bk}(q, c^i) - \Pi_{bk}(q, c^{i\prime}) = -(c^i - c^{i\prime})(1 + r_f) \cdot \frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)}$ is decreasing in q since $\frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)}$ is increasing in q. For a non-borrowing supplier i, $\Pi_{bk}(q, c^i) - \Pi_{bk}(q, c^{i\prime}) = -(c^i - c^{i\prime}) \cdot \frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)}$, which decreases in q. Following a similar argument, we can show that $\Pi_{bk}(q, r_f)$ is submodular in (q, r_f) and $\Pi_{bk}(q, p)$ is supermodular in (q, p). Note that $\forall q$, $\frac{\partial \Pi_{bk}(q)}{\partial c^i} < 0$, $\frac{\partial \Pi_{bk}(q)}{\partial r_f} < 0$, and $\frac{\partial \Pi_{bk}(q)}{\partial p} > 0$. By Envelope Theorem, we have $\Pi_{bk}(q_{bk}^*)$ decreases in c^i and r_f and increases in p.

(2) We next prove $\Pi_{bk}(q_{bk}^*)$ is increasing in $k^i, \forall i \in \{1, 2, \cdots, N\}$. Suppose $k^{i'} < k^{i''}$, Let $q_{bk}^{'*}$ and $q_{bk}^{''*}$ denote the equilibrium production quantity when $k^i = k^{i'}$ and $k^i = k^{i''}$, respectively. It follows from Eq.(5) that $\Pi_{bk}(q, k^i)$ increases in k^i for any q. Therefore, $\Pi_{bk}(q_{bk}^{'*}, k^i = k^{i''}) \geq \Pi_{bk}(q_{bk}^{'*}, k^i = k^{i'})$. By optimality, $\Pi_{bk}(q_{bk}^{''*}, k^i = k^{i''}) \geq \Pi_{bk}(q_{bk}^{'*}, k^i = k^{i''}) \geq \Pi_{bk}(q_{bk}^{'*}, k^i = k^{i'})$. Q.E.D.

Lemma 7 *Suppose a non-negative random variable D has a density function, $f(x)$, with a support of $[A, B]$, where $A \geq 0$ and B can be ∞. If $\{x \in (A, \alpha) : f(x) \text{is continuous and } f(x) > 0\} \neq \emptyset$ for $\forall \alpha \in (A, B]$, then $\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} - q > 0$ for any positive $q \geq A$.*

Proof of Lemma 7: Note $q - \frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} = \frac{1}{\bar{F}(q)}[q\bar{F}(q) - \int_0^q \bar{F}(x)dx]$. To prove $\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} - q > 0$, we can instead show $\int_0^q \bar{F}(x)dx > q\bar{F}(q)$. If there exists $\epsilon \in (A, \alpha) \subset (A, q)$, such that $f(x) > 0$ and is continuous in x for $x \in [\epsilon - \lambda, \epsilon + \lambda]$ for any small $\lambda > 0$. We can then show $\bar{F}(\epsilon) = \bar{F}(\epsilon + \lambda) + (\bar{F}(\epsilon) - \bar{F}(\epsilon + \lambda)) = \bar{F}(\epsilon + \lambda) + \int_\epsilon^{\epsilon+\lambda} f(x)dx > \bar{F}(\epsilon + \lambda)$. The last inequality holds because $\int_\epsilon^{\epsilon+\lambda} f(x)dx = \lambda f(\xi) > 0$, where $\xi \in (\epsilon, \epsilon + \lambda)$. We thus have $\bar{F}(\epsilon) > \bar{F}(\epsilon + \lambda) \geq \bar{F}(q)$. We can then show $\int_0^q \bar{F}(x)dx = \int_0^\epsilon \bar{F}(x)dx + \int_\epsilon^q \bar{F}(x)dx \geq \epsilon\bar{F}(\epsilon) + (q - \epsilon)\bar{F}(q) = \epsilon(\bar{F}(\epsilon) - \bar{F}(q)) + q\bar{F}(q) > q\bar{F}(q)$, where the first inequality holds because $\bar{F}(x)$ is nonincreasing. Note that if $f(x)$ is IFR, the conditions in this lemma is satisfied. Q.E.D.

Lemma 8 *If the distribution of D is IFR, $\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} - q$ is strictly increasing convex in q.*

Proof of Lemma 8: Note $\partial[\frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)} - q]/\partial q = \frac{f(q)}{[\bar{F}(q)]^2}\mathbf{E_D}[min(q,D)] \geq 0$ and strictly increases in q. Therefore, $\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} - q$ is strictly increasing convex in q. Q.E.D.

Proof of Corollary 1: Note that $q^*_{bk} \geq \min_i\{\frac{k^i}{c^i}\} \geq q^*_n$. We have $\Pi_{bk}(q^*_{bk}) \geq \Pi_{bk}(q^*_n) = \Pi_n(q^*_n)$, i.e., the assembler is better off under bank finance than under no finance. (1) The profits of a borrowing supplier i under bank finance and no finance are $\pi^i_{bk}(q^*_{bk}) = c^i \cdot (1+r_f)(\frac{\mathbf{E_D}min(q^*_{bk},D)}{\bar{F}(q^*_{bk})} - q^*_{bk}) + (r_f - r^i_s)k^i$ and $\pi^i_n(q^*_n) = c^i \cdot (1+r^i_s)(\frac{\mathbf{E_D}min(q^*_n,D)}{\bar{F}(q^*_n)} - q^*_n)$, respectively. To compare $\pi^i_{bk}(q^*_{bk})$ with $\pi^i_n(q^*_n)$, note that $\frac{\mathbf{E_D}min(q,D)}{\bar{F}(q)} - q > 0$ (Lemma 7) and increases in q (Lemma 8). Given $q^*_{bk} \geq \min_i\{\frac{k^i}{c^i}\} \geq q^*_n$ and $r_f \geq r^i_s$, we have $\pi^i_{bk}(q^*_{bk}) \geq \pi^i_n(q^*_n)$. (2) For any non-borrowing supplier j, its profits under bank finance and no finance are, $\pi^j_{bk}(q^*_{bk}) = c^j \cdot (1+r^j_s)(\frac{\mathbf{E_D}min(q^*_{bk},D)}{\bar{F}(q^*_{bk})} - q^*_{bk})$ and $\pi^j_n(q^*_n) = c^j \cdot (1+r^j_s)(\frac{\mathbf{E_D}min(q^*_n,D)}{\bar{F}(q^*_n)} - q^*_n)$, respectively. Following Lemma 7 and Lemma 8, we have $\pi^j_{bk}(q^*_{bk}) \geq \pi^j_n(q^*_n)$.

We next show the equilibrium solution under bank finance is strictly Pareto better than that under no finance if $q^*_{bk} > \min_i\{\frac{k^i}{c^i}\}$. Note that when $q^*_{bk} > \min_i\{\frac{k^i}{c^i}\}$, $q^*_{bk} > q^*_n$. We have $\pi^i_{bk}(q^*_{bk}) > \pi^i_n(q^*_n)$ for any borrowing supplier i in part (1), and $\pi^j_{bk}(q^*_{bk}) > \pi^j_n(q^*_n)$ for any non-borrowing supplier j in part (2). Therefore, the result is true.Q.E.D.

Proof of Lemma 6: The proof follows the same argument for Lemma 2. Q.E.D.

Proof of Theorem 3: Rearranging the terms in $\Pi_{br}(r_b)$, we have $\Pi_{br}(r_b) = \sum_{i=1}^{j}[(c^iq - k^i) - \frac{c^i}{\bar{F}(q)} \cdot \mathbf{E}_D min(q,D)] \cdot (r_b - r^i_s) + (p - \frac{\sum_{i=1}^N c^i(1+r^i_s)}{\bar{F}(q)}) \cdot \mathbf{E}_D min(q,D) - \sum_{i=1}^{j} (r_a - r^i_s)(c^iq - k^i)$. Hence for any q, $\partial\Pi_{br}(r_b)/\partial r_b = \sum_{i=1}^{j}(c^iq - k^i) - \frac{\mathbf{E_D}min(q,D)}{\bar{F}(q)} \cdot \sum_{i=1}^{j} c^i < 0$, where the last inequality holds because $q - \frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} - \frac{k^i}{c^i} < 0$ (Lemma 7). Therefore, $\Pi_{br}(r_b)$ is decreasing in r_b for any q. Because $r_b \geq r^i_s, \forall i \in \{1,2,\cdots,N\}$, we have $r^*_b = \max_i\{r^i_s\}$. Q.E.D.

Proof of Corollary 2: Given $r^i = r_a, \forall i \in \{1,2,\cdots,N\}$, substituting $r^*_b = \max_i\{r^i_s\}$ into $\Pi_{br}(q, r^*_b)$, we have $\Pi_{br}(q, r^*_b) = \mathbf{E}_D\ (p - \frac{\Sigma_{i=1}^N c^i(1+r^i_s)}{\bar{F}(q)}) \cdot min(q,D)$, which is concave in q by Lemma 3. Hence q^*_{br} is uniquely solved from $p\bar{F}(q) = \sum_{i=1}^N c^i(1+r^i_s)(\frac{f(q)}{[\bar{F}(q)]^2}\int_0^q \bar{F}(x)dx + 1)$. Q.E.D.

Proof of the sensitivity results in buyer finance:

To prove q^*_{br} strictly decreases with c^i, we can show $\Pi_{br}(q, c^i) - \Pi_{br}(q, c^{i\prime})$ is strictly decreasing in q for $c^i > c^{i\prime}$. For a borrowing supplier i , $\Pi_{br}(q,c^i) - \Pi_{br}(q,c^{i\prime}) = (c^i - c^{i\prime}) \cdot [-r_aq - \frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)} + r_b(q - \frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)})]$, where in the square brackets the first term strictly decreases in q and both the second and third terms decrease in q by Lemma 8. Hence $\Pi_{br}(q,c^i) - \Pi_{br}(q,c^{i\prime})$ is strictly decreasing in q. For a non-borrowing supplier j, $\Pi_{br}(q,c^j) - \Pi_{br}(q,c^{j\prime}) = -(c^j - c^{j\prime}) \cdot \frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)}$, which strictly decreases in q because $\frac{\mathbf{E_D}[min(q,D)]}{\bar{F}(q)}$ strictly increases in q. To prove q^*_{br} strictly decreases in r_a, we can show that for $r_a^{(1)} > r_a^{(2)}$, $\Pi_{br}(q, r_a = r_a^{(1)}) - \Pi_{br}(q, r_a = r_a^{(2)}) = (r_a^{(2)} - r_a^{(1)}) \cdot \sum_{i=1}^{N} \delta(c^i q - k^i) \cdot (c^i q - k^i)$ is strictly decreasing in q because $\sum_{i=1}^{N} \delta(c^i q - k^i) \cdot (c^i q - k^i)$ is strictly increasing in q, $\forall q \in (\min_i\{\frac{k^i}{c^i}\}, \infty)$. Following a similar argument, we can show that q_{br}^* strictly increases in p. The optimal value $\Pi_{br}(q_{br}^*)$ increases in p, decreases in c^i, and decreases with r_a because $\Pi_{br}(q_{br})$ increases in p, decreases in c^i, and decreases with r_a for any q.

Proof of Corollary 3:

When $\max_i\{r_s^i\} \leq r_a$, following similar argument in part (2) of the proof for the sensitivity in bank finance, we can show that $\Pi_{br}(q_{br}^*)$ increases in k^i. When $\max_i\{r_s^i\} > r_a$, we introduce a special case: if all $r_s^i, i \in \{1, 2, \cdots, N\}$, are the same, $\Pi_{br} = (p - \frac{\Sigma_{i=1}^{N} c^i(1+r_s^i)}{\bar{F}(q)})\mathbf{E_D}[minD, q] + \Sigma_{i=1}^{N}(\max_i\{r_s^i\} - r_a)(c^i q - k^i)^+$, which decreases in k^i; hence, $\Pi_{br}(q_{br}^*)$ decreases in k^i. Q.E.D.

Proof of Corollary 4: The proof follows the same argument for Corollary 1.

Proof of Theorem 4: By Theorem 3, the optimal interest rate in buyer finance is $r_b^* = \max_i\{r_s^i\}$ for any given q. Define $h(q) = \Pi_{br}(q, r_b = r_b^*) - \Pi_{bk}(q)$. We have $h(q) = (r_f - \max_i\{r_s^i\})\ \frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)}\Sigma_{i=1}^{j} c^i - (r_a - \max_i\{r_s^i\})\Sigma_{i=1}^{j}(c^i q - k^i)$. If $r_a \leq r_f$, we have $h(q) \geq (r_f - \max_i\{r_s^i\})[\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)}\Sigma_{i=1}^{j} c^i - \Sigma_{i=1}^{j}(c^i q - k^i)]$. Note that $r_f > r_s^i$ and $\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)}\Sigma_{i=1}^{j} c^i - \Sigma_{i=1}^{j}(c^i q - k^i) = (\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} - q)\Sigma_{i=1}^{j} c^i + \Sigma_{i=1}^{j} k^i > 0$ (using Lemma 7) imply $h(q) > 0$, $\forall q \geq 0$. Therefore, we have $\Pi_{br}(q_{br}^*) \geq \Pi_{br}(q_{bk}^*) > \Pi_{bk}(q_{bk}^*)$ when $r_a \leq r_f$. Note $\Pi_{bk}(q_{bk}^*)$ is not related to r_a and $\Pi_{br}(q_{br}^*)$ decreases in r_a because $\frac{\partial \Pi_{br}(q_{br}, r_a)}{\partial r_a} = -\Sigma_{i=1}^{j}(c^i q - k^i)^+ \leq 0$. We define $\bar{r}$ as the value of r_a such that $\Pi_{br}(q_{br}^*) = \Pi_{bk}(q_{bk}^*)$. Next if we show that there is a sufficient large r_a such that $\Pi_{br}(q_{br}^*) \leq \Pi_{bk}(q_{bk}^*)$, then $\bar{r}$ exists. Moreover, $\bar{r} \geq r_f$ because we have shown that $\Pi_{br}(q_{br}^*) > \Pi_{bk}(q_{bk}^*)$ when $r_a \leq r_f$.

Let $r_a^0 = \frac{p}{c^1} + max_i\{r_s^i\}$. Suppose when $r_a = r_a^0$, $q_{br}^* = \min_i\{k^i/c^i\} + \Delta^j$ for some $\Delta^j \geq 0$ and suppliers $i = 1, 2, \cdots, j$, borrow. Recall Eq. (8), we have $\Pi_{br}(q = \min_i\{k^i/c^i\} + \Delta^j) - \Pi_{br}(q = \min_i\{k^i/c^i\}) \leq p\Delta^j + (r_b - r_a)\Delta^j \Sigma_{i=1}^{j} \cdot c^i = \Delta^j(p + (max_i\{r_s^i\} - (\frac{p}{c^1} + max_i\{r_s^i\})) \cdot \Sigma_{i=1}^{j} \cdot c^i) = \Delta^j(p \cdot (1 - \frac{\Sigma_{i=1}^{j} c^i}{c^1})) \leq 0$. Therefore, $q_{br}^* \leq \min_i\{k^i/c^i\}$, and thus we have $\Pi_{br}(q_{br}^*) = \Pi_n(q_n^*) \leq \Pi_{bk}(q_{bk}^*)$, where the last inequality follows from Corollary 1. Q.E.D.

Proof of Corollary 5:

Follow the proof of Theorem 4, there exists $r_a = \bar{\bar{r}}$ such that $\Pi_{br}(q_{bk}^*)|_{r_a=\bar{\bar{r}}} = \Pi_{bk}(q_{bk}^*)$, where $\bar{\bar{r}} = \frac{(r_f - \max_i\{r_s^i\}) \cdot \mathbf{E_D} min(q_{bk}^*, D)}{\bar{F}(q_{bk}^*)(q_{bk}^* - \frac{\Sigma_J k^i}{\Sigma_J c^i})} + \max_i\{r_s^i\}$, and $J = \{i : c^i q_{bk}^* > k^i\}$. Substituting $r = \bar{\bar{r}}$ and $q = q_{bk}^*$ into $\Pi_{br}(q)$, we have $\Pi_{br}^*(q_{br}^*)|_{r_a=\bar{\bar{r}}} \geq \Pi_{br}(q_{bk}^*)|_{r_a=\bar{\bar{r}}} = \Pi_{bk}(q_{bk}^*)$. Because $\Pi_{br}^*(q_{br}^*)$ decreases in r_a (proved in Theorem 4), recall the definition of $\bar{r}$, we then have $\bar{\bar{r}} \leq \bar{r}$.

We next show $\bar{\bar{r}} = \bar{r}$ if $\max_i\{r_s^i\} = r_f$. Following the proof of Theorem 4, $\bar{r} = r_f$ if $r_f = \max_i\{r_s^i\}$. Substitute $r_f = \max_i\{r_s^i\}$ into the expression of $\bar{\bar{r}}$, we have $\bar{\bar{r}} = r_f$. Therefore, we have $\bar{\bar{r}} = \bar{r}$. Q.E.D.

Proof of Corollary 6: When $\frac{k^i}{c^i} = \frac{k^j}{c^j}$ $\forall i, j = 1, 2, \cdots, N$ and $\frac{k^i}{c^i} < q_{bk}^*$, the equilibrium quantity q_{bk}^* and q_{br}^* satisfy the following first-order conditions, respectively,

$$\frac{p\bar{F}(q_{bk}^*)}{\Sigma_{i=1}^N c^i} = (1 + r_f) + \frac{f(q_{bk}^*)(1 + r_f)\mathbf{E_D}min(q_{bk}^*, D)}{[\bar{F}(q_{bk}^*)]^2}, \tag{A-1}$$

$$\frac{p\bar{F}(q_{br}^*)}{\Sigma_{i=1}^N c^i} = (1 + r_a) + \frac{f(q_{br}^*)(1 + \max_i\{r_s^i\})\mathbf{E_D}min(q_{br}^*, D)}{[\bar{F}(q_{br}^*)]^2}. \tag{A-2}$$

Since $\Pi_{br}(q_{br})$ decreases with r_a, $\Pi_{br}(q_{br}^*)$ decreases with r_a. Let $r_a = \bar{r}$ such that $\Pi_{br}(q_{br}^*) = \Pi_{bk}(q_{bk}^*)$, we have $\bar{r} = \frac{[p - \frac{\Sigma_{i=1}^N c^i(1+\max_i\{r_s^i\})}{\bar{F}(q_{br}^*)}]\mathbf{E_D}min(q_{br}^*, D) - \Pi_{bk}(q_{bk}^*)}{\Sigma_{i=1}^N(c^i q_{br}^* - k^i)} + \max_i\{r_s^i\}$.

1. Suppose $\Sigma_{i=1}^N k^i = K$, by Theorem 4, there exists $r_a = \bar{r} > r_f$ such that $\Pi_{br}(q = q_{br}^*, r_a = \bar{r}, \Sigma_{i=1}^N k^i = K) = \Pi_{bk}(q_{bk}^*)$. By Eq. (A-1) and Eq. (A-2), both q_{bk}^* and q_{br}^* are independent of $k^i, \forall i \in \{1, 2, \cdots, N\}$. Then $\forall K' > K$, there is $\Pi_{br}(q = q_{br}^*, r_a = \bar{r}, \Sigma_{i=1}^N k^i = K') - \Pi_{bk}(q_{bk}^*) = (\bar{r} - \max_i\{r_s^i\})(K' - K) > 0$. Since $\Pi_{br}(q_{br}^*)$ decreases with r_a, there exists $\bar{r}' > \bar{r}$ such that $\Pi_{br}(q = {q_{br}^*}', r_a = \bar{r}', \Sigma_{i=1}^N k^i = K') - \Pi_{bk}(q_{bk}^*) = 0$. Therefore, $\bar{r}$ increases in $\Sigma_{i=1}^N k^i$.

2. Similarly to item 1, we can prove that $\bar{r}$ decreases in $\max_i\{r_s^i\}$.

3. By Eq. (A-1) and (A-2), both q_{bk}^* and q_{br}^* are independent of $k^i, \forall i \in \{1, 2, \cdots, N\}$. Therefore, $\Pi_{bk}(q_{bk}^*) = [p - \frac{\Sigma_{i=1}^N c^i(1+r_f)}{\bar{F}(q_{bk}^*)}]\mathbf{E_D}min(q_{bk}^*, D)$ is independent of k^i, while $\Pi_{br}(q_{br}^*) = [p - \frac{\Sigma_{i=1}^N c^i(1+\max_i\{r_s^i\})}{\bar{F}(q_{br}^*)}]\mathbf{E_D}min(q_{br}^*, D) + (\max_i\{r_s^i\} - r_a)\Sigma_{i=1}^N(c^i q_{br}^* - k^i)$, which increases in k^i if $r_a \geq \max_i\{r_s^i\}$. Hence, $\Pi_{br}(q_{br}^*) - \Pi_{bk}(q_{bk}^*)$ increases in $\Sigma_{i=1}^N k^i$ if $r_a \geq \max_i\{r_s^i\}$, decreases in $\Sigma_{i=1}^N k^i$ if $r_a < \max_i\{r_s^i\}$. Q.E.D

Proof of Theorem 5:

1. Given $r_s^1 = r_s^2 = r_s, c^1 = c^2 = \frac{C}{2}$, there is $\Pi_{br}(q, \theta) = (p - \frac{C(1+r_s) + \frac{C(r_f - r_s)}{2}[\delta(\frac{C}{2}q - \frac{\theta K}{1+\theta}) + \delta(\frac{C}{2}q - \frac{K}{1+\theta})]}{\bar{F}(q)}) \cdot \mathbf{E_D}[min(q, D)]$, which depends on θ only because the term $\delta(\frac{C}{2}q - \frac{\theta K}{1+\theta}) + \delta(\frac{C}{2}q - \frac{K}{1+\theta})$ depends on θ. Let $I(q, \theta) = \delta(\frac{C}{2}q - \frac{\theta K}{1+\theta}) + \delta(\frac{C}{2}q - \frac{K}{1+\theta})$, we focus on two scenarios: (1) $K < Cq_{bk}^*$. Since for $\forall \theta \in (0, 1]$, there is $\delta(\frac{C}{2}q_{bk}^* - \frac{\theta K}{1+\theta}) = 1$ and $\delta(\frac{C}{2}q_{bk}^* - \frac{K}{1+\theta})$ increases in θ, thus $I(q_{bk}^*, \theta)$ increases with θ. Therefore, $\Pi_{br}(q_{bk}^*, \theta)$ decreases in θ when $K < Cq_{bk}^*$. (2)$K \geq Cq_{bk}^*$. In this scenario, $\forall \theta \in (0, 1]$, there is $\delta(\frac{C}{2}q_{bk}^* - \frac{\theta K}{1+\theta})$ decreases in θ, while $\delta(\frac{C}{2}q_{bk}^* - \frac{K}{1+\theta}) = 0$, hence we have $I(q_{bk}^*, \theta)$ decreases with θ, and $\Pi_{br}(q_{bk}^*, \theta)$ increases in θ when $K \geq Cq_{bk}^*$.

2. Note that we can prove $\Pi^*_{br}(q^*_{br}, \theta)$ increases (decreases) in θ if we have $\Pi_{br}(q, \theta)$ increases (decreases) in θ, $\forall q$. Substitute $r^*_b = \max_i\{r^i_s\}$ into $\Pi_{br}(q)$, then $\Pi_{br}(q, \theta) = (p - \frac{\Sigma^2_{i=1} c^i(1+r^i_s)}{\bar{F}(q)}) \cdot \mathbf{E_D}[min(q, D)] + (\max_i\{r^i_s\} - r_a) \cdot ((c^1 q - \frac{\theta K}{1+\theta})^+ + (c^2 q - \frac{K}{1+\theta})^+)$, where $\theta \in (0, \frac{c^1}{c^2}]$. We next only consider $c^1 < c^2$ and $\theta \in (0, \frac{c^1}{c^2}]$, because if $c^2 < c^1$ we can simply switch the supplier's index. Let $B(q, \theta) = (c^1 q - \frac{\theta K}{1+\theta})^+ + (c^2 q - \frac{K}{1+\theta})^+$, then $\Pi_{br}(q, \theta) = (p - \frac{\Sigma^2_{i=1} c^i(1+r^i_s)}{\bar{F}(q)}) \cdot \mathbf{E_D}[min(q, D)] + (\max_i\{r^i_s\} - r_a) \cdot B(q, \theta)$. We next show $B(q, \theta)$ is monotone in θ for given q. With $\theta \in (0, \frac{c^1}{c^2}]$, $B(q, \theta)$ lies in one of the following three cases: (a) $B(q, \theta) = 0$; (b) $B(q, \theta) = c^1 q - \frac{\theta K}{1+\theta}$; (c) $B(q, \theta) = (c^1 + c^2)q - K$; For any $0 < \theta_1 < \theta_2 \leq \frac{c^1}{c^2}$, it is then easy to verify that $B(q, \theta_2) - B(q, \theta_1) \leq 0$, hence $B(q, \theta)$ deceases in θ . Therefore, $\Pi_{br}(q, \theta)$ increases in θ if $r_a \geq \max_i r^i_s$, but decreases in θ if $r_a < \max_i r^i_s$. Q.E.D.

Proof of Corollary 7: Let $C = \Sigma^N_{i=1} c^i$. When $\Sigma^N_{i=1} k^i = 0$, $\bar{\bar{r}} = \frac{(r_f - \max_i\{r^i_s\}) \cdot \mathbf{E_D} min(q^*_{bk}, D)}{\bar{F}(q^*_{bk}) q^*_{bk}} + \max_i\{r^i_s\}$. $\frac{\partial^2 \Pi_{bk}}{\partial q \partial C} = -(1 + r_f)[1 + \frac{f(q) \int_0^q \bar{F}(x)dx}{(\bar{F}(q))^2}] < 0$, hence q^*_{bk} decreases in C. It turns out that if we can show $G(q) = \frac{\int_0^q \bar{F}(x)dx}{q\bar{F}(q)}$ increases in q, then we have $\bar{\bar{r}}$ decreases in C. $G^{'}(q) = \frac{\partial G(q)}{\partial q} = \frac{q\bar{F}(q) - (1 - \frac{qf(q)}{\bar{F}(q)}) \int_0^q \bar{F}(x)dx}{q^2 \cdot \bar{F}(q)} = \frac{q\bar{F}(q) + (z(q) - 1) \int_0^q \bar{F}(x)dx}{q^2 \cdot \bar{F}(q)}$, where $z(q) = \frac{qf(q)}{\bar{F}(q)}$. Note $z^{'}(q) > 0$ because demand distribution follows IFR property, let $B(q) = q\bar{F}(q) + (z(q) - 1) \int_0^q \bar{F}(x)dx$, then $B^{'}(q) = z^{'}(q) \int_0^q \bar{F}(x)dx > 0$. Because $\forall q \in (-\infty, 0), B(q) = 0$ and $B(q)$ is continuous in q, then we have $B(q) \geq 0$ and thus $G(q)$ is increasing in q. Q.E.D

Proof of Theorem 6

1. When $N = 2$, $k^1 = k^2 = K/2$, $r^1_s = r^2_s = r_s$ and $\eta = \frac{c^1}{c^2} \in [0, 1]$, we have $\Pi_{bk}(q, \eta) = (p - \frac{C[(1+r_s) + \frac{1}{1+\eta}(r_f - r_s) \cdot \delta(\frac{C}{1+\eta} q - K/2) + \frac{\eta}{1+\eta}(r_f - r_s) \cdot \delta(\frac{\eta C}{1+\eta} q - K/2)]}{\bar{F}(q)}) \cdot \mathbf{E_D}[min(q, D)]$. Therefore, $\Pi_{bk}(q, \eta)$ depends on only η, because the term $\frac{\delta(\frac{C}{1+\eta} q - K/2)}{1+\eta} + \frac{\eta \delta(\frac{\eta C}{1+\eta} q - K/2)}{1+\eta}$ depends on η. Let $M(\eta) = \frac{\delta(\frac{C}{1+\eta} q - K/2)}{1+\eta} + \frac{\eta \delta(\frac{\eta C}{1+\eta} q - K/2)}{1+\eta}$. We then focus on two cases: (1) $K \geq C q^*_{bk}$. In this case, $M(\eta)$ decreases with η, and achieves its minimum at $\eta = 1$ because $\frac{k^1}{c^1} = \frac{k^2}{c^2} \geq q^*_{bk}$ and $\{\delta(\frac{C}{1+\eta} q - K/2)\}|_{\eta=1} = \{\delta(\frac{\eta C}{1+\eta} q - K/2)\}|_{\eta=1} = 0, \forall q \in [0, q^*_{bk}]$. Hence for $\eta \in [0, 1]$ and $\forall q \in [0, q^*_{bk}]$, $\Pi_{bk}(q, \eta)$ increases in η. (2) $K < C q^*_{bk}$. In this case, when $q = q^*_{bk}$, $M(\eta)$ achieves its maximum at $\eta = 1$ because $\frac{k^1}{c^1} = \frac{k^2}{c^2} < q^*_{bk}$ and $\{\delta(\frac{C}{1+\eta} q^*_{bk} - K/2)\}|_{\eta=1} = \{\delta(\frac{\eta C}{1+\eta} q^*_{bk} - K/2)\}|_{\eta=1} = 1$. Since $\delta(\frac{C}{1+\eta} q^*_{bk} - K/2) \geq \delta(\frac{\eta C}{1+\eta} q^*_{bk} - K/2)$, there exists $0 < \bar{\eta} \leq 1$ such that $\delta(\frac{C}{1+\eta} q^*_{bk} - K/2) = 1, \delta(\frac{\eta C}{1+\eta} q^*_{bk} - K/2) = 0$ when $\eta \in (0, \eta_1]$ and $\delta(\frac{C}{1+\eta} q^*_{bk} - K/2) = 1, \delta(\frac{\eta C}{1+\eta} q^*_{bk} - K/2) = 1$ for $\eta \in (\bar{\eta}, 1]$. Therefore, $M(\eta) = \frac{1}{1+\eta}$ for $\eta \in (0, \eta_1]$, and $M(\eta) = 1$ for $\eta \in (\bar{\eta}, 1]$. Hence $M(\eta)$ first decreases and then increases in η, while $\Pi_{bk}(q^*_{bk}, \eta)$ first increases and then decreases in η.

2. The proof of part 2 resembles the proof of part 2 of Theorem 5 and thus omitted. Q.E.D

Proof of Theorem 7: The proof of Theorem 8 (provided in the latter part of this appendix) shows that $q^*_{br} \geq q^*_{bk}$ if and only if $r_a \leq \hat{r}$, where $\hat{r} \geq r_f$. We next prove non-borrowing suppliers prefer buyer finance if and only if $r_a \leq \hat{r}$.

Note that $q^*_{bk} = \bar{F}^{-1}(\frac{c^i \cdot (1+r^i_s+(r_f-r^i_s)\cdot\delta(c^i q^*_{bk}-k^i))}{w^*_i})$, and $q^*_{br} = \bar{F}^{-1}(\frac{c^i \cdot (1+r^i_s+(r^*_b-r^i_s)\cdot\delta(c^i q^*_{br}-k^i))}{w^*_i})$. For a non-borrowing supplier i, $\delta(c^i q^*_{br} - k^i) = 0$ and $\delta(c^i q^*_{bk} - k^i) = 0$. Its profits in bank and buyer financing are $\pi^i_{bk}(q^*_{bk}) = \mathbf{E}_D[w^{i*}_{bk} \cdot min(q^*_{bk}, D) - c^i(1 + r^i_s)q^*_{bk}] = c^i \cdot (1 + r^i_s)(\frac{\mathbf{E_D}(q^*_{bk},D)}{\bar{F}(q^*_{bk})} - q^*_{bk})$, and $\pi^i_{br}(q^*_{br}) = \mathbf{E}_D[w^{i*}_{br} \cdot min(q^*_{br}, D) - c^i(1+r^i_s)q^*_{br}] = c^i \cdot (1+r^i_s)(\frac{\mathbf{E}_D min(q^*_{br},D)}{\bar{F}(q^*_{br})} - q^*_{br})$, respectively. Because $\frac{\mathbf{E}_D min(q,D)}{\bar{F}(q)} - q$ is strictly increasing in q (Lemma 8) and Theorem 8, we have $\pi^i_{bk}(q^*_{bk}) \leq \pi^i_{br}(q^*_{br})$ if and only if $r_a \leq \hat{r}$.

For a borrowing supplier i, $\pi^i_{bk}(q^*_{bk}) = c^i \cdot (1+r_f)(\frac{\mathbf{E_D}(q^*_{bk},D)}{\bar{F}(q^*_{bk})} - q^*_{bk}) + (r_f - r^i_s)k^i$ is independent of r_a because q^*_{bk} does not depend on r_a. $\pi^i_{br}(q^*_{br}) = c^i \cdot (1+\max_i\{r^i_s\})(\frac{\mathbf{E_D}(q^*_{br},D)}{\bar{F}(q^*_{br})} - q^*_{br}) + (\max_i\{r^i_s\} - r^i_s)k^i$ is decreasing in r_a because of Lemma 8 and $q^*_{br} \downarrow r_a$ (by the proof for sensitivity results in buyer finance). Let $\tilde{r}$ be the value of r_a such that $\pi^i_{bk}(q^*_{bk}) = \pi^i_{br}(q^*_{br})$. If $\tilde{r}$ exists, we have that if $r_a \leq \tilde{r}$, $\pi^i_{bk}(q^*_{bk}) \leq \pi^i_{br}(q^*_{br})$. Furthermore, we have $\tilde{r} < \hat{r}$. This is because when $r_a = \hat{r}$, we have $q^*_{bk} = q^*_{br}$ (By Theorem 8). From the above expression of $\pi^i_{bk}(q)$ and $\pi^i_{br}(q)$ for borrowing supplier i, we have $\pi^i_{bk}(q^*_{bk}) \geq \pi^i_{br}(q^*_{br})$ because of $r_f \geq \max_i\{r^i_s\}$.

We next show the existence of $\tilde{r}$. Note that $\pi_{br}(q^*_{br})$ is convex increasing in q^*_{br} because of Lemma 8 and $q^*_{br} \downarrow r_a$. If r_a increases such that $r_a > \hat{r}$, then $q^*_{br} < q^*_{bk}$, and we have $\pi^i_{br}(q^*_{br}) < \pi^i_{bk}(q^*_{bk})$. We next show that $\pi^i_{br}(q^*_{br}) > \pi^i_{bk}(q^*_{bk})$ if r_a goes below a value (possible a negative value). Let $\Pi'_{br}(q) = M(q) + \Sigma^N_{i=1} c^i \cdot \delta^i(\max_i\{r^i_s\} - r_a)$, where $M(q) = p\bar{F}(q) - \Sigma^N_{i=1} c^i(1 + r^i_s + (\max_i\{r^i_s\} - r^i_s) \cdot \delta^i) \cdot (1 + \frac{f(q)\mathbf{E}_D \min(q,D)}{(\bar{F}(q))^2})$ and $\delta^i = \delta(c^i q - k^i)$. Let $q = \bar{q}_{br}$ be the solution to $c^i \cdot (1 + \max_i\{r^i_s\})(\frac{\mathbf{E_D}(q,D)}{\bar{F}(q)} - q) + (\max_i\{r^i_s\} - r^i_s)k^i = \pi^i_{bk}(q^*_{bk})$. Substitute $q = \bar{q}_{br}$ into $\Pi'_{br}(q)$, we have, if $r_a < \frac{M(\bar{q}_{br}) + \Sigma^N_{i=1} c^i \cdot \delta^i \max_i\{r^i_s\}}{\Sigma^N_{i=1} c^i \cdot \delta^i}$, then $\Pi'_{br}(q)|_{q=\bar{q}_{br}} > 0$ and hence $q^*_{br} > \bar{q}_{br}$. By the definition of $\bar{q}_{br}$ and the above expression of $\pi^i_{br}(q^*_{br})$, we have $\pi^i_{br}(q^*_{br}) > [c^i \cdot (1 + \max_i\{r^i_s\})(\frac{\mathbf{E_D}(q,D)}{\bar{F}(q)} - q) + (\max_i\{r^i_s\} - r^i_s)k^i]|_{q=\bar{q}_{br}} = \pi^i_{bk}(q^*_{bk})$. Therefore, $\tilde{r}$ exists. When $\tilde{r}$ is negative, it means the borrowing always prefer bank finance when r_a is non-negative. Q.E.D.

Proof of Theorem 8: Consider $h(q) = \Pi_{br}(q) - \Pi_{bk}(q)$ as defined in the proof for Theorem 4. Note that $h(q) > 0$, $\forall q \geq 0$ if $r_a \leq r_f$ and $\sum_i k^i \geq 0$. Moreover, if $r_a \leq r_f$, $h(q)$ is also increasing in q because we have $h'(q) = [(r_f - \max_i\{r^i_s\}) \cdot (\frac{(f(q) \cdot \mathbf{E}_D\ min(q,D)}{(\bar{F}(q))^2} + 1) - (r_a - \max_i\{r^i_s\})]\Sigma_i c^i \ > 0$. If we show q^*_{br} cannot be in $[0, q^*_{bk})$, then there must be $q^*_{br} \geq q^*_{bk}$. By the optimality of q^*_{bk},

$\Pi_{bk}(q) \leq \Pi_{bk}(q^*_{bk})$ for any $q \in [0, q^*_{bk}]$. Furthermore, by the definition of $h(q)$, $\Pi_{br}(q) = \Pi_{bk}(q) + h(q) < \Pi_{bk}(q^*_{bk}) + h(q^*_{bk}) = \Pi_{br}(q^*_{bk})$ for any $q \in [0, q^*_{bk})$. That is, $\Pi_{br}(q) < \Pi_{br}(q^*_{bk})$ for any $q \in [0, q^*_{bk})$, hence $q^*_{br} \geq q^*_{bk}$. Furthermore, $\Pi'_{br}(q)|_{q=q^*_{bk}} = \Pi'_{bk}(q)|_{q=q^*_{bk}} + h'(q)|_{q=q^*_{bk}} = h'(q^*_{bk}) > 0$. Therefore, we must have $q^*_{br} > q^*_{bk}$ when $r_a \leq r_f$. Since q^*_{br} strictly decreases in r_a, we thus proved there exists $\hat{r} \geq r_f$ such that $q^*_{br} > q^*_{bk}$ if $r_a < \hat{r}$.

We next show that there exists a sufficient large r_a such that $q^*_{br} \leq q^*_{bk}$, which has been showed in the proof for Theorem 4, where we show $\exists r_a = \frac{p}{c^1} + max_i\{r^i_s\}$ such that $q^*_{br} = q^*_n \leq q^*_{bk}$.

For a centralized supply chain, the equilibrium production quantity $q^{*cen} = \bar{F}^{-1}(\frac{\Sigma c^i(1+r^i_s)}{p})$, we thus have $q^*_{br} = \bar{F}^{-1}(\frac{\Sigma^N_{i=1} c^i \cdot (1+r^i_s+(r^*_b - r^i_s)\cdot \delta(c^i q^*_{br} - k^i))}{\Sigma^N_{i=1} w^*_i}) \leq \bar{F}^{-1}(\frac{\Sigma^N_{i=1} c^i(1+r^i_s)}{\Sigma^N_{i=1} w^*_i}) \leq q^{*cen}$, where the last inequality holds because $\Sigma^N_{i=1} w^*_i < p$. Similarly, one can show $q^*_{bk} \leq q^{*cen}$. Q.E.D.

Proof of Corollary 9: When $\frac{k^i}{c^i} = \frac{k^j}{c^j}$ $\forall i, j = 1, 2, \cdots, N$ and $\frac{k^i}{c^i} < q^*_{bk}$, the equilibrium quantity q^*_{bk} and q^*_{br} , respectively, are solved from Eq. (A-1) and Eq. (A-2). By Eq. (A-2), q^*_{br} decreases in r_a. Let $\hat{r} = r_f + \frac{f(q^*_{bk})(r_f - \max_i\{r^i_s\})\cdot \mathbf{E_D} min(q^*_{bk}, D)}{[\bar{F}(q^*_{bk})]^2}$ and substitute $r_a = \hat{r}$ into Eq. (A-2). We then have $q^*_{br} = q^*_{bk}$. Since q^*_{br} is monotonically decreasing in r_a, we have $q^*_{br} \geq q^*_{bk}$ when $r_a \leq \hat{r} = r_f + \frac{f(q^*_{bk})(r_f - \max_i\{r^i_s\})\cdot \mathbf{E_D} min(q^*_{bk}, D)}{[\bar{F}(q^*_{bk})]^2}$.

1. From the above argument, when $\frac{k^i}{c^i} = \frac{k^j}{c^j}$, $\forall i, j = 1, 2, \cdots, N$, and $\frac{k^i}{c^i} < q^*_{bk}$, the equilibrium quantity in bank finance q^*_{bk} does not depend on $k^i, \forall i \in \{1, 2, \cdots, N\}$, hence, $\hat{r} = r_f + \frac{f(q^*_{bk})(r_f - \max_i\{r^i_s\})\cdot \mathbf{E_D} min(q^*_{bk}, D)}{[\bar{F}(q^*_{bk})]^2}$, which is also independent of k^i and $\Sigma^N_{i=1} k^i$.

2. We first show $\hat{r} = r_f + \frac{f(q^*_{bk})(r_f - \max_i\{r^i_s\})\cdot \mathbf{E_D} min(q^*_{bk}, D)}{[\bar{F}(q^*_{bk})]^2}$ is increasing in q^*_{bk}, which is true since both $\frac{f(q^*_{bk})}{\bar{F}(q^*_{bk})}$ and $\frac{\mathbf{E_D} min(q^*_{bk}, D)}{\bar{F}(q^*_{bk})}$ are increasing in q^*_{bk}. Note that $\hat{r}$ depends on $\Sigma^N_{i=1} c^i$ only because q^*_{bk} depends on $\Sigma^N_{i=1} c^i$, and by Eq. (A-1), q^*_{bk} decreases in $\Sigma^N_{i=1} c^i$. Therefore, $\hat{r}$ decreases in $\Sigma^N_{i=1} c^i$.

3. $\hat{r} = r_f + \frac{f(q^*_{bk})(r_f - \max_i\{r^i_s\})\cdot \mathbf{E_D} min(q^*_{bk}, D)}{[\bar{F}(q^*_{bk})]^2}$ decreases in $\max_i\{r^i_s\}$ because q^*_{bk} is independent of $\max_i\{r^i_s\}$. Q.E.D.