

Online Appendix

“Cross-Cluster Referrals and Subsidy Schemes in Multi-Cluster Healthcare Systems”

Liming Liu, Weixing Shang, Xiuli Chao, Shaohui Zheng

The Online Appendix contains all the proofs of the lemmas and propositions and presents additional technical results and discussion for the paper “Cross-Cluster Referrals and Subsidy Schemes in Multi-Cluster Healthcare Systems.”

PROOF OF PROPOSITION 1. Note that the patient incurs costs with actions A , S , and O as $h_i(x_i)$, $c_i + h_j(x_j)$, and r_i , respectively, where $h_i(x_i)$ is defined by Equation (1). The results can be obtained by comparing the three costs and examining the properties of the indifference curve for acceptance and switching. $\square$

To show Lemma 1 and Proposition 2, we need the following lemma.

Lemma A1. *In value iteration with (5), we have the following properties:*

- $P1'$. $V_n(\mathbf{x} + \mathbf{e}_i) \geq V_n(\mathbf{x})$;
- $P2'$. $V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x}) \geq V_n(\mathbf{x}) - V_n(\mathbf{x} - \mathbf{e}_i)$, if $\mathbf{x} \geq \mathbf{e}_i$;
- $P3'$. $V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - V_n(\mathbf{x} + \mathbf{e}_j) \geq V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x})$;
- $P4'$. $V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x}) \geq V_n(\mathbf{x} + \mathbf{e}_j) - V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j)$, if $\mathbf{x} \geq \mathbf{e}_i$.

PROOF OF LEMMA A1. These properties are true for $n = 0$ because $V_0(\mathbf{x}) = 0$ for all $\mathbf{x}$. Suppose that the function $V_n(\cdot)$ satisfies properties $P1'$ to $P4'$, by induction it remains to show that the function $V_{n+1}(\cdot)$ also satisfies these properties. From Equation (5), $V_{n+1}(\mathbf{x})$ is a linear combination of x_k , $A_k V_n(\mathbf{x})$, and $D_k V_n(\mathbf{x})$. It is clear that x_k satisfies $P1'$ to $P4'$, so does $D_k V_n(\mathbf{x})$ because of $P1'$ for $V_n(\cdot)$. It suffices to show that $P1'$ to $P4'$ are preserved under the operator A_k . It is easy to check from the definition that $A_k V_n(\mathbf{x} + \mathbf{e}_i) \geq A_k V_n(\mathbf{x})$, i.e., $P1'$ is preserved under A_k . In the following, we show that $P2'$ to $P4'$ are preserved under A_k .

We verify $P2'$ first. We only give the proof for $k = i$, and the proof for the case when $k = j$ is similar. We remain to show

$$\begin{aligned} & \min\{V_n(\mathbf{x} + 2\mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j), r_i + V_n(\mathbf{x} + \mathbf{e}_i)\} + \min\{V_n(\mathbf{x}), c_i + V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j), r_i + V_n(\mathbf{x} - \mathbf{e}_i)\} \\ & \geq \min\{V_n(\mathbf{x} + \mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_j), r_i + V_n(\mathbf{x})\} + \min\{V_n(\mathbf{x} + \mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_j), r_i + V_n(\mathbf{x})\}. \end{aligned}$$

Let LHS_{pq}/RHS_{pq} denote the q th term in the p th minimization on the left hand side (LHS)/right hand side (RHS) of the inequality, with $p = 1, 2$, and $q = 1, 2, 3$. Based on which terms are the minimizing terms on the LHS, we have nine possibilities. When $LHS = LHS_{1q} + LHS_{2q}$ for $q = 1, 2, 3$, it is clear that $LHS = LHS_{1q} + LHS_{2q} \geq RHS_{1q} + RHS_{2q} \geq RHS$. To verify $LHS \geq RHS$ in the other six cases, we need the following inequalities:

$$\begin{aligned} LHS_{11} + LHS_{22} & \geq RHS_{11} + RHS_{22}, & LHS_{11} + LHS_{23} & \geq RHS_{11} + RHS_{23}, \\ LHS_{12} + LHS_{21} & \geq RHS_{12} + RHS_{21}, & LHS_{12} + LHS_{23} & \geq RHS_{12} + RHS_{23}, \\ LHS_{13} + LHS_{21} & \geq RHS_{13} + RHS_{21}, & LHS_{13} + LHS_{22} & \geq RHS_{13} + RHS_{22}. \end{aligned}$$

The first inequality is true because

$$\begin{aligned} V_n(\mathbf{x} + 2\mathbf{e}_i) - V_n(\mathbf{x} + \mathbf{e}_j) & = [V_n(\mathbf{x} + 2\mathbf{e}_i) - V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j)] + [V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - V_n(\mathbf{x} + \mathbf{e}_j)] \\ & \geq [V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x} + \mathbf{e}_j)] + [V_n(\mathbf{x} + \mathbf{e}_j) - V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j)] & \text{(by } P2', P4') \\ & = V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j). \end{aligned}$$

The second inequality is from $P2'$ (twice), while the third one is from $P3'$. The fourth inequality is true because

$$\begin{aligned} V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - V_n(\mathbf{x} + \mathbf{e}_j) & \geq V_n(\mathbf{x} + \mathbf{e}_j) - V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) & \text{(by } P2') \\ & \geq V_n(\mathbf{x}) - V_n(\mathbf{x} - \mathbf{e}_i). & \text{(by } P3') \end{aligned}$$

The fifth inequality is trivial because it reduces to an equality. The last inequality is from $P4'$.

Next, we consider $P3'$. The subscripts i and j play the same role in the supermodularity property $P3'$, and in the following, we

assume $k = i$ without loss of generality. The statement that $P3'$ is preserved under A_k is equivalent to:

$$\min\{V_n(\mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j), c_i + V_n(\mathbf{x} + \mathbf{e}_i + 2\mathbf{e}_j), r_i\} + V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) + \min\{V_n(\mathbf{x} + \mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_j), r_i + V_n(\mathbf{x})\} \\ \geq \min\{V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j), c_i + V_n(\mathbf{x} + 2\mathbf{e}_j), r_i + V_n(\mathbf{x} + \mathbf{e}_j)\} + \min\{V_n(\mathbf{x} + 2\mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j), r_i + V_n(\mathbf{x} + \mathbf{e}_i)\}.$$

Use the same discussions and notations as that in the proof of $P2'$, we remain to show $LHS_{1q_1} + LHS_{2q_2} \geq RHS$ for all $q_1 \neq q_2$, which is implied by the following inequalities:

$$\begin{aligned} LHS_{11} + LHS_{22} &\geq RHS_{11} + RHS_{22}, & LHS_{11} + LHS_{23} &\geq RHS_{11} + RHS_{23}, \\ LHS_{12} + LHS_{21} &\geq RHS_{11} + RHS_{22}, & LHS_{12} + LHS_{23} &\geq RHS_{12} + RHS_{23}, \\ LHS_{13} + LHS_{21} &\geq RHS_{11} + RHS_{23}, & LHS_{13} + LHS_{22} &\geq RHS_{13} + RHS_{22}. \end{aligned}$$

The first and third inequalities are from $P2'$. The second inequality is true because

$$\begin{aligned} V_n(\mathbf{x} + 2\mathbf{e}_i + \mathbf{e}_j) - V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) &\geq V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - V_n(\mathbf{x} + \mathbf{e}_j) && \text{(by } P2') \\ &\geq V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x}). && \text{(by } P3') \end{aligned}$$

The fourth inequality is true because

$$\begin{aligned} V_n(\mathbf{x} + \mathbf{e}_i + 2\mathbf{e}_j) - V_n(\mathbf{x} + 2\mathbf{e}_j) &\geq V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) - V_n(\mathbf{x} + \mathbf{e}_j) && \text{(by } P3') \\ &\geq V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x}). && \text{(by } P3') \end{aligned}$$

The last two inequalities reduce to equalities, and hence they hold trivially.

Finally, we show $P4'$ is preserved under A_k . We only consider the case $k = i$, and the discussion for the case when $k = j$ is similar. We need to show

$$\begin{aligned} \min\{V_n(\mathbf{x} + 2\mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j), r_i + V_n(\mathbf{x} + \mathbf{e}_i)\} + \min\{V_n(\mathbf{x} + \mathbf{e}_j), c_i + V_n(\mathbf{x} - \mathbf{e}_i + 2\mathbf{e}_j), r_i + V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j)\} \\ \geq \min\{V_n(\mathbf{x} + \mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_j), r_i + V_n(\mathbf{x})\} + \min\{V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j), c_i + V_n(\mathbf{x} + 2\mathbf{e}_j), r_i + V_n(\mathbf{x} + \mathbf{e}_j)\}. \end{aligned}$$

It suffices to prove

$$\begin{aligned} LHS_{11} + LHS_{22} &\geq RHS_{12} + RHS_{21}, & LHS_{11} + LHS_{23} &\geq RHS_{11} + RHS_{23}, \\ LHS_{12} + LHS_{21} &\geq RHS_{12} + RHS_{21}, & LHS_{12} + LHS_{23} &\geq RHS_{12} + RHS_{23}, \\ LHS_{13} + LHS_{21} &\geq RHS_{11} + RHS_{23}, & LHS_{13} + LHS_{22} &\geq RHS_{12} + RHS_{23}. \end{aligned}$$

These inequalities lead to $LHS \geq RHS$, for $LHS = LHS_{1q_1} + LHS_{2q_2}$ with all $q_1 \neq q_2$. We remain to show the above six inequalities. The first inequality is true because

$$\begin{aligned} V_n(\mathbf{x} + 2\mathbf{e}_i) - V_n(\mathbf{x} + \mathbf{e}_i + \mathbf{e}_j) &\geq V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x} + \mathbf{e}_j) && \text{(by } P4') \\ &\geq V_n(\mathbf{x}) - V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) && \text{(by } P4') \\ &\geq V_n(\mathbf{x} + \mathbf{e}_j) - V_n(\mathbf{x} - \mathbf{e}_i + 2\mathbf{e}_j). && \text{(by } P4') \end{aligned}$$

The second inequality has been shown in the proof of $P2'$. The third and fifth inequalities are trivial because they reduce to equalities. The fourth inequality is from $P2'$. The sixth inequality is true because

$$\begin{aligned} V_n(\mathbf{x} + \mathbf{e}_i) - V_n(\mathbf{x} + \mathbf{e}_j) &\geq V_n(\mathbf{x}) - V_n(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) && \text{(by } P4') \\ &\geq V_n(\mathbf{x} + \mathbf{e}_j) - V_n(\mathbf{x} - \mathbf{e}_i + 2\mathbf{e}_j). && \text{(by } P4') \end{aligned}$$

Hence $P4'$ is preserved under A_k . □

PROOFS OF LEMMA 1 AND PROPOSITION 2. Recall that $V_n(\mathbf{x})$ satisfies properties $P1'$ to $P4'$ from Lemma A1, and that $V_n(\mathbf{x}) \rightarrow V(\mathbf{x})$ as $n \rightarrow \infty$. From Topkis (see Topkis, D.M. 1978. Minimizing a submodular function on a lattice. *Oper. Res.* **26**(2) 305–321), $V(\mathbf{x})$ satisfies properties $P1$ to $P4$, and Lemma 1 is true. Now consider Proposition 2. The conclusions for $x_i > \hat{y}_i(x_j)$, and otherwise with $x_i < \bar{y}_i(x_j)$, in statement (a), are obvious from the above definitions. The conclusion for $\bar{y}_i(x_j) \leq x_i \leq \hat{y}_i(x_j)$ is true because: opt-out cannot be optimal due to properties $P2$ and $P3$ of $V(\mathbf{x})$, while acceptance cannot be optimal from property $P4$ of $V(\mathbf{x})$. These validate statement (a). $\hat{y}_i(x_j)$ is non-increasing in x_j because both $V(\mathbf{x} + \mathbf{e}_i) - V(\mathbf{x})$ and $V(\mathbf{x} + \mathbf{e}_j) - V(\mathbf{x})$ are non-decreasing in x_j due to $P2$ and $P3$. $\bar{y}_i(x_j)$ is non-decreasing in x_j because $V(\mathbf{x} + \mathbf{e}_i) - V(\mathbf{x} + \mathbf{e}_j)$

is non-increasing in x_j from P4. Hence statement (b) is also true. $\square$

To show Propositions 3, 8, 7, and A1, we need the following lemma.

Lemma A2. *The waiting cost function $h_i(x_i)$ has the following properties: (a) $w = (\alpha + \mu_i)h_i(0)$; (b) $w + \mu_i h_i(x_i) = (\alpha + \mu_i)h_i(x_i + 1)$.*

PROOFS OF LEMMA A2. Part (a) follows from $(\alpha + \mu_i)h_i(0) = (\alpha + \mu_i)\frac{w}{\alpha}[1 - (\frac{\mu_i}{\alpha + \mu_i})] = w$. Part (b) is true because

$$(\alpha + \mu_i)h_i(x_i + 1) = (\alpha + \mu_i)\frac{w}{\alpha} \left[1 - \left(\frac{\mu_i}{\alpha + \mu_i} \right)^{x_i+2} \right] = w + \mu_i \frac{w}{\alpha} \left[1 - \left(\frac{\mu_i}{\alpha + \mu_i} \right)^{x_i+1} \right] = w + \mu_i h_i(x_i). \quad \square$$

PROOF OF PROPOSITION 3. We define an auxiliary function $\tilde{V}(\mathbf{x}) := \sum_{i=1}^2 \tilde{V}^{(i)}(x_i)$, where

$$\tilde{V}^{(i)}(x_i) = \frac{1}{\alpha + \lambda_i + \mu_i} \left(wx_i + \lambda_i \tilde{V}^{(i)}(x_i) + \mu_i \tilde{V}^{(i)}((x_i - 1)^+) \right).$$

We first prove that, by induction, for all $x_i \geq 0$ and $i = 1, 2$,

$$\tilde{V}^{(i)}(x_i + 1) - \tilde{V}^{(i)}(x_i) = h_i(x_i), \quad (12)$$

where $h_i(x_i)$ is the waiting cost function defined in Equation (1). Note that

$$\tilde{V}^{(i)}(x_i) = \frac{1}{\alpha + \mu_i} \left(wx_i + \mu_i \tilde{V}^{(i)}((x_i - 1)^+) \right).$$

It follows that $\tilde{V}^{(i)}(0) = 0$. Thus,

$$\tilde{V}^{(i)}(1) = \frac{w}{\alpha + \mu_i} \quad \text{and} \quad \tilde{V}^{(i)}(1) - \tilde{V}^{(i)}(0) = \frac{w}{\alpha + \mu_i} = h_i(0),$$

and (12) is satisfied when $x_i = 0$. Suppose (12) holds for x_i , then

$$\begin{aligned} \tilde{V}^{(i)}(x_i + 2) - \tilde{V}^{(i)}(x_i + 1) &= \frac{1}{\alpha + \mu_i} \left(w(x_i + 2) + \mu_i \tilde{V}^{(i)}(x_i + 1) \right) - \frac{1}{\alpha + \mu_i} \left(w(x_i + 1) + \mu_i \tilde{V}^{(i)}(x_i) \right) \\ &= \frac{1}{\alpha + \mu_i} \left[w + \mu_i \left(\tilde{V}^{(i)}(x_i + 1) - \tilde{V}^{(i)}(x_i) \right) \right] = \frac{w + \mu_i h_i(x_i)}{\alpha + \mu_i} = h_i(x_i + 1), \end{aligned}$$

where the last equality is by Lemma A2(b). This completes the induction argument that (12) holds for all $x_i \geq 0$. To prove the proposition, by result (12), we need to show, $\tilde{V}(\mathbf{x} + \mathbf{e}_i) - \tilde{V}(\mathbf{x}) \leq V(\mathbf{x} + \mathbf{e}_i) - V(\mathbf{x})$. To this end, we introduce $\tilde{V}_0^{(i)}(x_i) = 0$ for all x_i , and for $n > 0$, we define $\tilde{V}_n^{(i)}(x_i)$ recursively by

$$\tilde{V}_{n+1}^{(i)}(x_i) := \frac{1}{\alpha + \lambda_i + \mu_i} \left(wx_i + \lambda_i \tilde{V}_n^{(i)}(x_i) + \mu_i \tilde{V}_n^{(i)}((x_i - 1)^+) \right).$$

Then, it can be shown that, for any $x_i \geq 0$, $\tilde{V}_n^{(i)}(x_i) \rightarrow \tilde{V}^{(i)}(x_i)$ as n tends to infinity. Let $\tilde{V}_n(x_1, x_2) := \sum_{i=1}^2 \tilde{V}_n^{(i)}(x_i)$. We have, for all (x_1, x_2) , $\lim_{n \rightarrow \infty} \tilde{V}_n(x_1, x_2) = \tilde{V}(x_1, x_2)$, $i = 1, 2$.

We have shown that it suffices to prove, for all $\mathbf{x}' = (x'_i, x'_j) \geq (x_i, x_j) = \mathbf{x}$ and for any integer $n \geq 0$,

$$\tilde{V}_n(\mathbf{x}') - V_n(\mathbf{x}') \leq \tilde{V}_n(\mathbf{x}) - V_n(\mathbf{x}). \quad (13)$$

Without loss of generality, we assume that $\alpha + \beta = 1$. By definition, we have $\tilde{V}_0(\mathbf{x}) = 0$ for all $\mathbf{x}$ and $\tilde{V}_{n+1}(\mathbf{x}) = \sum_{i=1}^2 wx_i + \sum_{i=1}^2 \lambda_i \tilde{V}_n(\mathbf{x}) + \sum_{i=1}^2 \mu_i D_i \tilde{V}_n(\mathbf{x})$. Clearly, (13) is true for $n = 0$ as both sides equal 0. Suppose it is true for n , and consider the case of $n + 1$. We have

$$\begin{aligned} &\tilde{V}_{n+1}(\mathbf{x}') - V_{n+1}(\mathbf{x}') \\ &= \sum_{i=1}^2 \lambda_i [\tilde{V}_n(\mathbf{x}') - \min\{V_n(\mathbf{x}' + \mathbf{e}_i), c_i + V_n(\mathbf{x}' + \mathbf{e}_j), r_i + V_n(\mathbf{x}')\}] + \sum_{i=1}^2 \mu_i [D_i \tilde{V}_n(\mathbf{x}') - D_i V_n(\mathbf{x}')] \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^2 \lambda_i [\max\{\tilde{V}_n(\mathbf{x}') - V_n(\mathbf{x}' + \mathbf{e}_i), -c_i + \tilde{V}_n(\mathbf{x}') - V_n(\mathbf{x}' + \mathbf{e}_j), -r_i + \tilde{V}_n(\mathbf{x}') - V_n(\mathbf{x}')\}] \\
&\quad + \sum_{i=1}^2 \mu_i [D_i \tilde{V}_n(\mathbf{x}') - D_i V_n(\mathbf{x}')] \\
&\leq \sum_{i=1}^2 \lambda_i [\max\{\tilde{V}_n(\mathbf{x}) - V_n(\mathbf{x} + \mathbf{e}_i), -c_i + \tilde{V}_n(\mathbf{x}) - V_n(\mathbf{x} + \mathbf{e}_j), -r_i + \tilde{V}_n(\mathbf{x}) - V_n(\mathbf{x})\}] \\
&\quad + \sum_{i=1}^2 \mu_i [D_i \tilde{V}_n(\mathbf{x}) - D_i V_n(\mathbf{x})] \\
&= \sum_{i=1}^2 \lambda_i [\tilde{V}_n(\mathbf{x}) - \min\{V_n(\mathbf{x} + \mathbf{e}_i), c_i + V_n(\mathbf{x} + \mathbf{e}_j), r_i + V_n(\mathbf{x})\}] + \sum_{i=1}^2 \mu_i [D_i \tilde{V}_n(\mathbf{x}) - D_i V_n(\mathbf{x})] \\
&= \tilde{V}_{n+1}(\mathbf{x}) - V_{n+1}(\mathbf{x}).
\end{aligned}$$

The above inequality is true because: (i) $\tilde{V}_n(\mathbf{x}') - V_n(\mathbf{x}') \leq \tilde{V}_n(\mathbf{x}) - V_n(\mathbf{x})$ due to induction hypothesis, (ii) by P2 and P3 from Lemma 1,

$$\begin{aligned}
\tilde{V}_n(\mathbf{x}') - V_n(\mathbf{x}' + \mathbf{e}_i) &= [\tilde{V}_n(\mathbf{x}') - V_n(\mathbf{x}')] + [V_n(\mathbf{x}') - V_n(\mathbf{x}' + \mathbf{e}_i)] \\
&\leq [\tilde{V}_n(\mathbf{x}) - V_n(\mathbf{x})] + [V_n(\mathbf{x}) - V_n(\mathbf{x} + \mathbf{e}_i)] = \tilde{V}_n(\mathbf{x}) - V_n(\mathbf{x} + \mathbf{e}_i),
\end{aligned}$$

and (iii) similar argument shows $D_i \tilde{V}_n(\mathbf{x}') - D_i V_n(\mathbf{x}') \leq D_i \tilde{V}_n(\mathbf{x}) - D_i V_n(\mathbf{x})$. Hence (13) is established and the proof for Proposition 3 is complete. $\square$

PROOF OF PROPOSITION 4. Proposition 4 follows from $\mathcal{E}_i(\mathbf{x}) = V(\mathbf{x}) + h_i(x_i) - V(\mathbf{x} + \mathbf{e}_i)$ and Equation (13) for $\mathbf{x}' = \mathbf{x} + \mathbf{e}_i$, and details are omitted. $\square$

PROOF OF PROPOSITION 5. Propositions 5 follows from $\mathcal{E}_i(\mathbf{x}) = V(\mathbf{x}) + h_i(x_i) - V(\mathbf{x} + \mathbf{e}_i)$ and P3 of Lemma 1, respectively, and details are omitted. $\square$

PROOF OF PROPOSITION 6. Without the referral option, a patient from cluster i joins home station i if $x_i \leq \hat{y}_i^I$ and opts out otherwise. With referrals, the patient's decision is characterized in Proposition 1. The option of referrals alters the patient's decision in two scenarios. In the first scenario, $x_i > \hat{y}_i^I$ and $x_j \leq T_j^I$, and the option of referrals allows a patient who would otherwise opt out to switch stations. According to Proposition 2, this reduces system cost if $x_i \leq \hat{y}_i(x_j)$, i.e., $x_j \leq t_j(x_i) := \hat{y}_i^{-1}(x_i)$, which is well defined because $\hat{y}_i(x_j)$ is a non-increasing function; otherwise, it increases system cost. In the second scenario, $\bar{y}_i^I(x_j) < x_i \leq \hat{y}_i^I$, and the option of referrals induces a patient who would otherwise stay at the home station to switch. This reduces system cost if $x_i \geq \bar{y}_i(x_j)$ (i.e., if $x_j \leq t_j(x_i) := \min\{\bar{y}_i^{-1}(x_i), (\bar{y}_i^I)^{-1}(x_i)\}$, which is well defined because $\bar{y}_i(x_j)$ is a non-decreasing function), and increases system cost otherwise. Based on these two scenarios, we can obtain the results. $\square$

PROOF OF PROPOSITION 7. Consider part (a) first. Assuming switching is not allowed, we use $\bar{V}(\mathbf{x})$ to represent the system cost under individual optimization starting from the system state $\mathbf{x}$. Let $\bar{V}_0(x_1, x_2) = \sum_{m=0}^{x_1-1} h(m) + \sum_{k=0}^{x_2-1} h(k)$ for all (x_1, x_2) , and

$$\begin{aligned}
\mathcal{V}_{n+1}^c(0, 0) &= \frac{2\lambda \mathcal{V}_n^c(1, 0) + 2\mu \mathcal{V}_n^c(0, 0)}{\alpha + 2\lambda + 2\mu}, \\
\mathcal{V}_{n+1}^c(m, 0) &= \frac{mw + \lambda \mathcal{V}_n^c(m+1, 0) + \lambda \mathcal{V}_n^c(m, 1) + \mu \mathcal{V}_n^c(m, 0) + \mu \mathcal{V}_n^c(m-1, 0)}{\alpha + 2\lambda + 2\mu}, \quad 1 \leq m \leq q-1, \\
\mathcal{V}_{n+1}^c(q, 0) &= \frac{qw + 2\lambda \mathcal{V}_n^c(q, 1) + \lambda c + \mu \mathcal{V}_n^c(q, 0) + \mu \mathcal{V}_n^c(q-1, 0)}{\alpha + 2\lambda + 2\mu}, \\
\mathcal{V}_{n+1}^c(m, k) &= \frac{(m+k)w + \lambda \mathcal{V}_n^c(m+1, k) + \lambda \mathcal{V}_n^c(m, k+1) + \mu \mathcal{V}_n^c(m, k-1) + \mu \mathcal{V}_n^c(m-1, k)}{\alpha + 2\lambda + 2\mu}, \quad 1 \leq k \leq m \leq q-1, \\
\mathcal{V}_{n+1}^c(q, k) &= \frac{(q+k)w + \lambda \mathcal{V}_n^c(q, k) + \lambda r + \lambda \mathcal{V}_n^c(q, k+1) + \mu \mathcal{V}_n^c(q, k-1) + \mu \mathcal{V}_n^c(q-1, k)}{\alpha + 2\lambda + 2\mu}, \quad 1 \leq k \leq q-1,
\end{aligned}$$

$$\mathcal{V}_{n+1}^c(q, q) = \frac{2qw + 2\lambda(\mathcal{V}_n^c(q, q) + r) + 2\mu\mathcal{V}_n^c(q, q-1)}{\alpha + 2\lambda + 2\mu}.$$

Then $\mathcal{V}_n^c(\mathbf{x})$ converges to $\mathcal{V}^c(\mathbf{x})$ for all $\mathbf{x}$ as $n \rightarrow \infty$. We will show the following relation for all $n \geq 0$.

$$\text{U1: } \bar{\mathcal{V}}_n(m+1, k) \geq \bar{\mathcal{V}}_n(m, k) + h(m) \text{ and } \bar{\mathcal{V}}_n(m, k+1) \geq \bar{\mathcal{V}}_n(m, k) + h(k).$$

Note that $\bar{\mathcal{V}}_n(\mathbf{x})$ is symmetric across its two components, and we only need to demonstrate that $\bar{\mathcal{V}}_n(m+1, k) \geq \bar{\mathcal{V}}_n(m, k) + h(m)$ for $0 \leq k \leq m \leq q$ and $\bar{\mathcal{V}}_n(m, k+1) \geq \bar{\mathcal{V}}_n(m, k) + h(k)$ for $0 \leq k < m \leq q$. It is evident that U1 is satisfied for $n = 0$. Furthermore, when U1 is satisfied for n , we can use induction and U1 to show that it continues to hold for $n+1$. For instance, for $(m, k) = (0, 0)$:

$$\begin{aligned} \bar{\mathcal{V}}_{n+1}(1, 0) - \bar{\mathcal{V}}_{n+1}(0, 0) &= \frac{w + \lambda[\bar{\mathcal{V}}_n(2, 0) - \bar{\mathcal{V}}_n(1, 0)] + \lambda[\bar{\mathcal{V}}_n(1, 1) - \bar{\mathcal{V}}_n(1, 0)] + \mu[\bar{\mathcal{V}}_n(1, 0) - \bar{\mathcal{V}}_n(0, 0)]}{\alpha + 2\lambda + 2\mu} \\ &\geq \frac{w + \lambda h(1) + (\lambda + \mu)h(0)}{\alpha + 2\lambda + 2\mu} \quad (\text{by U1}) \\ &\geq \frac{(\alpha + \mu)h(0) + (2\lambda + \mu)h(0)}{\alpha + 2\lambda + 2\mu} = h(0); \end{aligned}$$

for $1 \leq k \leq m \leq q-2$:

$$\begin{aligned} &\bar{\mathcal{V}}_{n+1}(m+1, k) - \bar{\mathcal{V}}_{n+1}(m, k) \\ &= \frac{1}{\alpha + 2\lambda + 2\mu} [w + \lambda(\bar{\mathcal{V}}_n(m+2, k) - \bar{\mathcal{V}}_n(m+1, k)) + \lambda(\bar{\mathcal{V}}_n(m+1, k+1) - \bar{\mathcal{V}}_n(m, k+1)) \\ &\quad + \mu(\bar{\mathcal{V}}_n(m+1, k-1) - \bar{\mathcal{V}}_n(m, k-1)) + \mu(\bar{\mathcal{V}}_n(m, k) - \bar{\mathcal{V}}_n(m-1, k))] \\ &\geq \frac{1}{\alpha + 2\lambda + 2\mu} [w + \lambda h(m+1) + \lambda h(m) + \mu h(m) + \mu h(m-1)] \quad (\text{by U1}) \\ &\geq \frac{1}{\alpha + 2\lambda + 2\mu} [2\lambda h(m) + (\alpha + 2\mu)h(m)] = h(m). \end{aligned}$$

For other system states, we can similarly show that U1 continues to hold for $n+1$. As n goes to infinity, we have:

$$\text{U1': } \bar{\mathcal{V}}(m+1, k) \geq \bar{\mathcal{V}}(m, k) + h(m) \text{ and } \bar{\mathcal{V}}(m, k+1) \geq \bar{\mathcal{V}}(m, k) + h(k).$$

Then, we have:

$$\begin{aligned} \bar{\mathcal{V}}(q, 1) - \bar{\mathcal{V}}(q, 0) &= \frac{w + \lambda[\bar{\mathcal{V}}(q, 1) - \bar{\mathcal{V}}(q, 0)] + \lambda[\bar{\mathcal{V}}(q, 2) - \bar{\mathcal{V}}(q, 1)] + \mu[\bar{\mathcal{V}}(q-1, 1) - \bar{\mathcal{V}}(q-1, 0)]}{\alpha + 2\lambda + 2\mu} \\ &\geq \frac{w + \lambda h(0) + \lambda h(1) + \mu h(0)}{\alpha + 2\lambda + 2\mu} = h(0) + \frac{\lambda[h(1) - h(0)]}{\alpha + 2\lambda + 2\mu}. \quad (\text{by U1'}) \quad (14) \end{aligned}$$

Consider the case where switching is allowed and there is a switching cost c . Suppose $h(q-1) < r < h(q)$ ($q \in \mathbb{Z}^+$). Then each patient prefers staying at the home station to opting out iff the on-arrival queue length is strictly lower than q . Assuming $h(q-1) - h(0) < c < r - h(0)$, an arriving patient will switch to the other station when the home station has a queue length of q and the other station is empty. When the home station has a queue length of q and the other station has a queue length of 1, a patient prefers to opt out rather than switch because $c + h(1) > h(q-1) + h(1) - h(0) > h(q-1) + h(q) - h(q-1) = h(q) > r$. Thus, a patient never switches when the other station is not empty. We can conclude that an arriving patient stays at the home station if the queue length of the home station is strictly lower than q , switches if the queue length of the home station is q and the other station is empty, and opts out of the system for outside services otherwise. We can obtain the system cost under individual optimization through value iterations by using $\mathcal{V}_0^c(x_1, x_2) = \bar{\mathcal{V}}(x_1, x_2)$ for all (x_1, x_2) , and

$$\begin{aligned} \mathcal{V}_{n+1}^c(0, 0) &= \frac{2\lambda\mathcal{V}_n^c(1, 0) + 2\mu\mathcal{V}_n^c(0, 0)}{\alpha + 2\lambda + 2\mu}, \\ \mathcal{V}_{n+1}^c(m, 0) &= \frac{mw + \lambda\mathcal{V}_n^c(m+1, 0) + \lambda\mathcal{V}_n^c(m, 1) + \mu\mathcal{V}_n^c(m, 0) + \mu\mathcal{V}_n^c(m-1, 0)}{\alpha + 2\lambda + 2\mu}, \quad 1 \leq m \leq q-1, \\ \mathcal{V}_{n+1}^c(q, 0) &= \frac{qw + 2\lambda\mathcal{V}_n^c(q, 1) + \lambda c + \mu\mathcal{V}_n^c(q, 0) + \mu\mathcal{V}_n^c(q-1, 0)}{\alpha + 2\lambda + 2\mu}, \end{aligned}$$

$$\begin{aligned}
\mathcal{V}_{n+1}^c(m, k) &= \frac{(m+k)w + \lambda \mathcal{V}_n^c(m+1, k) + \lambda \mathcal{V}_n^c(m, k+1) + \mu \mathcal{V}_n^c(m, k-1) + \mu \mathcal{V}_n^c(m-1, k)}{\alpha + 2\lambda + 2\mu}, \quad 1 \leq k \leq m \leq q-1, \\
\mathcal{V}_{n+1}^c(q, k) &= \frac{(q+k)w + \lambda \mathcal{V}_n^c(q, k) + \lambda r + \lambda \mathcal{V}_n^c(q, k+1) + \mu \mathcal{V}_n^c(q, k-1) + \mu \mathcal{V}_n^c(q-1, k)}{\alpha + 2\lambda + 2\mu}, \quad 1 \leq k \leq q-1, \\
\mathcal{V}_{n+1}^c(q, q) &= \frac{2qw + 2\lambda[\mathcal{V}_n^c(q, q) + r] + 2\mu \mathcal{V}_n^c(q, q-1)}{\alpha + 2\lambda + 2\mu}.
\end{aligned}$$

Then $\mathcal{V}_n^c(\mathbf{x})$ converges to $\mathcal{V}^c(\mathbf{x})$ for all $\mathbf{x}$ as $n \rightarrow \infty$. We will show the following relation for all $n \geq 0$.

$$\text{U2: } \mathcal{V}_n^c(m, k) \geq \bar{\mathcal{V}}(m, k).$$

Obviously, U2 is satisfied for $n = 0$. If U2 is true for n , we now show it continues to hold for $n + 1$. Actually,

$$\begin{aligned}
\mathcal{V}_{n+1}^c(q, 0) - \bar{\mathcal{V}}(q, 0) &= \frac{1}{\alpha + 2\lambda + 2\mu} [qw + 2\lambda \mathcal{V}_n^c(q, 1) + \lambda c + \mu \mathcal{V}_n^c(q, 0) + \mu \mathcal{V}_n^c(q-1, 0)] \\
&\quad - \frac{1}{\alpha + 2\lambda + 2\mu} [qw + \lambda \bar{\mathcal{V}}(q, 0) + \lambda r + \lambda \bar{\mathcal{V}}(q, 1) + \mu \bar{\mathcal{V}}(q, 0) + \mu \bar{\mathcal{V}}(q-1, 0)] \\
&\geq \frac{\lambda}{\alpha + 2\lambda + 2\mu} [\mathcal{V}_n^c(q, 1) + c - \bar{\mathcal{V}}(q, 0) - r] \quad (\text{by U2}) \\
&\geq \frac{\lambda}{\alpha + 2\lambda + 2\mu} [\bar{\mathcal{V}}(q, 1) + c - \bar{\mathcal{V}}(q, 0) - r] \quad (\text{by U2}) \\
&\geq \frac{\lambda}{\alpha + 2\lambda + 2\mu} [h(0) + \frac{\lambda}{\alpha + 2\lambda + 2\mu} (h(1) - h(0)) + c - r], \quad (\text{by (14)}) \quad (15)
\end{aligned}$$

which is positive when $c > r - h(0) - \frac{\lambda(h(1)-h(0))}{\alpha+2\lambda+2\mu}$. For $(m, k) \neq (q, 0)$ or $(0, q)$, obviously U2 is satisfied for $n + 1$. Then, as n goes to infinity, we have:

$$\text{U2': } \mathcal{V}^c(m, k) \geq \bar{\mathcal{V}}(m, k).$$

Furthermore, we can check that

$$\begin{aligned}
\mathcal{V}^c(q, 0) - \bar{\mathcal{V}}(q, 0) &\geq \frac{1}{\alpha + 2\lambda + 2\mu} [\lambda \mathcal{V}^c(q, 1) + \lambda c - \lambda \bar{\mathcal{V}}(q, 0) - \lambda r] \quad (\text{by U2'}) \\
&\geq \frac{\lambda}{\alpha + 2\lambda + 2\mu} [\bar{\mathcal{V}}(q, 1) + c - \bar{\mathcal{V}}(q, 0) - r] \quad (\text{by U2'}) \\
&\geq \frac{\lambda}{\alpha + 2\lambda + 2\mu} [h(0) + \frac{\lambda(h(1) - h(0))}{\alpha + 2\lambda + 2\mu} + c - r] > 0. \quad (\text{by (14)})
\end{aligned}$$

Assuming $\mathcal{V}^c(m+1, 0) > \bar{\mathcal{V}}(m+1, 0)$, we now prove that $\mathcal{V}^c(m, 0) > \bar{\mathcal{V}}(m, 0)$. Actually, for $1 \leq m \leq q-1$:

$$\begin{aligned}
\mathcal{V}^c(m, 0) - \bar{\mathcal{V}}(m, 0) &\geq \frac{1}{\alpha + 2\lambda + 2\mu} [\lambda \mathcal{V}^c(m+1, 0) + \lambda \mathcal{V}^c(m, 1) + \mu \mathcal{V}^c(m, 0) + \mu \mathcal{V}^c(m-1, 0)] \\
&\quad - \frac{1}{\alpha + 2\lambda + 2\mu} [\lambda \bar{\mathcal{V}}(m+1, 0) + \lambda \bar{\mathcal{V}}(m, 1) + \mu \bar{\mathcal{V}}(m, 0) + \mu \bar{\mathcal{V}}(m-1, 0)] \\
&\geq \frac{\lambda}{\alpha + 2\lambda + 2\mu} [\mathcal{V}^c(m+1, 0) - \bar{\mathcal{V}}(m+1, 0)] > 0. \quad (\text{by U2'})
\end{aligned}$$

$$\begin{aligned}
\text{For } m = 0: \quad \mathcal{V}^c(0, 0) - \bar{\mathcal{V}}(0, 0) &= \frac{1}{\alpha + 2\lambda + 2\mu} [2\lambda \mathcal{V}^c(1, 0) - 2\lambda \bar{\mathcal{V}}(1, 0) + 2\mu \mathcal{V}^c(0, 0) - 2\mu \bar{\mathcal{V}}(0, 0)] \\
&\geq \frac{2\lambda}{\alpha + 2\lambda + 2\mu} [\mathcal{V}^c(1, 0) - \bar{\mathcal{V}}(1, 0)] > 0. \quad (\text{by U2'})
\end{aligned}$$

Based on the above, we conclude that, when $h(q-1) < r < h(q)$ ($q \in \mathbb{Z}^+$) and $c_q := \max\{h(q-1) - h(0), r - h(0) - \frac{\lambda[h(1)-h(0)]}{\alpha+2\lambda+2\mu}\} < c < r - h(0)$ (i.e., c is in a non-empty open interval), the system cost under individual optimization with switching option is strictly higher than that without, i.e., $\mathcal{V}^c(0, 0) > \bar{\mathcal{V}}(0, 0)$.

Now we consider part (b). We assume $h(1) < r < h(2)$, which results in a maximum queue length of 2 for each station, regardless of whether switching is allowed or not. When switching is allowed and costless (i.e., $c = 0$), a patient switches only if the other station has a shorter queue. From the system cost equations

$$\begin{aligned} \mathcal{V}^0(0,0) &= \frac{2\lambda\mathcal{V}^0(1,0)}{\alpha + 2\lambda}, & \mathcal{V}^0(1,0) &= \frac{w + 2\lambda\mathcal{V}^0(1,1) + \mu\mathcal{V}^0(0,0)}{\alpha + 2\lambda + \mu}, \\ \mathcal{V}^0(1,1) &= \frac{2w + 2\lambda\mathcal{V}^0(2,1) + 2\mu\mathcal{V}^0(1,0)}{\alpha + 2\lambda + 2\mu}, & \mathcal{V}^0(2,0) &= \frac{2w + 2\lambda\mathcal{V}^0(2,1) + \mu\mathcal{V}^0(1,0)}{\alpha + 2\lambda + \mu}, \\ \mathcal{V}^0(2,1) &= \frac{3w + 2\lambda\mathcal{V}^0(2,2) + \mu(\mathcal{V}^0(1,1) + V^{IR}(2,0))}{\alpha + 2\lambda + 2\mu}, & \mathcal{V}^0(2,2) &= \frac{4w + 2\lambda(r + \mathcal{V}^0(2,2)) + 2\mu\mathcal{V}^0(2,1)}{\alpha + 2\lambda + 2\mu}, \end{aligned}$$

we can obtain:

$$\mathcal{V}^0(0,0) = z_w w + \frac{32\lambda^5}{\alpha d} r,$$

where $z_w = A/B$, and

$$\begin{aligned} A &= 2\lambda(\alpha^4 + 10\alpha^3\lambda + 7\alpha^3\mu + 40\alpha^2\lambda^2 + 44\alpha^2\lambda\mu + 18\alpha^2\mu^2 + 80\alpha\lambda^3 \\ &\quad + 96\alpha\lambda^2\mu + 62\alpha\lambda\mu^2 + 20\alpha\mu^3 + 64\lambda^4 + 80\lambda^3\mu + 56\lambda^2\mu^2 + 28\lambda\mu^3 + 8\mu^4), \\ B &= \alpha(\alpha + 2\lambda + \mu)d, \\ d &= \alpha^4 + 8\alpha^3\lambda + 24\alpha^2\lambda^2 + 32\alpha\lambda^3 + 16\lambda^4 + 7\alpha^3\mu + 32\alpha^2\lambda\mu + 44\alpha\lambda^2\mu \\ &\quad + 16\lambda^3\mu + 18\alpha^2\mu^2 + 38\alpha\lambda\mu^2 + 16\lambda^2\mu^2 + 20\alpha\mu^3 + 12\lambda\mu^3 + 8\mu^4. \end{aligned}$$

Similarly, when switching is not allowed, we can use the system cost equations to obtain the system cost

$$\bar{\mathcal{V}}(0,0) = \frac{2\lambda((\alpha + 2\lambda + \mu)w + \lambda^2 r)}{\alpha(\alpha^2 + 2\alpha\lambda + \lambda^2 + 2\alpha\mu + \lambda\mu + \mu^2)}.$$

We can check that $\mathcal{V}^0(0,0)$ with $c = 0$ is strictly higher than $\bar{\mathcal{V}}(0,0)$ when

$$\alpha < \sqrt{2}\mu, \quad h(1) \leq r < \min\left(h(2), \frac{2w(\alpha + \mu)}{\alpha(2\alpha + 3\mu)}\right),$$

and λ is greater than $\bar{\lambda}$, the second root of the equation $k_4x^4 + k_3x^3 + k_2x^2 + k_1x + k_0 = 0$, where

$$\begin{aligned} k_4 &= -8\alpha(2\alpha + 3\mu)r + 16(\alpha + \mu)w, \\ k_3 &= -4(2\alpha + 3\mu)(3\alpha^2 + 6\alpha\mu + 2\mu^2)r + 8(3\alpha^2 + 8\alpha\mu + 7\mu^2)w, \\ k_2 &= -2(\alpha + \mu)^2(\alpha + 2\mu)(5\alpha + 7\mu)r + 2(5\alpha^3 + 18\alpha^2\mu + 27\alpha\mu^2 + 18\mu^3)w, \\ k_1 &= -(\alpha + \mu)(\alpha + 2\mu) \times ((\alpha + \mu)(\alpha + 2\mu)^2 r - (\alpha^2 - 2\alpha\mu - 2\mu^2)w), \\ k_0 &= -\mu(\alpha + \mu)(\alpha + 2\mu)^3 w. \end{aligned}$$

The details are omitted. □

PROOF OF PROPOSITION 8. We consider a symmetric system where $h(q-1) < r < h(q)$ for some $q \in \mathbb{Z}^+$ and $l \in \{0, 1, \dots, q-1\}$, and $c = r - h(l)$. Note that $h(m)$ is strictly increasing and strictly concave in m . Therefore, $h(q) - h(q-1) < h(q-1) - h(q-2) < \dots < h(1) - h(0)$. We further require $\delta > 0$ to be sufficiently small such that, when the switching cost increases from c to $c + \delta$, a patient alters their decision iff the queue lengths at the home station and the alternative station upon arrival are q and l , respectively. Specifically, we require $\delta < \bar{\delta} := \min\{h(l) - h(l-1), \sup\{\delta > 0 : c + \delta + h(k) < h(m) \text{ whenever } c + h(k) < h(m), \forall m, k \in \{0, 1, \dots, q-1\}\}\}$, where the first minimum is omitted when $l = 0$.

We define the following sets of system states based on queue lengths:

$$\begin{aligned} \Omega_1^c &= \{(m, k) : 0 \leq m, k \leq q-1, h(m) \leq c + h(k), h(k) \leq c + h(m)\}, \\ \Omega_2^c &= \{(m, k) : 0 \leq m, k \leq q, c + h(i) \leq r \text{ and } c + h(i) < h(j) \text{ for } (i, j) = (m, k) \text{ or } (k, m)\}, \\ \Omega_3^c &= \{(m, k) : 0 \leq m \leq q-1, k = q, r < c + h(m); \text{ or } m = q, 0 \leq k \leq q-1, r < c + h(k)\}, \end{aligned}$$

$$\Omega_4^c = \{(q, q)\}.$$

Suppose an arriving patient observes the queue lengths (m, k) . If $(m, k) \in \Omega_1^c$, the patient always accepts service at their home station. If $(m, k) \in \Omega_2^c$, the patient joins the station with the shorter queue by choosing either “Accept” or “Switch,” depending on which station’s queue length is shorter. If $(m, k) \in \Omega_3^c$, the patient opts out of the system if the home station queue length is q , and accepts service at the home station otherwise. Finally, if $(m, k) \in \Omega_4^c$, the patient always chooses to opt out.

Given a switching cost $c = r - h(l)$, the system cost under individual optimization can be obtained through value iteration. For all (x_1, x_2) , we define $\mathcal{V}_0^c(x_1, x_2) = \sum_{m=0}^{x_1-1} h(m) + \sum_{k=0}^{x_2-1} h(k)$. For $(m, k) \in \Omega_1^c$, we define

$$\mathcal{V}_{n+1}^c(m, k) = \frac{(m+k)w + \lambda[\mathcal{V}_n^c(m+1, k) + \mathcal{V}_n^c(m, k+1)] + \mu[\mathcal{V}_n^c((m-1)^+, k) + \mathcal{V}_n^c(m, (k-1)^+)]}{\alpha + 2\lambda + 2\mu}.$$

For $(m, k) \in \Omega_2^c$ with $m > k$ (noting that for $m < k$, $\mathcal{V}_{n+1}^c(m, k) = \mathcal{V}_{n+1}^c(k, m)$ by symmetry):

$$\mathcal{V}_{n+1}^c(m, k) = \frac{(m+k)w + \lambda[c + 2\mathcal{V}_n^c(m, k+1)] + \mu[\mathcal{V}_n^c(m-1, k) + \mathcal{V}_n^c(m, (k-1)^+)]}{\alpha + 2\lambda + 2\mu}.$$

For $(q, k) \in \Omega_3^c$ (similarly, for $(m, q) \in \Omega_3^c$, $\mathcal{V}_{n+1}^c(m, q) = \mathcal{V}_{n+1}^c(q, m)$):

$$\mathcal{V}_{n+1}^c(q, k) = \frac{(q+k)w + \lambda[r + \mathcal{V}_n^c(q, k) + \mathcal{V}_n^c(q, k+1)] + \mu[\mathcal{V}_n^c(q-1, k) + \mathcal{V}_n^c(q, (k-1)^+)]}{\alpha + 2\lambda + 2\mu}.$$

Finally, for $(m, k) = (q, q) \in \Omega_4^c$:

$$\mathcal{V}_{n+1}^c(q, q) = \frac{2qw + 2\lambda[r + \mathcal{V}_n^c(q, q)] + 2\mu\mathcal{V}_n^c(q, q-1)}{\alpha + 2\lambda + 2\mu}.$$

As $n \rightarrow \infty$, $\mathcal{V}_n^c(\mathbf{x})$ converges to $\mathcal{V}^c(\mathbf{x})$ for all $\mathbf{x}$.

For a switching cost $c' = c + \delta$, we similarly define the sets $\Omega_1^{c'}, \Omega_2^{c'}, \Omega_3^{c'}$, and $\Omega_4^{c'}$, as well as the cost functions $\mathcal{V}_n^{c'}(\mathbf{x})$ for $n \geq 0$. By the choice of δ , the system state $(m, k) \in \Omega_i^c$ iff $(m, k) \in \Omega_i^{c'}$ for $i \in \{1, 2, 3, 4\}$, with one exception: $(q, l), (l, q) \in \Omega_2^c$, whereas $(q, l), (l, q) \in \Omega_3^{c'}$. As before, the system cost under individual optimization can be obtained via value iteration. Consequently, $\mathcal{V}_n^{c'}(\mathbf{x})$ converges to $\mathcal{V}^{c'}(\mathbf{x})$ for all $\mathbf{x}$ as $n \rightarrow \infty$.

We will demonstrate the following relations for $n \geq 0$:

$$\text{S1: } \mathcal{V}_n^{c'}(m, k) \leq \mathcal{V}_n^c(m, k) + \frac{\lambda\delta}{\alpha};$$

$$\text{S2: } \mathcal{V}_n^c(m+1, k) \geq \mathcal{V}_n^c(m, k) + h(m) \text{ and } \mathcal{V}_n^c(m, k+1) \geq \mathcal{V}_n^c(m, k) + h(k).$$

Note that $\mathcal{V}_n^{c'}(\mathbf{x})$ and $\mathcal{V}_n^c(\mathbf{x})$ are symmetric with respect to their two components. Due to this symmetry, for S1, it suffices to show the inequality for $0 \leq k \leq m \leq q$. Similarly, for S2, we only need to show $\mathcal{V}_n^c(m+1, k) \geq \mathcal{V}_n^c(m, k) + h(m)$ for $0 \leq k \leq m \leq q-1$ and $\mathcal{V}_n^c(m, k+1) \geq \mathcal{V}_n^c(m, k) + h(k)$ for $0 \leq k < m \leq q$. Clearly, S1 and S2 hold for $n = 0$. In the following, assuming that S1 and S2 are satisfied for n , we show they continue to hold for $n+1$.

Consider S1 first. When $(m, k) \in \Omega_1^c \cup \Omega_3^c \cup \Omega_4^c$, it is clear that:

$$\mathcal{V}_{n+1}^{c'}(m, k) - \mathcal{V}_{n+1}^c(m, k) \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[2(\lambda + \mu) \frac{\lambda\delta}{\alpha} \right] \leq \frac{\lambda\delta}{\alpha}. \quad (\text{by S1})$$

When $(m, k) \in \Omega_2^c$, $m > k$, and $(m, k) \neq (q, l)$:

$$\mathcal{V}_{n+1}^{c'}(m, k) - \mathcal{V}_{n+1}^c(m, k) \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[2\lambda(\mathcal{V}_n^{c'}(m, k+1) - \mathcal{V}_n^c(m, k+1)) + \lambda\delta + 2\mu \frac{\lambda\delta}{\alpha} \right] \quad (\text{by S1})$$

$$\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[2(\lambda + \mu) \frac{\lambda\delta}{\alpha} + \lambda\delta \right] = \frac{\lambda\delta}{\alpha}. \quad (\text{by S1})$$

Finally, for $(m, k) = (q, l) \in \Omega_2^c$:

$$\mathcal{V}_{n+1}^{c'}(q, l) - \mathcal{V}_{n+1}^c(q, l)$$

$$\begin{aligned}
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda(\mathcal{V}_n^{c'}(q, l) + r + \mathcal{V}_n^{c'}(q, l + 1)) - 2\lambda\mathcal{V}_n^c(q, l + 1) - \lambda c + 2\mu \frac{\lambda\delta}{\alpha} \right] & (\text{by S1}) \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda(\mathcal{V}_n^{c'}(q, l) - \mathcal{V}_n^c(q, l + 1) + r - c) + (\lambda + 2\mu) \frac{\lambda\delta}{\alpha} \right] & (\text{by S1}) \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda(r - c - h(l)) + 2(\lambda + \mu) \frac{\lambda\delta}{\alpha} \right] & (\text{by S1 and S2}) \\
&= \frac{1}{\alpha + 2\lambda + 2\mu} \left[2(\lambda + \mu) \frac{\lambda\delta}{\alpha} \right] \leq \frac{\lambda\delta}{\alpha}.
\end{aligned}$$

Thus, S1 continues to hold for $\mathcal{V}_{n+1}^{c'}$ and $\mathcal{V}_{n+1}^c$.

Next, we consider S2. (a) We first show that $\mathcal{V}_{n+1}^c(m + 1, k) \geq \mathcal{V}_{n+1}^c(m, k) + h(m)$ for $0 \leq k \leq m \leq q - 1$. From Proposition 1, it suffices to consider the following four cases for the recurrent system states: (a.i) When both $(m + 1, k) \in \Omega_1^c$ and $(m, k) \in \Omega_1^c$ with $m > 0$:

$$\begin{aligned}
\mathcal{V}_{n+1}^c(m + 1, k) - \mathcal{V}_{n+1}^c(m, k) &\geq \frac{w + \lambda[h(m + 1) + h(m)] + \mu[h(m) + h(m - 1)]}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\
&= \frac{\lambda[h(m + 1) + h(m)] + \mu h(m) + (\alpha + \mu)h(m)}{\alpha + 2\lambda + 2\mu} \geq h(m). & (\text{by Lemma A2(b)})
\end{aligned}$$

When both $(m + 1, k) \in \Omega_1^c$ and $(m, k) \in \Omega_1^c$ with $m = 0$ (implying $m = k = 0$):

$$\begin{aligned}
\mathcal{V}_{n+1}^c(1, 0) - \mathcal{V}_{n+1}^c(0, 0) &\geq \frac{w + \lambda[h(1) + h(0)] + \mu h(0)}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\
&= \frac{(\alpha + \mu)h(0) + \lambda[h(1) + h(0)] + \mu h(0)}{\alpha + 2\lambda + 2\mu} \geq h(0). & (\text{by Lemma A2(a)})
\end{aligned}$$

(a.ii) When both $(m + 1, k) \in \Omega_2^c$ and $(m, k) \in \Omega_2^c$: It is clear that $m > 0$ (otherwise, $m = k = 0$ would imply $(m, k) \in \Omega_1^c$). Thus, by S2 and Lemma A2(b):

$$\mathcal{V}_{n+1}^c(m + 1, k) - \mathcal{V}_{n+1}^c(m, k) \geq \frac{w + 2\lambda h(m) + \mu[h(m) + h(m - 1)]}{\alpha + 2\lambda + 2\mu} = \frac{2\lambda h(m) + \mu h(m) + (\alpha + \mu)h(m)}{\alpha + 2\lambda + 2\mu} = h(m).$$

(a.iii) When $(m + 1, k) \in \Omega_2^c$, $(m, k) \in \Omega_1^c$, and $m > 0$:

$$\begin{aligned}
\mathcal{V}_{n+1}^c(m + 1, k) - \mathcal{V}_{n+1}^c(m, k) &\geq \frac{w + \lambda[h(m) + c + h(k)] + \mu[h(m) + h(m - 1)]}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\
&\geq \frac{\lambda[h(m) + h(m)] + \mu h(m) + [w + \mu h(m - 1)]}{\alpha + 2\lambda + 2\mu} & (\text{as } (m, k) \in \Omega_1^c) \\
&= \frac{2\lambda h(m) + \mu h(m) + (\alpha + \mu)h(m)}{\alpha + 2\lambda + 2\mu} = h(m).
\end{aligned}$$

When $(m + 1, k) \in \Omega_2^c$, $(m, k) \in \Omega_1^c$, and $m = 0$ (implying $m = k = 0$), by Lemma A2(a):

$$\mathcal{V}_{n+1}^c(1, 0) - \mathcal{V}_{n+1}^c(0, 0) \geq \frac{w + \lambda[2h(0) + c] + \mu h(0)}{\alpha + 2\lambda + 2\mu} \geq \frac{(\alpha + \mu)h(0) + (2\lambda + \mu)h(0)}{\alpha + 2\lambda + 2\mu} = h(0).$$

(a.iv) When $(m + 1, k) \in \Omega_3^c$ (requiring $m = q - 1$ and $k \leq q - 1$) and $(m, k) \in \Omega_1^c$: If $q \geq 2$,

$$\begin{aligned}
\mathcal{V}_{n+1}^c(q, k) - \mathcal{V}_{n+1}^c(q - 1, k) &\geq \frac{w + \lambda[r + h(q - 1)] + \mu[h(q - 1) + h(q - 2)]}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\
&\geq \frac{\lambda[h(q - 1) + h(q - 1)] + \mu h(q - 1) + [w + \mu h(q - 2)]}{\alpha + 2\lambda + 2\mu} & (\text{as } r > h(q - 1)) \\
&= \frac{2\lambda h(q - 1) + \mu h(q - 1) + (\alpha + \mu)h(q - 1)}{\alpha + 2\lambda + 2\mu} = h(q - 1).
\end{aligned}$$

If $q = 1$, we can similarly prove the inequality. Thus, we have $\mathcal{V}_{n+1}^c(m+1, k) \geq \mathcal{V}_{n+1}^c(m, k) + h(m)$ for $0 \leq k \leq m \leq q$.

(b) We next show that $\mathcal{V}_{n+1}^c(m, k+1) \geq \mathcal{V}_{n+1}^c(m, k) + h(k)$ for $0 \leq k < m \leq q$ by considering the following cases:

(b.i) If $(m, k+1), (m, k) \in \Omega_1^c$ and $k > 0$:

$$\begin{aligned} \mathcal{V}_{n+1}^c(m, k+1) - \mathcal{V}_{n+1}^c(m, k) &\geq \frac{w + \lambda[h(k) + h(k+1)] + \mu[h(k) + h(k-1)]}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\ &= \frac{\lambda[h(k) + h(k+1)] + \mu h(k) + (\alpha + \mu)h(k)}{\alpha + 2\lambda + 2\mu} \geq h(k). & (\text{by Lemma A2(b)}) \end{aligned}$$

If $(m, k+1), (m, k) \in \Omega_1^c$ and $k = 0$:

$$\begin{aligned} \mathcal{V}_{n+1}^c(m, 1) - \mathcal{V}_{n+1}^c(m, 0) &\geq \frac{w + \lambda[h(0) + h(1)] + \mu h(0)}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\ &= \frac{(\alpha + \mu)h(0) + \lambda[h(0) + h(1)] + \mu h(0)}{\alpha + 2\lambda + 2\mu} \geq h(0). & (\text{by Lemma A2(a)}) \end{aligned}$$

(b.ii) If $(m, k+1), (m, k) \in \Omega_2^c$ and $k \geq 1$, by S2:

$$\mathcal{V}_{n+1}^c(m, k+1) - \mathcal{V}_{n+1}^c(m, k) \geq \frac{w + 2\lambda h(k+1) + \mu[h(k) + h(k-1)]}{\alpha + 2\lambda + 2\mu} = \frac{2\lambda h(k+1) + \mu h(k) + (\alpha + \mu)h(k)}{\alpha + 2\lambda + 2\mu} \geq h(k).$$

If $(m, k+1), (m, k) \in \Omega_2^c$ and $k = 0$, by S2:

$$\mathcal{V}_{n+1}^c(m, 1) - \mathcal{V}_{n+1}^c(m, 0) \geq \frac{w + 2\lambda h(1) + \mu h(0)}{\alpha + 2\lambda + 2\mu} = \frac{2\lambda h(1) + \mu h(0) + (\alpha + \mu)h(0)}{\alpha + 2\lambda + 2\mu} \geq h(0).$$

(b.iii) If $(m, k+1), (m, k) \in \Omega_3^c$ (which requires $m = q$, $k < q-1$, and $k > l$):

$$\begin{aligned} \mathcal{V}_{n+1}^c(q, k+1) - \mathcal{V}_{n+1}^c(q, k) &\geq \frac{w + \lambda[h(k) + h(k+1)] + \mu[h(k) + h(k-1)]}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\ &\geq \frac{\lambda[h(k) + h(k+1)] + \mu h(k) + (\alpha + \mu)h(k)}{\alpha + 2\lambda + 2\mu} \geq h(k). \end{aligned}$$

(b.iv) If $(m, k+1) \in \Omega_4^c$ (requiring $m = q$ and $k = q-1$), $(m, k) \in \Omega_2^c$ (requiring $l = q-1$), and $q > 1$:

$$\begin{aligned} \mathcal{V}_{n+1}^c(q, q) - \mathcal{V}_{n+1}^c(q, q-1) &\geq \frac{w + \lambda(2r - c) + \mu[h(q-2) + h(q-1)]}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\ &= \frac{\lambda r + \lambda(r - c) + (\alpha + \mu)h(q-1) + \mu h(q-1)}{\alpha + 2\lambda + 2\mu} \\ &\geq \frac{2\lambda h(q-1) + (\alpha + \mu)h(q-1) + \mu h(q-1)}{\alpha + 2\lambda + 2\mu} = h(q-1). \end{aligned}$$

If $(m, k+1) \in \Omega_4^c$, $(m, k) \in \Omega_2^c$, and $q = 1$, by S2:

$$\mathcal{V}_{n+1}^c(1, 1) - \mathcal{V}_{n+1}^c(1, 0) \geq \frac{w + \lambda(2r - c) + \mu h(0)}{\alpha + 2\lambda + 2\mu} = \frac{(\alpha + \mu)h(0) + \lambda(r + h(0)) + \mu h(0)}{\alpha + 2\lambda + 2\mu} \geq h(0).$$

(b.v) If $(m, k+1) \in \Omega_4^c$ ($m = q$, $k = q-1$) and $(m, k) \in \Omega_3^c$ ($l < q-1$): For $q > 1$,

$$\begin{aligned} \mathcal{V}_{n+1}^c(q, q) - \mathcal{V}_{n+1}^c(q, q-1) &\geq \frac{w + \lambda[h(q-1) + r] + \mu[h(q-2) + h(q-1)]}{\alpha + 2\lambda + 2\mu} & (\text{by S2}) \\ &\geq \frac{[w + \mu h(q-2)] + 2\lambda h(q-1) + \mu h(q-1)}{\alpha + 2\lambda + 2\mu} \\ &= \frac{(\alpha + \mu)h(q-1) + 2\lambda h(q-1) + \mu h(q-1)}{\alpha + 2\lambda + 2\mu} = h(q-1). \end{aligned}$$

For $q = 1$, we can similarly show the above inequality. (b.vi) If $(m, k + 1) \in \Omega_3^c$ ($m = q, k > l - 1$) and $(m, k) \in \Omega_2^c$ ($k = l$), for $l > 0$:

$$\begin{aligned}\mathcal{V}_{n+1}^c(q, l + 1) - \mathcal{V}_{n+1}^c(q, l) &\geq \frac{w + \lambda[h(l + 1) + r - c] + \mu[h(l) + h(l - 1)]}{\alpha + 2\lambda + 2\mu} \quad (\text{by S2}) \\ &= \frac{[w + \mu h(l - 1)] + \lambda[h(l + 1) + h(l)] + \mu h(l)}{\alpha + 2\lambda + 2\mu} \\ &\geq \frac{(\alpha + \mu)h(l) + 2\lambda h(l) + \mu h(l)}{\alpha + 2\lambda + 2\mu} = h(l).\end{aligned}$$

If $l = 0$, by S2:

$$\mathcal{V}_{n+1}^c(q, 1) - \mathcal{V}_{n+1}^c(q, 0) \geq \frac{w + \lambda[h(1) + r - c] + \mu h(0)}{\alpha + 2\lambda + 2\mu} = \frac{(\alpha + \mu)h(0) + \lambda[h(1) + h(0)] + \mu h(0)}{\alpha + 2\lambda + 2\mu} \geq h(0).$$

(b.vii) If $(m, k + 1) \in \Omega_1^c$, $(m, k) \in \Omega_2^c$, and $k \geq 1$:

$$\begin{aligned}\mathcal{V}_{n+1}^c(m, k + 1) - \mathcal{V}_{n+1}^c(m, k) &\geq \frac{w + \lambda[h(m) - c + h(k + 1)] + \mu[h(k) + h(k - 1)]}{\alpha + 2\lambda + 2\mu} \quad (\text{by S2}) \\ &\geq \frac{(\alpha + \mu)h(k) + \lambda[h(k) + h(k + 1)] + \mu h(k)}{\alpha + 2\lambda + 2\mu} \geq h(k). \quad (\text{as } (m, k) \in \Omega_2^c)\end{aligned}$$

If $k = 0$, by S2:

$$\mathcal{V}_{n+1}^c(m, 1) - \mathcal{V}_{n+1}^c(m, 0) \geq \frac{w + \lambda[h(m) - c + h(1)] + \mu h(0)}{\alpha + 2\lambda + 2\mu} \geq \frac{(\alpha + \mu)h(0) + \lambda[h(0) + h(1)] + \mu h(0)}{\alpha + 2\lambda + 2\mu} \geq h(0).$$

Thus, $\mathcal{V}_{n+1}^c(m, k + 1) \geq \mathcal{V}_{n+1}^c(m, k) + h(k)$ holds for all $0 \leq k < m \leq q$.

Based on parts (a) and (b), we conclude that S1 and S2 hold for $n + 1$. As $n \rightarrow \infty$, we obtain:

$$\text{S1': } \mathcal{V}^c(m, k) \leq \mathcal{V}^c(m, k) + \frac{\lambda\delta}{\alpha};$$

$$\text{S2': } \mathcal{V}^c(m + 1, k) \geq \mathcal{V}^c(m, k) + h(m) \text{ and } \mathcal{V}^c(m, k + 1) \geq \mathcal{V}^c(m, k) + h(k).$$

We next show that $\mathcal{V}^c(l + 1, l) - \mathcal{V}^c(l + 1, l) \leq -\psi_l + \tilde{\psi}_l\delta$, where $\psi_l > 0$ and $\tilde{\psi}_l > 0$ are independent of δ . We consider the following two cases. (a) Suppose that $l = q - 1$. If $q > 1$, we have:

$$\begin{aligned}\mathcal{V}^c(q, q) - \mathcal{V}^c(q, q - 1) &= \frac{2qw + 2\lambda[\mathcal{V}^c(q, q) + r] + 2\mu\mathcal{V}^c(q, q - 1)}{\alpha + 2\lambda + 2\mu} \\ &\quad - \frac{(2q - 1)w + \lambda[2\mathcal{V}^c(q, q) + c] + \mu\mathcal{V}^c(q, q - 2) + \mu\mathcal{V}^c(q - 1, q - 1)}{\alpha + 2\lambda + 2\mu} \\ &\geq \frac{w + \lambda(2r - c) + \mu h(q - 2) + \mu h(q - 1)}{\alpha + 2\lambda + 2\mu} \quad (\text{by S2'}) \\ &= \frac{\lambda c + 2\lambda(r - c) + (\alpha + \mu)h(q - 1) + \mu h(q - 1)}{\alpha + 2\lambda + 2\mu} \\ &= h(q - 1) + \frac{\lambda c}{\alpha + 2\lambda + 2\mu}, \quad \text{and}\end{aligned}$$

$$\begin{aligned}\mathcal{V}^c(q, q - 1) - \mathcal{V}^c(q, q - 1) &= \frac{(2q - 1)w + \lambda[\mathcal{V}^c(q, q - 1) + r + \mathcal{V}^c(q, q)] + \mu[\mathcal{V}^c(q, q - 2) + \mathcal{V}^c(q - 1, q - 1)]}{\alpha + 2\lambda + 2\mu} \\ &\quad - \frac{(2q - 1)w + \lambda[2\mathcal{V}^c(q, q) + c] + \mu[\mathcal{V}^c(q, q - 2) + \mathcal{V}^c(q - 1, q - 1)]}{\alpha + 2\lambda + 2\mu} \\ &\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda(\mathcal{V}^c(q, q - 1) - \mathcal{V}^c(q, q) + r - c) + 2(\lambda + \mu)\frac{\lambda\delta}{\alpha} \right] \quad (\text{by S1'})\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda \left(r - c - h(q-1) - \frac{\lambda c}{\alpha + 2\lambda + 2\mu} \right) + 2(\lambda + \mu) \frac{\lambda \delta}{\alpha} \right] \\
&= \frac{1}{\alpha + 2\lambda + 2\mu} \left[-\frac{\lambda^2 c}{\alpha + 2\lambda + 2\mu} + 2(\lambda + \mu) \frac{\lambda \delta}{\alpha} \right].
\end{aligned}$$

When $l = q - 1$ and $q = 1$, the above two inequalities can be shown similarly based on Lemma A2(a). Thus, by setting $\psi_l = \frac{\lambda^2 c}{(\alpha + 2\lambda + 2\mu)^2}$ and $\tilde{\psi}_l = \frac{2\lambda(\lambda + \mu)}{\alpha(\alpha + 2\lambda + 2\mu)}$, it follows that $\mathcal{V}^{c'}(l+1, l) - \mathcal{V}^c(l+1, l) \leq -\psi_l + \tilde{\psi}_l \delta$.

(b) Consider the case where $l < q - 1$. If $l \geq 1$, we have

$$\begin{aligned}
\mathcal{V}^c(q, l+1) - \mathcal{V}^c(q, l) &= \frac{(q+l+1)w + \lambda[\mathcal{V}^c(q, l+1) + r + \mathcal{V}^c(q, l+2)] + \mu[\mathcal{V}^c(q-1, l+1) + \mathcal{V}^c(q, l)]}{\alpha + 2\lambda + 2\mu} \\
&\quad - \frac{(q+l)w + \lambda[2\mathcal{V}^c(q, l+1) + c] + \mu[\mathcal{V}^c(q-1, l) + \mathcal{V}^c(q, l-1)]}{\alpha + 2\lambda + 2\mu} \\
&\geq \frac{w + \lambda[h(l+1) + r - c] + \mu[h(l) + h(l-1)]}{\alpha + 2\lambda + 2\mu} \quad (\text{by S2'}) \\
&= \frac{\lambda[h(l+1) + h(l)] + (\alpha + \mu)h(l) + \mu h(l)}{\alpha + 2\lambda + 2\mu} \quad (\text{by Lemma A2(b)}) \\
&= h(l) + \frac{\lambda[h(l+1) - h(l)]}{\alpha + 2\lambda + 2\mu}, \text{ and} \\
\mathcal{V}^{c'}(q, l) - \mathcal{V}^c(q, l) &= \frac{(q+l)w + \lambda[\mathcal{V}^{c'}(q, l) + r + \mathcal{V}^{c'}(q, l+1)] + \mu[\mathcal{V}^{c'}(q, l-1) + \mathcal{V}^{c'}(q-1, l)]}{\alpha + 2\lambda + 2\mu} \\
&\quad - \frac{(q+l)w + \lambda[2\mathcal{V}^c(q, l+1) + c] + \mu[\mathcal{V}^c(q, l-1) + \mathcal{V}^c(q-1, l)]}{\alpha + 2\lambda + 2\mu} \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda(\mathcal{V}^c(q, l) - \mathcal{V}^c(q, l+1) + r - c) + 2(\lambda + \mu) \frac{\lambda \delta}{\alpha} \right] \quad (\text{by S1'}) \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda \left(r - c - h(l) - \frac{\lambda[h(l+1) - h(l)]}{\alpha + 2\lambda + 2\mu} \right) + 2(\lambda + \mu) \frac{\lambda \delta}{\alpha} \right] \\
&= \frac{1}{\alpha + 2\lambda + 2\mu} \left[-\frac{\lambda^2[h(l+1) - h(l)]}{\alpha + 2\lambda + 2\mu} + 2(\lambda + \mu) \frac{\lambda \delta}{\alpha} \right].
\end{aligned}$$

If $l = 0$, we can similarly prove the above inequalities based on Lemma A2(a). Thus, we have $\mathcal{V}^{c'}(q, l) - \mathcal{V}^c(q, l) \leq -\xi_q + \tilde{\xi}_q \delta$, where $\xi_q = \frac{\lambda^2[h(l+1) - h(l)]}{(\alpha + 2\lambda + 2\mu)^2} > 0$ and $\tilde{\xi}_q = \frac{2\lambda(\lambda + \mu)}{\alpha(\alpha + 2\lambda + 2\mu)} > 0$ are independent of δ .

For $m = q - 1, q - 2, \dots, l + 1$, we have:

$$\begin{aligned}
\mathcal{V}^{c'}(m, l) - \mathcal{V}^c(m, l) &= \frac{(m+l)w + \lambda[\mathcal{V}^{c'}(m+1, l) + \mathcal{V}^{c'}(m, l+1)] + \mu[\mathcal{V}^{c'}(m, (l-1)^+) + \mathcal{V}^{c'}(m-1, l)]}{\alpha + 2\lambda + 2\mu} \\
&\quad - \frac{(m+l)w + \lambda[\mathcal{V}^c(m+1, l) + \mathcal{V}^c(m, l+1)] + \mu[\mathcal{V}^c(m, (l-1)^+) + \mathcal{V}^c(m-1, l)]}{\alpha + 2\lambda + 2\mu} \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda(\mathcal{V}^{c'}(m+1, l) - \mathcal{V}^c(m+1, l)) + (\lambda + 2\mu) \frac{\lambda \delta}{\alpha} \right] \quad (\text{by S1'}) \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda(-\xi_{m+1} + \tilde{\xi}_{m+1} \delta) + (\lambda + 2\mu) \frac{\lambda \delta}{\alpha} \right] \\
&= \frac{1}{\alpha + 2\lambda + 2\mu} \left[-\lambda \xi_{m+1} + \left(\tilde{\xi}_{m+1} + \frac{\lambda + 2\mu}{\alpha} \right) \lambda \delta \right].
\end{aligned}$$

That is, $\mathcal{V}^{c'}(m, l) - \mathcal{V}^c(m, l) \leq -\xi_m + \tilde{\xi}_m \delta$, where $\xi_m = \frac{\lambda \xi_{m+1}}{\alpha + 2\lambda + 2\mu} > 0$ and $\tilde{\xi}_m = \frac{\lambda}{\alpha + 2\lambda + 2\mu} \left(\tilde{\xi}_{m+1} + \frac{\lambda + 2\mu}{\alpha} \right) > 0$ are independent of δ . Consequently, $\mathcal{V}^{c'}(l+1, l) - \mathcal{V}^c(l+1, l) \leq -\psi_l + \tilde{\psi}_l \delta$ for positive values ψ_l and $\tilde{\psi}_l$ that are independent of δ , by setting $\psi_l = \xi_{l+1}$ and $\tilde{\psi}_l = \tilde{\xi}_{l+1}$.

Based on the two cases above, we have $\mathcal{V}^{c'}(l+1, l) - \mathcal{V}^c(l+1, l) \leq -\psi_l + \tilde{\psi}_l \delta$, where $\psi_l > 0$ and $\tilde{\psi}_l > 0$ are independent of δ . Next, we aim to demonstrate that for $m = l, l-1, \dots, 0$, there exist positive values $\psi_m, \tilde{\psi}_m, \varphi_m$, and $\tilde{\varphi}_m$, all independent of δ , such that the following properties hold:

$$\text{T1: } \mathcal{V}^{c'}(m+1, m) - \mathcal{V}^c(m+1, m) \leq -\psi_m + \tilde{\psi}_m \delta,$$

$$\text{T2: } \mathcal{V}^{c'}(m, m) - \mathcal{V}^c(m, m) \leq -\varphi_m + \tilde{\varphi}_m \delta.$$

Having established T1 for $m = l$, we now show that T1 implies T2 for a given m . Specifically,

$$\begin{aligned} \mathcal{V}^{c'}(m, m) - \mathcal{V}^c(m, m) &= \frac{2\lambda[\mathcal{V}^{c'}(m+1, m) - \mathcal{V}^c(m+1, m)] + 2\mu[\mathcal{V}^{c'}(m, m-1) - \mathcal{V}^c(m, m-1)]}{\alpha + 2\lambda + 2\mu} \\ &\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[2\lambda(-\psi_m + \tilde{\psi}_m \delta) + 2\mu \frac{\lambda \delta}{\alpha} \right] \quad (\text{by T1 and S1'}) \\ &= \frac{2\lambda}{\alpha + 2\lambda + 2\mu} \left[-\psi_m + \left(\tilde{\psi}_m + \frac{\mu}{\alpha} \right) \delta \right]. \end{aligned}$$

Hence, T2 holds for m , where $\varphi_m = \frac{2\lambda\psi_m}{\alpha+2\lambda+2\mu} > 0$ and $\tilde{\varphi}_m = \frac{2\lambda}{\alpha+2\lambda+2\mu} \left(\tilde{\psi}_m + \frac{\mu}{\alpha} \right) > 0$ are independent of δ .

We then demonstrate that T2 for m implies T1 for $m-1$. Indeed,

$$\begin{aligned} \mathcal{V}^{c'}(m, m-1) - \mathcal{V}^c(m, m-1) &\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda \left(\mathcal{V}^{c'}(m, m) - \mathcal{V}^c(m, m) \right) + \lambda \left(\delta + \frac{\lambda \delta}{\alpha} \right) + 2\mu \frac{\lambda \delta}{\alpha} \right] \quad (\text{by S1'}) \\ &\leq \frac{\lambda}{\alpha + 2\lambda + 2\mu} \left[-\varphi_m + \left(\tilde{\varphi}_m + 1 + \frac{\lambda + 2\mu}{\alpha} \right) \delta \right]. \quad (\text{by T2}) \end{aligned}$$

Hence, T1 holds for $m-1$, where $\psi_{m-1} = \frac{\lambda\varphi_m}{\alpha+2\lambda+2\mu} > 0$ and $\tilde{\psi}_{m-1} = \frac{\lambda}{\alpha+2\lambda+2\mu} \left(\tilde{\varphi}_m + 1 + \frac{\lambda+2\mu}{\alpha} \right) > 0$ are independent of δ . By induction, T1 and T2 hold for $m = l, l-1, \dots, 0$.

Based on T2 for $m = 0$, we have $\mathcal{V}^{c'}(0, 0) - \mathcal{V}^c(0, 0) \leq -\varphi_0 + \tilde{\varphi}_0 \delta$. It follows that $\mathcal{V}^{c'}(0, 0) < \mathcal{V}^c(0, 0)$ provided that $\delta < \min\{\varphi_0/\tilde{\varphi}_0, \bar{\delta}\}$. $\square$

Next, we present the following proposition, which demonstrates that even in situations where patients never opt out (i.e., $r \geq h(\infty)$), the system cost under individual optimization can still decrease as the switching cost increases.

Proposition A1. *For a symmetric system with the following conditions: $r \geq h(\infty)$, $c = h(1) - h(0)$, $\alpha > \mu$, and $\delta > 0$ is sufficiently small, we have $\mathcal{V}^{c'}(0, 0) < \mathcal{V}^{c-\delta}(0, 0)$, i.e., when the switching cost is reduced from c to $c - \delta$, the system cost under individual optimization strictly increases.*

PROOF OF PROPOSITION A1. We assume that $r \geq h(\infty)$, $h(1) - h(0) > h(\infty) - h(1)$ (equivalent to $\alpha > \mu$), and $c = h(1) - h(0)$. Additionally, we require $c - \delta > h(\infty) - h(1)$, which is equivalent to $\delta < 2h(1) - h(0) - h(\infty) = w\mu(\alpha - \mu)/[\alpha(\alpha + \mu)^2]$. The above conditions imply that under switching cost $c - \delta$, each arriving patient will switch iff the home station is non-empty and the other station is empty. Under switching cost c , each arriving patient will switch iff the queue length of the home station is 2 or higher, and the other station is empty.

When the switching cost is $c' = c - \delta$, the system cost under individual optimization can be obtained via value iteration with $\mathcal{V}_0^{c'}(\mathbf{x}) = 0$ for all $\mathbf{x}$ and

$$\begin{aligned} \mathcal{V}_{n+1}^{c'}(0, 0) &= \frac{2\lambda\mathcal{V}_n^{c'}(1, 0) + 2\mu\mathcal{V}_n^{c'}(0, 0)}{\alpha + 2\lambda + 2\mu}, \\ \mathcal{V}_{n+1}^{c'}(1, 0) &= \frac{w + 2\lambda\mathcal{V}_n^{c'}(1, 1) + \lambda c' + \mu\mathcal{V}_n^{c'}(1, 0) + \mu\mathcal{V}_n^{c'}(0, 0)}{\alpha + 2\lambda + 2\mu}, \\ \mathcal{V}_{n+1}^{c'}(m, 0) &= \frac{mw + 2\lambda\mathcal{V}_n^{c'}(m, 1) + \lambda c' + \mu\mathcal{V}_n^{c'}(m, 0) + \mu\mathcal{V}_n^{c'}(m-1, 0)}{\alpha + 2\lambda + 2\mu}, \quad m \geq 2, \\ \mathcal{V}_{n+1}^{c'}(m, k) &= \frac{(m+k)w + \lambda\mathcal{V}_n^{c'}(m+1, k) + \lambda\mathcal{V}_n^{c'}(m, k+1) + \mu\mathcal{V}_n^{c'}(m, k-1) + \mu\mathcal{V}_n^{c'}(m-1, k)}{\alpha + 2\lambda + 2\mu}, \quad m, k \geq 1. \end{aligned}$$

Then $\mathcal{V}_n^{c'}(\mathbf{x}) \rightarrow \mathcal{V}^{c'}(\mathbf{x})$ for all $\mathbf{x}$ as $n \rightarrow \infty$.

Similarly, when the switching cost is c , the system cost under individual optimization can be obtained via value iteration with $\mathcal{V}_0^c(\mathbf{x}) = 0$ for all $\mathbf{x}$ and

$$\begin{aligned}\mathcal{V}_{n+1}^c(0,0) &= \frac{2\lambda\mathcal{V}_n^c(1,0) + 2\mu\mathcal{V}_n^c(0,0)}{\alpha + 2\lambda + 2\mu}, \\ \mathcal{V}_{n+1}^c(1,0) &= \frac{w + \lambda\mathcal{V}_n^c(2,0) + \lambda\mathcal{V}_n^c(1,1) + \mu\mathcal{V}_n^c(1,0) + \mu\mathcal{V}_n^c(0,0)}{\alpha + 2\lambda + 2\mu}, \\ \mathcal{V}_{n+1}^c(m,0) &= \frac{mw + 2\lambda\mathcal{V}_n^c(m,1) + \lambda c + \mu\mathcal{V}_n^c(m,0) + \mu\mathcal{V}_n^c(m-1,0)}{\alpha + 2\lambda + 2\mu}, \quad m \geq 2, \\ \mathcal{V}_{n+1}^c(m,k) &= \frac{(m+k)w + \lambda\mathcal{V}_n^c(m+1,k) + \lambda\mathcal{V}_n^c(m,k+1) + \mu\mathcal{V}_n^c(m,k-1) + \mu\mathcal{V}_n^c(m-1,k)}{\alpha + 2\lambda + 2\mu}, \quad m, k \geq 1.\end{aligned}$$

Then $\mathcal{V}_n^c(\mathbf{x}) \rightarrow \mathcal{V}^c(\mathbf{x})$ for all $\mathbf{x}$ as $n \rightarrow \infty$.

We define:

$$\begin{aligned}\tau_a &= \frac{\alpha\lambda(\alpha+\mu)(\alpha^2+4\lambda^2+4\mu^2+4\alpha\lambda+4\alpha\mu+6\lambda\mu)c + (16\lambda^4+8(4\alpha+\mu)\lambda^3+4\alpha(6\alpha+7\mu)\lambda^2+2\alpha(\alpha+\mu)(4\alpha+7\mu)\lambda+\alpha(\alpha+\mu)(\alpha+2\mu)^2)w}{\alpha(\alpha+2\lambda+\mu)(8\lambda^3+4(3\alpha+2\mu)\lambda^2+2(3\alpha^2+3\mu^2+7\alpha\mu)\lambda+(\alpha+\mu)(\alpha+2\mu)^2)}, \\ \tau_b &= \frac{-\alpha\lambda(\alpha^2+4\lambda^2+4\alpha\lambda+2\alpha\mu+2\lambda\mu)c + (8\lambda^3+4(3\alpha+\mu)\lambda^2+2\alpha(3\alpha+5\mu)\lambda+\alpha(\alpha+2\mu)^2)w}{\alpha(8\lambda^3+4(3\alpha+2\mu)\lambda^2+2(3\alpha^2+3\mu^2+7\alpha\mu)\lambda+(\alpha+\mu)(\alpha+2\mu)^2)}, \\ \tau_c &= \frac{\alpha\lambda\mu(\alpha^2-4\lambda^2+4\mu^2+4\alpha\mu+2\lambda\mu)c + (\alpha+2\lambda+2\mu)(8\lambda^3+4(3\alpha+\mu)\lambda^2+2\alpha(3\alpha+5\mu)\lambda+\alpha(\alpha+2\mu)^2)w}{\alpha(\alpha+2\lambda+\mu)(8\lambda^3+4(3\alpha+2\mu)\lambda^2+2(3\alpha^2+3\mu^2+7\alpha\mu)\lambda+(\alpha+\mu)(\alpha+2\mu)^2)}, \text{ and} \\ \tau_d &= \frac{-\alpha\lambda\mu(\alpha^2+4\lambda^2-2\mu^2+4\alpha\lambda+2\lambda\mu)c + (16\lambda^4+8(4\alpha+3\mu)\lambda^3+8(3\alpha^2+2\mu^2+6\alpha\mu)\lambda^2+2(4\alpha^3+2\mu^3+16\alpha\mu^2+15\alpha^2\mu)\lambda+\alpha(\alpha+2\mu)^3)w}{\alpha(\alpha+2\lambda+\mu)(8\lambda^3+4(3\alpha+2\mu)\lambda^2+2(3\alpha^2+3\mu^2+7\alpha\mu)\lambda+(\alpha+\mu)(\alpha+2\mu)^2)}.\end{aligned}$$

$h(1) - h(0) = w\mu/(\alpha + \mu)^2$, we can easily verify that $\tau_a, \tau_b, \tau_c, \tau_d < \frac{w}{\alpha}$. Under the conditions $\alpha > \mu$ and $c = h(1) - h(0)$, we can verify that $\frac{\lambda c + \mu \tau_c}{\alpha + 2\lambda + 2\mu} < c$. We further require that $\tau_i + \frac{\lambda \delta}{\alpha + \mu} < \frac{w}{\alpha}$ (for $i = a, b, c, d$), i.e., $\delta < \frac{(\alpha + \mu)[w - \alpha \max\{\tau_a, \tau_b, \tau_c, \tau_d\}]}{\alpha \lambda}$.

We will show the following relations for $n \geq 0$:

$$\begin{aligned}\text{R1: } \mathcal{V}_n^c(m, k) &\leq \mathcal{V}_n^{c'}(m, k) + \frac{\lambda \delta}{\alpha}; & \text{R2: } \mathcal{V}_n^{c'}(1, 0) &\leq \mathcal{V}_n^{c'}(0, 0) + \tau_a + \frac{\lambda \delta}{\alpha + \mu}; \\ \text{R3: } \mathcal{V}_n^{c'}(1, 1) &\leq \mathcal{V}_n^{c'}(1, 0) + \tau_b + \frac{\lambda \delta}{\alpha + \mu}; & \text{R4: } \mathcal{V}_n^{c'}(2, 0) &\leq \mathcal{V}_n^{c'}(1, 0) + \tau_c + \frac{\lambda \delta}{\alpha + \mu}; \\ \text{R5: } \mathcal{V}_n^{c'}(2, 1) &\leq \mathcal{V}_n^{c'}(1, 1) + \tau_d + \frac{\lambda \delta}{\alpha + \mu}; & \text{R6: } \mathcal{V}_n^{c'}(m+1, k) &\leq \mathcal{V}_n^{c'}(m, k) + \frac{w}{\alpha}; \\ \text{R7: } \mathcal{V}_n^{c'}(2, 0) &\leq \mathcal{V}_n^{c'}(1, 1) + \frac{\lambda c + \mu \tau_c}{\alpha + 2\lambda + 2\mu}.\end{aligned}$$

It is clear that R1 to R7 are satisfied for $n = 0$. In the following, assuming R1, R2, ..., R7 are satisfied for n , we show they continue to hold for $n + 1$.

Consider R1 first. Clearly R1 is true for $(m, k) = (0, 0)$, and $m \geq 1$ and $k \geq 1$. For $(m, k) = (1, 0)$:

$$\begin{aligned}\mathcal{V}_{n+1}^c(1, 0) - \mathcal{V}_{n+1}^{c'}(1, 0) &\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda \left(\mathcal{V}_n^c(2, 0) - \mathcal{V}_n^{c'}(1, 1) + \frac{\lambda \delta}{\alpha} \right) - \lambda c' + 2\mu \frac{\lambda \delta}{\alpha} \right] & (\text{by R1}) \\ &\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda \left(\mathcal{V}_n^{c'}(2, 0) - \mathcal{V}_n^{c'}(1, 1) \right) - \lambda c' + (2\lambda + 2\mu) \frac{\lambda \delta}{\alpha} \right] & (\text{by R1}) \\ &< \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda c - \lambda(c - \delta) + (2\lambda + 2\mu) \frac{\lambda \delta}{\alpha} \right] = \frac{\lambda \delta}{\alpha}. & (\text{by R7})\end{aligned}$$

For (m, k) where $m \geq 2$ and $k = 0$:

$$\mathcal{V}_{n+1}^c(m, 0) - \mathcal{V}_{n+1}^{c'}(m, 0) \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[(2\lambda + 2\mu) \frac{\lambda \delta}{\alpha} + \lambda \delta \right] = \frac{\lambda \delta}{\alpha},$$

where the inequality is by R1 for n . Thus, R1 continues to hold for $n + 1$.

Next, we consider R2, R3, R4, and R5. We can check that τ_a, τ_b, τ_c , and τ_d satisfy:

$$\tau_a = \frac{w + \lambda c + 2\lambda \tau_b + \mu \tau_a}{\alpha + 2\lambda + 2\mu}, \tau_b = \frac{w - \lambda c + 2\lambda \tau_d + \mu \tau_a}{\alpha + 2\lambda + 2\mu}, \tau_c = \frac{w + 2\lambda \tau_d + \mu \tau_a + \mu \tau_c}{\alpha + 2\lambda + 2\mu}, \tau_d = \frac{w + \frac{2\lambda w}{\alpha} + \mu \tau_b + \mu \tau_c}{\alpha + 2\lambda + 2\mu}.$$

Then, we have

$$\begin{aligned}
\mathcal{V}_{n+1}^{c'}(1, 0) - \mathcal{V}_{n+1}^{c'}(0, 0) &= \frac{w + 2\lambda[\mathcal{V}_n^{c'}(1, 1) - \mathcal{V}_n^{c'}(1, 0)] + \lambda c' + \mu[\mathcal{V}_n^{c'}(1, 0) - \mathcal{V}_n^{c'}(0, 0)]}{\alpha + 2\lambda + 2\mu} \\
&\leq \frac{w + 2\lambda\tau_b + \lambda c' + \mu\tau_a + (2\lambda + \mu)\lambda\delta/(\alpha + \mu)}{\alpha + 2\lambda + 2\mu} \leq \tau_a + \frac{\lambda\delta}{\alpha + \mu}; \quad (\text{by R2 and R3}) \\
\mathcal{V}_{n+1}^{c'}(1, 1) - \mathcal{V}_{n+1}^{c'}(1, 0) &= \frac{w + 2\lambda[\mathcal{V}_n^{c'}(2, 1) - \mathcal{V}_n^{c'}(1, 1)] - \lambda c' + \mu[\mathcal{V}_n^{c'}(1, 0) - \mathcal{V}_n^{c'}(0, 0)]}{\alpha + 2\lambda + 2\mu} \\
&\leq \frac{w + 2\lambda\tau_d - \lambda c' + \mu\tau_a + (2\lambda + \mu)\lambda\delta/(\alpha + \mu)}{\alpha + 2\lambda + 2\mu} = \tau_b + \frac{\lambda\delta}{\alpha + \mu}; \quad (\text{by R2 and R5}) \\
\mathcal{V}_{n+1}^{c'}(2, 0) - \mathcal{V}_{n+1}^{c'}(1, 0) &= \frac{w + 2\lambda(\mathcal{V}_n^{c'}(2, 1) - \mathcal{V}_n^{c'}(1, 1)) + \mu(\mathcal{V}_n^{c'}(1, 0) - \mathcal{V}_n^{c'}(0, 0)) + \mu(\mathcal{V}_n^{c'}(2, 0) - \mathcal{V}_n^{c'}(1, 0))}{\alpha + 2\lambda + 2\mu} \\
&\leq \frac{w + 2\lambda\tau_d + \mu\tau_a + \mu\tau_c + 2(\lambda + \mu)\lambda\delta/(\alpha + \mu)}{\alpha + 2\lambda + 2\mu} \leq \tau_c + \frac{\lambda\delta}{\alpha + \mu}; \quad (\text{by R2, R4, and R5}) \\
\mathcal{V}_{n+1}^{c'}(2, 1) - \mathcal{V}_{n+1}^{c'}(1, 1) &\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[w + 2\lambda\frac{w}{\alpha} + \mu(\mathcal{V}_n^{c'}(1, 1) - \mathcal{V}_n^{c'}(0, 1)) + \mu(\mathcal{V}_n^{c'}(2, 0) - \mathcal{V}_n^{c'}(1, 0)) \right] \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left(w + 2\lambda\frac{w}{\alpha} + \mu\tau_b + \mu\tau_c + 2\mu\lambda\delta/(\alpha + \mu) \right) = \tau_d + \frac{\lambda\delta}{\alpha + \mu}, \quad (\text{by R3 and R4})
\end{aligned}$$

where the first inequality is true because, according to R6, we have $\mathcal{V}_n^{c'}(3, 1) - \mathcal{V}_n^{c'}(2, 1) \leq w/\alpha$ and $\mathcal{V}_n^{c'}(2, 2) - \mathcal{V}_n^{c'}(1, 2) \leq w/\alpha$. Thus, R2, R3, R4, and R5 continue to hold for $n + 1$.

Then, we consider R6. For $(m, k) = (0, 0)$, the conclusion is from R2 and the fact that $\tau_a < w/\alpha$. For (m, k) where $m \geq 1$ and $k = 0$:

$$\mathcal{V}_{n+1}^{c'}(m + 1, 0) - \mathcal{V}_{n+1}^{c'}(m, 0) \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[w + (2\lambda + 2\mu)\frac{w}{\alpha} \right] = \frac{w}{\alpha},$$

where the inequality is true due to R6 for n . For (m, k) , where $m = 0$ and $k \geq 1$:

$$\begin{aligned}
&\mathcal{V}_{n+1}^{c'}(1, k) - \mathcal{V}_{n+1}^{c'}(0, k) \\
&= \frac{w + \lambda(\mathcal{V}_n^{c'}(2, k) - \mathcal{V}_n^{c'}(1, k)) + \lambda(\mathcal{V}_n^{c'}(1, k + 1) - \mathcal{V}_n^{c'}(1, k)) - \lambda c' + \mu(\mathcal{V}_n^{c'}(1, k - 1) - \mathcal{V}_n^{c'}(0, k - 1))}{\alpha + 2\lambda + 2\mu} \\
&\leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[w + (2\lambda + \mu)\frac{w}{\alpha} - \lambda c' \right] \leq \frac{w}{\alpha},
\end{aligned}$$

where the first inequality is true due to R6 for n . For (m, k) where $m \geq 1$ and $k \geq 1$:

$$\mathcal{V}_{n+1}^{c'}(m + 1, k) - \mathcal{V}_{n+1}^{c'}(m, k) \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[w + (2\lambda + 2\mu)\frac{w}{\alpha} \right] = \frac{w}{\alpha},$$

where the inequality is from R6 for n . Hence, R6 continues to hold for $n + 1$.

Now we consider R7. It continues to hold for $n + 1$ as

$$\begin{aligned}
\mathcal{V}_{n+1}^{c'}(2, 0) - \mathcal{V}_{n+1}^{c'}(1, 1) &= \frac{\lambda c' + \mu[\mathcal{V}_n^{c'}(2, 0) - \mathcal{V}_n^{c'}(1, 0)]}{\alpha + 2\lambda + 2\mu} \leq \frac{\lambda(c - \delta) + \mu(\tau_c + \frac{\lambda\delta}{\alpha + \mu})}{\alpha + 2\lambda + 2\mu} \quad (\text{by R4}) \\
&\leq \frac{\lambda c + \mu\tau_c}{\alpha + 2\lambda + 2\mu}.
\end{aligned}$$

Based on the above, as n goes to infinity, we have

$$\text{R1': } \mathcal{V}^c(m, k) \leq \mathcal{V}^{c'}(m, k) + \frac{\lambda\delta}{\alpha}; \quad \text{R7': } \mathcal{V}^{c'}(2, 0) \leq \mathcal{V}^{c'}(1, 1) + \frac{\lambda c + \mu\tau_c}{\alpha + 2\lambda + 2\mu}.$$

Then, we have

$$\mathcal{V}^c(1, 0) - \mathcal{V}^{c'}(1, 0) = \frac{1}{\alpha + 2\lambda + 2\mu} [w + \lambda\mathcal{V}^c(2, 0) + \lambda\mathcal{V}^c(1, 1) + \mu\mathcal{V}^c(1, 0) + \mu\mathcal{V}^c(0, 0)]$$

$$\begin{aligned}
& - \frac{1}{\alpha + 2\lambda + 2\mu} \left[w + 2\lambda \mathcal{V}^{c'}(1, 1) + \lambda c' + \mu \mathcal{V}^{c'}(1, 0) + \mu \mathcal{V}^{c'}(0, 0) \right] \\
& \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda \left(\mathcal{V}^c(2, 0) - \mathcal{V}^{c'}(1, 1) + \frac{\lambda \delta}{\alpha} \right) - \lambda c' + 2\mu \frac{\lambda \delta}{\alpha} \right] \quad (\text{by R1'}) \\
& \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\lambda \left(\mathcal{V}^{c'}(2, 0) - \mathcal{V}^{c'}(1, 1) \right) - \lambda c' + 2(\lambda + \mu) \frac{\lambda \delta}{\alpha} \right] \quad (\text{by R1'}) \\
& \leq \frac{1}{\alpha + 2\lambda + 2\mu} \left[\frac{\lambda(\lambda c + \mu \tau_c)}{\alpha + 2\lambda + 2\mu} - \lambda(c - \delta) + 2(\lambda + \mu) \frac{\lambda \delta}{\alpha} \right] \quad (\text{by R7'}) \\
& = \frac{\lambda}{\alpha + 2\lambda + 2\mu} \left[- \left(\frac{(\alpha + \lambda + 2\mu)c - \mu \tau_c}{\alpha + 2\lambda + 2\mu} \right) + (\alpha + 2\lambda + 2\mu) \frac{\delta}{\alpha} \right].
\end{aligned}$$

Note that $\frac{\lambda c + \mu \tau_c}{\alpha + 2\lambda + 2\mu} < c$, we have $\mathcal{V}^c(1, 0) < \mathcal{V}^{c'}(1, 0)$ if $\delta < \frac{\alpha[(\alpha + \lambda + 2\mu)c - \mu \tau_c]}{(\alpha + 2\lambda + 2\mu)^2}$. Therefore, $\mathcal{V}^c(0, 0) = \frac{2\lambda}{\alpha + 2\lambda} \mathcal{V}^c(1, 0) < \frac{2\lambda}{\alpha + 2\lambda} \mathcal{V}^{c'}(1, 0) = \mathcal{V}^{c'}(0, 0)$ when $\delta < \min \left\{ \frac{w\mu(\alpha - \mu)}{\alpha(\alpha + \mu)^2}, \frac{(\alpha + \mu)[w - \alpha \max\{\tau_a, \tau_b, \tau_c, \tau_d\}]}{\alpha\lambda}, \frac{\alpha[(\alpha + \lambda + 2\mu)c - \mu \tau_c]}{(\alpha + 2\lambda + 2\mu)^2} \right\}$. $\square$

PROOF OF PROPOSITION 9. Suppose a patient arrives at station i and observes state $\mathbf{x}$. Under the subsidies $s_i^O(\mathbf{x}) = -\mathcal{E}_i(\mathbf{x})$, and $s_i^S(\mathbf{x}) = -\mathcal{E}_i(\mathbf{x}) + \mathcal{E}_j(\mathbf{x})$, the utilities of staying, switching, and opting out become $h_i(x_i)$, $c_i - s_i^S(\mathbf{x}) + h_j(x_j)$, and $r_i - s_i^O(\mathbf{x})$, respectively. Note that $c_i - s_i^S(\mathbf{x}) + h_j(x_j) = c_i + h_i(x_i) + V(\mathbf{x} + \mathbf{e}_j) - V(\mathbf{x} + \mathbf{e}_i)$ and $r_i - s_i^O(\mathbf{x}) = r_i + h_i(x_i) + V(\mathbf{x}) - V(\mathbf{x} + \mathbf{e}_i)$, the patient's individual choice is

$$\arg \min \left\{ h_i(x_i), c_i - s_i^S(\mathbf{x}) + h_j(x_j), r_i - s_i^O(\mathbf{x}) \right\} = \arg \min \left\{ V(\mathbf{x} + \mathbf{e}_i), c_i + V(\mathbf{x} + \mathbf{e}_j), r_i + V(\mathbf{x}) \right\},$$

which is the system-optimal solution. $\square$

PROOF OF PROPOSITION 10. We first consider part (a). Let $\tilde{V}_0(\mathbf{x}) = 0$ for all $\mathbf{x}$. For $n \geq 0$, define $\tilde{V}_{n+1}(\mathbf{x}) = \frac{1}{\alpha + \beta} [\sum_{i=1}^2 \sum_{k=1}^M \lambda_i p_{ik} \min\{w_k \varsigma_i(x_i) + \tilde{V}_n(\mathbf{x} + \mathbf{e}_i), c_i + w_k \varsigma_j(x_j) + \tilde{V}_n(\mathbf{x} + \mathbf{e}_j), r_i + \tilde{V}_n(\mathbf{x})\} + \sum_{i=1}^2 \mu_i \tilde{V}_n(\mathbf{x} - \mathbf{e}_i) \mathbf{1}[x_i > 0]]$. We prove by induction that $\tilde{V}_n$ is non-decreasing in each component of $\mathbf{x}$. The claim is trivial for $n = 0$ by the definition of $\tilde{V}_0$. Suppose that $\tilde{V}_n(\mathbf{x} + \mathbf{e}_i) \geq \tilde{V}_n(\mathbf{x})$, $i = 1, 2$. Since ς_i, ς_j , and $\tilde{V}_n$ are non-decreasing in each component of $\mathbf{x}$, every term on the right-hand side of the recursion is non-decreasing in x_i and x_j . Therefore, $\tilde{V}_{n+1}(\mathbf{x} + \mathbf{e}_i) \geq \tilde{V}_{n+1}(\mathbf{x})$, $i = 1, 2$. Hence, $\tilde{V}_n$ is non-decreasing in each component for all n . Passing to the limit as $n \rightarrow \infty$, we obtain $\tilde{V}(\mathbf{x} + \mathbf{e}_i) \geq \tilde{V}(\mathbf{x})$, $i = 1, 2$.

Suppose that an arriving patient prefers to opt out, i.e.,

$$r_i < w_k \varsigma_i(x_i) \text{ and } r_i < c_i + w_k \varsigma_j(x_j).$$

Then, from $\tilde{V}(\mathbf{x}) \leq \tilde{V}(\mathbf{x} + \mathbf{e}_i)$ and $\tilde{V}(\mathbf{x}) \leq \tilde{V}(\mathbf{x} + \mathbf{e}_j)$, we have

$$r_i + \tilde{V}(\mathbf{x}) < w_k \varsigma_i(x_i) + \tilde{V}(\mathbf{x} + \mathbf{e}_i) \text{ and } r_i + \tilde{V}(\mathbf{x}) < c_i + w_k \varsigma_j(x_j) + \tilde{V}(\mathbf{x} + \mathbf{e}_j).$$

Thus, it follows from the optimality equation that the system-optimal decision is to opt out the patient.

We then consider part (b). Suppose a type- k patient arrives at station i and observes state $\mathbf{x}$. Under an opt-out subsidy level $s_i^O(\mathbf{x}) = \tilde{V}(\mathbf{x} + \mathbf{e}_i) - \tilde{V}(\mathbf{x})$ and a switching subsidy level $s_i^S(\mathbf{x}) = \tilde{V}(\mathbf{x} + \mathbf{e}_i) - \tilde{V}(\mathbf{x} + \mathbf{e}_j)$, the utilities of staying, switching, and opting out become $w_k \varsigma_i(x_i)$, $c_i - s_i^S(\mathbf{x}) + w_k \varsigma_j(x_j)$, and $r_i - s_i^O(\mathbf{x})$, respectively. Note that $c_i - s_i^S(\mathbf{x}) + w_k \varsigma_j(x_j) = c_i + w_k \varsigma_j(x_j) + \tilde{V}(\mathbf{x} + \mathbf{e}_j) - \tilde{V}(\mathbf{x} + \mathbf{e}_i)$ and $r_i - s_i^O(\mathbf{x}) = r_i + \tilde{V}(\mathbf{x}) - \tilde{V}(\mathbf{x} + \mathbf{e}_i)$, the patient's individual choice is

$$\begin{aligned}
& \arg \min \left\{ w_k \varsigma_i(x_i), c_i - s_i^S(\mathbf{x}) + w_k \varsigma_j(x_j), r_i - s_i^O(\mathbf{x}) \right\} \\
& = \arg \min \left\{ w_k \varsigma_i(x_i) + \tilde{V}(\mathbf{x} + \mathbf{e}_i), c_i + w_k \varsigma_j(x_j) + \tilde{V}(\mathbf{x} + \mathbf{e}_j), r_i + \tilde{V}(\mathbf{x}) \right\},
\end{aligned}$$

which is the system-optimal solution. $\square$