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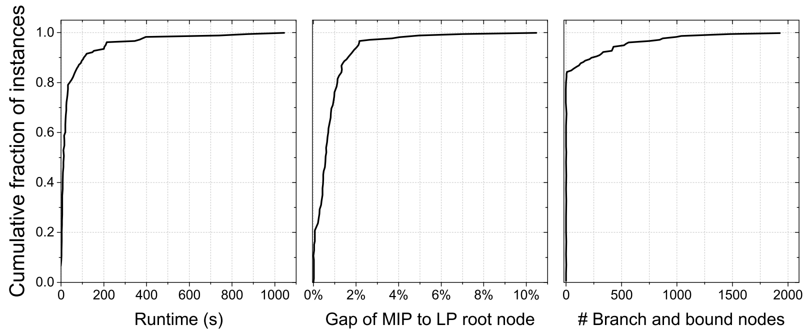

Figure EC.1 Distribution of the computational runtimes, the percentage gap of the MIP optimal solution with respect to its LP root node, and number of branch and bound nodes.

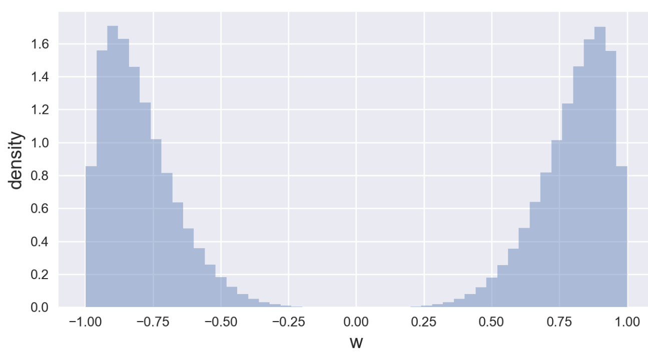

Figure EC.2 The distribution of the demand uncertainty factors used in the simulations.

EC.1. Proofs

EC.1.1. Proof of Lemma 1

Note that V^1 defined by the Bellman equations (3.1)–(3.2) gives the optimal expected profit of a pricing policy that sets the same price for customers arriving in each period t. An upper bound for V^1 is the optimal expected profit for a pricing policy that can set a different price and fulfillment for each arriving customer. We will develop an upper bound on the optimal expected profit of this more flexible policy.

Suppose period $t \in [T]$ channel-zone demand has a multinomial distribution with N^t as the number of trials and $\psi^t_{mz}(p_e, p_{bz})$ as the probability of a demand occuring in zone $z \in [Z]$ and channel $m \in \{e, b\}$ given the online price p_e and store price p_{bz}. Suppose that $\psi^t_{ez}(\infty, p) = \psi^t_{bz}(p, \infty) = 0$ for any p. The demand in period t is equivalent to subdividing the period into N^t subperiods, where in each subperiod there is no more than one arrival (across all zones) and the probability of an arrival to zone z making a purchase in channel m given prices (p_e, p_b) is $\psi^t_{mz}(p_e, p_b)$. Let p^{tn}_e be the online price and p^{tn}_{bz} be the store price observed by an arrival $n \in [N^t]$ to zone $z \in [Z]$ during period $t \in [T]$. Define $\tilde{D}^{tn}_{mz}(p^{tn}_e, p^{tn}_{bz})$ as the stochastic demand for channel $m = e, b$ in zone z during subperiod $n \in [N^t]$ of period t. Note that $E[\tilde{D}^{tn}_{mz}(p^{tn}_e, p^{tn}_{bz})] = \psi^t_{mz}(p^{tn}_e, p^{tn}_{bz})$, and that

$$\sum_{z\in[Z]} \sum_{m=e,b} \psi^t_{mz}(p^{tn}_e, p^{tn}_{bz}) \leq 1, \quad \forall n \in [N^t], \forall t \in [T]. \tag{A.1}$$

Let us assume that there exists a $p_\infty \in \Omega$ such that $\tilde{D}^{tn}_{bz}(p_e, p_\infty) = 0$ for any $p_e \in \Omega$. Define $\tilde{R}^{tn}_{ez}(p^{tn}_e, p^{tn}_{bz}) = p^t_e \tilde{D}^{tn}_{ez}(p^{tn}_e, p^{tn}_{bz})$ and $\tilde{R}^{tn}_{bz}(p^{tn}_e, p^{tn}_{bz}) = p^t_{bz} \tilde{D}^{tn}_{bz}(p^{tn}_e, p^{tn}_{bz})$ as the stochastic revenue from the online and store channel, respectively, in zone z during subperiod $n \in [N^t]$ of period t.

Let $\tilde{\lambda}^{tnz}_{ij}$ be a binary random variable which equals to 1 if and only if a zone z arrival in subperiod n of period t is presented by price p_i online and p_j in the store. Let $\tilde{\mu}^{tne}_i$ be a binary random variable which equals to 1 if and only if a zone z arrival in subperiod n of period t is presented with price p_i online. Further, let $p_{I+1} := \infty$. Let $\tilde{y}^{tn}_{ez}$ and $\tilde{y}^{tn}_{z'z}$ be the random fulfillment from the EFC and the store in zone z', respectively, of an arrival in zone z during subperiod n of period t. Let $\tilde{s}^{tn}_{bz}$ be the random store sales in zone z during subperiod n of period t. Therefore, $V^1 \le V^*$, where

$$V^* = \underset{\lambda, \mu, y, s}{\text{maximize}} \quad E\left[\sum_{t\in[T]}\sum_{n\in[N^t]}\sum_{z\in[Z]}\sum_{m=e,b}\sum_{i\in[I+1]}\sum_{j\in[I+1]} \tilde{\lambda}^{tnz}_{ij}\tilde{R}^{tn}_{mz}(p_i,p_j)\right] + q\left(x_e + \sum_{z\in[Z]} x_{bz}\right) \tag{A.2a}$$

$$- E\left[\sum_{t\in[T]}\sum_{n\in[N^t]}\sum_{z\in[Z]}\left(q\tilde{s}^{tn}_{bz} + (c_{ez}+q)\tilde{y}^{tn}_{ez} + \sum_{z'\in[Z]}(c_{z'z}+q)\tilde{y}^{tn}_{z'z}\right)\right] \tag{A.2b}$$

subject to

$$\tilde{y}^{tn}_{ez} + \sum_{z'\in[Z]} \tilde{y}^{tn}_{z'z} = \sum_{i\in[I+1]}\sum_{j\in[I+1]} \tilde{\lambda}^{tnz}_{ij}\tilde{D}^{tn}_{ez}(p_i,p_j), \quad \forall z\in[Z], \forall t\in[T], \forall n\in[N^t], \tag{A.2c}$$

$$\tilde{s}^{tn}_{bz} = \sum_{i\in[I+1]}\sum_{j\in[I+1]} \tilde{\lambda}^{tnz}_{ij}\tilde{D}^{tn}_{bz}(p_i,p_j), \quad \forall z\in[Z], \forall t\in[T], \forall n\in[N^t], \tag{A.2d}$$

$$\sum_{t\in[T]}\sum_{n\in[N^t]} \tilde{y}^{tn}_{ez} \le x_e, \tag{A.2e}$$

$$\sum_{t\in[T]}\sum_{n\in[N^t]}\left(\tilde{s}^{tn}_{bz} + \sum_{z'\in[Z]}\tilde{y}^{tn}_{z'z}\right) \le x_{bz}, \quad \forall z\in[Z], \tag{A.2f}$$

$$\sum_{j\in[I+1]} \tilde{\lambda}^{tnz}_{ij} = \tilde{\mu}^{tne}_i, \quad \forall t\in[T],\ n\in[N^t],\ z\in[Z],\ i\in[I+1], \tag{A.2g}$$

$$\sum_{i\in[I+1]} \tilde{\mu}^{tne}_i = 1, \quad \forall t\in[T],\ n\in[N^t], \tag{A.2h}$$

$$\tilde{y}\ge 0,\ \tilde{s}\ge 0,\ \tilde{\lambda}\in\{0,1\},\ \tilde{\mu}\in\{0,1\} \tag{A.2i}$$

Let $P_{zt} : \Omega\times\Omega \mapsto [0,1]$ be the marginal probability distribution of zone z prices under the optimal policy π^* to (A.2). Let us define $\bar{\psi}^t_{mz}$ and $\bar{r}^t_{mz}$ be the optimal expected demand rate and revenue rate, respectively, during period t in channel m of zone z. Note that

$$\bar{\psi}^t_{mz} = \sum_{i\in[I+1]}\sum_{j\in[I+1]} \psi^t_{mz}(p_i,p_j)P_{zt}(p_i,p_j), \quad m=e,b,\ \forall t\in[T], \forall n\in[N^t], \forall z\in[Z], \tag{A.3}$$

$$\bar{r}^t_{mz} = \sum_{i\in[I+1]}\sum_{j\in[I+1]} r^t_{mz}(p_i,p_j)P_{zt}(p_i,p_j), \quad m=e,b,\ \forall t\in[T], \forall n\in[N^t], \forall z\in[Z] \tag{A.4}$$

Also, $\sum_{j\in[I+1]} P_{zt}(p_i,p_j) = \sum_{j\in[I+1]} P_{z't}(p_i,p_j) = P_t(p_i)$ for some $P_t(p_i)\in[0,1]$ and any $z, z'\in[Z]$ and $p_i\in\Omega$. Finally, given the optimal demand rate, the probability distribution of fulfillment and store sales under π^* achieves an expected fulfillment cost equal to:

$$
\begin{aligned}
FC^* = \underset{s\geq 0,\, y\geq 0}{\text{minimize}} \quad & \sum_{t\in[T]}\sum_{n\in[N^t]}\sum_{z\in[Z]}\left(qs_{bz}^{tn}+(c_{ez}+q)y_{ez}^{tn}+\sum_{z\in[Z]}y_{z'z}^{tn}\right) && \text{(A.5a)}\\
\text{subject to} \quad & y_{ez}^{tn}+\sum_{z'\in[Z]}y_{z'z}^{tn}=\bar{\psi}_{ez}^{t}, \qquad \forall z\in[Z], t\in[T], n\in[N^t], && \text{(A.5b)}\\
& s_{bz}^{tn}=\bar{\psi}_{bz}^{t}, \qquad \forall z\in[Z], t\in[T], n\in[N^t], && \text{(A.5c)}\\
& \sum_{t\in[T]}\sum_{n\in[N^t]}y_{ez}^{tn}\leq x_e, && \text{(A.5d)}\\
& \sum_{t\in[T]}\sum_{n\in[N^t]}\left(s_{bz}^{tn}+\sum_{z'\in[Z]}y_{z'z}^{tn}\right)\leq x_{bz}, \quad \forall z\in[Z] && \text{(A.5e)}
\end{aligned}
$$

Defining $r_{ez}^t(p_i,p_j):=p_i\psi_{ez}^t(p_i,p_j)$ and $r_{bz}^t(p_i,p_j):=p_j\psi_{bz}^t(p_i,p_j)$, the deterministic LP counterpart is:

$$
\begin{aligned}
V^{LP} = \underset{\lambda,\mu,y,s}{\text{maximize}} \quad & \sum_{t\in[T]}\sum_{n\in[N^t]}\sum_{z\in[Z]}\sum_{m=e,b}\sum_{i\in[I+1]}\sum_{j\in[I+1]}\lambda_{ij}^{tnz}r_{mz}^t(p_i,p_j)+q\left(x_e+\sum_{z\in[Z]}x_{bz}\right) && \text{(A.6a)}\\
& -\sum_{t\in[T]}\sum_{n\in[N^t]}\sum_{z\in[Z]}\left(qs_{bz}^{tn}+(c_{ez}+q)y_{ez}^{tn}+\sum_{z'\in[Z]}(c_{z'z}+q)y_{z'z}^{tn}\right) && \text{(A.6b)}
\end{aligned}
$$

subject to

$$
\begin{aligned}
& y_{ez}^{tn}+\sum_{z'\in[Z]}y_{z'z}^{tn}=\sum_{i\in[I+1]}\sum_{j\in[I+1]}\lambda_{ij}^{tnz}\psi_{ez}^t(p_i,p_j), \quad \forall z\in[Z],\ \forall t\in[T],\ \forall n\in[N^t], && \text{(A.6c)}\\
& s_{bz}^{tn}=\sum_{i\in[I+1]}\sum_{j\in[I+1]}\lambda_{ij}^{tnz}\psi_{bz}^t(p_i,p_j), \quad \forall z\in[Z],\ \forall t\in[T],\ \forall n\in[N^t], && \text{(A.6d)}\\
& \sum_{t\in[T]}\sum_{n\in[N^t]}y_{ez}^{tn}\leq x_e, && \text{(A.6e)}\\
& \sum_{t\in[T]}\sum_{n\in[N^t]}\left(s_{bz}^{tn}+\sum_{z'\in[Z]}y_{z'z}^{tn}\right)\leq x_{bz}, \quad \forall z\in[Z], && \text{(A.6f)}\\
& \sum_{j\in[I+1]}\lambda_{ij}^{tnz}=\mu_i^{tne}, \quad \forall t\in[T],\ n\in[N^t],\ z\in[Z],\ i\in[I+1], && \text{(A.6g)}\\
& \sum_{i\in[I+1]}\mu_i^{tne}=1, \quad \forall t\in[T],\ n\in[N^t], && \text{(A.6h)}\\
& y\geq 0,\ s\geq 0,\ \lambda\geq 0,\ \mu\geq 0 && \text{(A.6i)}
\end{aligned}
$$

We next construct a feasible solution (λ,μ,s,y) to (A.6) which achieves an objective value equal to V^*. Let

$$
\lambda_{ij}^{tnz}=P_{zt}(p_i,p_j), \quad \forall t\in[T],\forall n\in[N^t],\forall z\in[Z],\forall i\in[I+1],\forall j\in[I+1] \tag{A.7}
$$

$$
\mu_i^{tne}=P_t(p_i), \qquad \forall t\in[T],\forall n\in[N^t],\forall i\in[I+1]. \tag{A.8}
$$

It is easy to verify that (λ,μ) satisfy constraints (A.6g)–(A.6i). We let (s,y) be the solution to a linear program of the following form:

$$FC' = \underset{s\geq 0,\ y\geq 0}{\text{minimize}} \quad \sum_{t\in[T]}\sum_{n\in[N^t]}\sum_{z\in[Z]}\left(qs^{tn}_{bz} + (c_{ez}+q)y^{tn}_{ez} + \sum_{z\in[Z]} y^{tn}_{z'z}\right) \tag{A.9a}$$

subject to

$$y^{tn}_{ez} + \sum_{z'\in[Z]} y^{tn}_{z'z} = \sum_{i\in[I+1]}\sum_{j\in[I+1]} P_{zt}(p_i,p_j)\psi^t_{ez}(p_i,p_j), \quad \forall z\in[Z], t\in[T], n\in[N^t], \tag{A.9b}$$

$$s^{tn}_{bz} = \sum_{i\in[I+1]}\sum_{j\in[I+1]} P_{zt}(p_i,p_j)\psi^t_{bz}(p_i,p_j), \quad \forall z\in[Z], t\in[T], n\in[N^t], \tag{A.9c}$$

$$\sum_{t\in[T]}\sum_{n\in[N^t]} y^{tn}_{ez} \leq x_e, \tag{A.9d}$$

$$\sum_{t\in[T]}\sum_{n\in[N^t]}\left(s^{tn}_{bz} + \sum_{z'\in[Z]} y^{tn}_{z'z}\right) \leq x_{bz}, \quad \forall z\in[Z] \tag{A.9e}$$

Note that due to (A.9b)–(A.9e), we have that (λ,μ,s,y) also satisfy (A.6c)–(A.6f). Additionally, from definition (A.3), the right-hand side of (A.9b) and (A.9c) are equal to $\bar{\psi}^t_{ez}$ and $\bar{\psi}^t_{bz}$, respectively. Hence, $FC^* = FC'$. Therefore,

$$V^{LP} \geq \sum_{t\in[T]}\sum_{n\in[N^t]}\sum_{z\in[Z]}\sum_{m=e,b}\underbrace{\sum_{i\in[I+1]}\sum_{j\in[I+1]} P_{zt}(p_i,p_j)r^t_{mz}(p_i,p_j)}_{=\bar{r}^t_{mz}} + q\left(x_e + \sum_{z\in[Z]} x_{bz}\right) - FC' = V^* \geq V^1.$$

What is left to prove Lemma 1 is to show that in each period t, a stationary solution is optimal for (A.6). Note that for any optimal solution (λ,μ,y,s), a stationary solution $(\bar{\lambda},\bar{\mu},\bar{y},\bar{s})$ is also optimal, where for any $t\in[T]$ and $n\in[N^t]$, we set

$$\bar{\lambda}^{tnz}_{ij} = \frac{1}{N^t}\sum_{n'\in[N^t]}\lambda^{tn'z}_{ij}, \quad \bar{y}^{tn}_{ez} = \frac{1}{N^t}\sum_{n'\in[N^t]} y^{tn'}_{ez}, \quad \bar{y}^{tn}_{z'z} = \frac{1}{N^t}\sum_{n'\in[N^t]} y^{tn'}_{z'z}, \quad \bar{s}^{tn}_{bz} = \frac{1}{N^t}\sum_{n'\in[N^t]} s^{tn'}_{bz}. \quad \square$$

EC.1.2. Proof of Lemma 2

To prove the first part of the lemma, it suffices to prove that $V^t_D(x^t) = \bar{V}^*$, where

$$\bar{V}^* := \underset{\lambda,\mu,s,y,u}{\text{maximize}} \quad \sum_{k=t}^{T}\sum_{z\in[Z]}\sum_{m=e,b}\sum_{i\in[I]} p_i s^k_{mzi} - \sum_{z\in[Z]} c_{ez}y_{ez} - \sum_{z\in[Z]}\sum_{z'\in[Z]} c_{zz'}y_{zz'} + q\left(u_e + \sum_{z\in[Z]} u_{bz}\right) \tag{A.10a}$$

subject to Constraints (3.4c) – (3.4e),

$$\sum_{i\in[I]} s^k_{mzi} \leq \sum_{i\in[I]}\sum_{j\in[I]}\lambda^{kz}_{ij} d^k_{mz}(p_i,p_j), \quad k=t,\dots,T,\ m=e,b,\ z\in[Z], \tag{A.10b}$$

$$s^k_{ezi} \leq \sum_{j\in[I]}\lambda^{kz}_{ij} d^k_{ez}(p_i,p_j), \quad k=t,\dots,T,\ z\in[Z],\ i\in[I], \tag{A.10c}$$

$$s^k_{bzj} \leq \sum_{i\in[I]}\lambda^{kz}_{ij} d^k_{bz}(p_i,p_j), \quad k=t,\dots,T,\ z\in[Z],\ i\in[I], \tag{A.10d}$$

$$\sum_{i\in[I]}\mu^{ke}_i = 1, \quad \sum_{j\in[I]}\mu^{kz}_j = 1, \quad \forall z\in[Z], \tag{A.10e}$$

$$\lambda^{kz}_{ij} \leq \mu^{ke}_i, \quad \lambda^{kz}_{ij} \leq \mu^{kz}_j \quad \forall z\in[Z],\ i\in[I],\ j\in[I], \tag{A.10f}$$

$$s\geq 0, y\geq 0, u\geq 0,\ \lambda,\mu\in\{0,1\} \tag{A.10g}$$

Since the feasible price set is discrete, then (3.4) is equivalent to the optimization model with the objective (A.10a), and with constraints (3.4c)–(3.4e) and (A.10b)–(A.10g). Let us denote by $(\hat{\lambda}, \hat{\mu}, \hat{s}, \hat{y}, \hat{u})$ the optimal solution to this model, which has an objective value of $V_D^t(x^t)$. Defining

$$s_{ezi}^k = \hat{\mu}_i^{ke} \hat{s}_{ez}^k, \qquad k = t, \ldots, T, \, \forall z \in [Z], \, \forall i \in [I],$$
$$s_{bzj}^k = \hat{\mu}_j^{kz} \hat{s}_{bz}^k, \qquad k = t, \ldots, T, \, \forall z \in [Z], \, \forall j \in [I],$$

we can easily check that $(\hat{\lambda}, \hat{\mu}, s, \hat{y}, \hat{u})$ is a feasible solution to (A.10) and achieves the objective value $V_D^t(x^t)$. Hence, $V_D^t(x^t) \le \bar{V}^*$.

We next show that $V_D^t(x^t) \ge \bar{V}^*$. Let us denote the maximizer of (A.10) as $(\bar{\lambda}, \bar{\mu}, \bar{s}, \bar{y}, \bar{u})$, which achieves the optimal value $\bar{V}^*$. From the binary constraints of the MIP, and from constraints (A.10e), it follows that for time k, there exist price indices $(i_{ke}, j_{kz_1}, \ldots, j_{kz_n})$ such that:

$$\bar{\mu}_{i_{ke}}^{ke} = 1, \qquad\qquad \bar{\mu}_i^{ke} = 0, \quad \forall i \ne i_{ke}, \tag{A.11}$$
$$\bar{\mu}_{j_{kz}}^{kz} = 1, \quad \forall z \in [Z], \qquad \bar{\mu}_j^{kz} = 0, \quad \forall j \ne j_{kz}, \forall z \in [Z]. \tag{A.12}$$

These then imply from constraints (A.10f) that $\bar{\lambda}_{ij}^{kz} = 0$ for all i, j where $i \ne i_{ke}$ or $j \ne j_{kz}$, for all $z \in [Z]$. Therefore, from constraint (A.10c), it follows that $\bar{s}_{ezi}^k = 0$ for all $i \ne i_{ke}$, $z \in [Z]$. Similarly, from constraint (A.10d), it follows that $\bar{s}_{bzj}^k = 0$ for all $j \ne j_{kz}$, $z \in [Z]$. Define

$$s_{ez}^k = \bar{s}_{ezi_{ke}}^k, \qquad k = t, \ldots, T, \, \forall z \in [Z],$$
$$s_{bz}^k = \bar{s}_{bzj_{kz}}^k, \qquad k = t, \ldots, T, \, \forall z \in [Z]$$

Based on the definitions (A.11)–(A.12), we have that $\bar{s}_{ezi}^k = \bar{\mu}_i^{ke} s_{ez}^k$ and $\bar{s}_{bzj}^k = \bar{\mu}_j^{kz} s_{bz}^k$ for any $k \in [T]$, $z \in [Z]$ and $i \in [I]$. Therefore $(\bar{\lambda}, \bar{\mu}, s, \bar{y}, \bar{u})$ is a feasible solution to (3.4) with an objective value $\bar{V}^*$. Hence, $\bar{V}^* \le V_D^t(x^t)$, which proves the lemma. □

To prove the second part of the lemma, assume the MNL model (3.5) of choice for arrivals in zone $z \in [Z]$. Under this demand model, assuming that N_z^t is the expected number of zone z arrivals in period t, $d_{mz}^t(p_i, p_j) = N_z^t \cdot \theta_{mz}(p_i, p_j)$ for all $i, j \in [I]$. Let us introduce the following constants:

$$\gamma_{ezi}^k := \exp\left(\alpha_{ez} - \beta_{ez} p_i\right), \quad k = t, \ldots, T, \, \forall z \in [Z], \, \forall i \in [I], \tag{A.13}$$
$$\gamma_{bzj}^k := \exp\left(\alpha_{bz} - \beta_{bz} p_j\right), \quad k = t, \ldots, T, \, \forall z \in [Z], \, \forall j \in [I]. \tag{A.14}$$

Therefore, by introducing binary decision variables for the price decisions, we can reformulate optimization model (3.4) as:

$$V_D^t(x^t) = \underset{\mu, s, y, u}{\text{maximize}} \quad \sum_{k=t}^{T} \sum_{z \in [Z]} \sum_{i \in [I]} p_i \left(s_{ez}^k \mu_i^{ke} + s_{bz}^k \mu_i^{kz}\right) - \sum_{z \in [Z]} c_{ez} y_{ez} - \sum_{z, z' \in [Z]} c_{zz'} y_{zz'} + q \left(u_e + \sum_{z \in [Z]} u_{bz}\right) \tag{A.15a}$$

$$\text{subject to} \quad \text{Constraints } (3.4\text{c}) - (3.4\text{f}), (\text{A.10e}),$$

$$s_{ez}^k \le \frac{N_z^t \sum_{i \in [I]} \gamma_{ezi}^k \mu_i^{ke}}{1 + \sum_{i \in [I]} \gamma_{ezi}^k \mu_i^{ke} + \sum_{j \in [I]} \gamma_{bzj}^k \mu_j^{kz}}, \quad k = t, \ldots, T, \, \forall z \in [Z], \tag{A.15b}$$

$$s_{bz}^k \le \frac{N_z^t \sum_{j \in I_{bz}} \gamma_{bzj}^k \mu_j^{kz}}{1 + \sum_{i \in I_e} \gamma_{ezi}^k \mu_i^{ke} + \sum_{j \in I_{bz}} \gamma_{bzj}^k \mu_j^{kz}}, \quad k = t, \ldots, T, \, \forall z \in [Z], \tag{A.15c}$$

$$\mu_i^{ke} \in \{0, 1\}, \mu_i^{kz} \qquad k = t, \ldots, T, \, \forall z \in [Z], \, \forall i \in [I] \tag{A.15d}$$

Hence, it suffices to prove the following lemma.

LEMMA EC.1. *Under a multinomial logit demand model, $V_D^t(x^t) = \bar{V}_2^*$, where*

$$\bar{V}_2^* = \underset{\mu, s, y, u, g, h}{\text{maximize}} \quad \sum_{k=t}^{T} \sum_{z \in [Z]} \sum_{m=e,b} \sum_{i \in [I]} p_i s_{mzi}^k - \sum_{z \in [Z]} c_{ez} y_{ez} - \sum_{z, z' \in [Z]} c_{zz'} y_{zz'} + q \left(u_e + \sum_{z \in [Z]} u_{bz} \right) \tag{A.16a}$$

subject to Constraints (3.4c) – (3.4f), (A.10e),

$$\sum_{i \in [I]} s_{mzi}^k \le N_z^t \sum_{i \in [I]} \gamma_{mzi}^k h_{mzi}^k, \qquad k = t, \ldots, T,\ m = e, b,\ \forall z \in [Z], \tag{A.16b}$$

$$s_{mzi}^k \le N_z^k \gamma_{mzi}^k h_{mzi}^k, \qquad k = t, \ldots, T,\ m = e, b,\ \forall z \in [Z], \forall i \in [I], \tag{A.16c}$$

$$\sum_{i \in [I]} h_{mzi}^k = g_z^k, \qquad k = t, \ldots, T,\ m = e, b,\ \forall z \in [Z], \tag{A.16d}$$

$$h_{ezi}^k \le \mu_i^{ke}, \qquad k = t, \ldots, T,\ \forall z \in [Z], \forall i \in [I], \tag{A.16e}$$

$$h_{bzj}^k \le \mu_j^{kz}, \qquad k = t, \ldots, T,\ \forall z \in [Z], \forall j \in [I], \tag{A.16f}$$

$$g_z^k + \sum_{i \in I_e} \gamma_{ezi}^k h_{ezi}^k + \sum_{j \in I_{bz}} \gamma_{bzj}^k h_{bzj}^k = 1 \qquad k = t, \ldots, T,\ \forall z \in Z, \tag{A.16g}$$

$$g \ge 0,\ h \ge 0,\ \mu \in \{0, 1\} \tag{A.16h}$$

Proof. Let us denote by $(\hat{\mu}, \hat{s}, \hat{y}, \hat{u})$ as the optimal solution to (A.15) with objective value $V_D^t(x^t)$. Defining

$$s_{ezi}^k = \hat{\mu}_i^{ke} \hat{s}_{ez}^k, \qquad k = t, \ldots, T, \forall z \in [Z], \forall i \in [I],$$

$$s_{bzj}^k = \hat{\mu}_j^{kz} \hat{s}_{bz}^k, \qquad k = t, \ldots, T, \forall z \in [Z], \forall j \in [I],$$

$$g_z^k = \frac{1}{1 + \sum_{i \in [I]} \gamma_{ezi}^k \hat{\mu}_i^{ke} + \sum_{j \in [I]} \gamma_{bzj}^k \hat{\mu}_j^{kz}}, \qquad k = t, \ldots, T, \forall z \in [Z],$$

$$h_{ezi}^k = \hat{\mu}_i^{ke} g_z^k, \qquad k = t, \ldots, T, \forall z \in [Z], \forall i \in [I],$$

$$h_{bzj}^k = \hat{\mu}_j^{kz} g_z^k, \qquad k = t, \ldots, T, \forall z \in [Z], \forall j \in [I],$$

we can easily check that $(\hat{\mu}, s, \hat{y}, \hat{u}, g, h)$ is a feasible solution to the mixed integer program (A.16) and achieves the objective value $V_D^t(x^t)$. Hence, $V_D^t(x^t) \le \bar{V}_2^*$.

We next show that $V_D^t(x^t) \ge \bar{V}_2^*$. Let us denote the maximizer of model (A.16) as $(\bar{\mu}, \bar{s}, \bar{y}, \bar{u}, \bar{g}, \bar{h})$, which achieves the optimal value $\bar{V}_2^*$. From the binary constraints of (A.16), and from constraints (A.10e), it follows that for time k, there exist price indices $(i_{ke}, j_{kz_1}, \ldots, j_{kz_n})$ such that:

$$\bar{\mu}_{i_{ke}}^{ke} = 1, \qquad \bar{\mu}_i^{ke} = 0, \quad \forall i \ne i_{ke}, \tag{A.17}$$

$$\bar{\mu}_{j_{kz}}^{kz} = 1, \quad \forall z \in [Z], \qquad \bar{\mu}_j^{kz} = 0, \quad \forall j \ne j_{kz}, \forall z \in [Z]. \tag{A.18}$$

From constraints (A.17) and (A.16c), it follows that $\bar{s}_{ezi}^k = 0$ for all $i \ne i_{ke}$, $z \in [Z]$. Similarly, from constraints (A.18) and (A.16c), it follows that $\bar{s}_{bzj}^k = 0$ for all $j \ne j_{kz}$, $z \in [Z]$. We set $s_{ez}^k = \bar{s}_{ezi_{ke}}^k$ and $s_{bz}^k = \bar{s}_{bzj_{kz}}^k$ for all $z \in [Z]$. Thus, it is easy to check based on the definitions (A.17)–(A.18) that:

$$\bar{s}_{ezi}^k = \bar{\mu}_i^{ke} s_{ez}^k, \qquad k = t, \ldots, T, \forall z \in [Z], \forall i \in [I], \tag{A.19}$$

$$\bar{s}_{bzj}^k = \bar{\mu}_j^{kz} s_{bz}^k, \qquad k = t, \ldots, T, \forall z \in [Z], \forall j \in [I], \tag{A.20}$$

From constraints (A.17) and (A.16e), it follows that $\bar{h}_{ezi}^k = 0$ for all $i \ne i_{ke}$, $z \in [Z]$. Similarly, from constraints (A.18) and (A.16f), it follows that $\bar{h}_{bzj}^k = 0$ for all $j \ne j_{kz}$, $z \in [Z]$. Thus, these imply from constraints (A.16d) that $\bar{h}^k_{ezi_{ke}} = \bar{g}^k_z$ for all $z \in [Z]$, and that $\bar{h}^k_{bzj_{kz}} = \bar{g}^k_z$ for all $z \in [Z]$. Thus, it is easy to check based on the definitions (A.17)–(A.18) that we have the following relationships:

$$\bar{h}^k_{ezi} = \bar{\mu}^{ke}_i \bar{g}^k_z, \qquad k = t, \dots, T, \forall z \in [Z], \forall i \in [I], \tag{A.21}$$

$$\bar{h}^k_{bzj} = \bar{\mu}^{kz}_j \bar{g}^k_z, \qquad k = t, \dots, T, \forall z \in [Z], \forall j \in [I], \tag{A.22}$$

From (A.21)–(A.22), and from (A.16g), it follows that $1 = \bar{g}^k_z + \gamma^k_{ezi_{ke}} \bar{g}^k_z + \gamma^k_{bzj_{kz}} \bar{g}^k_z$. Thus,

$$\bar{g}^k_z = \frac{1}{1 + \gamma^k_{ezi_{ke}} + \gamma^k_{bzj_{kz}}} = \frac{1}{1 + \sum_{i\in[I]} \gamma^k_{ezi} \bar{\mu}^{ke}_i + \sum_{j\in[I]} \gamma^k_{bzj} \bar{\mu}^{kz}_j}. \tag{A.23}$$

Substituting (A.21)–(A.22) into (A.16b), and using the relationship (A.23), we have

$$s^t_{ez} \le N^k_z \gamma^k_{ezi_{ke}} \bar{g}^k_z = \frac{N^k_z \sum_{i\in[I]} \gamma^k_{ezi} \bar{\mu}^{ke}_i}{1 + \sum_{i\in[I]} \gamma^k_{ezi} \bar{\mu}^{ke}_i + \sum_{j\in[I]} \gamma^k_{bzj} \bar{\mu}^{kz}_j}, \tag{A.24}$$

$$s^t_{bz} \le N^k_z \gamma^k_{bzj_{kz}} \bar{g}^k_z = \frac{N^k_z \sum_{j\in[I]} \gamma^k_{bzj} \bar{\mu}^{kz}_j}{1 + \sum_{i\in[I]} \gamma^k_{ezi} \bar{\mu}^{ke}_i + \sum_{j\in[I]} \gamma^k_{bzj} \bar{\mu}^{kz}_j}. \tag{A.25}$$

Thus, (A.24)–(A.25) proves that $(s, \bar{y}, \bar{u}, \bar{\mu})$ is feasible for model (A.15). Moreover, since we have (A.19)–(A.20), then this solution achieves the objective value $\bar{V}^*_2$. Thus, $\bar{V}^*_2 \le V^t_D(x^t)$, proving the lemma. □

EC.1.3. Proof of Proposition 1

If w and p are fixed, then (3.8) is a linear program. Hence, by strong LP duality, $U^t(w, p; x^t)$ is equivalent to a minimization LP. Thus, the worst-case retailer profit is equivalent to:

$$\begin{aligned}
U^t_R(p; x^t) = \underset{\alpha, \beta, w}{\text{minimize}} \quad & \sum_{k=t}^{T} \sum_{z\in[Z]} \sum_{m=e,b} d^k_{mz}(p^k_e, p^k_{bz}) \left(1 + \delta^k_{mz} w^k_{mz}\right) \alpha^k_{mz} + \beta_e x^t_e + \sum_{z\in[Z]} \beta_{bz} x^t_{bz} && \text{(A.26a)}\\
\text{subject to} \quad & \alpha^k_{bz} + \beta^k_{bz} \ge p^k_{bz}, \qquad k = 1, \dots, T, \ \forall z \in [Z], && \text{(A.26b)}\\
& \alpha^k_{ez} + \beta^k_e \ge p^k_e - c_{ez}, \qquad k = 1, \dots, T, \ \forall z \in [Z], && \text{(A.26c)}\\
& \alpha^k_{ez} + \beta^k_{bz'} \ge p^k_e - c_{z'z}, \qquad k = 1, \dots, T, \ \forall z, z' \in [Z], && \text{(A.26d)}\\
& \alpha \ge 0, \ \beta \ge q, \ w \in W^t_{\Gamma,\Delta} && \text{(A.26e)}
\end{aligned}$$

where α, β are the variables in the dual of $U^t(w, p; x^t)$. To prove Proposition 1, we need the following result:

Lemma EC.2. *If $(\bar{\alpha}, \bar{\beta}, \bar{w})$ is the optimal solution for* (A.26)*, then $\bar{\alpha}^k_{bz} \le p^k_{bz} - q$ and $\bar{\alpha^k_{ez}} \le p^k_e - c^{\min}_z - q$ for all $z \in [Z]$, where $c^{\min}_z := \min\left\{c_{ez}, \min_{z'\in[Z]} c_{z'z}\right\}$.*

Proof. Since the demands are nonnegative for all realizations, then given $\bar{\beta}$, $\bar{\alpha}$ must take the smallest feasible value allowed by constraints (A.26b)–(A.26d). Thus, $\bar{\alpha}^k_{bz} = \max\left(0, \, p^k_{bz} - \bar{\beta}_{bz}\right)$, and $\bar{\alpha}^k_{ez} = \max\left(0, \, p^k_e - c_{ez} - \bar{\beta}_e, \max_{z'\in[Z]} \left(p^k_e - c_{z'z} - \bar{\beta}_{bz'}\right)\right)$. Since the $\bar{\beta}$ variables are bounded below by q, we have the following upper bounds for the $\bar{\alpha}$ variables: $\bar{\alpha}^k_{bz} \le p^k_{bz} - q$ and $\bar{\alpha}^k_{ez} \le p^k_e - c^{\min}_z - q$ for all $z \in [Z]$. □

Note that α variables are the shadow prices for the demand constraints of $U^t(w, p; x^t)$, while the β variables are the shadow prices for its inventory constraints. Thus the upper bounds in Lemma EC.2 are natural because α^k_{bz} is the marginal increase in value with an additional unit of store demand, which cannot exceed the marginal value of a store sale. Similarly, α^k_{ez} is the marginal increase in value with an additional unit of online demand, which cannot exceed the maximum marginal value of an online sale (i.e., using the cheapest fulfillment). Moreover, due to Assumption 1, these upper bounds on α are nonnegative.

Optimization problem (A.26) is nonconvex due to the bilinear term $\alpha^\top w$ in the objective. Note however that the w variables are bounded between -1 and 1. Moreover, if we define the parameters $p^{\max} := \max\{p : p \in \Omega\}$, $A^k_{bz} := p^{\max} - q$, and $A^k_{ez} := p^{\max} - c^{\min}_z - q$, then from Lemma EC.2, we can add the constraints $\alpha^k_{mz} \le A^k_{mz}$ to (A.26) without changing its optimal value. Hence the α variables are also bounded. Therefore, using these bounds on w and α, we can "lift" the optimization problem (A.26) onto a higher dimensional space by introducing variables $\eta^k_{mz} = \alpha^k_{mz} w^k_{mz}$, which linearizes the objective. This results in a linear program whose optimal value $U^t_l(p;x^t)$ is a lower bound to $U^t_R(p;x^t)$, where

$$
\begin{aligned}
U^t_l(p;x^t) := \underset{\alpha,\beta,w,\nu,\eta}{\text{minimize}} \quad & \sum_{k=t}^{T}\sum_{z\in[Z]}\sum_{m=e,b} d^k_{mz}(p^k_e,p^k_{bz})\left(\alpha^k_{mz} + \delta^k_{mz}\eta^k_{mz}\right) + \beta_e x^t_e + \sum_{z\in[Z]} \beta_{bz} x^t_{bz} && \text{(A.27a)}\\
\text{subject to} \quad & \alpha^k_{bz} + \beta^k_{bz} \ge p^k_{bz}, \qquad k = 1,\dots,T,\ \forall z\in[Z], && \text{(A.27b)}\\
& \alpha^k_{ez} + \beta^k_{e} \ge p^k_{e} - c_{ez}, \qquad k = 1,\dots,T,\ \forall z\in[Z], && \text{(A.27c)}\\
& \alpha^k_{ez} + \beta^k_{bz'} \ge p^k_{e} - c_{z'z}, \qquad k = 1,\dots,T,\ \forall z,z'\in[Z], && \text{(A.27d)}\\
& \alpha \ge 0,\ \beta \ge q,\ w \in W^t_{\Gamma,\Delta}, && \text{(A.27e)}\\
& \left|\eta^k_{mz}\right| \le \alpha^k_{mz}, \qquad m=e,b,\ k=1,\dots,T,\ \forall z\in[Z], && \text{(A.27f)}\\
& \left|A^k_{mz} w^k_{mz} - \eta^k_{mz}\right| \le A^k_{mz} - \alpha^k_{mz}, \qquad m=e,b,\ k=1,\dots,T,\ \forall z\in[Z] && \text{(A.27g)}
\end{aligned}
$$

Note that constraints (A.27f), (A.27g) and (3.6) can be linearized. The new constraints (A.27f) and (A.27g) are valid inequalities that are satisfied by any feasible solution to the linearized problem. We next derive the form (3.9) in Proposition 1. Note that due to strong duality, we can reformulate (A.27) as:

$$
\begin{aligned}
U^t_l(p;x^t) = \underset{\substack{s,y,u,r,R,g,G,\\ h,H,\phi,\Phi,l,L,f}}{\text{maximize}} \quad & \sum_{k=t}^{T}\sum_{z\in[Z]}\left(p^k_{bz}s^k_{bz} + \left(p^k_e - c_{ez}\right)y^k_{ez} + \sum_{z'\in[Z]}\left(p^k_e - c_{z'z}\right)y^k_{z'z}\right) && \text{(A.28a)}\\
& - \sum_{k=t}^{T}\left(\Gamma^k f^k + \sum_{z\in[Z]}\sum_{m=e,b}\left(\phi^k_{mz} + \Phi^k_{mz}\right) + \sum_{z\in[Z]}\sum_{m=e,b} A^k_{mz}\left(g^k_{mz} + G^k_{mz}\right)\right) && \text{(A.28b)}\\
& - \sum_{k=t}^{T}\Delta^k\left(l^k + L^k\right) + q\left(u_e + \sum_{z\in[Z]} u_{bz}\right) && \text{(A.28c)}
\end{aligned}
$$

subject to

$$
\begin{aligned}
& s^k_{bz} + r^k_{bz} + R^k_{bz} - g^k_{bz} - G^k_{bz} \le d^k_{bz}(p^k_e,p^k_{bz}), && k=t,\dots,T,\ \forall z\in[Z], \\
& && \text{(A.28d)}\\
& y^k_{ez} + \sum_{z'\in[Z]} y^k_{z'z} + r^k_{ez} + R^k_{ez} - g^k_{ez} - G^k_{ez} \le d^k_{ez}(p^k_e,p^k_{bz}), && k=t,\dots,T,\ \forall z\in[Z], \\
& && \text{(A.28e)}\\
& \sum_{k=t}^{T}\sum_{z\in[Z]} y^k_{ez} + u_e = x^t_e, && \text{(A.28f)}\\
& \sum_{k=t}^{T} s^k_{bz} + \sum_{k=t}^{T}\sum_{z'\in[Z]} y^k_{zz'} + u_{bz} = x^t_{bz}, && \forall z\in[Z], \quad \text{(A.28g)}
\end{aligned}
$$

$$\phi_{mz}^k - \Phi_{mz}^k + h_{mz}^k - H_{mz}^k - A_{mz}^k \left(g_{mz}^k - G_{mz}^k\right) + a_{mz}^k \left(l^k - L^k\right) = 0, \quad m = e, b,\ k = t, \ldots, T,\ \forall z \in [Z], \tag{A.28h}$$

$$- f^k + h_{mz}^k + H_{mz}^k = 0, \quad m = e, b,\ k = t, \ldots, T,\ \forall z \in [Z], \tag{A.28i}$$

$$r_{mz}^k - R_{mz}^k + g_{mz}^k - G_{mz}^k = \delta_{mz}^k d_{mz}^k(p_e^k, p_{bz}^k),, \tag{A.28j}$$

$$s, y, u, r, R, g, G, h, H, \phi, \Phi, l, L \geq 0. \tag{A.28k}$$

Define the following variables: $s_{ez}^k = y_{ez}^k + \sum_{z'} y_{z'z}^k$, $y_{ez} = \sum_{k=t}^T y_{ez}^k$, and $y_{z'z} = \sum_{k=t}^T y_{z'z}^k$. Thus,

$$U_l^t(p; x^t) = \underset{\substack{s,y,u,r,R,g,G,\\ h,H,\phi,\Phi,l,L,f}}{\text{maximize}} \quad \sum_{k=t}^T \sum_{z \in [Z]} \left(p_e^k s_{ez}^k + p_{bz}^k s_{bz}^k\right) - \sum_{z \in [Z]} \left(c_{ez} y_{ez} + \sum_{z' \in [Z]} c_{z'z} y_{z'z}\right) \tag{A.29a}$$

$$- \sum_{k=t}^T \left(\Gamma^k f^k + \sum_{z \in [Z]} \sum_{m=e,b} \left(\phi_{mz}^k + \Phi_{mz}^k\right) + \sum_{z \in [Z]} \sum_{m=e,b} A_{mz}^k \left(g_{mz}^k + G_{mz}^k\right)\right) \tag{A.29b}$$

$$- \sum_{k=t}^T \Delta^k \left(l^k + L^k\right) + q \left(u_e + \sum_{z \in [Z]} u_{bz}\right) \tag{A.29c}$$

subject to

Constraints (3.4c)–(3.4f), (A.28h)–(A.28k),

$$s_{mz}^k + r_{mz}^k + R_{mz}^k - g_{mz}^k - G_{mz}^k \leq d_{mz}^k(p_e^k, p_{bz}^k), \quad m = e, b,\ k = t, \ldots, T,\ z \in [Z] \tag{A.29d}$$

Let us introduce the following variable transformations:

$$\begin{aligned}
\chi_{mz}^k &= \phi_{mz}^k - \Phi_{mz}^k, & X_{mz}^k &= \phi_{mz}^k + \Phi_{mz}^k,\\
\upsilon_{mz}^k &= g_{mz}^k - G_{mz}^k, & \Lambda_{mz}^k &= g_{mz}^k + G_{mz}^k,\\
\psi_{mz}^k &= r_{mz}^k - R_{mz}^k, & \Psi_{mz}^k &= r_{mz}^k + R_{mz}^k,\\
\upsilon_{mz}^k &= h_{mz}^k - H_{mz}^k, & \Upsilon_{mz}^k &= h_{mz}^k + H_{mz}^k,\\
\vartheta^k &= l^k - L^k, & \Theta^k &= l^k + L^k.
\end{aligned}$$

Therefore, by replacing these new variables into model (A.29), we have its equivalent formulation:

$$U_l^t(p; x^t) = \underset{\substack{s,y,u,\chi,X,\upsilon,\Lambda,\\ \psi,\Psi,\upsilon,\Upsilon,\vartheta,\Theta,f}}{\text{maximize}} \quad \sum_{k=t}^T \sum_{z \in [Z]} \left(p_e^k s_{ez}^k + p_{bz}^k s_{bz}^k\right) - \sum_{z \in [Z]} \left(c_{ez} y_{ez} + \sum_{z' \in [Z]} c_{z'z} y_{z'z}\right) \tag{A.30a}$$

$$- \sum_{k=t}^T \left(\Gamma^k f^k + \Delta^k \Theta^k + \sum_{z \in [Z]} \sum_{m=e,b} \left(X_{mz}^k + A_{mz}^k \Lambda_{mz}^k\right)\right) + q \left(u_e + \sum_{z \in [Z]} u_{bz}\right) \tag{A.30b}$$

subject to Constraints (3.4c)–(3.4f) , (A.30c)

$$s_{mz}^k \leq d_{mz}^k(p_e^k, p_{bz}^k) + \Lambda_{mz}^k - \Psi_{mz}^k, \quad m = e, b,\ k = t, \ldots, T,\ z \in [Z], \tag{A.30d}$$

$$\chi_{mz}^k + \upsilon_{mz}^k - A_{mz}^k \upsilon_{mz}^k + a_{mz}^t \vartheta^k = 0, \quad m = e, b,\ k = t, \ldots, T,\ z \in [Z], \tag{A.30e}$$

$$\Upsilon_{mz}^k = f^k, \quad m = e, b,\ k = t, \ldots, T,\ z \in [Z], \tag{A.30f}$$

$$\upsilon_{mz}^k + \psi_{mz}^k = \delta_{mz}^k d_{mz}^k(p_e^k, p_{bz}^k), \quad m = e, b,\ k = t, \ldots, T,\ z \in [Z], \tag{A.30g}$$

$$X_{mz}^k \geq |\chi_{mz}^k|, \quad m = e, b,\ k = t, \ldots, T,\ z \in [Z], \tag{A.30h}$$

$$\Lambda_{mz}^k \geq |\upsilon_{mz}^k|, \qquad m=e,b,\ k=t,\dots,T,\ z\in[Z], \tag{A.30i}$$

$$\Psi_{mz}^k \geq |\psi_{mz}^k|, \qquad m=e,b,\ k=t,\dots,T,\ z\in[Z], \tag{A.30j}$$

$$\Upsilon_{mz}^k \geq |\upsilon_{mz}^k|, \qquad m=e,b,\ k=t,\dots,T,\ z\in[Z], \tag{A.30k}$$

$$\Theta^k \geq |\vartheta^k|, \qquad k=t,\dots,T \tag{A.30l}$$

Note that since $\Upsilon_{mz}^k = f^k$, then we can eliminate the Υ_{mz}^k variables by replacing the constraint (A.30k) by $f^k \geq |\upsilon_{mz}^k|$. By equation (A.30e), we know $\upsilon_{mz}^k = A_{mz}^k \upsilon_{mz}^k - \chi_{mz}^k - a_{mz}^t \vartheta^k$. Due to the maximizing objective, in the optimal solution, we have $X_{mz}^k = |\chi_{mz}^k|$ and $\Theta^k = |\vartheta^k|$. Next, because the coefficient of s_{mz}^k in the objective is positive and Ψ_{mz}^k reduces the upper bound for s_{mz}^k without impacting the objective, an alternative optimal solution is obtained when Ψ_{mz}^k is reduced and set equal to $|\psi_{mz}^k|$. Lastly, note that the coefficient of Λ_{mz}^k is negative, and it increases the bound on s_{mz}^k through equation (A.30d). However, since A_{mz}^k is an upper bound on the marginal profit for every unit of sale in channel m and zone z, it is optimal to decrease Λ_{mz}^k to the smallest feasible value and incur the least penalty A_{mz}^k. Hence, in the optimal solution $\Lambda_{mz}^k = |\upsilon_{mz}^k|$. Hence, we have that model (A.30) is equivalent to (3.9). □

EC.1.4. Proof of Lemma 4

Consider a clairvoyant who knows the future demand factors $w = (w_{mz})_{mz}$ prior to making decisions, and thus is able to earn the highest profit with his or her price, sales, and fulfillment decisions by solving $U_{PF}(w) := \max_{p=(p^1,\dots,p^T)} U^1(w,p;x)$. On each sample path w, the "perfect foresight" realized profit $U_{PF}(w)$ is an upper bound on the realized profit of any pricing policy. Hence, the expected perfect foresight profit V_{PF} is an upper bound on the optimal expected profit. □

EC.2. Demand estimation for Business Value Assessment

We geo-spatially clustered the retailer's stores into 50 zones using a k-means (k=50) algorithm on the store coordinates. Fig. EC.3 shows the 50 zones used for the experiments. We geo-tag all transaction data using zones based on the origin of the demand and the fulfillment location. We ignored the buy-online-pickup-in-store option since we observed few such transactions in the data for the items analyzed. Fig. EC.3 also shows the zonal distribution of sales (the volume is proportional to the pie size) for one of the product category in our data for the retailer. The pie in each zone illustrates the relative frequency of brick-and-mortar sales and e-commerce sales in the zone. Note the heterogeneity of the e-commerce channel share across zones (e.g., 4% to 11%), which can result in certain zones having relatively more ship-from-store activity. Aside from the ability to model geographic-based heterogeneity, another advantage of zone tagging is the ability to tractably capture cross-channel effect (Harsha et al. 2015). We use the zone-tagged data to estimate SKU-zone level demand models described in this section.

We use the MNL function Eq. (3.5) to model the aggregate consumer behavior across channels. The time series sales data exhibit a distinct product lifecycle (PLC) representing the baseline popularity of a product over its selling season that begins at time t_{start} and has a pre-planned exit date of t_{end}. We estimate the PLC curve by fitting a beta distribution which encompasses a variety of curve shapes, as well as other prediction coefficients using the procedure described later in this section. Model selection and cross-validation on a

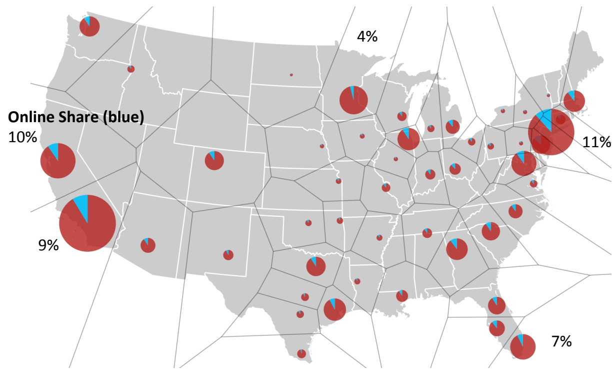

Figure EC.3 **Distribution of sales over 50 zones for a product category. Sales volume is proportional to the pie size. A pie shows the relative frequency of brick-and-mortar sales and e-commerce sales.**

variety of training instances yielded the following market-size model that predicts the customer arrival rate for any week t in the selling season:

$$\begin{aligned}\log(\text{Market Size}_t) =& \gamma_0 + \gamma_1 \log(1+t-t_{\text{start}}) + \gamma_2 \log(1+t_{\text{end}}-t) \\ &+ \sum_k \gamma_{3,k} \text{HOLIDAY-VARIABLES}_{k,t},\end{aligned} \tag{B.1}$$

and the following market-share model to predict the channel shares in week t:

$$\begin{aligned}\log(\text{Channel Attraction}_t) =& \beta_0 + \beta_1 \text{PRICE}_t + \sum_k \beta_{2,k} \text{PROMOTION-VARIABLES}_{k,t} \\ &+ \sum_j \beta_{3,j} \text{COMPETITOR-PRICE-VARIABLES (optional)}_{j,t}.\end{aligned} \tag{B.2}$$

Holiday spikes, if any, are addressed using holiday indicator variables. It was also beneficial to add channel-specific temporal lag effects prior to the holiday weeks in order to model the spike in online gift orders placed earlier due to the lead time of delivery. Promotional indicators, which include whether the product was advertised that week, were also useful. Competitor prices are introduced as channel-specific attributes, whenever they are available. Future competitor price data are generally not available, but we can use time series methods to forecast competitor prices based on historical trends.

Since the clearance period occurs during the final 10-12 weeks of the product lifecycle, the end-of-life sales decay (measured by decay coefficient γ_2) is a key prediction component for clearance period demand. This decay can occur due to factors such as the waning popularity of the product towards the end of life. Due to the broken assortment effect, this decay may be amplified by inventory depletion, since the item is less visible to store customers. To gauge the incremental impact of low inventory levels on store sales during the markdown period, we experimented with several threshold-based inventory-effect models (Smith and Achabal 1998, Caro and Gallien 2012). However, we did not observe any significant improvement in prediction quality after incorporating such inventory effects, and a PLC-based market-size prediction model was adequate for our application. A review of the in-store display procedures followed by sales associates indicated that the categories we analyzed were unlikely to be influenced by the broken assortment effect and the 'store-presentation' effects. Nevertheless, incorporating inventory effects in an omnichannel environment can be useful for relevant product categories such as fashion apparel.

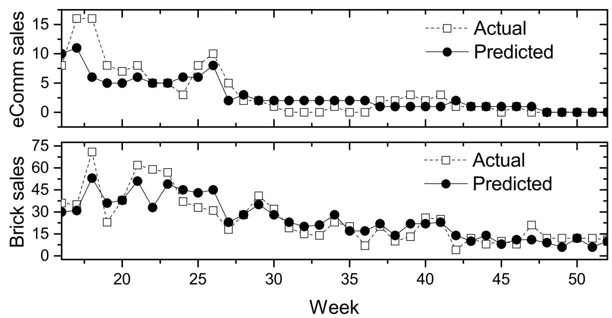

Figure EC.4 Predicted and actual e-commerce sales, and brick-and-mortar sales for a specific SKU-zone.

EC.2.1. Estimation procedure.

The γ and β coefficients in Eqs. (B.1–B.2) are estimated using real historical sales and price data. The goal is to predict future end-of-life sales by channel and location using partial (early and mid-season) TLOG data from the current selling season. There are two challenges.

First, the standard methods to estimate discrete choice models require historical information about every choice, which in our setting, includes censored lost sales. We employ an integrated mixed-integer programming (MIP) approach that jointly estimates market size and the market share parameters in the presence of censored lost sales data proposed by Subramanian and Harsha (2017). Their method performs imputations endogenously in the MIP by estimating optimal values for the probabilities of the unobserved censored choice. Under mild assumptions, they show the method is asymptotically consistent. Besides being a computationally fast single step method, this estimation approach is capable of calibrating market-size covariates (e.g., $\gamma_1, \gamma_2, \gamma_3$), a critical feature with real data. We incorporated model enhancements such as regularization using lasso and ridge penalties and sign constraints on price coefficients to enable an automated demand estimation environment.

The second challenge is in estimating the decay coefficient, γ_2, for the PLC curve without the full sales history of an item (e.g., future end-of-life sales trajectory can be convex, concave or affine). To overcome this problem, we employed the following two-phase procedure to estimate the parameters of the demand model. In the first phase, the average end-of-life sales decay coefficient γ_2 for a representative SKU from the category was estimated using a learning procedure, and employed as a 'prior' desired value in the second phase of estimation that is done at a SKU-zone level for all SKUs. Such priors can also be estimated using historical values of like-SKUs in the same category.

The training sales data was used to estimate parameters of the channel attraction models and the market size model. As we move closer toward the end of the season, and more end-of-life sales data becomes available, the prediction model is recalibrated on a weekly basis using the most recent data, updating all coefficients including the decay coefficient γ_2 with its previous estimate used as a prior, to produce improved sales forecasts for the remaining weeks.

EC.2.2. Prediction accuracy.

We next discuss the achieved model fit and prediction quality using the retailer's actual sales data and prices. The prediction results presented here is the 12-week look-ahead forecast for the entire clearance

Table EC.1 Average same-channel and cross-channel price elasticities for Tablets Category.

Channel Sales	Elasticity to brick-and-mortar price	Elasticity to e-commerce price
Brick-and-mortar sales	-1.3	0.7
E-commerce sales	2.8	-3.9

period as opposed to rolling horizon weekly sales predictions. We present the look-ahead forecast because the OCPX model at the start of the clearance period requires an estimate of demand for all future periods until the planned end date. As time progresses, the demand predictions for the remaining weeks will need to be revised each period. The forecast quality was measured in terms of the volume weighted mean absolute percentage error (WMAPE). We observed this to be largely dependent on the sales rates and hence, the level of disaggregation at which the model calibration was performed, which is consistent with the observations in Caro and Gallien (2012). The achieved out-of-sample WMAPE at the category-chain level was about 22%. This WMAPE value is in close proximity to that observed by Caro and Gallien (2012) who report a 23.8% WMAPE at the category-chain level. At the lowest level of aggregation (SKU-zone level), the average sales rate across the SKUs analyzed during the clearance period (10 per week for brick and 2 for online) was more than 10 times lower than the mid-season sales rate, resulting in the predicted weekly sales deviating from actual sales by ± 5 units for stores and ± 1.2 units for online. Fig. EC.4 is a sample graphical plot of the model fit (for training data) and the look-ahead predictions (for test, the final 12 weeks of sales).

To measure the impact of cross-channel causals to prediction accuracy, we compared the estimated model to a baseline which uses the data to estimate channel demand models without cross-channel causals. We observed the omnichannel demand model reduces SKU-zone level WMAPEs by 1.5 percentage points for the brick channel, and 5 percentage points for the digital channel over the baseline. The incremental gain in prediction accuracy was higher when compared to the partner retailers incumbent single-channel demand forecasting system. Although the benefit of incorporating of cross-channel effects varies by product category, channel price differential, and selling season, for the categories we tested, we observed that the online sales prediction (in general, across categories and across multiple retailers) tends to improve after incorporating cross-channel price and promotion effects. Overall, our demand model and estimated parameters result in prediction qualities consistent with the goals set by the retailer, and was embedded within our proposed optimization framework to calculate optimal prices and inventory partitions.

EC.2.3. Estimated price elasticities.

We present the average price same-channel and cross-channel elasticity values evaluated at the average channel price for the Tablets category in Table EC.1. These relatively high elasticity values are typical of markdown settings. Note that the cross-elasticities are asymmetric in that the impact of brick prices on the online sales is different from (and tends to be higher than) the impact of the online prices on brick sales. It is indicative of the heterogeneity of the customers shopping in the different channels as well as the volume share of these channels (the absolute change in volume of brick sales is much higher than that for the online channel).

Additional References

Subramanian, S., P. Harsha. 2017. Demand modeling in the presence of unobserved lost sales. Under review.