

Online Appendix for “Modeling Payment Timing in Multiechelon Inventory Systems with Applications to Supply Chain Coordination”

Proof of Lemma 1: Rewriting (1) gives that for any j ,

$$\begin{aligned}
 \mathbf{S}_j(t) &= -B_j(t) + D_j(t) \\
 \mathbf{R}_j(t) &= I_j(t) + \mathbf{S}_j(t) \\
 D_{j+1}(t) &= IO_j(t) + \mathbf{R}_j(t) \\
 \mathbf{S}_{j+1}(t) &= IT_j(t) + \mathbf{R}_j(t)
 \end{aligned} \tag{8}$$

We prove by induction. For $j = 1$, (2) becomes $\mathbf{S}_1(t) = -B_1(t) + D_1(t)$; $\mathbf{R}_1(t) = IN_1(t) + D_1(t)$; $D_1(t) = D_1(t)$. The first equation directly follows from that in (8). The second equation is from the first two equations in (8). Now, if (2) holds for j , we have from (8) and (2)

$$\begin{aligned}
 \mathbf{S}_{j+1}(t) &= IT_j(t) + \mathbf{R}_j(t) = IT_j(t) + IN_j(t) + \sum_{i=1}^{j-1} IOP_i(t) + D_1(t) \\
 &= -B_{j+1}(t) + \sum_{i=1}^j IOP_i(t) + D_1(t)
 \end{aligned} \tag{9}$$

Similarly, from (8) and (2)

$$\begin{aligned}
 D_{j+1}(t) &= IO_j(t) + \mathbf{R}_j(t) = IO_j(t) + IN_j(t) + \sum_{i=1}^{j-1} IOP_i(t) + D_1(t) \\
 &= \sum_{i=1}^j IOP_i(t) + D_1(t)
 \end{aligned}$$

and finally, from (8) and (9),

$$\begin{aligned}
 \mathbf{R}_{j+1}(t) &= I_{j+1}(t) + \mathbf{S}_{j+1}(t) = I_{j+1}(t) - B_{j+1}(t) + \sum_{i=1}^j IOP_i(t) + D_1(t) \\
 &= IN_{j+1}(t) + \sum_{i=1}^j IOP_i(t) + D_1(t)
 \end{aligned}$$

Proof of Proposition 1: Follows directly from the definition of payment processes and fundamental processes.

Proof of Proposition 2: We thank an anonymous reviewer for his/her suggestions regarding this proof. First, we establish the following lemma:

Lemma (A1). *Consider an n -stage serial system and let $B_j(t)$ and $\tilde{B}_j(t)$, respectively, be the local backorders at stage j under two local base stock policies \mathbf{y} and $\tilde{\mathbf{y}}$. Then, assuming the system begins empty just before time 0, for any sample path, we have $|B_j(t) - \tilde{B}_j(t)| \leq |\tilde{y}_n - y_n| + \dots + |\tilde{y}_j - y_j|$.*

Proof. By induction. Let $D_j^{L_j}$ = demand at stage j during stage j 's lead time from $t - L_j$ to t . For

$j = n$ we have reliable supply, so $B_n(t) = [D_n^{L_n} - y_n]^+$ and $\tilde{B}_n(t) = [D_n^{L_n} - \tilde{y}_n]^+$. Suppose $y_n \geq \tilde{y}_n$. Then, for any given sample path, one of three cases occurs:

$$\begin{aligned} D_n^{L_n} \leq \tilde{y}_n \leq y_n &\Rightarrow B_n(t) = \tilde{B}_n(t) = 0 &\Rightarrow B_n(t) - \tilde{B}_n(t) = 0 \\ \tilde{y}_n \leq D_n^{L_n} \leq y_n &\Rightarrow B_n(t) = 0, \tilde{B}_n(t) = D_n^{L_n} - \tilde{y}_n \leq y_n - \tilde{y}_n &\Rightarrow \tilde{y}_n - y_n \leq B_n(t) - \tilde{B}_n(t) \leq 0 \\ \tilde{y}_n \leq y_n \leq D_n^{L_n} &\Rightarrow B_n(t) = D_n^{L_n} - y_n, \tilde{B}_n(t) = D_n^{L_n} - \tilde{y}_n &\Rightarrow B_n(t) - \tilde{B}_n(t) = \tilde{y}_n - y_n \leq 0 \end{aligned}$$

So for all cases, $\tilde{y}_n - y_n \leq B_n(t) - \tilde{B}_n(t) \leq 0$. If instead $y_n \leq \tilde{y}_n$, similar arguments give $0 \leq B_n(t) - \tilde{B}_n(t) \leq \tilde{y}_n - y_n$. So combining the two yields $|B_n(t) - \tilde{B}_n(t)| \leq |\tilde{y}_n - y_n|$. Now suppose the result holds for stages $n, \dots, j+1$. The key difference is that shipments to stage j may be delayed due to backorders at stage $j+1$, so at time t the inventory transfer positions for stage j one lead time prior to the current time t (i.e., at time $t - L_j$) under the respective policies are given by $ITP_j(t - L_j) = y_j - B_{j+1}(t - L_j)$ and $\widetilde{ITP}_j(t - L_j) = \tilde{y}_j - \tilde{B}_{j+1}(t - L_j)$. Note that

$$\begin{aligned} B_j(t) &= [D_1^{L_j} - ITP_j(t - L_j)]^+ = [D_1^{L_j} - y_j + B_{j+1}(t - L_j)]^+ \\ \tilde{B}_j(t) &= [D_1^{L_j} - \widetilde{ITP}_j(t - L_j)]^+ = [D_1^{L_j} - \tilde{y}_j + \tilde{B}_{j+1}(t - L_j)]^+ \end{aligned}$$

By considering both $y_j - B_{j+1}(t - L_j) \geq \tilde{y}_j - \tilde{B}_{j+1}(t - L_j)$ and $y_j - B_{j+1}(t - L_j) \leq \tilde{y}_j - \tilde{B}_{j+1}(t - L_j)$, and considering three different cases regarding the ordering of these two quantities and $D_1^{L_j}$ for each inequality, arguments similar to those above imply $|B_j(t) - \tilde{B}_j(t)| \leq |\tilde{y}_j - y_j + B_{j+1}(t - L_j) - \tilde{B}_{j+1}(t - L_j)|$. Using properties of the absolute values of sums, along with the inductive hypothesis, this yields

$$|B_j(t) - \tilde{B}_j(t)| \leq |\tilde{y}_j - y_j| + |B_{j+1}(t - L_j) - \tilde{B}_{j+1}(t - L_j)| \leq |\tilde{y}_j - y_j| + |\tilde{y}_{j+1} - y_{j+1}| + \dots + |\tilde{y}_n - y_n|.$$

□

Returning to Proposition 2, consider base-stock policies \mathbf{y} and $\tilde{\mathbf{y}}$. Here, quantities with a tilde are associated with policy $\tilde{\mathbf{y}}$; those without correspond to policy \mathbf{y} . We can write $\pi_j(t)$ as

$$\begin{aligned} \pi_j(t) &= - \sum_{k=1}^{\infty} \{w_j \cdot 1\{\tau_j^k \leq t\} + \int_0^t \alpha_j w_j \cdot 1\{\tau_j^k \leq u\} du\} \\ &\quad + \sum_{k=1}^{\infty} \{w_{j-1} \cdot 1\{\tau_{j-1}^k \leq t\} + \int_0^t \alpha_j w_{j-1} \cdot 1\{\tau_{j-1}^k \leq u\} du\} \\ &= -\{w_j N_j(t) + \alpha_j w_j \int_0^t N_j(u) du\} + \{w_{j-1} N_{j-1}(t) + \alpha_j w_{j-1} \int_0^t N_{j-1}(u) du\} \\ &= \alpha_j (w_{j-1} - w_j) \int_0^t D_1(u) du + w_{j-1} N_{j-1}(t) - w_j N_j(t) \\ &\quad - \alpha_j \int_0^t [(w_{j-1} - w_j) (D_1(u) - N_{j-1}(u)) + w_j (N_j(u) - N_{j-1}(u))] du. \end{aligned}$$

So,

$$\begin{aligned}
\pi_j(t) - \tilde{\pi}_j(t) &= w_{j-1} [N_{j-1}(t) - \tilde{N}_{j-1}(t)] - w_j [N_j(t) - \tilde{N}_j(t)] \\
&\quad - \int_0^t \alpha_j [(w_{j-1} - w_j) (D_1(u) - N_{j-1}(u)) + w_j (N_j(u) - N_{j-1}(u))] du \\
&\quad + \int_0^t \alpha_j [(w_{j-1} - w_j) (D_1(u) - \tilde{N}_{j-1}(u)) + w_j (\tilde{N}_j(u) - \tilde{N}_{j-1}(u))] du.
\end{aligned} \tag{10}$$

Expressions for key quantities such as $N_j(t) - \tilde{N}_j(t)$, $N_j(t) - N_{j-1}(t)$ and $D_1(t) - N_{j-1}(t)$ depend on the specific payment timing scheme being used. We show detailed arguments establishing the result for one relatively general category of payment timing - very similar arguments establish the result for other schemes. Specifically, suppose stage j makes payments when stage k ships, and stage $j-1$ makes payments when stage m ships. Define echelon base stock levels $Y_i = y_1 + \dots + y_i$ for any stage i , and assume that the system starts empty just before time 0. Then by time t , stage j has ordered $Y_j + D_1(t)$ units, stage k has received orders for $Y_{k-1} + D_1(t)$ units, and stage k has shipped $Y_{k-1} + D_1(t) - B_k(t)$ units, so $N_j(t) = Y_{k-1} + D_1(t) - B_k(t)$. Similarly, $N_{j-1}(t) = Y_{m-1} + D_1(t) - B_m(t)$. Therefore

$$N_j(t) - \tilde{N}_j(t) = Y_{k-1} - \tilde{Y}_{k-1} - B_k(t) + \tilde{B}_k(t), \tag{11}$$

$$N_{j-1}(t) - \tilde{N}_{j-1}(t) = Y_{m-1} - \tilde{Y}_{m-1} - B_m(t) + \tilde{B}_m(t), \tag{12}$$

$$N_j(t) - N_{j-1}(t) = Y_{k-1} - Y_{m-1} - B_k(t) + B_m(t), \tag{13}$$

$$D_1(t) - N_{j-1}(t) = -Y_{m-1} + B_m(t). \tag{14}$$

From (10), assuming the relevant limits exist (which will be established later), we have

$$\begin{aligned}
\lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} (\pi_j(t) - \tilde{\pi}_j(t)) \right\} &= \lim_{t \rightarrow \infty} \frac{1}{t} \{ w_{j-1} E[N_{j-1}(t) - \tilde{N}_{j-1}(t)] - w_j E[N_j(t) - \tilde{N}_j(t)] \} \\
&\quad - \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j [(w_{j-1} - w_j) (D_1(u) - N_{j-1}(u)) + w_j (N_j(u) - N_{j-1}(u))] du \right\} \\
&\quad + \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j [(w_{j-1} - w_j) (D_1(u) - \tilde{N}_{j-1}(u)) + w_j (\tilde{N}_j(u) - \tilde{N}_{j-1}(u))] du \right\}.
\end{aligned}$$

By (11) and (12), the argument of the first limit is

$$w_{j-1} [Y_{m-1} - \tilde{Y}_{m-1} - B_m(t) + \tilde{B}_m(t)] - w_j [Y_{k-1} - \tilde{Y}_{k-1} - B_k(t) + \tilde{B}_k(t)].$$

From Lemma A1, this quantity is uniformly bounded below by the constant

$$\begin{aligned}
LB &= w_{j-1} [Y_{m-1} - \tilde{Y}_{m-1} - |\tilde{y}_m - y_m| - |\tilde{y}_{m-1} - y_{m-1}| - \dots - |\tilde{y}_1 - y_1|] \\
&\quad - w_j [Y_{k-1} - \tilde{Y}_{k-1} + |\tilde{y}_k - y_k| + |\tilde{y}_{k-1} - y_{k-1}| + \dots + |\tilde{y}_1 - y_1|]
\end{aligned}$$

and uniformly bounded above by the constant

$$\begin{aligned}
UB &= w_{j-1} \left[Y_{m-1} - \tilde{Y}_{m-1} + |\tilde{y}_n - y_n| + |\tilde{y}_{n-1} - y_{n-1}| + \cdots + |\tilde{y}_m - y_m| \right] \\
&\quad - w_j \left[Y_{k-1} - \tilde{Y}_{k-1} - |\tilde{y}_n - y_n| - |\tilde{y}_{n-1} - y_{n-1}| - \cdots - |\tilde{y}_k - y_k| \right].
\end{aligned}$$

Since $\lim_{t \rightarrow \infty} \frac{1}{t} LB = \lim_{t \rightarrow \infty} \frac{1}{t} UB = 0$, the first limit above exists and

$$\lim_{t \rightarrow \infty} \frac{1}{t} \left\{ w_{j-1} E \left[N_{j-1}(t) - \tilde{N}_{j-1}(t) \right] - w_j E \left[N_j(t) - \tilde{N}_j(t) \right] \right\} = 0.$$

Based on (11)-(14), these remaining limit terms can be written as

$$\begin{aligned}
& \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j \left[(w_{j-1} - w_j) \left(D_1(u) - \tilde{N}_{j-1}(u) \right) + w_j \left(\tilde{N}_j(u) - \tilde{N}_{j-1}(u) \right) \right] du \right\} \\
& - \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j \left[(w_{j-1} - w_j) \left(D_1(u) - N_{j-1}(u) \right) + w_j \left(N_j(u) - N_{j-1}(u) \right) \right] du \right\} \\
= & \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j \left[(w_{j-1} - w_j) \left(-\tilde{Y}_{m-1} + \tilde{B}_m(u) \right) + w_j \left(\tilde{Y}_{k-1} - \tilde{Y}_{m-1} - \tilde{B}_k(u) + \tilde{B}_m(u) \right) \right] du \right\} \\
& - \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j \left[(w_{j-1} - w_j) \left(-Y_{m-1} + B_m(u) \right) + w_j \left(Y_{k-1} - Y_{m-1} - B_k(u) + B_m(u) \right) \right] du \right\} \\
= & \alpha_j \left[- (w_{j-1} - w_j) \tilde{Y}_{m-1} + w_j \left(\tilde{Y}_{k-1} - \tilde{Y}_{m-1} \right) \right] \\
& + \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j \left[(w_{j-1} - w_j) \tilde{B}_m(u) + w_j \left(-\tilde{B}_k(u) + \tilde{B}_m(u) \right) \right] du \right\} \\
& - \alpha_j \left[- (w_{j-1} - w_j) Y_{m-1} + w_j \left(Y_{k-1} - Y_{m-1} \right) \right] \\
& - \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j \left[(w_{j-1} - w_j) B_m(u) + w_j \left(-B_k(u) + B_m(u) \right) \right] du \right\}.
\end{aligned}$$

It has been established (see, e.g., Zipkin 2000, pp. 180-181) that under a fixed policy $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t B_i(u) du$ exists (with probability 1). Specifically, the backorder process's long run frequency distribution exists (with probability 1), the backorder process has a limiting distribution, and the two distributions are equal. If we denote the random variable with this distribution \underline{B}_i^ℓ , then $\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t B_i(u) du = E[\underline{B}_i^\ell]$,

which we denote \underline{B}_j . Therefore the limits above exist and we have

$$\begin{aligned}
& \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} (\pi_j(t) - \tilde{\pi}_j(t)) \right\} \\
&= \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j [(w_{j-1} - w_j) (D_1(u) - \tilde{N}_{j-1}(u)) + w_j (\tilde{N}_j(u) - \tilde{N}_{j-1}(u))] du \right\} \\
&\quad - \lim_{t \rightarrow \infty} E \left\{ \frac{1}{t} \int_0^t \alpha_j [(w_{j-1} - w_j) (D_1(u) - N_{j-1}(u)) + w_j (N_j(u) - N_{j-1}(u))] du \right\} \\
&= \alpha_j \left[-(w_{j-1} - w_j) \tilde{Y}_{m-1} + w_j (\tilde{Y}_{k-1} - \tilde{Y}_{m-1}) \right] + \alpha_j (w_{j-1} - w_j) \tilde{\underline{B}}_m + \alpha_j w_j (\tilde{\underline{B}}_m - \tilde{\underline{B}}_k) \\
&\quad - \alpha_j \left[-(w_{j-1} - w_j) Y_{m-1} + w_j (Y_{k-1} - Y_{m-1}) \right] - \alpha_j (w_{j-1} - w_j) \underline{B}_m - \alpha_j w_j (\underline{B}_m - \underline{B}_k) \\
&= \alpha_j \left[(w_{j-1} - w_j) (-\tilde{Y}_{m-1} + \tilde{\underline{B}}_m) + w_j (\tilde{Y}_{k-1} - \tilde{Y}_{m-1} - \tilde{\underline{B}}_k + \tilde{\underline{B}}_m) \right] \\
&\quad - \alpha_j \left[(w_{j-1} - w_j) (-Y_{m-1} + \underline{B}_m) + w_j (Y_{k-1} - Y_{m-1} - \underline{B}_k + \underline{B}_m) \right] \\
&= \left[\alpha_j (w_{j-1} - w_j) \tilde{B}_j^f(\tilde{\mathbf{y}}) + \alpha_j w_j \tilde{I}_j^f(\tilde{\mathbf{y}}) \right] - \left[\alpha_j (w_{j-1} - w_j) B_j^f(\mathbf{y}) + \alpha_j w_j I_j^f(\mathbf{y}) \right] \\
&= C_j^f(\tilde{\mathbf{y}}) - C_j^f(\mathbf{y}).
\end{aligned}$$

Proof of Proposition 3: For any pair of local base-stock levels (y_2, y_1) , from the definition of base stock policies and the definition of IOP_j , we have $IT_2 + I_2 = y_2 + B_2$ and $IT_1 + I_1 = y_1 + B_1 - B_2$. As a result, $C^f(y_2, y_1)$ can be expressed as

$$\begin{aligned}
C^f(y_2, y_1) &= \alpha c [y_2 + B_2(y_2) + y_1 + B_1(y_2, y_1) - B_2(y_2)] + \alpha(p - c)B_1(y_2, y_1) \\
&= \alpha c(y_1 + y_2) + \alpha p \left[F_2(y_2) \int_{y_1}^{\infty} (x - y_1) f_1(x) dx \right. \\
&\quad \left. + \int_{y_2}^{\infty} \int_{y_1 + y_2 - v}^{\infty} (u - y_1 - y_2 + v) f_1(u) f_2(v) dudv \right].
\end{aligned}$$

For any (y_2, y_1) with $y_2 > 0$ and any $\epsilon \in [0, y_2]$, we can write

$$\begin{aligned}
C^f(y_2 - \epsilon, y_1 + \epsilon) &= \alpha c(y_2 + y_1) + \alpha p \left[F_2(y_2 - \epsilon) \int_{y_1 + \epsilon}^{\infty} (x - y_1 - \epsilon) f_1(x) dx \right. \\
&\quad \left. + \int_{y_2 - \epsilon}^{\infty} \int_{y_1 + y_2 - v}^{\infty} (u - y_1 - y_2 + v) f_1(u) f_2(v) dudv \right]
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial C^f(y_2 - \epsilon, y_1 + \epsilon)}{\partial \epsilon} &= -f_2(y_2 - \epsilon) \int_{y_1 + \epsilon}^{\infty} (x - y_1 - \epsilon) f_1(x) dx + F_2(y_2 - \epsilon) \int_{y_1 + \epsilon}^{\infty} -f_1(x) dx \\
&\quad + \int_{y_1 + \epsilon}^{\infty} (u - y_1 - \epsilon) f_1(u) f_2(y_2 - \epsilon) du \\
&= -F_2(y_2 - \epsilon) (1 - F_1(y_1 + \epsilon)) < 0,
\end{aligned}$$

total supply chain costs decrease as ϵ drops to 0 while keeping $y_1 + y_2$ constant. (Note that reducing y_2 further while keeping $y_1 + y_2$ constant does not affect either inventory levels or backorders at the retailer, and thus does not affect total supply chain costs.) Therefore $y_2^* = 0$ is optimal. Given no

inventory at the supplier, the problem reduces to a standard single-location base stock model with leadtime $L_2 + L_1$. Therefore, the optimal base stock level at the retailer is $F_{1+2}^{-1}(\frac{p-c}{p})$.

Proof of Proposition 4: The supplier's best response is independent of the retailer's choice of y_1 . It is equal to zero when $w = c$ and increasing in w . The retailer's best response is $y_1(y_2) = \arg \min_{y_1} \left\{ F_2(y_2)g_1(y_1) + \int_{y_2}^{\infty} g_1(y_1 + y_2 - x)f_2(x)dx \right\}$. To see that $y_1(y_2)$ is decreasing in y_2 , we check for supermodularity. First, we confirm that $\frac{\partial C_1(y_2, y_1)}{\partial y_2} = \int_{y_2}^{\infty} g_1'(y_1 + y_2 - x)f_2(x)dx$, and $g_1'(y) = \alpha_1 F_1(y) - \alpha_1(p - w)$. So, $\frac{\partial C_1(y_2, y_1)}{\partial y_2} = \alpha_1 p_1 \int_{y_2}^{\infty} F_1(y_1 + y_2 - x)f_2(x)dx - \alpha_1(p - w)(1 - F_2(y_2))$. Then, $\frac{\partial^2 C_1(y_2, y_1)}{\partial y_2 \partial y_1} = \alpha_1 p_1 \int_{y_2}^{\infty} f_1(y_1 + y_2 - x)f_2(x)dx > 0$. So, $y_1(y_2)$ is decreasing in y_2 . It approaches $F_1^{-1}\left(\frac{p-w}{p}\right)$ from above as $y_2 \rightarrow \infty$. To derive an upper bound, note that when $y_2 = 0$ the retailer's best response is $y_1(0) = \arg \min_{y_1} \left\{ \int_0^{\infty} f_2(x)g_1(y_1 - x)dx \right\} = F_{1+2}^{-1}\left(\frac{p-w}{p}\right)$. So, $y_1^*(y_2)$ is less than $F_{1+2}^{-1}\left(\frac{p-w}{p}\right)$ so long as $w > c$.

Proof of Proposition 5: To determine equilibrium behavior, note that from C_1^f , the retailer only incurs costs based on backorders. Therefore, $\partial C_1^f(y_2, y_1)/\partial y_1 < 0$ for any value of y_2 , and the retailer will simply try to order as much as possible, i.e., $y_1(y_2) = \Upsilon$. Notice that the supplier's cost function has the same structure as the supply chain's cost function, except that it has cost $\alpha(w - c)$ associated with the backorders instead of $\alpha(p - c)$. Therefore, from Proposition 3, the supplier's costs are minimized when $(y_2, y_1) = (0, F_{1+2}^{-1}(\frac{w-c}{w}))$. Although the supplier cannot force this specific solution, the exact same inventory levels and retailer backorders (and thus the same costs) can be achieved with any base-stock pair $(y_2, y_1) = (-x, F_{1+2}^{-1}(\frac{w-c}{w}) + x)$ with $x \geq 0$. Given the retailer's choice, the supplier can accomplish this by choosing $y_2(\Upsilon) = F_{1+2}^{-1}(\frac{w-c}{w}) - \Upsilon < 0$, yielding the equilibrium $(y_2, y_1) = (F_{1+2}^{-1}(\frac{w-c}{w}) - \Upsilon, \Upsilon)$, and the same inventory and retailer backorders as $(y_2, y_1) = (0, F_{1+2}^{-1}(\frac{w-c}{w}))$. Since the centrally optimal base-stock levels are $(y_2, y_1) = (0, F_{1+2}^{-1}(\frac{p-c}{p}))$, and $F_{1+2}^{-1}(\frac{w-c}{w}) \leq F_{1+2}^{-1}(\frac{p-c}{p})$, the inventory at the retailer is lower than optimal. Furthermore, since $F_{1+2}^{-1}(\frac{w-c}{w})$ is increasing in w , the deviation from optimality is decreasing in w .

The additional technical requirement that an upper bound Υ limiting each party's base-stock choice to reasonable levels warrants some discussion. Without any additional technical restrictions, this case is not well-behaved - i.e., a Nash equilibrium does not exist. With no penalty for ordering, the retailer would choose an arbitrarily large base-stock level to avoid any chance of backorders. Such behavior is not reasonable to expect in any practical setting, and this anomalous theoretical behavior can be prevented with any one of a variety of minor technical adjustments. The approach we choose is to simply add an upper bound Υ limiting each party's base-stock choice to reasonable levels. (Other approaches that would lead to the same outcome include using echelon base-stock levels as decision variables, or allowing sequential base-stock level choices where the supplier acts second. All of these

approaches capture the practical fact that the supplier is in a position to control how much inventory reaches the retailer - a feature that was observed in the Motivating Story in the introduction.)

Proof of Proposition 6: Let $\bar{\gamma} = \frac{c(p-w)}{w(p-c)}$. Under $\bar{\gamma}$ the retailer's cost can be rewritten

$$\begin{aligned} C_1(y_2, y_1) &= \bar{\gamma}\alpha_1 w [IT_1 + I_1(y_2, y_1)] + \alpha_1(p-w)B_1(y_2, y_1) \\ &= \frac{c(p-w)}{w(p-c)}\alpha_1 w [IT_1 + I_1(y_2, y_1)] + \alpha_1(p-w)B_1(y_2, y_1) \\ &= \frac{\alpha_1 c(p-w)}{p-c} [IT_1 + I_1(y_2, y_1)] + \alpha_1(p-w)B_1(y_2, y_1) \\ &= \frac{p-w}{p-c} \{ \alpha_1 c [IT_1 + I_1(y_2, y_1)] + \alpha_1(p-c)B_1(y_2, y_1) \}. \end{aligned}$$

When $y_2 = 0$, the quantity in brackets is equal to the total supply chain cost, so the retailer's best response is $y_1(0) = F_{1+2}^{-1}\left(\frac{p-c}{p}\right)$. It remains to show that $y_2\left(F_{1+2}^{-1}\left(\frac{p-c}{p}\right)\right) = 0$. Note that

$$\bar{\gamma}w - c = \frac{c(p-w)}{w(p-c)}w - c = \frac{c(p-w) - c(p-c)}{p-c} = \left(\frac{w-c}{p-c}\right)c,$$

the supplier's average cost under $\bar{\gamma}$ can be rewritten as

$$\begin{aligned} C_2(y_2, y_1) &= \alpha_2 c [IT_2 + I_2(y_2)] - \alpha_2(\bar{\gamma}w - c) [IT_1 + I_1(y_2, y_1)] + \alpha_2(w-c)B_1(y_2, y_1) \\ &= \alpha_2 c [IT_2 + I_2(y_2)] + \left(\frac{\alpha_2 c(w-c)}{p-c}\right) [IT_1 + I_1(y_2, y_1)] + \alpha_2(w-c)B_1(y_2, y_1) \\ &= \alpha_2 c [IT_2 + I_2(y_2)] + \frac{w-c}{p-c} \{ \alpha_2 c [IT_1 + I_1(y_2, y_1)] + \alpha_2(p-c)B_1(y_2, y_1) \} \\ &= \frac{p-w}{p-c}\alpha_2 c IT_2 + \frac{p-w}{p-c}\alpha_2 c I_2(y_2) \\ &\quad + \frac{w-c}{p-c} \{ \alpha_2 c [IT_2 + I_2(y_2) + IT_1 + I_1(y_2, y_1)] + \alpha_2(p-c)B_1(y_2, y_1) \}. \end{aligned}$$

Now, we know that the first term does not depend on base stock levels. The second term is minimized and equal to zero for any $y_2 \leq 0$. To show that $y_2 = 0$, we look at the third term, which is simply $\frac{w-c}{p-c}$ times the total supply chain costs if the supply chain incurred interest rate α_2 . We know that the optimal solution to the total supply chain costs is $\left(0, F_{1+2}^{-1}\left(\frac{p-c}{p}\right)\right)$, which implies that the supplier's best response is $y_2\left(F_{1+2}^{-1}\left(\frac{p-c}{p}\right)\right) = 0$.

Proof of Proposition 7: We will show that when the centralized optimal policy is to hold zero inventory at the supplier, then the γ above coordinates the supply chain. To find sufficient conditions under which it is optimal to hold zero inventory at the supplier in the centralized solution, we follow the arguments outlined by Zipkin (2000 p.317). First, we can establish that $y_1^\alpha \equiv F_1^{-1}\left(\frac{\alpha p + b^p + h_2^p}{\alpha p + b^p + h_1^p}\right)$ is the solution to the retailer's cost function under the traditional "bottom-up" recursion algorithm to evaluating the optimal policy for multiechelon supply chains. That cost function is traditionally expressed as $C_1^\alpha(y) = h_1' E(y - D_2)^+ + (b + h_2') E(y - D_2)^-$, which has solution $y_1^\alpha \equiv F_1^{-1}\left(\frac{b + h_2'}{b + h_1' + h_2'}\right)$ and we can make the substitutions $h_2' = \alpha c + h_2^p$, $h_1' = h_1^p - h_2^p$, $b = \alpha(p-c) + b^p$. Next, we

can also establish that $y_1^b \equiv F_{1+2}^{-1}\left(\frac{\alpha(p-c)+b^p}{\alpha p+b^p+h_1^p}\right)$ is the optimal *echelon* base stock policy at the *supplier* assuming zero local inventory at the supplier. This follows from the fact that the solution to $C_2^b(y) = h_2'\mu_1 + h_1(y - \mu_1 - \mu_2) + (b + h_1)E(y - D_{1+2})^-$ is $y_1^b \equiv F_{1+2}^{-1}\left(\frac{b}{b+h_1}\right)$, and we can make the substitutions $h_1 = \alpha c + h_1^p$, $b = \alpha(p - c) + b^p$. Thus y_1^b determines the maximum level of inventory that the supplier should allow the retailer to maintain. Therefore, if $y_1^b \leq y_1^a$, then the supplier holds zero inventory and the optimal policy in local terms is $y_2^* = 0$, $y_1^* = F_{1+2}^{-1}\left(\frac{\alpha(p-c)+b^p}{\alpha p+b^p+h_1^p}\right)$. Next, we show that this centralized solution is a Nash equilibrium under $\gamma = \frac{c(\alpha p+h_1^p)-w(\alpha c+h_1^p)-\frac{b^p}{\alpha}(h_1^p-\phi(\alpha c+h_1^p))}{w(\alpha(p-c)+b^p)}$. The retailer's average cost is

$$C_1(y_2, y_1) = (\gamma\alpha w + h_1^p) [IT_1 + I_1(y_2, y_1)] + (\alpha(p - w) + \phi b^p) B_1(y_2, y_1)$$

When $y_2 = 0$, the retailer's best response is $y_1(0) = F_{1+2}^{-1}\left(\frac{\alpha(p-w)+\phi b^p}{\alpha(p-w)+\phi b^p+\gamma\alpha w+h_1^p}\right)$. Now, under $\gamma = \frac{c(\alpha p+h_1^p)-w(\alpha c+h_1^p)-\frac{b^p}{\alpha}(h_1^p-\phi(\alpha c+h_1^p))}{w(\alpha(p-c)+b^p)}$ and with some algebra, the retailer's critical ratio can be rewritten as $\frac{\alpha(p-w)+\phi b^p}{\alpha(p-w)+\phi b^p+\gamma\alpha w+h_1^p} = \frac{\alpha(p-c)+b^p}{\alpha p+b^p+h_1^p}$. So, the retailer's best response function is equal to the centralized base-stock level. What is left to show is that the supplier's best response is also equal to the centralized base stock level, $y_2^* \left(F_{1+2}^{-1}\left(\frac{\alpha(p-c)+b^p}{\alpha p+b^p+h_1^p}\right)\right) = 0$. Under $\gamma = \frac{c(\alpha p+h_1^p)-w(\alpha c+h_1^p)-\frac{b^p}{\alpha}(h_1^p-\phi(\alpha c+h_1^p))}{w(\alpha(p-c)+b^p)}$ and with some algebra, the supplier's overall cost can be expressed as

$$\begin{aligned} C_2(y_2, y_1) &= (\alpha c + h_2^p) [IT_2 + I_2(y_2)] - \alpha(\gamma w - c) [IT_1 + I_1(y_2, y_1)] + [\alpha(w - c) + (1 - \phi)b^p] B_1(y_2, y_1) \\ &= (\alpha c + h_2^p) [IT_2 + I_2] + \frac{\alpha(w - c) + (1 - \phi)b^p}{\alpha(p - c) + b^p} [(\alpha c + h_1^p)[IT_1 + I_1] + (\alpha(p - c) + b^p)B_1] \\ &= \frac{\alpha(p - w) + \phi b^p}{\alpha(p - c) + b^p} (\alpha c + h_2^p) [IT_2 + I_2] \\ &\quad + \frac{\alpha(w - c) + (1 - \phi)b^p}{\alpha(p - c) + b^p} \{(\alpha c + h_2^p) [IT_2 + I_2] + (\alpha c + h_1^p)[IT_1 + I_1] + (\alpha(p - c) + b^p)B_1\} \end{aligned}$$

The first term is equal to zero for any $y_2 \leq 0$. The second term is proportional to the supply chain total costs, which we know is minimized as the centralized optimal solution of $y_2 = 0$. Thus, we know the base stock levels $y_2^* = 0$, $y_1^* = F_{1+2}^{-1}\left(\frac{\alpha(p-c)+b^p}{\alpha p+b^p+h_1^p}\right)$ are a Nash equilibrium.