Electronic Companion to “Management and Effects of In-Store Promotional Displays”

EC.1. Proofs of Analytical Results

Proof of Proposition 1. Suppose we have two products, i and j, such that $Y_i \geq Y_j$ and $m_i \geq m_j$, or equivalently, product i dominates product j. Our aim is to show that product j is never optimal to promote.

Let $K = \arg\max_{i \in S}\{m_i\}$. Note that K is not necessarily a singleton. Hence, let $k = \arg\max_{i \in K}\{Y_i\}$. In words, k is the most popular product among the products having the highest margin. The product k is well defined because by assumption no pair of products have the same popularity and the profit margin. We can assume w.l.o.g. that products i, j, and k are distinct. By definition of product k, the dominated product (j) is different from product k. If products i and k are the same, the following arguments still hold.

We argue that $\max\{\pi^i, \pi^k\} \geq \pi^j$. This implies that there is always a more profitable option (either promoting product i or product k) than promoting product j. Let $M = P_A^0 \cdot \big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big)$. We can write π^x as a convex combination of m_x and M for all $x \in S$ as follows:

$$\pi^x = P_D^x \cdot m_x + (1 - P_D^x) \cdot M \tag{EC.1}$$

We consider two cases separately: ($m_i > M$) and ($m_i \leq M$). If ($m_i > M$), we have

$$\pi^j = P_D^j \cdot m_j + (1 - P_D^j) \cdot M \leq P_D^j \cdot m_i + (1 - P_D^j) \cdot M \leq P_D^i \cdot m_i + (1 - P_D^i) \cdot M = \pi^i \tag{EC.2}$$

The first inequality follows because $m_i \geq m_j$ and the second follows because $P_D^i \geq P_D^j$ and $m_i > M$. Note also that at least one of the above inequalities is strict because we have assumed no two products i and j have $m_i = m_j$ and $Y_i = Y_j$. Therefore we either have $m_i > m_j$ or $P_D^i > P_D^j$, or both. Hence, $\pi^i > \pi^j$ if $m_i > M$. Next, consider the case where ($m_i \leq M$). By Assumption 1, we have $m_k > M$. Since $m_j \leq m_i \leq M < m_k$, it follows from (EC.1) that $\pi^k > \pi^j$. □

Proof of Proposition 2. We can write π^x as follows:

$$\pi^x = (1 - P_A^x) \cdot \theta_x + P_A^x \cdot \big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big) \tag{EC.3}$$

Consider two products, i and j, such that $Y_i < Y_j$ and $\theta_i \geq \theta_j$. We have

$$\begin{aligned} \pi^j = (1 - P_A^j) \cdot \theta_j + P_A^j \cdot \big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big) &< (1 - P_A^i) \cdot \theta_j + P_A^i \cdot \big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big) && \text{(EC.4)} \\ &\leq (1 - P_A^i) \cdot \theta_i + P_A^i \cdot \big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big) = \pi^i && \text{(EC.5)} \end{aligned}$$

The first inequality follows by Assumption 2 and the fact that $P_A^i > P_A^j$. The second inequality follows since $\theta_i \geq \theta_j$. □

Proof of Proposition 3. Consider two products, $j \in E'$ and $k \in E'$, such that $Y_j > Y_k$.

Part 1: We consider $\pi^j(r)$ and $\pi^k(r)$ as functions of r. Notice that they are linearly increasing in r. Since $P_A^j < P_A^k$, $\pi^k(r)$ increases at a faster rate than $\pi^j(r)$. This implies that $\pi^j(r)$ and $\pi^k(r)$ intersect at a unique value of r above which $\pi^k(r) > \pi^j(r)$. Hence, the optimal choice of product to promote, $i^*(r)$, always switches from a more popular product to a less popular one within the set E' (or stays the same) as r increases. Since the products in E' are indexed in decreasing order of their popularities, $i^*(r)$ is non-decreasing in r.

Part 2: We consider $\pi^j(c)$ and $\pi^k(c)$ as functions of c. We first solve the equation $\pi^j(c) = \pi^k(c)$ for c and argue that the solution is unique, implying that $\pi^j(c)$ and $\pi^k(c)$ intersect at a unique value of c. Some manipulation of the equation $\pi^j(c) = \pi^k(c)$ yields the following:

$$e^{-c+\lambda I} = \frac{e^{Y_0}(e^{Y_j} \cdot m_j - e^{Y_k} \cdot m_k) + e^{Y_j+Y_k} \cdot (m_j - m_k)}{e^{Y_k} \cdot m_k - e^{Y_j} \cdot m_j + (e^{Y_j} - e^{Y_k})\big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big)} \tag{EC.6}$$

By continuity and monotonicity of the LHS in c, there exists a unique value of c that satisfies $\pi^i(c) = \pi^j(c)$. Hence, $\pi^j(c)$ and $\pi^k(c)$ intersect only once. Next, we show that $\pi^j(c) > \pi^k(c)$ for sufficiently large c. Since j and k are picked arbitrarily, this is sufficient to show that the the optimal choice of product to promote, $i^*(c)$, always switches from a less popular product to a more popular one within the set E' (or stays the same) as c increases.

In order to show that $\pi^j(c) > \pi^k(c)$ for sufficiently large c, we write π^x as a convex combination of θ_x and $\big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big)$ for $x \in S$ as in EC.3. It is straightforward to show that $\lim_{c\to\infty} P_A^x = 0$ for all $x \in S$. Then, in the limit, $\pi^j(c) > \pi^k(c)$, because $\theta_j > \theta_k$ by Proposition 2. □

Proof of Proposition 4. Assume without loss of generality that $\arg\max_{i\in S}\{\pi^i(S)\} = j$ and $\arg\max_{i\in S}\{\pi^i(S^+)\} = k$ where $j \neq k$. (Otherwise, the optimal product is the same for both assortments, and the proposition statement is satisfied with equality.) By Proposition 2, both of these products are in the set E' corresponding to assortment S. Our aim is to show that $\theta_j > \theta_k$, which implies that $j < k$ due to the order of products in the set E'.

Fix $M = \bar{m}(\mathbf{Y}_S, \mathbf{m}_S, \lambda) = \bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda)$. Define the following functions of x:

$$\tilde{\pi}^j(x) = P_D^j(x) \cdot m_j + P_A^j(x) \cdot (M + r), \tag{EC.7}$$

$$\tilde{\pi}^k(x) = P_D^k(x) \cdot m_k + P_A^k(x) \cdot (M + r), \tag{EC.8}$$

where $P_D^j(x)$, $P_A^j(x)$, $P_D^k(x)$, $P_A^k(x)$ are the choice probabilities obtained by replacing $\lambda \cdot I(\mathbf{Y}, \lambda)$ with x. After solving $\tilde{\pi}^j(x) - \tilde{\pi}^k(x) = 0$, we get

$$e^{-c+x} = \frac{e^{Y_0}(e^{Y_j} \cdot m_j - e^{Y_k} \cdot m_k) + e^{Y_j+Y_k} \cdot (m_j - m_k)}{e^{Y_k} \cdot m_k - e^{Y_j} \cdot m_j + (e^{Y_j} - e^{Y_k}) \cdot (M + r)} \tag{EC.9}$$

Since the LHS of the above equation is continuous and monotonically increasing in x, $\tilde{\pi}^i(x)$ and $\tilde{\pi}^j(x)$ intersect at a unique value of x, say x'. Furthermore, $x' \in [\lambda \cdot I(\mathbf{Y}_S, \lambda), \lambda \cdot I(\mathbf{Y}_{S^+}, \lambda)]$ because $\tilde{\pi}^j\big(\lambda \cdot I(\mathbf{Y}_S, \lambda)\big) = \pi^j(S) > \pi^k(S) = \tilde{\pi}^k\big(\lambda \cdot I(\mathbf{Y}_S, \lambda)\big)$, and $\tilde{\pi}^k\big(\lambda \cdot I(\mathbf{Y}_{S^+}, \lambda)\big) = \pi^k(S^+) > \pi^j(S^+) = \tilde{\pi}^j\big(\lambda \cdot I(\mathbf{Y}_{S^+}, \lambda)\big)$ by definitions of j and k. This implies that we have $\tilde{\pi}^j(x) > \tilde{\pi}^k(x)$ for all $x < x'$, and $\tilde{\pi}^j(x) < \tilde{\pi}^k(x)$ for all $x > x'$. As $x \to -\infty$, we know that $\tilde{\pi}^j(x) > \tilde{\pi}^k(x)$ iff $\theta_j > \theta_k$ because $\lim_{x\to-\infty} \tilde{\pi}^i(x) = \theta_i$ for all $i \in S$. Hence, we conclude that $\theta_j > \theta_k$. □

Proof of Lemma 1. It suffices to prove that $\lambda \cdot I(\mathbf{Y},\lambda)$ is increasing in λ in the interval $(0,1]$. To that end, we calculate the first derivative of $\lambda \cdot I(\mathbf{Y},\lambda)$ with respect to λ as follows:

$$\frac{d\left(\lambda \cdot I(\mathbf{Y},\lambda)\right)}{d\lambda} = \ln\left(\sum_{k\in S\cup\{0\}} \mathrm{e}^{Y_k/\lambda}\right) - \frac{\sum_{q\in S\cup\{0\}} \frac{Y_q}{\lambda}\cdot \mathrm{e}^{Y_q/\lambda}}{\sum_{q\in S\cup\{0\}} \mathrm{e}^{Y_q/\lambda}} = \frac{\sum_{q\in S\cup\{0\}} \mathrm{e}^{Y_q/\lambda}\left(\ln\left(\sum_{k\in S\cup\{0\}} \mathrm{e}^{Y_k/\lambda}\right) - \frac{Y_q}{\lambda}\right)}{\sum_{q\in S\cup\{0\}} \mathrm{e}^{Y_q/\lambda}}$$

The denominator is clearly positive. Since $\left(\ln\left(\sum_{k\in S\cup\{0\}} \mathrm{e}^{Y_k/\lambda}\right) - \frac{Y_q}{\lambda}\right) > 0$ for all $q \in S\cup\{0\}$ and for all $\lambda \in (0,1]$, the numerator in the above expression is also positive. Hence, we have $\frac{d\left(\lambda \cdot I(\mathbf{Y},\lambda)\right)}{d\lambda} > 0$, implying that $\lambda \cdot I(\mathbf{Y},\lambda)$ is strictly increasing in $\lambda \in (0,1]$. □

Proof of Proposition 5. Assume without loss of generality that there exists $\lambda' \in (0,1]$ at which the optimal choice of product switches from one product to another. (If λ' does not exist, then the optimal product is the same for all $\lambda \in (0,1]$, and thus we are done.) Suppose these two products are indexed by j and k, or more formally, $\arg\max_{i\in S}\{\pi^i(\lambda'-\epsilon)\} = j$ and $\arg\max_{i\in S}\{\pi^i(\lambda'+\epsilon)\} = k$ for sufficiently small $\epsilon > 0$. By Proposition 2, both of these products are in the set E'. Our aim is to show that $\theta_j > \theta_k$, which implies that $j < k$ due to the order of products in the set E'.

Fix $M = \bar{m}(\mathbf{Y},\mathbf{m},\lambda')$. Define $\tilde{\pi}^j(x)$ and $\tilde{\pi}^k(x)$ as in EC.7 and EC.8. Following similar arguments as in EC.9, we can show that $\tilde{\pi}^j(x)$ and $\tilde{\pi}^k(x)$ intersect at a unique value of x. This intersection occurs at $x = \lambda' \cdot I(\mathbf{Y},\lambda')$ because

$$\tilde{\pi}^j\left(\lambda' \cdot I(\mathbf{Y},\lambda')\right) = \pi^j(\lambda') = \pi^k(\lambda') = \tilde{\pi}^k\left(\lambda' \cdot I(\mathbf{Y},\lambda')\right). \tag{EC.10}$$

We have $\lim_{x\to-\infty} \tilde{\pi}^i(x) = \theta_i$ for all $i \in S$. Hence, we have either one of the following cases:

1. $\theta_j > \theta_k$ and therefore $\tilde{\pi}^j(x) > \tilde{\pi}^k(x)\ \forall x < \lambda' \cdot I(\mathbf{Y},\lambda')$ and $\tilde{\pi}^j(x) < \tilde{\pi}^k(x)\ \forall x > \lambda' \cdot I(\mathbf{Y},\lambda')$.
2. $\theta_j < \theta_k$ and therefore $\tilde{\pi}^j(x) < \tilde{\pi}^k(x)\ \forall x < \lambda' \cdot I(\mathbf{Y},\lambda')$ and $\tilde{\pi}^j(x) > \tilde{\pi}^k(x)\ \forall x > \lambda' \cdot I(\mathbf{Y},\lambda')$.

We claim that case 2 cannot be true, and we show it by contradiction. Suppose that $\theta_j < \theta_k$ and therefore $\tilde{\pi}^j(x) > \tilde{\pi}^k(x)$ for all $x > \lambda' \cdot I(\mathbf{Y},\lambda')$. Then, consider the following equalities:

$$\pi^j(\lambda) = \tilde{\pi}^j\left(\lambda \cdot I(\mathbf{Y},\lambda)\right) + P_A^j\left(\lambda \cdot I(\mathbf{Y},\lambda) \cdot \left(\bar{m}(\mathbf{Y},\mathbf{m},\lambda) - M\right)\right. \tag{EC.11}$$

$$\pi^k(\lambda) = \tilde{\pi}^k\left(\lambda \cdot I(\mathbf{Y},\lambda) + P_A^k\left(\lambda \cdot I(\mathbf{Y},\lambda) \cdot \left(\bar{m}(\mathbf{Y},\mathbf{m},\lambda) - M\right)\right.\right. \tag{EC.12}$$

Since $\bar{m}(\mathbf{Y},\mathbf{m},\lambda)$ is non-decreasing in λ by the proposition assumption, the expression $\left(\bar{m}(\mathbf{Y},\mathbf{m},\lambda) - M\right)$ in the above equations are non-negative for any $\lambda > \lambda'$. Moreover, $\theta_j < \theta_k$ implies that $Y_j < Y_k$ by definition of E', and therefore $P_A^j\left(\lambda \cdot I(\mathbf{Y},\lambda)\right) > P_A^k\left(\lambda \cdot I(\mathbf{Y},\lambda)\right)$ for any $\lambda \in (0,1]$. Finally, case 2 implies that $\tilde{\pi}^j\left((\lambda'+\epsilon) \cdot I(\mathbf{Y},(\lambda'+\epsilon))\right) > \tilde{\pi}^k\left((\lambda'+\epsilon) \cdot I(\mathbf{Y},(\lambda'+\epsilon))\right)$. As a result, plugging $\lambda = (\lambda'+\epsilon)$ in (EC.11) and (EC.12) reveals that $\pi^j(\lambda'+\epsilon) > \pi^k(\lambda'+\epsilon)$, which contradicts with the definition of j and k. Hence, case 2 cannot be true, implying that $\theta_j > \theta_k$. □

Proof of Proposition 6. We prove in two steps. First, we show that $\Delta^j(S) - \Delta^k(S^+)$ is decreasing linearly in $\beta \geq 0$, which implies that $\Delta^j(S) < \Delta^k(S^+)$ for some large enough $\beta \geq 0$. Second, we show that $\Delta^j(S) > \Delta^k(S^+)$ when $\beta = 0$. These together imply the existence of the threshold $\beta' \geq 0$ and completes the proof.

1. $\Delta^j(S) - \Delta^k(S^+)$ is decreasing linearly in $\beta \geq 0$.

We express $\Delta^j(S) - \Delta^k(S^+)$ as follows:

$$\begin{aligned}\Delta^j(S) - \Delta^k(S^+) &= \Pi^j(S) - \Pi^0(S) - \left(\Pi^k(S^+) - \Pi^0(S^+)\right)\\ &= (1+\beta)\left(\pi^j(S) - \pi^k(S^+)\right) - \left(P_A^0(S)\cdot \bar{m}(\mathbf{Y}_S, \mathbf{m}_S, \lambda) - P_A^0(S^+)\cdot \bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda)\right).\end{aligned}$$

We must show that $\pi^j(S) < \pi^k(S^+)$. To that end, consider the following inequalities:

$$\begin{aligned}\pi^k(S^+) \geq \pi^j(S^+) &= \left(1 - P_A^j(S^+)\right)\cdot\theta_j + P_A^j(S^+)\cdot\left(\bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda) + r\right) && \text{(EC.13)}\\ &> \left(1 - P_A^j(S)\right)\cdot\theta_j + P_A^j(S)\cdot\left(\bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda) + r\right) && \text{(EC.14)}\\ &\geq \left(1 - P_A^j(S)\right)\cdot\theta_j + P_A^j(S)\cdot\left(\bar{m}(\mathbf{Y}_S, \mathbf{m}_S, \lambda) + r\right) = \pi^j(S) && \text{(EC.15)}\end{aligned}$$

where (EC.14) follows from Assumption 2 and the fact that $P_A^i(S^+) > P_A^i(S)$ for all $i \in S$, and (EC.15) follows because $\bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda) \geq \bar{m}(\mathbf{Y}_S, \mathbf{m}_S, \lambda)$.

2. $\Delta^j(S) > \Delta^k(S^+)$ when $\beta = 0$.

Let $\beta = 0$. By Assumption 1, we know that $\Delta^j(S) > 0$, i.e., promoting the optimal product in S is more profitable than no-promotion. If $\Delta^k(S^+) < 0$, then $\Delta^j(S) > 0 > \Delta^k(S^+)$ and we are done. Hence, we consider the case when $\Delta^k(S^+) > 0$.

Since $\Delta^j(S) \geq \Delta^k(S)$ by the optimality of product j when the assortment is S, showing $\Delta^k(S) > \Delta^k(S^+)$ will be sufficient to prove that $\Delta^j(S) > \Delta^k(S^+)$. To that end, we write $\Delta^k(S)$ as

$$\begin{aligned}\Delta^k(S) &= \frac{e^{Y_k}}{e^{Y_0} + e^{Y_k} + e^{-c+\lambda\cdot I(S)}}\cdot m_k - \left(\frac{e^{-c+\lambda\cdot I(S)}}{e^{Y_0} + e^{-c+\lambda\cdot I(S)}} - \frac{e^{-c+\lambda\cdot I(S)}}{e^{Y_0} + e^{Y_k} + e^{-c+\lambda\cdot I(S)}}\right)\cdot\left(\bar{m}(\mathbf{Y}_S, \mathbf{m}_S, \lambda) + r\right)\\ &= \frac{e^{Y_0}\cdot e^{Y_k}\cdot m_k + e^{Y_k}\cdot\left(m_k - \bar{m}(\mathbf{Y}_S, \mathbf{m}_S, \lambda) - r\right)\cdot e^{-c+\lambda I(S)}}{\left(e^{Y_0} + e^{Y_k} + e^{-c+\lambda\cdot I(S)}\right)\cdot\left(e^{Y_0} + e^{-c+\lambda\cdot I(S)}\right)}\end{aligned}$$

Fix $M = \bar{m}(\mathbf{Y}_S, \mathbf{m}_S, \lambda)$. Define the function $\bar{\Delta}(S)$ as follows:

$$\bar{\Delta}(S) := \frac{e^{Y_0}\cdot e^{Y_k}\cdot m_k + e^{Y_k}\cdot(m_k - M - r)\cdot e^{-c+\lambda\cdot I(S)}}{\left(e^{Y_0} + e^{Y_k} + e^{-c+\lambda\cdot I(S)}\right)\cdot\left(e^{Y_0} + e^{-c+\lambda\cdot I(S)}\right)} \tag{EC.16}$$

We will show that $\bar{\Delta}(S)$ decreases as the assortment S expands. To that end, consider the function $f : [I(\mathbf{Y}_S), I(\mathbf{Y}_{S^+})] \to [e^{Y_0} + e^{-c+\lambda\cdot I(S)}, e^{Y_0} + e^{-c+\lambda\cdot I(S^+)}]$ such that $x = f(I) = e^{Y_0} + e^{-c+\lambda\cdot I(\cdot)}$. Notice that x increases monotonically as $I(\cdot)$ increases, or equivalently as the assortment expands. Hence, $f(\cdot)$ is a one-to-one correspondence between the sets $[I(\mathbf{Y}_S), I(\mathbf{Y}_{S^+})]$ and $[e^{Y_0} + e^{-c+\lambda\cdot I(S)}, e^{Y_0} + e^{-c+\lambda\cdot I(S^+)}]$. By plugging x in the RHS of (EC.16) we get:

$$\frac{e^{Y_0}\cdot e^{Y_k}\cdot m_k + e^{Y_k}\cdot(m_k - M - r)\cdot(x - e^{Y_0})}{(e^{Y_k} + x)\cdot x} \tag{EC.17}$$

In order to show that $\bar{\Delta}(S)$ decreases as $I(\cdot)$ increases in the interval $[I(\mathbf{Y}_S), I(\mathbf{Y}_{S^+})]$, we will show that (EC.17) is decreasing in x in the interval $[e^{Y_0} + e^{-c+\lambda\cdot I(S)}, e^{Y_0} + e^{-c+\lambda\cdot I(S^+)}]$. To that end, we compute the first derivative of (EC.17) with respect to x as:

$$-\frac{e^{Y_k}x\left[x(m_k - M - r) + 2e^{Y_0}(M + r)\right] + e^{2Y_k+Y_0}(M + r)}{(e^{Y_k} + x)^2\cdot x^2} \tag{EC.18}$$

The denominator is clearly positive. We will argue that the numerator is also positive for all $x \in [\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S)}, \mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}]$. To that end, it suffices to show that

$$\left[x(m_k - M - r) + 2\mathrm{e}^{Y_0}(M+r)\right] > 0 \text{ for all } x \in [\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S)}, \mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}] \tag{EC.19}$$

Since we consider the case when $\Delta^k(S^+) > 0$ we have the following:

$$\begin{aligned}\Delta^k(S^+) =& \frac{\mathrm{e}^{Y_k}}{\mathrm{e}^{Y_0} + \mathrm{e}^{Y_k} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}} \cdot m_k \\ &- \left(\frac{\mathrm{e}^{-c+\lambda \cdot I(S^+)}}{\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}} - \frac{\mathrm{e}^{-c+\lambda \cdot I(S^+)}}{\mathrm{e}^{Y_0} + \mathrm{e}^{Y_k} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}} \right) \cdot (\bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda) + r) > 0\end{aligned}$$

Since $\bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda) \geq M$, by replacing $\bar{m}(\mathbf{Y}_{S^+}, \mathbf{m}_{S^+}, \lambda)$ by M in the above inequality, we get

$$\frac{\mathrm{e}^{Y_k}}{\mathrm{e}^{Y_0} + \mathrm{e}^{Y_k} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}} \cdot m_k - \left(\frac{\mathrm{e}^{-c+\lambda \cdot I(S^+)}}{\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}} - \frac{\mathrm{e}^{-c+\lambda \cdot I(S^+)}}{\mathrm{e}^{Y_0} + \mathrm{e}^{Y_k} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}} \right) \cdot (M + r) > 0$$

By plugging $\bar{x} = \mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}$ in the above inequality and some algebraic manipulation, we get $m_k > \frac{(\bar{x} - \mathrm{e}^{Y_0})}{\bar{x}} \cdot (M + r)$. This implies that $\bar{x} \cdot m_k - (\bar{x} - \mathrm{e}^{Y_0}) \cdot (M + r) > 0$. Then, we can write

$$\bar{x} \cdot (m_k - M - r) + \mathrm{e}^{Y_0}\,(M + r) > 0 \tag{EC.20}$$

Inequality (EC.20) implies that (EC.19) holds for all $x \leq \bar{x}$. To clarify, note that (EC.19) holds for all x if $(m_k - M - r) \geq 0$. Otherwise, it holds for all $x \leq \bar{x}$ because (EC.20) guarantees it for $x = \bar{x}$, and the LHS of (EC.19) is decreasing in x. As a result, we showed that the expression in (EC.17) is decreasing in $x \in [\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S)}, \mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda \cdot I(S^+)}]$, which in turn implies that $\bar{\Delta}(\cdot)$ decreases as $I(\cdot)$ increases in the interval $[I(\mathbf{Y}_S), I(\mathbf{Y}_{S^+})]$ (i.e., as the assortment S expands).

Thus we have $\Delta^j(S) \geq \Delta^k(S) = \bar{\Delta}(S) > \bar{\Delta}(S^+) \geq \Delta^k(S^+)$ when $\beta = 0$. ☐

Proof of Corollary 1 Let $\beta'(\cdot)$ denote the threshold β' in Proposition 6 as a function of the two assortments being compared. Choose $\beta_1 = \min\{\beta'(S, S^+), \beta'(S, S^{++})\}$ and $\beta_2 = \max\{\beta'(S, S^{++}), \beta'(S^+, S^{++})\}$. Part 1 and 3 follow by definitions of β_1 and β_2. For $\beta > \beta_1$, $\max\{\Delta^k(S^+), \Delta^l(S^{++})\} > \Delta^j(S)$. For $\beta < \beta_2$, $\max\{\Delta^j(S), \Delta^k(S^+)\} > \Delta^l(S^{++})$. Combining two, we have $\max\{\Delta^j(S), \Delta^k(S^+), \Delta^l(S^{++})\} = \Delta^k(S^+)$ for $\beta_1 < \beta < \beta_2$, so part 2 holds.

Proof of Proposition 7. Let $i^*(\underline{\lambda}) = j$ and $i^*(\bar{\lambda}) = k$ for ease of exposition. We will follow a similar procedure to the proof of Proposition 6 and prove in two steps.

1. $\Delta^j(\underline{\lambda}) - \Delta^k(\bar{\lambda})$ is decreasing linearly in $\beta \geq 0$. It follows the same arguments as the first part of the proof of Proposition 6 by replacing S and S^+ with $\underline{\lambda}$ and $\bar{\lambda}$, respectively.
2. $\Delta^j(\underline{\lambda}) > \Delta^k(\bar{\lambda})$ when $\beta = 0$.

Assume w.l.o.g. that $\exists\ \lambda' \in [\underline{\lambda}, \bar{\lambda}]$ such that $\arg\max_{i \in S}\{\pi^i(\lambda)\} = j$ for all $\lambda \in [\underline{\lambda}, \lambda']$, i.e., the optimal product to promote is j within the sub-interval $[\underline{\lambda}, \lambda']$, and it switches to another product at $\lambda = \lambda'$, denoted by $i^*(\lambda')$. We will show that $\Delta^j(\underline{\lambda}) > \Delta^j(\lambda') = \Delta^{i^*(\lambda')}(\lambda')$, where the equality holds because $\pi^j(\lambda') = \pi^{i^*(\lambda')}(\lambda')$. ($i^*(\lambda')$ and k are not necessarily the same product.) Since the same argument holds for any sub-interval in $[\underline{\lambda},\bar{\lambda}]$ and because $\Delta^{(\cdot)}(\lambda)$ is continuous, we conclude that $\Delta^j(\underline{\lambda}) > \Delta^k(\bar{\lambda})$. Define the following functions:

$$\underline{\Delta}(\lambda) := \frac{\mathrm{e}^{Y_0}\cdot \mathrm{e}^{Y_j}\cdot m_j + \mathrm{e}^{Y_j}\cdot (m_j - \bar{m}(\lambda') - r)\cdot \mathrm{e}^{-c+\lambda\cdot I(\lambda)}}{(\mathrm{e}^{Y_0}+\mathrm{e}^{Y_j}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)})\cdot(\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)})} \tag{EC.21}$$

$$\bar{\Delta}(\lambda) := \frac{\mathrm{e}^{Y_0}\cdot \mathrm{e}^{Y_j}\cdot m_j + \mathrm{e}^{Y_j}\cdot (m_j - \bar{m}(\underline{\lambda}) - r)\cdot \mathrm{e}^{-c+\lambda\cdot I(\lambda)}}{(\mathrm{e}^{Y_0}+\mathrm{e}^{Y_j}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)})\cdot(\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)})} \tag{EC.22}$$

Note that $\underline{\Delta}(\lambda) \le \bar{\Delta}(\lambda)$ for any $\lambda \in [\underline{\lambda},\bar{\lambda}]$, because $\bar{m}(\lambda)$ is non-decreasing in λ for $\lambda \in [\underline{\lambda},\bar{\lambda}]$ by the assumption given in the proposition. We will now show that $\bar{\Delta}(\lambda)$ is decreasing in λ for $\lambda \in [\underline{\lambda},\lambda']$. To that end, consider the function $f : [\underline{\lambda},\lambda'] \to [\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}, \mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda'\cdot I(\lambda')}]$ such that $x = f(\lambda) = \mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)}$. Notice that x increases monotonically in λ due to Lemma 1. Hence, $f(\lambda)$ is a one-to-one correspondence between the sets $[\underline{\lambda},\lambda']$ and $[\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}, \mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda'\cdot I(\lambda')}]$. By plugging x in (EC.22) we get:

$$\bar{\Delta}(x) = \frac{\mathrm{e}^{Y_0}\cdot \mathrm{e}^{Y_j}\cdot m_j + \mathrm{e}^{Y_j}\cdot (m_j - \bar{m}(\underline{\lambda}) - r)\cdot (x-\mathrm{e}^{Y_0})}{(\mathrm{e}^{Y_j}+x)\cdot x} \tag{EC.23}$$

In order to show that $\bar{\Delta}(\lambda)$ decreases in λ for $\lambda \in [\underline{\lambda},\lambda']$, we will show that $\bar{\Delta}(x)$ decreases in x for $x \in [\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}, \mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda'\cdot I(\lambda')}]$. The first derivative of (EC.23) with respect to x is:

$$\frac{d\bar{\Delta}(x)}{dx} = -\frac{\mathrm{e}^{Y_j}x\,[x(m_j-\bar{m}(\underline{\lambda})-r)+2\mathrm{e}^{Y_0}(\bar{m}(\underline{\lambda})+r)] + \mathrm{e}^{2Y_j+Y_0}(\bar{m}(\underline{\lambda})+r)}{(\mathrm{e}^{Y_j}+x)^2\cdot x^2} \tag{EC.24}$$

The denominator is clearly positive. We will argue that the numerator is also positive and so $\frac{d\bar{\Delta}(x)}{dx} < 0$ because of the negative sign in front. To that end, it suffices to show that

$$\left[x(m_j-\bar{m}(\underline{\lambda})-r)+2\mathrm{e}^{Y_0}(\bar{m}(\underline{\lambda})+r)\right] > 0 \text{ for all } x \in [\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}, \mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda'\cdot I(\lambda')}] \tag{EC.25}$$

By Assumption 1, we know that $\Delta^j(\lambda) > 0$ for $\lambda \in [\underline{\lambda},\lambda']$, i.e., promoting the optimal product is more profitable than no-promotion. It implies the following:

$$\Delta^j(\lambda) = \frac{\mathrm{e}^{Y_j}\cdot m_j}{\mathrm{e}^{Y_0}+\mathrm{e}^{Y_j}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)}} - \frac{\mathrm{e}^{Y_j}\cdot \mathrm{e}^{-c+\lambda\cdot I(\lambda)}}{(\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)})\cdot(\mathrm{e}^{Y_0}+\mathrm{e}^{Y_j}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)})}\cdot(\bar{m}(\lambda)+r) > 0 \text{ for all } \lambda \in [\underline{\lambda},\lambda']$$

By some algebraic manipulation, we get

$$m_j > \frac{\mathrm{e}^{-c+\lambda\cdot I(\lambda)}}{\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)}}\cdot(\bar{m}(\lambda)+r) \;\text{ for all } \lambda \in [\underline{\lambda},\lambda'] \tag{EC.26}$$

Since $\bar{m}(\underline{\lambda}) \le \bar{m}(\lambda)$ for all $\lambda \in [\underline{\lambda},\lambda']$, inequality (EC.26) implies

$$m_j > \frac{\mathrm{e}^{-c+\lambda\cdot I(\lambda)}}{\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda\cdot I(\lambda)}}\cdot(\bar{m}(\underline{\lambda})+r) \;\text{ for all } \lambda \in [\underline{\lambda},\lambda'] \tag{EC.27}$$

By plugging x in (EC.27) and some algebraic manipulation, we get

$$x\cdot\left(m_j-\bar{m}(\underline{\lambda})-r\right)+\mathrm{e}^{Y_0}\left(\bar{m}(\underline{\lambda})+r\right) > 0 \text{ for all } x \in [\mathrm{e}^{Y_0}+\mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}, \mathrm{e}^{Y_0}+\mathrm{e}^{-c+\lambda'\cdot I(\lambda')}] \tag{EC.28}$$

Inequality (EC.28) implies that (EC.25) holds for all $x \in [\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}, \mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda'\cdot I(\lambda')}]$. Hence, $\bar{\Delta}(x)$ is decreasing in x in that interval. Then we have the following sequence of inequalities:

$$\Delta^j(\underline{\lambda}) = \bar{\Delta}\big(\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}\big) > \bar{\Delta}\big(\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda'\cdot I(\lambda')}\big) \geq \underline{\Delta}\big(\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda'\cdot I(\lambda')}\big) = \Delta^j(\lambda') = \Delta^{i^*(\lambda')}(\lambda').$$

The first inequality follows as $\bar{\Delta}(x)$ is decreasing in x and $\big(\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\underline{\lambda}\cdot I(\underline{\lambda})}\big) < \big(\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda'\cdot I(\lambda')}\big)$. The second inequality follows by the definition of $\bar{\Delta}(\cdot)$ and $\underline{\Delta}(\cdot)$. Since the same arguments can be made for all such sub-intervals in $[\underline{\lambda}, \bar{\lambda}]$, we conclude that $\Delta^j(\underline{\lambda}) > \Delta^k(\bar{\lambda})$ when $\beta = 1$. $\square$

Proof of Proposition 8. Let $i^*(\underline{c}) = j$, $i^*(\bar{c}) = k$, and $\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) = M$ for ease of exposition. We will follow a similar procedure to the proof of Proposition 6 and prove in two steps.

1. $\Delta^j(\underline{c}) - \Delta^k(\bar{c})$ is increasing linearly in $\beta \geq 0$. It follows the same arguments as the first part of the proof of Proposition 6 by replacing S and S^+ with $\underline{c}$ and $\bar{c}$, respectively.
2. $\Delta^j(\underline{c}) < \Delta^k(\bar{c})$ when $\beta = 0$.

Assume w.l.o.g. that $\exists\ c' \in [\underline{c}, \bar{c}]$ such that $\arg\max_{i\in S}\{\pi^i(c)\} = j\ \ \forall c \in [\underline{c}, c']$ and $\arg\max_{i\in S}\{\pi^i(c)\} = i^*(c) \neq j$ for $c \in [c', c' + \epsilon]$ for some (small) $\epsilon > 0$. We will show that $\Delta^j(\underline{c}) < \Delta^j(c') = \Delta^{i^*(c')}(c')$, where the equality holds because $\pi^j(c') = \pi^{i^*(c')}(c')$. (Notice that $i^*(c')$ and k are not necessarily the same product.) Since the same argument holds for any sub-interval in $[\underline{c}, \bar{c}]$ and $\Delta^{(\cdot)}(c)$ is continuous, we can conclude that $\Delta^j(\underline{c}) < \Delta^k(\bar{c})$. Define the following function:

$$\bar{\Delta}(c) := \frac{\mathrm{e}^{Y_0} \cdot \mathrm{e}^{Y_j} \cdot m_j + \mathrm{e}^{Y_j} \cdot (m_j - M - r) \cdot \mathrm{e}^{-c+\lambda\cdot I}}{(\mathrm{e}^{Y_0} + \mathrm{e}^{Y_j} + \mathrm{e}^{-c+\lambda\cdot I}) \cdot (\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda\cdot I})}, \tag{EC.29}$$

which equals $\Delta^j(c)$ for $\beta = 1$. We will show that $\bar{\Delta}(c)$ is increasing in c for $c \in [\underline{c}, c']$. To that end, consider the function $f : [\underline{c}, c'] \to [\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}, \mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I}]$ such that $x = f(c) = \mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda\cdot I}$. Notice that x decreases monotonically in c. Hence, $f(c)$ is a one-to-one correspondence between the sets $[\underline{c}, c']$ and $[\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}, \mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I}]$. By plugging x in (EC.29) we get:

$$\bar{\Delta}(x) = \frac{\mathrm{e}^{Y_0} \cdot \mathrm{e}^{Y_j} \cdot m_j + \mathrm{e}^{Y_j} \cdot (m_j - M - r) \cdot (x - \mathrm{e}^{Y_0})}{(\mathrm{e}^{Y_j} + x) \cdot x} \tag{EC.30}$$

In order to show that $\bar{\Delta}(c)$ increases in c for $c \in [\underline{c}, c']$, we will show that $\bar{\Delta}(x)$ decreases in x for $x \in [\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}, \mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I}]$. To that end, we compute the first derivative of (EC.30) as:

$$\frac{d\bar{\Delta}(x)}{dx} = -\frac{\mathrm{e}^{Y_j} x\,[x(m_j - M - r) + 2\mathrm{e}^{Y_0}(M + r)] + \mathrm{e}^{2Y_j + Y_0}(M + r)}{(\mathrm{e}^{Y_j} + x)^2 \cdot x^2} \tag{EC.31}$$

The denominator is clearly positive. We will argue that the numerator is also positive and so $\dfrac{d\bar{\Delta}(x)}{dx} < 0$ because of the negative sign in front. To that end, it suffices to show that

$$\big[x(m_j - M - r) + 2\mathrm{e}^{Y_0}(M + r)\big] > 0 \text{ for all } x \in [\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}, \mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I}] \tag{EC.32}$$

It follows from Assumption 1 that $m_j \geq \dfrac{\mathrm{e}^{-c+\lambda\cdot I}}{\mathrm{e}^{Y_0} + \mathrm{e}^{-c+\lambda\cdot I}} \cdot (M + r)$ for all $c \in [\underline{c}, c']$. By plugging x in and some algebraic manipulation, we get $x \cdot \big(m_j - M - r\big) + \mathrm{e}^{Y_0}\big(M + r\big) \geq 0$ for all $x \in [\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}, \mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I}]$. This implies that (EC.32) holds for all x in the interval $[\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}, \mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I}]$. Hence, $\bar{\Delta}(x)$ is decreasing in x in that interval. Then, we can write

$\Delta^j(\underline{c}) = \bar{\Delta}\big(\mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I}\big) < \bar{\Delta}\big(\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}\big) = \Delta^j(c') = \Delta^{i^*(c')}(c')$. The inequality follows because $\bar{\Delta}(x)$ is decreasing in x, and $(\mathrm{e}^{Y_0} + \mathrm{e}^{-c'+\lambda\cdot I}) < (\mathrm{e}^{Y_0} + \mathrm{e}^{-\underline{c}+\lambda\cdot I})$. Since the same arguments can be made for all such sub-intervals in $[\underline{c}, \bar{c}]$, we conclude that $\Delta^j(\underline{c}) < \Delta^k(\bar{c})$ when $\beta = 1$. $\square$

Proof of Proposition 9. Suppose we have two products, i and j, such that $Y_i \geq Y_j$ and $m_i \geq m_j$, or equivalently, product i dominates product j. Our aim is to show that product j is never optimal to promote.

Let $\Pi^0 = P_A^0 \cdot \big(\bar{m}(\mathbf{Y}, \mathbf{m}, \lambda) + r\big)$. We have $\tilde{\Pi}^x = (1 + \beta\phi_x)\,(P_D^x \cdot m_x + (1 - P_D^x) \cdot \Pi^0)\ \forall x \in S$.

We consider two cases separately: $(m_i > \Pi^0)$ and $(m_i \leq \Pi^0)$. If $(m_i > \Pi^0)$, we have

$$\tilde{\Pi}^j = (1 + \beta\phi_j)\,\big(P_D^j \cdot m_j + (1 - P_D^j) \cdot \Pi^0\big) \leq (1 + \beta\phi_i)\,\big(P_D^i \cdot m_i + (1 - P_D^i) \cdot \Pi^0\big) = \tilde{\Pi}^i \qquad \text{(EC.33)}$$

The inequality holds because $m_i \geq m_j$, $P_D^i \geq P_D^j$, $m_i > \Pi^0$, and $\phi_i \geq \phi_j$, and the inequality is strict because we have assumed no two products i and j have $m_i = m_j$ and $Y_i = Y_j$. Therefore we either have $m_i > m_j$ or $P_D^i > P_D^j$, or both.

Next consider the case where $(m_i \leq \Pi^0)$. Since $m_j \leq m_i \leq \Pi^0$, we have $\pi^j \leq \Pi^0$. This also implies that $\tilde{\Pi}^j = (1 + \beta\phi_j)\pi^j \leq (1 + \beta\phi_{max})\Pi^0 < \tilde{\Pi}^k$ where product k is as defined in Assumption 3. Hence, it is never optimal to promote the dominated product j. $\square$

Proof of Proposition 10. Because of the optimality of $i^*(r_1)$ for $r = r_1$ and of $i^*(r_2)$ for $r = r_2$, we have $\tilde{\Pi}^{i^*(r_1)}(r_1) - \tilde{\Pi}^{i^*(r_2)}(r_1) \geq 0$ and $\tilde{\Pi}^{i^*(r_2)}(r_2) - \tilde{\Pi}^{i^*(r_1)}(r_2) \geq 0$. Adding these inequalities, we get $\tilde{\Pi}^{i^*(r_1)}(r_1) - \tilde{\Pi}^{i^*(r_2)}(r_1) + \tilde{\Pi}^{i^*(r_2)}(r_2) - \tilde{\Pi}^{i^*(r_1)}(r_2) \geq 0$. By substituting using the definition of $\tilde{\Pi}$ and simplifying, we find that the following is an equivalent inequality:

$$r_1\left[P_A^{i^*(r_1)}(1 + \beta\phi_{i^*(r_1)}) - P_A^{i^*(r_2)}(1 + \beta\phi_{i^*(r_2)})\right] + r_2\left[P_A^{i^*(r_2)}(1 + \beta\phi_{i^*(r_2)}) - P_A^{i^*(r_1)}(1 + \beta\phi_{i^*(r_1)})\right] \geq 0.$$

This implies that $(r_1 - r_2)P_A^{i^*(r_1)}(1 + \beta\phi_{i^*(r_1)}) \geq (r_1 - r_2)P_A^{i^*(r_2)}(1 + \beta\phi_{i^*(r_2)})$ and therefore that $P_A^{i^*(r_1)}(1 + \beta\phi_{i^*(r_1)}) \geq P_A^{i^*(r_2)}(1 + \beta\phi_{i^*(r_2)})$ because $r_1 > r_2$. $\square$

EC.2. Simulation Support for the Results in Section 5

We ran simulations to verify the results in Section 5 (Propositions 6,7,8) when comparing categories that are roughly similar but not identical. In particular, we considered all pairs of the categories we used in our earlier numerical analysis in Section 4 and Appendix B. Specifically, we obtained $\binom{100}{2} = 4950$ pairs of categories. Then, we considered difference in one parameter (namely assortment size $|S|$, customer heterogeneity λ, and transit cost c) at a time in the following way: For each pair, we set the value of the parameter under consideration high for one of the categories and low for the other one. All the other parameters are set to the medium level as described in Appendix B.2. Proposition 6 holds for more than 95% of the instances, whereas Propositions 7 and 8 hold for all instances. Hence, the choice of category to promote pivots on the demand expansion parameter β as suggested by these results even if the categories are *roughly* similar.