Appendix D: Proof of Results in Section 5

D.1. Proof of Proposition 1

To show the result, we use Lemma 2. That is, we maximize the virtual profit subject to IC and IR constraints. We start with ignoring both IC and IR constraints and for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, we characterize $\arg\max_{t\in[0,T]} R(\theta,t)$. We show that for any $\theta \geq \theta_L^T$, $\arg\max_{t\in[0,T]} R(\theta,t) = \mathfrak{t}_T(\theta)$ and for $\theta < \theta_L^T$, $\arg\max_{t\in[0,T]} R(\theta,t) = T$ and $R(\theta,T) < 0$. To complete the proof, we show that the mechanism that only sells the item to customer of type $\theta > \theta_L^T$ at time $\mathfrak{t}_T(\theta)$ and price $\mathfrak{p}(\theta)$, defined in the proposition, is IR and IC. Thus, it is optimal.

First of all, using the proof of Theorem 1, it is easy to show that $\arg\max_{t\in[0,T]} R(\theta,t) = \mathfrak{t}_T(\theta)$ for $\theta < \theta_L^T$. Thus, in the following, we show that for $\theta < \theta_L^T$, $\arg\max_{t\in[0,T]} R(\theta,t) = T$ and $R(\theta,T) < 0$. Observe that for any $\theta < \theta_L^T$, $\mathfrak{t}_f(\theta) = \arg\max_{t\geq 0} R(\theta,t) \geq T$. This follows from the fact that $\mathfrak{t}_f(\theta_H^T) = T$ and $\mathfrak{t}_f(\cdot)$ is decreasing; see Lemma 7. Then, as we show in the proof of Lemma 6, $R(\theta,t)$ has an inverted u-shape in t. This implies that the unique maximum of $R(\theta,t)$, $\mathfrak{t}_f(\theta)$, is greater than T, and as a result, $\arg\max_{t\in[0,T]} R(\theta,t) = T$, for any $\theta < \theta_L^T$. Next, we show that $R(\theta,T) < 0$ when $\theta < \theta_L^T$. To do so, we confirm that $e^{g(\theta)T}R(\theta,T)$ is increasing in θ. Then, the result follows because by definition, $R(\theta_L^T,T) = 0$.

The derivative of $e^{g(\theta)T}R(\theta,T)$ w.r.t. θ is given by

$$\frac{\partial(e^{g(\theta)T}R(\theta,T))}{\partial\theta} = \frac{\partial(\theta + \alpha(\theta)(1-g'(\theta)\theta T))}{\partial\theta} = 1 + \alpha'(\theta)(1-g'(\theta)\theta T) - \alpha(\theta)(g'(\theta)\theta)'T > 0,$$

where the inequality holds because by Assumption 1, $g'(\theta)\theta$ is increasing. By the monotonicity of $g'(\theta)\theta$ and the fact that $(1-g'(\theta_L^T)\theta_L^T T) \geq 0$, we have $(1-g'(\theta)\theta T) > 0$ for any $\theta < \theta_L$.

So far, we characterized $\arg\max_{t\in[0,T]} R(\theta,t)$. To complete the proof, we need to show the mechanism is IR and IC. We start with showing the mechanism is IR. Given the payment rule of the mechanism, the mechanism is IR if $1-g'(\theta)\theta\mathfrak{t}_T(\theta) \geq 0$ for any $\theta \geq \theta_L^T$. To show this, we consider the following cases:

- Case 1 ($\theta \geq \theta_H$): For this range of θ, $1-g'(\theta)\theta\mathfrak{t}_T(\theta) = 1 > 0$.
- Case 2 ($\theta \in [\theta_H^T, \theta_H]$:) For this range of θ, $1-g'(\theta)\theta\mathfrak{t}_T(\theta) = 1-g'(\theta)\theta\mathfrak{t}_f(\theta)$. To show the result, we make use of Lemma 4, where we have that $1-g'(\theta)\theta\mathfrak{t}_f(\theta) \geq 0$ for any $\theta \in [\theta_L, \theta_H]$. The result then follows by the fact that $\theta_H^T \geq \theta_L$.
- Case 3 ($\theta \in [\theta_L^T, \theta_H^T]$:) For this range of θ, $1-g'(\theta)\theta\mathfrak{t}_T(\theta) = 1-g'(\theta)\theta T$. By Case 2 and the fact that $\mathfrak{t}_f(\theta_H^T) = T$, we have $1-g'(\theta_H^T)\theta_H^T T \geq 0$. Then, the monotonicity of $g'(\theta)\theta$ implies that $1-g'(\theta)\theta T \geq 0$ for any $\theta \in [\theta_L^T, \theta_H^T]$.

Finally, the mechanism is IC because for any $\theta \geq \theta_L^T$, $1-g'(\theta)\theta\mathfrak{t}_T(\theta) \geq 0$ and $\mathfrak{t}_T(\cdot)$ is decreasing; see the proof of Theorem 1 for details.

D.2. Proof of Proposition 2

First observe that mechanism $\mathcal{M}_T$ is IR and IC because $\mathfrak{t}_T(\cdot)$ is decreasing and $1-g'(\theta)\theta\mathfrak{t}_T(\theta) \geq 0$ for any $\theta \geq \theta_L^T$; see the proof of Proposition 1. Next, we show that mechanism $\mathcal{M}_T$ is approximately optimal. To this aim, we dualize the IR constraints to construct an upper bound on the profit of the optimal mechanism and we then compare the profit of mechanism $\mathcal{M}_T$ with the upper bound.

To dualize the IR constraints, we use $\lambda_g(\cdot)$, defined in Eq. (19). Following the proof of Theorem 1, one can show that

$$\begin{aligned}\mathsf{Rev}_{opt} - \mathsf{Rev}_{\mathcal{M}_T} &\le \int_{\underline{\Theta}}^{\theta_L^T} \max_{t\in[0,T]} \left\{ e^{-g(z)t}\left(z - (1-g'(z)tz)\frac{g(z)}{g'(z)}\right) f(z)\right\} dz \\ &\le \int_{\underline{\Theta}}^{\theta_L^T} e^{-g(z)T}\left(z - (1-g'(z)Tz)\frac{g(z)}{g'(z)}\right) f(z) dz \;\le\; \int_{\underline{\Theta}}^{\theta_L^T} z e^{-\frac{g(z)}{g'(z)z}} f(z) dz\,, \end{aligned} \tag{22}$$

where the equality follows because $\arg\max_{t\in[0,T]}\left\{e^{-g(z)t}\left(z-(1-g'(z)tz)\frac{g(z)}{g'(z)}\right)\right\} = T$ and the second inequality holds because $\arg\max_{t\ge 0}\left\{e^{-g(z)t}\left(z-(1-g'(z)tz)\frac{g(z)}{g'(z)}\right)\right\} = \frac{1}{g'(z)z}$. To get the desired bound, we show that $z \mapsto z e^{-\frac{g(z)}{g'(z)z}}$ is increasing. This implies that $\mathsf{Rev}_{opt} - \mathsf{Rev}_{\mathcal{M}_T} \le \theta_L^T \exp\left(-\frac{g(\theta_L^T)}{g'(\theta_L^T)\theta_L^T}\right) F(\theta_L^T)$.

The derivative of $z e^{-\frac{g(z)}{g'(z)z}}$ w.r.t. z is given by

$$e^{-\frac{g(z)}{g'(z)z}}\left(1 - z\frac{(g'(z))^2 z - (g'(z)z)' g(z)}{(g'(z)z)^2}\right) \;=\; z\frac{(g'(z)z)'g(z)}{(g'(z)z)^2} \ge 0\,,$$

where the inequality holds because by Assumption 1, $g(z)\ge 0$ and $g'(z)z$ is increasing.

D.3. Proof of Theorem 2

We need to find an optimal solution of Problem OPT where the objective function is replaced with $\mathrm{E}\left[\varsigma(\theta)(R(\theta,\mathsf{t}(\theta)) - c) - u(\underline{\Theta},\underline{\Theta})\right]$.

The proof of this theorem is similar to that of Theorem 1. We first relax the problem by ignoring the interval conditions. We will show that in an optimal solution of the relaxed problem, $\varsigma(\theta) = 1$ for $\theta \ge \theta_c$, and is zero otherwise. Here, θ_c solves $R(\theta_c, \mathsf{t}_g(\theta_c)) = c$. We further show that for customers with type $\theta \ge \theta_c$, the optimal allocation time is $\mathsf{t}_g(\theta)$.

To show the result, we make use of Lemma 8, stated at the end of this section, where we show that $R(\theta, \mathsf{t}_g(\theta))$ is an increasing function of $\theta \in [\underline{\Theta}, \bar{\Theta}]$. Then, to complete the proof, we show that the optimal solution of the relaxed problem satisfies the envelope conditions. This part of the proof is similar to that of Theorem 1. Thus, it is omitted.

LEMMA 8. *$R(\theta, \mathsf{t}_g(\theta))$ is increasing in θ where $R(\theta,t)$ and $\mathsf{t}_g(\theta)$ are defined in Equations (4) and (6).*

Proof of Lemma 8 By definition, we have

$$R(\theta,\mathsf{t}_g(\theta)) = \begin{cases} \theta + \alpha(\theta) & \text{if } \theta \ge \theta_H; \\ R(\theta, \mathsf{t}_f(\theta)) & \text{if } \theta \in [\theta_L, \theta_H]; \\ e^{-\frac{g(\theta)}{g'(\theta)\theta}}\theta & \text{if } \theta \in [\underline{\Theta}, \theta_L]; \end{cases}$$

$R(\theta, \mathsf{t}_g(\theta))$ is obviously increasing in θ when $\theta \ge \theta_H$. Furthermore, given that $\theta \le \theta_L$, then $R(\theta, \mathsf{t}_g(\theta))$ is also increasing. To see why note that for any $\theta \le \theta_L$,

$$\frac{dR(\theta,\mathsf{t}_g(\theta))}{d\theta} \;=\; \frac{d(e^{-\frac{g(\theta)}{g'(\theta)\theta}}\theta)}{d\theta} \;=\; e^{-\frac{g(\theta)}{g'(\theta)\theta}}\frac{\theta g(\theta)(g'(\theta)\theta)'}{(g'(\theta)\theta)^2} \;\ge\; 0\,,$$

where the inequality holds because $g'(\theta)\theta$ is increasing. Furthermore, observe that $R(\theta, \mathsf{t}_g(\theta))$ is a continuous function of θ because $\mathsf{t}_g(\theta)$ is continuous. Thus, it suffices to show that $R(\theta, \mathsf{t}_g(\theta))$ is increasing in $\theta \in [\theta_L, \theta_H]$.

Recall that $\mathsf{t}_g(\theta) = \mathsf{t}_f(\theta)$ for $\theta \in [\theta_L, \theta_H]$. That is, $\mathsf{t}_g(\theta)$ is the FOC solution. Thus, by the Envelope theorem, the derivative of $R(\theta, \mathsf{t}_f(\theta))$ w.r.t. θ is given by

$$\begin{aligned}\frac{\partial\big(R(\theta,\mathsf{t}_f(\theta))\big)}{\partial\theta} &= e^{-g(\theta)\mathsf{t}_f(\theta)}\Big(-g'(\theta)\mathsf{t}_f(\theta)\big(\theta+\alpha(\theta)(1-g'(\theta)\mathsf{t}_f(\theta)\theta)\big)\\&+\ 1+\alpha'(\theta)(1-g'(\theta)\mathsf{t}_f(\theta)\theta)) - \mathsf{t}_f\alpha(\theta)(g'(\theta)\theta)'\Big)\\&= e^{-g(\theta)\mathsf{t}_f(\theta)}\Big((1-g'(\theta)\mathsf{t}_f(\theta)\theta)\big(\alpha'(\theta)-\alpha(\theta)\big)-\mathsf{t}_f(\theta)\alpha(\theta)(g'(\theta)\theta)'\Big)\ \geq\ 0,\end{aligned}$$

where the inequality holds because, as we show in Lemma 4, $(1 - g'(\theta)\mathsf{t}_f(\theta)\theta) \geq 0$ for any $\theta \in [\theta_L, \theta_H]$, and by Assumption 1, $g'(\theta)\theta$ is increasing in θ. □

D.4. Proof of Proposition of 3

Recall that $\mathsf{Rev}^1_{opt} = \mathsf{Rev}(\mathbf{q})$ and $\mathsf{Rev}^2_{opt}(\tilde{\mathbf{q}}) = \mathsf{Rev}(\tilde{\mathbf{q}})$. Therefore, any feasible solution of Problem $\mathsf{Rev}(\mathbf{q})$ is a feasible solution of Problem $\mathsf{Rev}(\tilde{\mathbf{q}})$ and vice versa. In fact, for any feasible solution of Problem Rev, we have

$$\begin{aligned}\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}}) &= \sum_{k\in[K]} \varsigma_k p_k (q_k - \tilde{q}_k)\\ \Rightarrow\quad \mathsf{Rev}(\mathbf{q}) - \mathrm{E}[\mathsf{Rev}(\tilde{\mathbf{q}}) \mid M] &= \sum_{k\in[K]} \varsigma_k p_k\ \mathrm{E}[(q_k - \tilde{q}_k) \mid M]\ =\ 0,\end{aligned}$$

where the above equation is the desired result.

Next, we prove claim (10). Let x_{ik} be 1 if customer i is of type k and zero otherwise. Note that for any $i \in [M]$, we have $\sum_{k\in[K]} x_{ik} = 1$. Then, for any feasible solution of Problem $\mathsf{Rev}(\tilde{\mathbf{q}})$, we get

$$\begin{aligned}\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}}) &= \sum_{k\in[K]} \varsigma_k p_k (q_k - \tilde{q}_k) \ = \ \sum_{k\in[K]} \varsigma_k p_k \Big(q_k - \frac{1}{M}\sum_{i\in[M]} x_{ik}\Big)\\ &= \sum_{k\in[K]} \varsigma_k p_k \Big(\mathrm{E}[x_{ik}] - \frac{1}{M}\sum_{i\in[M]} x_{ik}\Big) \ = \ \sum_{k\in[K]} \varsigma_k \frac{p_k}{M}\sum_{i\in[M]}\Big(\mathrm{E}[x_{ik}] - x_{ik}\Big)\\ &= \frac{1}{M}\sum_{i\in[M]}\sum_{k\in[K]} \varsigma_k p_k \big(\mathrm{E}[x_{ik}] - x_{ik}\big).\end{aligned}$$

Define $y_i = \sum_{k\in[K]} \varsigma_k p_k \big(\mathrm{E}[x_{ik}] - x_{ik}\big)$. Note that $\mathrm{E}[y_i] = 0$, $i \in [M]$. For any M, let $\epsilon = \bar{\Theta}\sqrt{\frac{\log(M)}{2M}}$ and define the following event

$$\mathcal{A} = \Big\{\frac{1}{M}\sum_{i\in[M]} y_i - \mathrm{E}[y_i] \geq \epsilon\Big\}$$

We will show that $\Pr\big(\mathcal{A} \mid M\big) \leq \frac{1}{M}$. This implies that with high probability $\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}}) \leq \bar{\Theta}\sqrt{\frac{\log(M)}{2M}}$. Then, we get

$$\begin{aligned}\mathrm{E}\Big[\mathrm{E}\left[|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})|\big| M\right]\Big] &= \mathrm{E}\Big[\mathrm{E}\left[|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})|\mathbb{1}\{\mathcal{A}\} + |\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})|\mathbb{1}\{\mathcal{A}^c\}\big| M\right]\Big]\\ &\leq \mathrm{E}\left[\frac{\bar{\Theta}}{M} + \Theta\sqrt{\frac{\log(M)}{2M}}\right] \ \leq\ \Theta\sqrt{\frac{\log(n)}{2n}} + \frac{\bar{\Theta}}{n},\end{aligned} \tag{23}$$

where $\mathcal{A}^c$ is the complement of event $\mathcal{A}$. To obtain the first inequality, we used the fact that (i) under event $\mathcal{A}^c$, $|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})| \leq \epsilon = \Theta\sqrt{\frac{\log(M)}{2M}}$, (ii) $|\mathsf{Rev}(\mathbf{q}) - \mathsf{Rev}(\tilde{\mathbf{q}})| \leq \bar{\Theta}$, and (iii) $\Pr(\mathcal{A}|M) \leq \frac{1}{M}$. Further, the second inequality, which is the desired result, holds because $M \geq n$ a.s.

To complete the proof, we show that $\Pr\left(\mathcal{A} \mid M\right) \le \frac{1}{M}$. To do so, we use the Azuma-Hoeffding inequality:

$$\Pr\left(\mathcal{A} \mid M\right) \;=\; \Pr\Big(\frac{1}{M}\sum_{i\in[M]} y_i - \mathrm{E}[y_i] \ge \epsilon \mid M\Big) \;\le\; \exp\left(-\frac{2M\epsilon^2}{\max_{i\in[M]}\; |y_i|^2}\right) \tag{24}$$

In the following, we present an upper bound on $\max_{i\in[M]}|y_i|^2$. Let $k' \in [K]$ be the type of customer i; that is $x_{ik'}=1$ and $x_{ik}=0$ for $k\in k'$. Then,

$$\begin{aligned} y_i \;&=\; \sum_{k\in[K]} \varsigma_k p_k\big(\mathrm{E}[x_{ik}] - x_{ik}\big) \;=\; \sum_{k\in[K]} \varsigma_k p_k q_k - \varsigma_{k'} p_{k'} \\ \Rightarrow\quad |y_i| \;&\le\; \max_{k\in[K]}\; p_k \;\le\; \max_{k\in[K]}\; \theta_k \;=\; \bar{\Theta}, \end{aligned} \tag{25}$$

where the last inequality follows from the IR constraints. Applying (25) in (24), we get, $\Pr\left(\mathcal{A}|M\right) \;\le\; \exp\left(-\frac{2M\epsilon^2}{\bar{\Theta}^2}\right) \;=\; \frac{1}{M}$.

Appendix E: Proofs and Additional Result for Sections 3 and 4

Proof of Lemma 1 The proof falls naturally into two parts. In the first part, we show that in an incentive-compatible mechanism conditions in Equations (1) and (2) hold. In the second part, we show that if Equations (1) and (2) hold, the mechanism is IC.

First Part: Consider a customer with type θ that reports $\hat\theta$. Without loss of generality, we assume that $\theta \ge \hat\theta$. Then, the utility of the customer is given by $u(\theta,\hat\theta) = \varsigma(\hat\theta).\big(V(\theta,\mathfrak{t}(\hat\theta)) - \mathfrak{p}(\hat\theta)\big)$. Incentive compatibility implies that

$$\begin{aligned} u(\theta,\theta) - u(\hat\theta,\hat\theta) \;&\le\; u(\theta,\theta) - u(\hat\theta,\theta) \;=\; \varsigma(\theta).\big(V(\theta,\mathfrak{t}(\theta)) - V(\hat\theta,\mathfrak{t}(\theta))\big) \\ &=\; \int_{z=\hat\theta}^{\theta} \varsigma(\theta)\partial_1 V(z,\mathfrak{t}(\theta))dz \;=\; \int_{z=\hat\theta}^{\theta} \varsigma(\theta) e^{-g(z)\mathfrak{t}(\theta)}(1-g'(z)\mathfrak{t}(\theta)z)dz\ , \end{aligned} \tag{26}$$

and

$$\begin{aligned} u(\theta,\theta) - u(\hat\theta,\hat\theta) \;&\ge\; u(\theta,\hat\theta) - u(\hat\theta,\hat\theta) \;=\; \varsigma(\hat\theta).\big(V(\theta,\mathfrak{t}(\hat\theta)) - V(\hat\theta,\mathfrak{t}(\hat\theta))\big) \\ &=\; \int_{z=\hat\theta}^{\theta} \varsigma(\hat\theta)\partial_1 V(z,\mathfrak{t}(\hat\theta))dz \;=\; \int_{z=\hat\theta}^{\theta} \varsigma(\hat\theta) e^{-g(z)\mathfrak{t}(\hat\theta)}(1-g'(z)\mathfrak{t}(\hat\theta)z)dz\ , \end{aligned} \tag{27}$$

where $\partial_1 V(\theta,t) = \frac{\partial V(\theta,t)}{\partial\theta}$. Then, using the above equations, we have

$$\begin{aligned} \frac{\int_{z=\hat\theta}^{\theta}\varsigma(\hat\theta)e^{-g(z)\mathfrak{t}(\hat\theta)}(1-g'(z)\mathfrak{t}(\hat\theta)z)dz}{\theta-\hat\theta} &\le \frac{u(\theta,\theta)-u(\hat\theta,\hat\theta)}{\theta-\hat\theta}\ , \\ \frac{\int_{z=\hat\theta}^{\theta}\varsigma(\theta)e^{-g(z)\mathfrak{t}(\theta)}(1-g'(z)\mathfrak{t}(\theta)z)dz}{\theta-\hat\theta} &\ge \frac{u(\theta,\theta)-u(\hat\theta,\hat\theta)}{\theta-\hat\theta}\ . \end{aligned}$$

Finally by taking the limit as $\hat\theta \to \theta^-$, we get Eq. (1).[16] Then, by Equations (1), (26), and (27), we get the second condition, given in Eq. (2).

Second Part: Here, we will show that if in a mechanism Equations (1) and (2) hold, the mechanism is IC. By Eq. (1),

$$\begin{aligned} u(\theta,\theta) - u(\hat\theta,\hat\theta) \;&=\; \int_{z=\hat\theta}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)}(1-g'(z)\mathfrak{t}(z)z)dz \;\ge\; \int_{z=\hat\theta}^{\theta} \varsigma(\hat\theta) e^{-g(z)\mathfrak{t}(\hat\theta)}(1-g'(z)\mathfrak{t}(\hat\theta)z)dz \\ &=\; \varsigma(\hat\theta)\left(\theta e^{-g(\theta)\mathfrak{t}(\hat\theta)} - \hat\theta e^{-g(\hat\theta)\mathfrak{t}(\hat\theta)}\right) \;=\; u(\theta,\hat\theta) - u(\hat\theta,\hat\theta)\ , \end{aligned} \tag{28}$$

[16] By Theorem 2 in (1), to satisfy Eq. (1), $\mathfrak{t}(\cdot)$ is not required to be continuous.

where the inequality follows from Eq. (2). The final equation implies that $u(\theta,\theta) \geq u(\theta,\hat{\theta})$. Similarly,

$$\begin{aligned} u(\theta,\theta) - u(\hat{\theta},\hat{\theta}) &= \int_{z=\hat{\theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)} (1 - g'(z)\mathfrak{t}(z)z) dz \leq \int_{z=\hat{\theta}}^{\theta} \varsigma(\theta) e^{-g(z)\mathfrak{t}(\theta)} (1 - g'(z)\mathfrak{t}(\theta)z) dz \\ &= \varsigma(\theta) \left(\theta e^{-g(\theta)\mathfrak{t}(\theta)} - \hat{\theta} e^{-g(\hat{\theta})\mathfrak{t}(\theta)} \right) = u(\theta,\theta) - u(\hat{\theta},\theta) \ , \end{aligned} \tag{29}$$

That is, $u(\hat{\theta},\hat{\theta}) \geq u(\hat{\theta},\theta)$. The above equation along with Eq. (29) imply that the mechanism is IC.

□

Proof of Lemma 2 Consider any IC mechanism. Then, the expected profit of the firm from selling to a single customer is given by

$$\mathrm{E}[\varsigma(\theta)(\mathfrak{p}(\theta) - h\mathfrak{t}(\theta) - c)] = \mathrm{E}\left[\varsigma(\theta)\big(\theta e^{-g(\theta)\mathfrak{t}(\theta)} - u(\theta,\theta) - h\mathfrak{t}(\theta) - c\big)\right], \tag{30}$$

where the expectation is with respect to the customer type θ. In the following, we compute $\mathrm{E}[u(\theta,\theta)]$. By Lemma 1

$$\begin{aligned} \mathrm{E}[u(\theta,\theta)] &= u(\underline{\Theta},\underline{\Theta}) + \int_{\theta=\underline{\Theta}}^{\bar{\Theta}} dF(\theta) \int_{z=\underline{\Theta}}^{\theta} \varsigma(z) e^{-g(z)\mathfrak{t}(z)} (1 - g'(z)\mathfrak{t}(z)z) dz \\ &= u(\underline{\Theta},\underline{\Theta}) + \int_{z=\underline{\Theta}}^{\bar{\Theta}} \int_{\theta=z}^{\bar{\Theta}} dF(\theta) \varsigma(z) e^{-g(z)\mathfrak{t}(z)} (1 - g'(z)\mathfrak{t}(z)z) dz \\ &= u(\underline{\Theta},\underline{\Theta}) + \int_{z=\underline{\Theta}}^{\bar{\Theta}} (1 - F(z)) \varsigma(z) e^{-g(z)\mathfrak{t}(z)} (1 - g'(z)\mathfrak{t}(z)z) dz \\ &= u(\underline{\Theta},\underline{\Theta}) + \mathrm{E}\left[\frac{(1-F(\theta))}{f(\theta)} \varsigma(\theta) e^{-g(\theta)\mathfrak{t}(\theta)} (1 - g'(\theta)\mathfrak{t}(\theta)\theta) \right] . \end{aligned} \tag{31}$$

By replacing Eq. (31) in Eq. (30), we get the desired result. □

E.1. Lower Bound on the Profit Gain of the Dynamic Pricing Policy

Here, we compare the profit of the optimal mechanism given in Theorem 1 with that of the optimal FP policy when $g(\theta) = \theta^a$, $a \geq 0$. We note that the FP policy is optimal when $a = 0$. Under the FP policy, the firm only sells to customers with type $\theta \geq \theta_0$ at time zero by posting a fixed price of θ_0 where θ_0 solves $\theta_0 + \alpha(\theta_0) = 0$.

Lemma 9 **(Lower Bound on the Profit Gain of DP)**. *Suppose that $g(\theta) = \theta^a$. Then, we have*

$$\frac{\mathsf{Rev}_{opt} - \mathsf{Rev}_f}{\mathsf{Rev}_f} \geq e^{-\frac{1}{a}} \frac{\mathrm{E}\left[\theta \mathbb{1}\{\theta \leq \theta_0\}\right]}{\theta_0(1 - F(\theta_0))},$$

where Rev_f and Rev_{opt} are the expected profit of the firm under the FP policy and optimal DP policy, respectively, and θ_0 solves $\theta_0 + \alpha(\theta_0) = 0$.

Proof of Lemma 9 is given at the end of this section.

Assume that the customer type θ is drawn from the uniform distribution in the range of $[0,1]$; that is, $\theta \sim U(0,1)$. Then, $\theta_0 = \frac{1}{2}$, and $\mathsf{Rev}_f = \frac{1}{4}$. Lemma 9 implies that the firm can increase its profit by more than $100.e^{-\frac{1}{a}}.\frac{\int_0^{\theta_0} x dx}{\theta_0(1-\theta_0)} = 50.e^{-\frac{1}{a}}$ percent by using DP. The profit gain of the DP (in %) for $a = 0.5, 1, 1.5$, and 2 is at least 6.8, 18.4, 25.7, and 30.3, respectively.

Proof of Lemma 9 By Lemma 2, under the FP policy ($a=0$),

$$\mathsf{Rev}_f \;=\; \mathrm{E}\left[\varsigma(\theta)R(\theta,0)\right] \;=\; \mathrm{E}\left[(\theta+\alpha(\theta))\mathbb{1}\{\theta \geq \theta_0\}\right]\;, \tag{32}$$

where the last inequality holds because in the FP policy, $\mathsf{t}(\theta)=0$ for $\theta \geq \theta_0$ and $\varsigma(\theta)=1$ only for customers of type $\theta \geq \theta_0$.

Similarly, under the mechanism described in Theorem 1, we have

$$\begin{aligned}\mathsf{Rev}_{opt} &= \mathrm{E}\left[R(\theta,\mathsf{t}_g(\theta))\right] \;=\; \mathrm{E}\left[e^{-g(\theta)\mathsf{t}_g(\theta)}\big(\theta+\alpha(\theta)(1-g'(\theta)\mathsf{t}_g(\theta)\theta)\big)\right] \;=\; \mathrm{E}\left[e^{-\theta^a\mathsf{t}_g(\theta)}\big(\theta+\alpha(\theta)(1-a\theta^a\mathsf{t}_g(\theta))\big)\right]\\ &= \mathrm{E}\Big[(\theta+\alpha(\theta))\times\mathbb{1}\{\theta\geq\theta_H\}+e^{-\theta^a\mathsf{t}_g(\theta)}\big(\theta+\alpha(\theta)(1-a\theta^a\mathsf{t}_g(\theta))\big)\times\mathbb{1}\{\theta\in(\theta_L,\theta_H)\}+e^{-\frac{1}{a}}\theta\times\mathbb{1}\{\theta\leq\theta_L\}\Big]\,,\end{aligned}$$

where the second equation holds because $g(\theta)=\theta^a$ and the third equation follows from the time of purchase in the optimal DP policy, i.e., $\mathsf{t}_g(\cdot)$, which is given in Eq. (6).

We consider the following two cases.

- $\theta_L \leq \theta_0$: We start with rewriting Rev_{opt} as follows.

$$\begin{aligned}\mathsf{Rev}_{opt} \;=\; \mathrm{E}\Big[&(\theta+\alpha(\theta))\times\mathbb{1}\{\theta\geq\theta_H\}+\; e^{-\theta^a\mathsf{t}_g(\theta)}\big(\theta+\alpha(\theta)(1-a\theta^a\mathsf{t}_g(\theta))\big)\times\mathbb{1}\{\theta\in(\theta_0,\theta_H)\}\\ &+e^{-\theta^a\mathsf{t}_g(\theta)}\big(\theta+\alpha(\theta)(1-a\theta^a\mathsf{t}_g(\theta))\big)\times\mathbb{1}\{\theta\in(\theta_L,\theta_0)\}+\; e^{-\frac{1}{a}}\theta\times\mathbb{1}\{\theta\leq\theta_L\}\Big]\,.\end{aligned}$$

In the above equation, we broke down the middle term of Rev_{opt} into two terms. Considering that $\theta\in(\theta_L,\theta_H)$, $\mathsf{t}_g(\theta)$ is the FOC solution, i.e., $\mathsf{t}_g(\theta)=\arg\max_{t\geq 0}\{R(\theta,t)\}$, we get

$$\begin{aligned}\mathsf{Rev}_{opt} \;&\geq\; \mathrm{E}\Big[(\theta+\alpha(\theta))\times\mathbb{1}\{\theta\geq\theta_H\}+(\theta+\alpha(\theta))\times\mathbb{1}\{\theta\in(\theta_0,\theta_H)\}\\ &\qquad+e^{-\frac{1}{a}}\theta\times\mathbb{1}\{\theta\in(\theta_L,\theta_0)\}+\; e^{-\frac{1}{a}}\theta\times\mathbb{1}\{\theta\leq\theta_L\}\Big]\;,\\ &=\; \mathrm{E}\Big[(\theta+\alpha(\theta))\times\mathbb{1}\{\theta>\theta_0\}+e^{-\frac{1}{a}}\theta\times\mathbb{1}\{\theta\leq\theta_0\}\Big]\end{aligned}$$

By the above equation and Eq. (32), we get $\mathsf{Rev}_{opt}-\mathsf{Rev}_f \;\geq\; e^{-\frac{1}{a}}\mathrm{E}\left[\theta\times\mathbb{1}\{\theta\leq\theta_0\}\right]$. Then the result follows because $\mathsf{Rev}_f=\theta_0(1-F(\theta_0))$.

- $\theta_L>\theta_0$: Since for $\theta\in(\theta_L,\theta_H)$, $\mathsf{t}_g(\theta)$ is the FOC solution, we have

$$\mathsf{Rev}_{opt} \;\geq\; \mathrm{E}\Big[(\theta+\alpha(\theta))\times\mathbb{1}\{\theta>\theta_L\}+e^{-\frac{1}{a}}\theta\times\mathbb{1}\{\theta\leq\theta_L\}\Big]$$

This leads to

$$\mathsf{Rev}_{opt}-\mathsf{Rev}_f \;\geq\; e^{-\frac{1}{a}}\mathrm{E}\left[(\theta e^{-\frac{1}{a}}-\theta-\alpha(\theta))\times\mathbb{1}\{\theta\in(\theta_0,\theta_L]\}+\theta e^{-\frac{1}{a}}\mathbb{1}\{\theta\leq\theta_0\}\right]$$

To complete the proof, we show that for any $\theta\in(\theta_0,\theta_L]$, $(\theta e^{-\frac{1}{a}}-\theta-\alpha(\theta))\geq 0$. This gives us $\mathsf{Rev}_{opt}-\mathsf{Rev}_f \;\geq\; e^{-\frac{1}{a}}\mathrm{E}\left[\theta\times\mathbb{1}\{\theta\leq\theta_0\}\right]$, which is the desired result.

To show $(\theta e^{-\frac{1}{a}}-\theta-\alpha(\theta))\geq 0$ for any $\theta\in(\theta_0,\theta_L]$, we will verify that (i) $(\theta e^{-\frac{1}{a}}-\theta-\alpha(\theta))\geq 0$ at $\theta=\theta_L$, and (ii) $(\theta e^{-\frac{1}{a}}-\theta-\alpha(\theta))$ is decreasing in θ. By definition, $\mathsf{t}_g(\cdot)$ is continuous at $\theta=\theta_L$. This implies $\mathsf{t}_g(\theta)=\frac{1}{a\theta^a}$ when $\theta=\theta_L$. Then, considering the fact that $\mathsf{t}_g(\theta_L)$ is the FOC solution, we have $R(\theta_L,\mathsf{t}_g(\theta_L))\geq R(\theta_L,0)$. Thus, we get $\theta e^{-\frac{1}{a}}-\theta-\alpha(\theta)\geq 0$ at $\theta=\theta_L$. Finally, $\theta e^{-\frac{1}{a}}-\theta-\alpha(\theta)$ is decreasing in θ because $e^{-\frac{1}{a}}-1\leq 0$ and $\alpha(\cdot)$ is increasing.

□

Appendix F: Proof of Theorem 3 of Appendix A

The proof of Theorem is divided into three lemmas: Lemma 10, 11, and 12. In Lemma 10, 11, and 12, we characterize the optimal mechanism for when the holding cost is low, medium, and high, respectively.

To characterize the optimal mechanism, by Lemma 2, we should solve the following optimization problem.

$$\begin{aligned} \max_{\{u(\underline{\Theta},\underline{\Theta})\geq 0,\ (\mathfrak{t},\varsigma,\mathfrak{p})\}} \quad & \mathrm{E}\Big[\varsigma(\theta)(R(\theta,\mathfrak{t}(\theta)) - h\mathfrak{t}(\theta))\Big] - u(\underline{\Theta},\underline{\Theta}) \\ \text{s.t.} \quad & u(\theta,\theta) \;\geq\; u(\theta,\hat{\theta}) \qquad \theta,\hat{\theta}\in[\underline{\Theta},\bar{\Theta}] \qquad \text{(IC)} \\ & u(\theta,\theta) \;\geq\; 0 \qquad \theta\in[\underline{\Theta},\bar{\Theta}] \qquad \text{(IR)} \end{aligned} \tag{OPT-H}$$

Here, the objective function is the virtual profit and $R(\theta,\mathfrak{t}(\theta))$ is defined in Eq. (12). The first and second sets of constraints ensure that the mechanism is IC and IR, respectively.

Lemma 10 **(Low Holding Cost)**. *If Assumption 2 holds, the valuation function* $V(\theta,t)=\theta e^{-\theta t}$*, and the holding cost* $h\leq H_l$*, then the optimal mechanism sells to the customer of type* $\theta\geq\max\{\underline{\theta}_L,\underline{\Theta}\}$ *at time* $\mathfrak{t}_h(\theta)$ *and at price* $\mathfrak{p}(\theta)=V(\theta,\mathfrak{t}_h(\theta))-\int_{\max\{\underline{\theta}_L,\underline{\Theta}\}}^{\theta} e^{-\mathfrak{t}_h(z)z}(1-\mathfrak{t}_h(z)z)dz$ *where* H_l*,* $\mathfrak{t}_h(\cdot)$*, and* $\underline{\theta}_L$ *are defined in Equations (11) and (13). Further, for* $\theta<\max\{\underline{\theta}_L,\underline{\Theta}\}$*,* $\mathfrak{p}(\theta)=\infty$ *and* $\varsigma(\theta)=0$*, and for* $\theta\geq\max\{\underline{\theta}_L,\underline{\Theta}\}$*,* $\varsigma(\theta)=1$*.*

The proof of Lemma 10 is given in Appendix F.1.

Lemma 11 **(Medium Holding Cost)**. *If Assumption 2 holds, the valuation function* $V(\theta,t)=\theta e^{-\theta t}$*, the holding cost* $h\in[H_l,H_h]$*, and* $R(\theta,\mathfrak{t}_f(\theta))-h\mathfrak{t}_f(\theta)=0$ *has a unique solution, then the optimal mechanism sells to the customer of type* $\theta\geq\underline{\theta}_M$ *at time* $\mathfrak{t}_h(\theta)$ *and at price* $\mathfrak{p}(\theta)=V(\theta,\mathfrak{t}_h(\theta))-\int_{\underline{\theta}_M}^{\theta} e^{-\mathfrak{t}_h(z)z}(1-\mathfrak{t}_h(z)z)dz$ *where* $R(\theta,t)$ *is defined in Eq. (12) and* H_l*,* H_h*,* $\underline{\theta}_M$*,* $\mathfrak{t}_h(\cdot)$*, and the FOC solution* $\mathfrak{t}_f(\cdot)$ *are defined in Equations (11) and (13). Further, for* $\theta<\underline{\theta}_M$*,* $\mathfrak{p}(\theta)=\infty$ *and* $\varsigma(\theta)=0$*, and for* $\theta\geq\underline{\theta}_M$*,* $\varsigma(\theta)=1$*.*

The assumption in Lemma 11 is discussed in Appendix F.4, and the proof of Lemma 11 is provided in Appendix F.2.

Lemma 12 **(High Holding Cost)**. *If Assumption 2 holds, the valuation function* $V(\theta,t)=\theta e^{-\theta t}$*, the holding cost* $h\geq H_h$*, and* $R(\theta,\mathfrak{t}_f(\theta))-H_h\mathfrak{t}_f(\theta)=0$ *has a unique solution, then the optimal mechanism sells to customers with type* $\theta\geq\theta_0$ *at time zero and at price* $\mathfrak{p}(\theta)=\theta_0$ *where* θ_0 *solves* $\theta_0+\alpha(\theta_0)=0$*,* $R(\theta,t)$ *is defined in Eq. (12), and* H_l*,* H_h*, and the FOC solution* $\mathfrak{t}_f(\cdot)$ *are defined in Eq. (11). Further, for* $\theta<\theta_0$*,* $\mathfrak{p}(\theta)=\infty$ *and* $\varsigma(\theta)=0$*, and for* $\theta\geq\theta_0$*,* $\varsigma(\theta)=1$*.*

The proof is given in Appendix F.3.

F.1. Optimal Mechanism for a Low Holding Cost

In this section, we present the proof of Lemma 10. Throughout the proof, for convenience, we assume that $\underline{\theta}_L\geq\underline{\Theta}$. We need to show that the time of allocation in the optimal mechanism is given by

$$\mathfrak{t}^*(\theta):=\begin{cases}\mathfrak{t}_h(\theta) & \text{if } \theta\geq\max\{\underline{\theta}_L,\underline{\Theta}\};\\ \infty & \text{if } \theta<\max\{\underline{\theta}_L,\underline{\Theta}\}\end{cases}=\begin{cases}0 & \text{if } \theta\geq\theta_H^h;\\ \mathfrak{t}_f(\theta) & \text{if } \theta\in[\theta_L^h,\theta_H^h];\\ \frac{1}{\theta} & \text{if } \theta\in[\underline{\theta}_L,\theta_L^h];\\ \infty & \text{if } \theta<\underline{\theta}_L.\end{cases} \tag{33}$$

Note that $\mathfrak{t}^*(\theta) = \infty$ implies that the mechanism does not allocate the item to customers with type θ; that is $\varsigma(\theta) = 0$.

To characterize the optimal mechanism, by Lemma 2, we need to solve the optimization Problem OPT-H. That is, we need to maximize the expected virtual profit subject to IR and IC constrains. Lemma 1 shows that a mechanism is IC if and only if the interval and envelope conditions hold. In the following, we relax Problem OPT-H and only consider the IR and envelope conditions. We then show that the solution of the relaxed problem also satisfies the interval condition. Thus, it is optimal.

The relaxed problem can be formulated as follows.

$$\max_{\{u(\underline{\Theta},\underline{\Theta})\geq 0,\ (\mathfrak{t},\varsigma)\}} \mathrm{E}\left[\varsigma(\theta)(R(\theta,\mathfrak{t}(\theta)) - h\mathfrak{t}(\theta))\right] - u(\underline{\Theta},\underline{\Theta})$$
$$\text{s.t.} \quad u(\theta,\theta) \ = \ u(\underline{\Theta},\underline{\Theta}) + \int_{\underline{\Theta}}^{\theta} \varsigma(z) e^{-z\mathfrak{t}(z)}(1-\mathfrak{t}(z)z)dz \ \geq \ 0 \quad \text{for} \quad \theta \in [\underline{\Theta},\bar{\Theta}] \ , \quad \text{(IR)} \quad \text{(OPT-H-R)}$$

where the maximization is taken over the purchase time $\mathfrak{t}(\theta)$ and utility of a customer with type $\underline{\Theta}$, i.e., $u(\underline{\Theta},\underline{\Theta})$. Here, $R(\theta,t)$ is the virtual value of customer of type θ at time t, and is defined in Eq. (12).

The following lemma characterizes the optimal solution of the relaxed problem.

LEMMA 13. *Suppose that $V(\theta) = \theta e^{-\theta t}$. Then, if Assumptions 2 hold and the holding cost $h \leq H_l$, in an optimal solution of Problem* OPT-H-R*, $u(\underline{\Theta},\underline{\Theta}) = 0$, the optimal allocation rule is $\mathfrak{t}^*(\cdot)$ where $\mathfrak{t}^*(\cdot)$ is defined in Eq. (33).*

The proof is provided in Appendix F.1.1. In the proof, we first show that $\mathfrak{t}^*(\cdot)$ is a feasible solution of the relaxed problem. Then, we show that it is optimal.

To verify that $\mathfrak{t}^*(\cdot)$ is an optimal solution of Problem OPT-H, we show that the interval condition specified in Lemma 1 is fulfilled. That is, for any $\hat{\theta}, \theta \in [\underline{\Theta},\bar{\Theta}]$ such that $\hat{\theta} \leq \theta$,

$$\int_{\hat{\theta}}^{\theta} A(z,\mathfrak{t}^*(\hat{\theta}))dz \ \leq \ \int_{\hat{\theta}}^{\theta} A(z,\mathfrak{t}^*(z))dz \ ,$$
$$\int_{\hat{\theta}}^{\theta} A(z,\mathfrak{t}^*(z))dz \ \leq \ \int_{\hat{\theta}}^{\theta} A(z,\mathfrak{t}^*(\theta))d \ ,$$

where $A(z,t) = \partial_1 V(z,t) = e^{-zt}(1-zt)$. Note that $A(z,t) = 0$ when t goes to infinity. Thus, for any $z < \underline{\theta}_L$ and $t \geq 0$, we have $A(z,\mathfrak{t}^*(z)) = \varsigma(z)A(z,t)$. Further, for any $z \geq \underline{\theta}_L$, we have $A(z,\mathfrak{t}^*(z)) = \varsigma(z)A(z,\mathfrak{t}_h(z))$. Thus, showing the above equations is equivalent to verifying the interval conditions in Lemma 1. In addition, note that $\int_{\hat{\theta}}^{\theta} A(z,\mathfrak{t}^*(z))dz = u(\theta,\theta) - u(\hat{\theta},\hat{\theta})$. To this aim, we show that for any $z \geq \hat{\theta}$, $A(z,\mathfrak{t}^*(\hat{\theta})) \leq A(z,\mathfrak{t}^*(z))$ and for any $z \leq \theta$, $A(z,\mathfrak{t}^*(z)) \leq A(z,\mathfrak{t}^*(\theta))$.

We will make use of the following preliminary results.

LEMMA 14. *The FOC solution $\mathfrak{t}_f(\theta)$, defined in Eq. (11), is a decreasing function of θ as long as $R(\theta,\mathfrak{t}_f(\theta)) - h\mathfrak{t}_f(\theta) \geq 0$. In addition, for any $\theta \in [\underline{\Theta},\bar{\Theta}]$, $0 \leq A(\theta,\mathfrak{t}_h(\theta)) \leq 1$ where $A(z,t) = e^{-zt}(1-zt)$.*

LEMMA 15. *For any $h \geq 0$, $R(\theta,\mathfrak{t}_h(\theta)) - h\mathfrak{t}_h(\theta)$ is increasing in $\theta \geq \underline{\theta}_h$. Furthermore, $R(\theta,\mathfrak{t}_h(\theta)) - h\mathfrak{t}_h(\theta) \geq 0$ for any $\theta \geq \underline{\theta}_h$.*

Unless stated otherwise, the proof of all technical lemmas is given in Appendix H.

By Lemma 15, $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) \geq 0$ for any $\theta \in [\theta_L^h, \theta_H^h]$. This and Lemma 14 imply that $\mathsf{t}_h(\theta) = \mathsf{t}_f(\theta)$ is decreasing for any $\theta \in [\theta_L^h, \theta_H^h]$. Then, considering the fact that $\mathsf{t}_h(\theta) = 0$ for $\theta \geq \theta_H^h$, $\mathsf{t}_h(\theta) = \frac{1}{\theta}$ for $\theta \in [\underline{\theta}_L, \theta_L^h]$, $\mathsf{t}_h(\theta_H^h) = 0$, and $\mathsf{t}_h(\theta_L^h) = \frac{1}{\theta_L^h}$, we can conclude that $\mathsf{t}_h(\theta)$ is decreasing in $\theta \geq \underline{\theta}_L$.

Now, we are ready to show that the interval conditions are satisfied.

We first note that when $\hat{\theta} \leq \underline{\theta}_L$, it easy to show that for any $z \geq \hat{\theta}$, $A(z, \mathsf{t}^*(\hat{\theta})) \leq A(z, \mathsf{t}^*(z))$. This holds because $A(z, \mathsf{t}^*(\hat{\theta})) = 0$ and as shown in Lemma 14, $A(z, \mathsf{t}^*(z)) \geq 0$. In addition, when $\theta \leq \underline{\theta}_L$, we have $A(z, \mathsf{t}^*(z)) \leq A(z, \mathsf{t}^*(\theta))$ for any $z \leq \theta$. This follows from the fact that both $A(z, \mathsf{t}^*(z))$ and $A(z, \mathsf{t}^*(\theta))$ are both zero.

Next, we assume that both θ and $\hat{\theta}$ are greater than $\underline{\theta}_L$. Recall that for $\theta \geq \underline{\theta}_L$, $\mathsf{t}^*(\theta) = \mathsf{t}_h(\theta)$. We start with showing $A(z, \mathsf{t}_h(\hat{\theta})) \leq A(z, \mathsf{t}_h(z))$, $z \geq \hat{\theta}$. We consider two cases: 1- $(1 - \mathsf{t}_h(\hat{\theta})z) \leq 0$ and 2- $(1 - \mathsf{t}_h(\hat{\theta})z) > 0$. Assume that $(1 - \mathsf{t}_h(\hat{\theta})z) \leq 0$. Then, we have

$$e^{-z\mathsf{t}_h(\hat{\theta})}(1 - \mathsf{t}_h(\hat{\theta})z) \ \leq \ 0 \ \leq \ e^{-z\mathsf{t}_h(z)}(1 - \mathsf{t}_h(z)z) \ ,$$

where the second inequality follows from Lemma 14 where we show that $A(z, \mathsf{t}_h(z)) = e^{-z\mathsf{t}_h(z)}(1 - \mathsf{t}_h(z)z) \geq 0$. By the above equation, we get $A(z, \mathsf{t}_h(\hat{\theta})) \leq A(z, \mathsf{t}_h(z))$.

Now, assume that $(1 - \mathsf{t}_h(\hat{\theta})z) > 0$. Then, considering the fact that $\mathsf{t}_h(\cdot)$ is decreasing, for any $z \geq \hat{\theta}$, we have $(1 - \mathsf{t}_h(z)z) \geq (1 - \mathsf{t}_h(\hat{\theta})z)$, and $e^{-\mathsf{t}_h(z)z} \geq e^{-\mathsf{t}_h(\hat{\theta})z}$. By multiplying these two equations, we get $A(z, \mathsf{t}_h(\hat{\theta})) \leq A(z, \mathsf{t}_h(z))$.

Next, we will verify that $A(z, \mathsf{t}_h(z)) \leq A(z, \mathsf{t}_h(\theta))$. Given that $\mathsf{t}_h(\cdot)$ is decreasing, for any $z \geq \theta$, we have

$$0 \leq (1 - \mathsf{t}_h(z)z) \leq (1 - \mathsf{t}_h(\theta)z), \quad \text{and} \quad e^{-\mathsf{t}_h(z)z} \leq e^{-\mathsf{t}_h(\theta)z} \ ,$$

where the first inequality follows from Lemma 14 where we show $A(z, \mathsf{t}_h(z)) = e^{-z\mathsf{t}_h(z)}(1 - \mathsf{t}_h(z)z) \geq 0$. By multiplying these two equations, we have $A(z, \mathsf{t}_h(z)) \leq A(z, \mathsf{t}_h(\theta))$.

F.1.1. Proof of Lemma 13 Here, with some abuse of notation, we denote $\mathsf{t}^*(\cdot)$ with $\mathsf{t}_h(\cdot)$. Recall that $\mathsf{t}^*(\theta) = \mathsf{t}_h(\theta)$ when $\theta \geq \underline{\theta}_L$ and is ∞ otherwise. In addition, for simplicity, we denote $u(\theta, \theta)$ by $u(\theta)$.

The proof has two parts. In the first part, we show that the solution given in Lemma 13 is a feasible solution of Problem OPT-H-R. In the second part, we verify that this solution is an optimal solution of this problem.

Feasibility: To show that $\mathsf{t}_h(\cdot)$ is a feasible solution of Problem OPT-H-R, we will verify that $u(\theta) \geq 0$ for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$. For any $\theta \leq \theta_L^h$, it is easy to verify that $u(\theta) = u(\underline{\Theta}) = 0$. Thus, we only need to show that $u(\theta) \geq 0$ for any $\theta \geq \theta_L^h$. To prove that $u(\theta) \geq 0$ for $\theta \geq \theta_L^h$, we make use of Lemma 14 where we show that $e^{-\mathsf{t}_h(\theta)\theta}(1 - \mathsf{t}_h(\theta)\theta) \geq 0$. This implies that $u(\theta) \ = \ \int_{\underline{\Theta}}^{\theta} e^{-\mathsf{t}_h(z)z}(1 - \mathsf{t}_h(z)z)dz \ \geq \ 0$

Optimality: Here, we will show that the solution given in Lemma 13, is an optimal solution of Problem OPT-H-R. To this end, we find an upper bound for the optimal value of Problem OPT-H-R by dualizing the IR constraints. Then, we will show that the solution given in Lemma 13 achieves the upper bound and thus is optimal.

Upper Bound of OPT-H-R*:* For any purchase time $\mathfrak{t}(\cdot)$ and Lagrangian function $\lambda : [\underline{\Theta}, \bar{\Theta}] \to \mathbb{R}^+$, we define the following function.

$$L_h(\mathfrak{t}(\cdot), \lambda(\cdot), u(\underline{\Theta})) = \mathrm{E}[R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta) - u(\underline{\Theta})] + \int_{\underline{\Theta}}^{\bar{\Theta}} \lambda(z) u(z) dz \ ,$$

where $u(z) = \int_{\underline{\Theta}}^{z} e^{-\mathfrak{t}(\theta)\theta}(1 - \mathfrak{t}(\theta)\theta) d\theta + u(\underline{\Theta})$, and R is defined in Eq. (12). Note that $\mathrm{E}[\varsigma(\theta)(R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta)) - u(\underline{\Theta})]$ is the objective function of Problem OPT-H-R. However, here we remove $\varsigma(\theta)$ and instead, we assume that $R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta) = 0$ when $\mathfrak{t}(\theta) = \infty$. Recall that given that $\mathfrak{t}_h(\theta) = \infty$, we have $\varsigma(\theta) = 0$.

Then, considering the fact that $\lambda(\cdot) \geq 0$, for any $(\mathfrak{t}(\cdot), u(\underline{\Theta}))$ such that $u(\theta) = u(\underline{\Theta}) + \int_{\underline{\Theta}}^{\theta} e^{-z\mathfrak{t}(z)}(1 - z\mathfrak{t}(z)) \geq 0$, we have

$$L_h(\mathfrak{t}(\cdot), \lambda(\cdot), u(\underline{\Theta})) \ \geq \ \mathrm{E}[R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta) - u(\underline{\Theta})]$$

One can think of $\lambda(\theta)$ as a dual variable for the IR constraints. Therefore, for any $\lambda : [\underline{\Theta}, \bar{\Theta}] \to \mathbb{R}^+$,

$$\max_{(\mathfrak{t}(\cdot), u(\underline{\Theta})) \in \mathcal{T}} \{\mathrm{E}[R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta) - u(\underline{\Theta})]\} \ \leq \ \max_{(\mathfrak{t}(\cdot), u(\underline{\Theta})) \in \mathcal{T}} \{L_h(\mathfrak{t}(\cdot), \lambda(\cdot), u(\underline{\Theta}))\} \ , \tag{34}$$

where

$$\mathcal{T} = \Big\{(\mathfrak{t}(\cdot), u(\underline{\Theta})) : u(\underline{\Theta}) \geq 0, \ \mathfrak{t}(\theta) \geq 0, \ \text{and } u(\underline{\Theta}) + \textstyle\int_{\underline{\Theta}}^{\theta} e^{-z\mathfrak{t}(z)}(1 - z\mathfrak{t}(z)) \geq 0 \text{ for any } \theta \in [\underline{\Theta}, \bar{\Theta}]\Big\}$$

is the set of feasible solutions. In the following, we will characterize an upper bound for $\max_{(\mathfrak{t}(\cdot), u(\underline{\Theta})) \in \mathcal{T}} \{\mathrm{E}[R(\theta, \mathfrak{t}(\theta)) - h\mathfrak{t}(\theta) - u(\underline{\Theta})]\}$ by considering a specific Lagrangian function, defined below.

$$\lambda_h(\theta) = \begin{cases} 0 & \text{if } \theta > \theta_L^h; \\ \Big(f(\theta)(\theta + \alpha(\theta) + \frac{h}{\theta e^{-1}})\Big)' & \text{if } \theta \in [\underline{\theta}_L, \theta_L^h]; \\ \Big(f(\theta)(2\theta + \alpha(\theta))\Big)' & \text{if } \theta \in [\underline{\Theta}, \underline{\theta}_L]; \end{cases} \tag{35}$$

where $\big(f(\theta)(2\theta + \alpha(\theta))\big)'$ and $\big(f(\theta)(\theta + \alpha(\theta) + \frac{h}{\theta e^{-1}})\big)'$ are respectively the derivative of $\big(f(\theta)(2\theta + \alpha(\theta))\big)$ and $\big(f(\theta)(\theta + \alpha(\theta) + \frac{h}{\theta e^{-1}})\big)$ with respect to θ. The following lemma establishes that $\lambda_h(\theta) \geq 0$.

Lemma 16. *When $h \leq H_l$, for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$, $\lambda_h(\theta)$, defined in Eq. (35), is nonnegative.*

The following claim shows that $(\mathfrak{t}_h(\cdot), u(\underline{\Theta}) = 0)$ is an optimal solution of Problem OPT-H-R.

Claim: With a slight abuse of notation, let

$$(t_\lambda(\cdot), u_\lambda) = \arg \max_{(\mathfrak{t}(\cdot), u(\underline{\Theta})) \in \mathcal{T}} \{L_h(\mathfrak{t}(\cdot), \lambda_h(\cdot), u(\underline{\Theta}))\} \ .$$

Then, $t_\lambda(\cdot) = \mathfrak{t}_h(\theta)$ for any $\theta \in [\underline{\Theta}, \bar{\Theta}]$ and $u_\lambda = 0$. Furthermore, $L_h(\mathfrak{t}_h(\cdot), \lambda_h(\cdot), u_\lambda) = \mathrm{E}[R(\theta, \mathfrak{t}_h(\theta)) - h\mathfrak{t}_h(\theta) - u_\lambda]$.

Proof of the Claim: By definition, $\lambda_h(\theta) \ = \ 0$ for $\theta > \theta_L^h$. Thus, we get

$$\begin{aligned} L_h(\mathfrak{t}(\cdot), \lambda_h(\cdot), u(\underline{\Theta})) = & \int_{z=\theta_L^h}^{\bar{\Theta}} \big(R(z, \mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z) dz \\ & + \int_{z=\underline{\Theta}}^{\theta_L^h} \Big(\big(R(z, \mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z) + \lambda_h(z) u(z)\Big) dz - u(\underline{\Theta}) \ . \end{aligned} \tag{36}$$

From definition of $\lambda_h(\cdot)$, the last two terms of Eq. (36) can be written as

$$\int_{\underline{\theta}_L}^{\theta_L^h} \big(R(z,\mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z)dz + \int_{\underline{\theta}_L}^{\theta_L^h} u(z)d\left(f(z)(z+\alpha(z)+\frac{h}{ze^{-1}})\right)$$
$$\int_{\underline{\Theta}}^{\underline{\theta}_L} \Big(\big(R(z,\mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z)dz + \int_{\underline{\Theta}}^{\underline{\theta}_L} u(z)d\Big(f(z)(2z+\alpha(z))\Big) - u(\underline{\Theta}) \ . \tag{37}$$

We first focus on the first two terms where $z \in [\underline{\theta}_L, \theta_L^h]$. By integrating by part and using the definition of R, the first two terms can be rewritten as

$$\int_{\underline{\theta}_L}^{\theta_L^h} \big(e^{-\mathfrak{t}(z)z}\big(z+\alpha(z)(1-\mathfrak{t}(z)z) - h\mathfrak{t}(z)\big) f(z)dz$$
$$+ u(z)f(z)(z+\alpha(z)+\frac{h}{ze^{-1}})\Big|_{\underline{\theta}_L}^{\theta_L^h} - \int_{\underline{\theta}_L}^{\theta_L^h} e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big) f(z)\left(z+\alpha(z)+\frac{h}{ze^{-1}}\right)dz \ .$$

In the above equation, we use the fact that $\frac{du(z)}{dz} = e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big)$. Then, by definition of θ_L^h, i.e., the fact that $(\theta_L^h + \alpha(\theta_L^h) + \frac{h}{\theta_L^h e^{-1}}) = 0$, the above equation is simplified as

$$- u(\underline{\theta}_L) f(\underline{\theta}_L)(\underline{\theta}_L + \alpha(\underline{\theta}_L) + \frac{h}{\underline{\theta}_L e^{-1}})$$
$$+ \int_{\underline{\theta}_L}^{\theta_L^h} f(z)\left(e^{-\mathfrak{t}(z)z} z^2 \mathfrak{t}(z) - h\mathfrak{t}(z) - e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big)\frac{h}{ze^{-1}}\right)dz \ . \tag{38}$$

Now, we focus on the last three terms of Eq. (37). Again, by integrating by part and using definition of R, the last two terms of Eq. (37) can be rewritten as

$$\int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\left(e^{-\mathfrak{t}(z)z}\big(z+\alpha(z)(1-\mathfrak{t}(z)z)\big) - h\mathfrak{t}(z)\right)dz$$
$$+ u(z)f(z)(2z+\alpha(z))\Big|_{\underline{\Theta}}^{\underline{\theta}_L} - \int_{\underline{\Theta}}^{\underline{\theta}_L} e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big) f(z)\,(2z+\alpha(z))\,dz - u(\underline{\Theta})$$
$$= u(\underline{\theta}_L) f(\underline{\theta}_L)(2\underline{\theta}_L + \alpha(\underline{\theta}_L)) + u(\underline{\Theta})\Big(-1 - f(\underline{\Theta})(2\underline{\Theta}+\alpha(\underline{\Theta}))\Big)$$
$$+ \int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\left(ze^{-\mathfrak{t}(z)z}(-1+2\mathfrak{t}(z)z) - h\mathfrak{t}(z)\right)dz \ . \tag{39}$$

Note that the coefficient of $u(\underline{\Theta})$, i.e., $\big(-1 - f(\underline{\Theta})(2\underline{\Theta}+\alpha(\underline{\Theta}))\big)$, can be simplified as $-2\underline{\Theta} f(\underline{\Theta}) \le 0$. By plugging Equations (38) and (39) into Eq. (36), and by using definition of $\underline{\theta}_L$, we get

$$L_h(\mathfrak{t}(\cdot),\lambda_h(\cdot),u(\underline{\Theta})) = \int_{\theta_L^h}^{\bar{\Theta}} \big(R(z,\mathfrak{t}(z)) - h\mathfrak{t}(z)\big) f(z)dz$$
$$+ \int_{\underline{\theta}_L}^{\theta_L^h} f(z)\left(e^{-\mathfrak{t}(z)z} z^2 \mathfrak{t}(z) - h\mathfrak{t}(z) - e^{-\mathfrak{t}(z)z}\big(1-\mathfrak{t}(z)z\big)\frac{h}{ze^{-1}}\right)dz$$
$$+ \int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\left(ze^{-\mathfrak{t}(z)z}(-1+2\mathfrak{t}(z)z) - h\mathfrak{t}(z)\right)dz - 2\underline{\Theta} f(\underline{\Theta}) u(\underline{\Theta}) \ .$$

First of all, since the coefficient of $u(\underline{\Theta})$ is negative, to maximize the above equation, we need to set $u(\underline{\Theta})$ to zero. That is, $u_\lambda = 0$. Then, $\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{L_h(\mathfrak{t}(\cdot),\lambda_h(\cdot),u(\underline{\Theta}))\}$ can be upper-bounded as follows

$$\max_{(\mathfrak{t}(\cdot),u(\underline{\Theta}))\in\mathcal{T}}\{L_h(\mathfrak{t}(\cdot),\lambda_h(\cdot),u(\underline{\Theta})=0)\} \le \int_{\theta_L^h}^{\bar{\Theta}} f(z)\max_{t\ge 0}\big\{R(z,t) - ht\big\}dz$$
$$+ \int_{\underline{\theta}_L}^{\theta_L^h} f(z)\max_{t\ge 0}\left\{\left(e^{-tz}z^2 t - ht - e^{-tz}\big(1-tz\big)\frac{h}{ze^{-1}}\right)\right\}dz$$
$$+ \int_{\underline{\Theta}}^{\underline{\theta}_L} f(z)\max\left\{\max_{t\ge 0}\left\{(ze^{-tz}(-1+2tz) - ht)\right\},0\right\}dz \ . \tag{40}$$

We take advantage of the following lemma to simplify the first term of the above equation.

LEMMA 17. *If Assumption 2 holds and the holding cost $h \le H_l$, then for any $z \ge \theta_L^h$, we have $\arg\max_{t\ge 0}\{R(z,t)-ht\} = \mathsf{t}_h(z)$, where $R(\theta,t)$ is defined in Eq. (12).*

Note that the optimal solution characterized in Lemma 17 is the maximum of the FOC solution and zero. We now simplify the second term of Eq. (40). It is easy to verify that for any $z \in [\underline{\theta}_L, \theta_L^h]$, we have

$$\arg\max_{t\ge 0}\left\{\left(e^{-tz}z^2t - ht - e^{-tz}(1-tz)\frac{h}{ze^{-1}}\right)\right\} = \frac{1}{z} = \mathsf{t}_h(z)\ . \tag{41}$$

Finally, the following lemma characterizes an optimal solution of the third term of Eq. (40).

LEMMA 18. *If Assumption 2 holds and the holding cost $h \le H_l$, for any $z \le \underline{\theta}_L$, we have*

$$\max_{t\ge 0}\{(ze^{-tz}(-1+2tz)-ht)\} \le 0\ .$$

Lemmas 17, 18, and Eq. (41) show that $t_\lambda(\theta) = \mathsf{t}_h(\theta)$ and $u_\lambda = 0$. Then, the proof is completed by observing that $L_h(\mathsf{t}_h(\cdot), \lambda_h(\cdot), 0) = \mathrm{E}[R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)]$.

F.2. Optimal Mechanism for a Medium Holding Cost

Here, we present the proof for Lemma 11. We show that in the optimal solution of Problem OPT-H, the time of purchase is given by

$$\mathsf{t}^*(\theta) = \begin{cases} \mathsf{t}_h(\theta) & \text{if } \theta \ge \underline{\theta}_M; \\ \infty & \text{if } \theta < \underline{\theta}_M \end{cases} = \begin{cases} 0 & \text{if } \theta \ge \theta_H^h, \\ \mathsf{t}_f(\theta) & \text{if } \theta \in [\underline{\theta}_M, \theta_H^h], \\ \infty & \text{if } \theta < \underline{\theta}_M, \end{cases} \tag{42}$$

and $u(\theta,\theta) = \int_{\underline{\theta}_M}^{\theta} e^{-z\mathsf{t}_h(z)}(1 - z\mathsf{t}_h(z))dz$. Here, $\mathsf{t}^*(\theta) = \infty$ implies that customer with type θ does not purchase the item; that is, $\varsigma(\theta) = 0$.

The proof has three main steps. In the first step, we relax the problem by ignoring both IC and IR constraints and we find an allocation rule that maximizes the virtual profit. Then, we show that the solution of this relaxed problem can construct a mechanism that satisfy the IR and envelope conditions. Finally, we show that the aforementioned solution also satisfies the interval conditions, as a result, it is optimal.

- Maximizing virtual profit without IC and IR constraints: Consider that following optimization problem.

$$\max_{\{\mathsf{t}(\theta)\ge 0:\ \theta\in[\underline{\Theta},\bar{\Theta}]\}} \mathrm{E}\left[R(\theta, \mathsf{t}(\theta)) - h\mathsf{t}(\theta)\right], \tag{OPT-H-1}$$

where $R(\theta,t)$ is defined in Eq. (12).[17] The following lemma shows that $\mathsf{t}^*(\cdot)$, given in Eq. (42), is an optimal solution of Problem OPT-H-1.

LEMMA 19. *The optimal solution of Problem* OPT-H-1 *is given by* $\mathsf{t}^*(\cdot)$ *where* $\mathsf{t}^*(\cdot)$ *is defined in Eq. (42).*

The proof is similar to the proof of Lemma 17; thus, it is omitted. The main idea of the proof is to show that $R(\theta,t) - ht$ as a function of t has an inverted u-shape. Thus, it obtains its maximum at $\max\{0, \mathsf{t}_f(\theta)\}$, where $\mathsf{t}_f(\theta)$ is the FOC solution. Note that to show Lemma 19, we need the assumption that $\underline{\theta}_M$, i.e., the solution of $R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) = 0$, is unique. By this assumption, for any $\theta < \underline{\theta}_M$ we get

$$\max_{t\ge 0}\{R(\theta,t) - ht\} = R(\theta, \mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) < 0 \quad \text{for } \theta < \underline{\theta}_M\ .$$

This implies that it is optimal not to allocate the item to customers with type $\theta < \underline{\theta}_M$ and set $\varsigma(\theta) = \infty$.

[17] Again, we assume that $R(\theta, \mathsf{t}(\theta)) - h\mathsf{t}(\theta) = 0$ when $\mathsf{t}(\theta) = \infty$.

- Maximizing virtual profit with IR and envelope constraints: Here, we show that the purchase time $\mathfrak{t}^*(\cdot)$ is an optimal solution of Problem OPT-H-R. [18] To this aim, we verify that

$$u(\theta,\theta) \;=\; \int_{\underline{\theta}_M}^{\theta} e^{-\mathfrak{t}_h(z)z}(1-\mathfrak{t}_h(z)z)dz \;\geq\; 0\;.$$

Particularly, we show that for any $\theta \geq \underline{\theta}_M$, $(1-\theta\mathfrak{t}_h(\theta)) \geq 0$. Since $\mathfrak{t}_h(\theta)=0$ for $\theta \geq \theta_H^h$, it suffices to show that $(1-\theta\mathfrak{t}_h(\theta)) \geq 0$ for any $\theta \in [\underline{\theta}_M, \theta_H^h]$.

Lemma 20. *For any $h \in [H_l, H_h]$ and $\theta \in [\underline{\theta}_M, \theta_H^h]$, we have $1-\theta\mathfrak{t}_h(\theta) \geq 0$.*

In the proof, we show that when $h \geq H_l$ and $\theta \geq \tilde{\theta}$, we have $1-\mathfrak{t}_f(\theta)\theta \geq 0$. Then, we show that $\underline{\theta}_M \geq \tilde{\theta}$. This implies that $1-\mathfrak{t}_f(\theta)\theta \geq 0$ for any $\theta \in [\underline{\theta}_M, \theta_H^h]$, which is the desired result.

- Maximizing virtual profit with IR and IC constraints: Here, we need to show that the time of purchase $\mathfrak{t}^*(\cdot)$ and its associated payment, given in Lemma 11, satisfy the interval conditions presented in Lemma 1. This part of the proof is very similar to that of Lemma 10. Thus, we do not repeat it here.

F.3. Optimal Mechanism for a High Holding Cost

In this section, we present the proof of Lemma 12.

In the following, we show that $\max_{t\geq 0}\{R(\theta,t)-ht\} \;=\; R(\theta,0) \;=\; \theta+\alpha(\theta) \;\geq\; 0$ for $\theta \geq \theta_0$, and for any $\theta < \theta_0$, $\max_{t\geq 0}\{R(\theta,t)-ht\} \;<\; 0$ where R is defined in Eq. (12). This implies that in the optimal mechanism, the firm only sells to customers with type $\theta \geq \theta_0$.

We first show that for any $\theta \geq \theta_0$, $\arg\max_{t\geq 0}\{R(\theta,t)-ht\} = 0$. To this aim, we will verify that $\frac{\partial(R(\theta,t)-ht)}{\partial t}\big|_{t=0} \leq 0$. This will give us the desired result because as we show in Lemma 17, $R(\theta,t)-ht$ as a function of t has an inverted u-shape. Therefore, if $\frac{\partial(R(\theta,t)-ht)}{\partial t}\big|_{t=0} \leq 0$, we have $\arg\max_{t\geq 0}\{R(\theta,t)-ht\} = 0$.

By definition,

$$\frac{\partial\,(R(\theta,t)-ht)}{\partial t} \;=\; -\theta e^{-t\theta}(\theta+\alpha(\theta)(2-\theta t))-h\;,$$

and at $t=0$ and for any $\theta \geq \theta_0$, we have

$$\frac{\partial\,(R(\theta,t)-ht)}{\partial t}\Big|_{t=0} \;=\; -\theta(\theta+2\alpha(\theta))-h \;\leq\; 0\;, \tag{43}$$

where the inequality holds because

$$h \;\geq\; H_h \;=\; \theta_0^2 \;=\; -\theta_0(\theta_0+2\alpha(\theta_0)) \;=\; \max_{\theta\in[\theta_0,\bar{\Theta}]}\{-\theta(\theta+2\alpha(\theta))\}\;.$$

The first equality follows because $\theta_0+\alpha(\theta_0)=0$ and last equality holds because $\arg\max_{\theta\in[\theta_0,\bar{\Theta}]}\{-\theta(\theta+2\alpha(\theta))\} = \theta_0$. To see why the latter holds note that

$$\big(-\theta(\theta+2\alpha(\theta))\big)' \;=\; -2(\theta+\alpha(\theta))-2\theta\alpha'(\theta) \;\leq\; 0\;,$$

where the inequality follows because for any $\theta \geq \theta_0$, we have $(\theta+\alpha(\theta)) \geq 0$.

Next, we will verify that for any $\theta < \theta_0$, $\max_{t\geq 0}\{R(\theta,t)-ht\} \;<\; 0$. Note that it suffices to show that $\max_{t\geq 0}\{R(\theta,t)-H_h t\} \;<\; 0$ considering the fact that $R(\theta,t)-ht$ is decreasing in h.

[18] It is easy to observe that in an optimal solution of Problem OPT-H-R, we need to set $u(\underline{\Theta},\underline{\Theta})$ to zero.

By Eq. (43), at $h = H_h$ we have $\mathsf{t}_f(\theta_0) = 0$, and

$$\max_{t\geq 0}\{R(\theta,t)-H_h t\} = R(\theta_0,\mathsf{t}_f(\theta_0))-H_h\mathsf{t}_f(\theta_0) = \theta_0+\alpha(\theta_0) = 0 .$$

Then, by our assumption that $R(\theta,\mathsf{t}_f(\theta))-H_h\mathsf{t}_f(\theta)=0$ has unique solution, we have

$$R(\theta,\mathsf{t}_f(\theta))-H_h\mathsf{t}_f(\theta) = \max_{t\geq 0}\{R(\theta,t)-H_h t\} < 0 \text{ for any } \theta<\theta_0 .$$

F.4. Discussing the Assumption in Theorem 3

In this section, we discuss the assumption in Theorem 3. This assumption requires that the solution of equation $R(\theta,\mathsf{t}_f(\theta))-h\mathsf{t}_f(\theta)=0$ to be unique, where $R(\theta,t)$ is defined in Eq. (12).

The following lemma shows that for any $h\in[H_l,H_h]$, $R(\theta,\mathsf{t}_f(\theta))-h\mathsf{t}_f(\theta)=0$ has a unique solution if the solution of $R(\theta,\mathsf{t}_f(\theta))-H_l\mathsf{t}_f(\theta)=0$ is unique. In addition, it shows that $R(\theta,\mathsf{t}_f(\theta))-H_l\mathsf{t}_f(\theta)=0$ has a unique solution when $\alpha'(\theta)$ is small enough.

LEMMA 21. *If the solution of $R(\theta,\mathsf{t}_f(\theta))-H_l\mathsf{t}_f(\theta)=0$ is unique, then, for any $h\in[H_l,H_h]$, $R(\theta,\mathsf{t}_f(\theta))-h\mathsf{t}_f(\theta)=0$ has a unique solution. Furthermore, the solution of $R(\theta,\mathsf{t}_f(\theta))-H_l\mathsf{t}_f(\theta)=0$ is unique if $\alpha'(\theta)\leq\frac{(\sqrt{5}+1)^2}{2}\approx 5.2$ for any $\theta\leq\tilde{\theta}$ where $\tilde{\theta}$ solves $2\tilde{\theta}+\alpha(\tilde{\theta})=0$ and H_l and the FOC solution $\mathsf{t}_f(\cdot)$ are defined in Eq. (11).*

The proof of Lemma 21 is given at the end of this section. Note that for the uniform and exponential distributions, we have $\alpha'(\theta)\leq 5.2$. In fact, for the uniform distribution $U(a,b)$, we have $\alpha'(\theta)=1$ for any $\theta\in[a,b]$ where $a<b$ and $a,b\in\mathbb{R}$. For the exponential distribution with rate $\lambda\geq 0$, $\alpha'(\theta)=0$ for any $\theta\geq 0$. Furthermore, for a truncated normal distribution with mean μ, standard deviation σ, and cut-off greater than $\mu-\sigma$, we have $\alpha'(\theta)\leq 4.48$ for any $\theta\geq(\mu-\sigma)$. Note that the domain of the truncated normal distribution with cut-off $\mathcal{C}$ is $[\mathcal{C},\infty)$.

Proof of Lemma 21 First, we show that if the solution of Eq. (44) is unique at $h=H_l$, then this equation has a unique solution for any $h\in[H_l,H_h]$.

$$R(\theta,\mathsf{t}_f(\theta))-h\mathsf{t}_f(\theta)=0 . \tag{44}$$

By Lemma 11, $\tilde{\theta}$ solves $R(\tilde{\theta},\mathsf{t}_f(\tilde{\theta}))-H_l\mathsf{t}_f(\tilde{\theta})=0$ where $2\tilde{\theta}+\alpha(\tilde{\theta})=0$ and $1-\mathsf{t}_f(\tilde{\theta})\tilde{\theta}=0$. Then, by our assumption, $\tilde{\theta}$ is the unique solution of Eq. (44) at $h=H_l$. This assumption and the proof of Lemma 20 imply that for any $h>H_l$, any solutions of Eq. (44) satisfy the following property: $1-\theta\mathsf{t}_f(\theta)\geq 0$.

Next, we use this property to show that for any $h\in[H_l,H_h]$, there is only one solution to Eq. (44). Let θ^* solve Eq. (44). By the Envelope theorem, the derivative of $R(\theta,\mathsf{t}_f(\theta))-h\mathsf{t}_f(\theta)$ w.r.t. θ at θ^* is given by

$$\begin{aligned}\frac{\partial\big(R(\theta,\mathsf{t}_f(\theta))-h\mathsf{t}_f(\theta)\big)}{\partial\theta}\Big|_{\theta=\theta^*} &= -\mathsf{t}_f(\theta^*)e^{-\mathsf{t}_f(\theta^*)\theta^*}(\theta^*+\alpha(\theta^*)(2-\mathsf{t}_f(\theta^*)\theta^*))\\ &+ e^{-\mathsf{t}_f(\theta^*)\theta^*}(1+\alpha'(\theta^*)(1-\mathsf{t}_f(\theta^*)\theta^*))\\ &= h\frac{\mathsf{t}_f(\theta^*)}{\theta^*}+e^{-\mathsf{t}_f(\theta^*)\theta^*}(1+\alpha'(\theta^*)(1-\mathsf{t}_f(\theta^*)\theta^*)) > 0 ,\end{aligned} \tag{45}$$

where the second equality follows from the FOC, i.e., $\frac{\partial(R(\theta^*,t)-ht)}{\partial t}\Big|_{\mathsf{t}_f(\theta^*)}=0$ and the inequality holds because $(1-\mathsf{t}_f(\theta^*)\theta^*)\geq 0$. By the above equation, the derivative of $R(\theta^*,\mathsf{t}_f(\theta^*))-h\mathsf{t}_f(\theta^*)$ w.r.t. θ^* is always positive. This implies that Eq. (44) has a unique solution.

Next, we show that at $h = H_l$, the solution of Eq. (44) is unique if for any $\theta \le \tilde{\theta}$, $\alpha'(\theta) \le \frac{(\sqrt{5}+1)^2}{2} \approx 5.2$.

We first argue that any $\theta > \tilde{\theta}$ cannot solve Eq. (44). To this end, we use the proof of Lemma 11 where we show $1 - \mathsf{t}_f(\theta)\theta \ge 0$ for any $\theta \ge \tilde{\theta}$. The fact that $1 - \mathsf{t}_f(\theta)\theta \ge 0$ for any $\theta \ge \tilde{\theta}$ implies that $\frac{\partial\left(R(\theta,\mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta)\right)}{\partial\theta} > 0$; see Eq. (45). Then, considering the fact that $R(\tilde{\theta}, \mathsf{t}_f(\tilde{\theta})) - H_l \mathsf{t}_f(\tilde{\theta}) = 0$, we have $R(\theta, \mathsf{t}_f(\theta)) - H_l \mathsf{t}_f(\theta) > 0$ for any $\theta \ge \tilde{\theta}$.

Next, we show that any $\theta < \tilde{\theta}$ cannot solve Eq. (44). Let $\zeta(\theta) = \theta \mathsf{t}_f(\theta)$. For simplicity, we denote $\zeta(\theta)$ by ζ. Then, Eq. (44) at $h = H_l$ can be written as

$$G(\theta,\zeta) \;:=\; \theta e^{-\zeta}(\theta + \alpha(\theta)(1-\zeta)) - H_l\zeta \;=\; 0\ .$$

We assume, contrary to our result, that there exists $\theta^* < \tilde{\theta}$ that solves Eq. (44). Then, we show that we have $\frac{\partial G}{\partial \theta}\big|_{\theta=\theta^*} > 0$ and $\frac{\partial G}{\partial \theta}\big|_{\theta=\tilde{\theta}} > 0$. This implies that there cannot exist $\theta^* < \tilde{\theta}$ that solves Eq. (44).

We consider the following two cases: i- $1-\zeta \ge 0$ and ii- $1-\zeta < 0$.

Case i- By the FOC, we have $\frac{\partial G}{\partial \zeta} = 0$. This implies that

$$\begin{aligned}\frac{\partial G}{\partial \theta}\Big|_{\theta=\theta^*} &= e^{-\zeta}(\theta^* + \alpha(\theta^*)(1-\zeta)) + \theta^* e^{-\zeta}(1 + \alpha'(\theta^*)(1-\zeta)) \\ &= \frac{H_l\zeta}{\theta^*} + \theta^* e^{-\zeta}(1+\alpha'(\theta^*)(1-\zeta)) \;\ge\; 0\ ,\end{aligned} \tag{46}$$

where the second equation holds because $G(\theta^*,\zeta) = 0$ and the inequality holds because $1-\zeta \ge 0$. Note that the above equation also implies that $\frac{\partial G(\theta,\zeta)}{\partial\theta}\big|_{\theta=\tilde{\theta}} > 0$ considering the fact that at $\theta = \tilde{\theta}$, we have $1-\zeta \;=\; 1 - \tilde{\theta}\mathsf{t}_f(\tilde{\theta}) \;=\; 0$.

Case ii- Next we focus on the case of $1-\zeta < 0$. In the following, we show when $1-\zeta < 0$ and $\alpha'(\theta) \le \frac{(\sqrt{5}+1)^2}{2} \approx 5.2$ for any $\theta \le \tilde{\theta}$, we get $\frac{\partial G}{\partial\theta}\big|_{\theta=\theta^*} > 0$. This implies that θ^* does not exist.

By definition,

$$\begin{aligned}\frac{\partial G}{\partial \theta}\Big|_{\theta=\theta^*} &= e^{-\zeta}\left(2\theta^* + (\theta^*\alpha'(\theta^*) + \alpha(\theta^*))(1-\zeta)\right) \\ &\ge e^{-\zeta}\left(2\theta^* + (\theta^*\alpha'(\theta^*) - 2\theta^*)(1-\zeta))\right) \\ &= e^{-\zeta}\theta^*\left(\alpha'(\theta^*)(1-\zeta) + 2\zeta\right)\ .\end{aligned} \tag{47}$$

The inequality holds because $1-\zeta < 0$ and $\theta^* \le \tilde{\theta}$. Note that for any $\theta^* \le \tilde{\theta}$, $\alpha(\theta^*) \le -2\theta^*$. To complete the proof, we show that $(\alpha'(\theta^*)(1-\zeta) + 2\zeta) \ge 0$ when $\alpha'(\theta^*) \le \frac{(\sqrt{5}+1)^2}{2}$.

First assume that $\alpha'(\theta^*) \le 2$. Then, we get

$$(\alpha'(\theta^*)(1-\zeta) + 2\zeta) \;\ge\; 2(1-\zeta) + 2\zeta \;=\; 2 \;>\; 0\ ,$$

where the first inequality holds because $1-\zeta < 0$.

Now, assume that $\alpha'(\theta^*) \in [2, \frac{(\sqrt{5}+1)^2}{2}]$. We make use of the following claim.

Claim: Let $\theta^* < \tilde{\theta}$ solve Eq. (44). Then, $\zeta \;=\; 1 - \mathsf{t}_f(\theta^*)\theta^* \;\le\; \frac{1+\sqrt{5}}{2}$.

The proof of the claim is given at the end of the proof of this lemma.

Given that $\alpha'(\theta^*) \in [2, \frac{(\sqrt{5}+1)^2}{2}]$, then $\zeta \mapsto (\alpha'(\theta)(1-\zeta)+2\zeta)$ is decreasing. Then, by the claim, we get

$$\alpha'(\theta)(1-\zeta)+2\zeta \;\geq\; \alpha'(\theta)\left(1-\frac{1+\sqrt{5}}{2}\right)+2\left(\frac{1+\sqrt{5}}{2}\right) \;\geq\; 0\ ,$$

where the second inequality holds because $\alpha'(\theta) \;\leq\; \frac{(\sqrt{5}+1)^2}{2} \;\approx\; 5.2$.

Proof of Claim: Since θ^* solves Eq. (44) and $\mathfrak{t}_f(\theta^*)$ is the FOC solution, we get

$$\theta^* e^{-\zeta}(\theta^*+\alpha(\theta^*)(1-\zeta)) \;=\; H_l\zeta \qquad \text{and} \qquad -\theta^* e^{-\zeta}(\theta^*+\alpha(\theta^*)(2-\zeta)) \;=\; H_l\ ,$$

where the second equation implies that $\zeta \leq 2$. By dividing these two equations, we get $-\frac{(\theta^*+\alpha(\theta^*)(1-\zeta))}{(\theta^*+\alpha(\theta^*)(2-\zeta))}-\zeta=0$. This can be simplified as

$$\theta^*+\alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta} \;=\; 0\ ,$$

where for any $\zeta \in [1,2]$, $\zeta \mapsto \frac{1+\zeta-\zeta^2}{1+\zeta}$ is decreasing, $\frac{1+\zeta-\zeta^2}{1+\zeta}$ crosses zero at $\frac{1+\sqrt{5}}{2}$, and at $\zeta=1$, $\frac{1+\zeta-\zeta^2}{1+\zeta}\Big|_{\zeta=1}=\frac{1}{2}$. We note that $\theta^*+\alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta}\Big|_{\zeta=1}=\theta^*+\frac{1}{2}\alpha(\theta^*) \;<\; 0$ for any $\theta^*<\tilde{\theta}$, and $\theta^*+\alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta}\Big|_{\zeta=\frac{1+\sqrt{5}}{2}} \;=\; \theta^* \;\geq\; 0$. Then, we can conclude that ζ that solves $\theta^*+\alpha(\theta^*)\frac{1+\zeta-\zeta^2}{1+\zeta} \;=\; 0$ should be less than $\frac{1+\sqrt{5}}{2}$.

□

Appendix G: Proof of Supporting Results of Appendix C

G.1. Proof of Lemma 4

It is easy to verify that $(1-g'(\theta)\mathfrak{t}_g(\theta)\theta)=1$ for any $\theta>\theta_H$, and it is zero for any $\theta \leq \theta_L$. Thus, it suffices to show that $(1-g'(\theta)\mathfrak{t}_g(\theta)\theta)\geq 0$ when $\theta \in [\theta_L,\theta_H]$.

By definition, for any $\theta \in [\theta_L,\theta_H]$, we have

$$(1-g'(\theta)\mathfrak{t}_g(\theta)\theta) \;=\; -\theta\left(\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}\right)\ .$$

Since $\theta>0$, to show $(1-g'(\theta)\mathfrak{t}_g(\theta)\theta)>0$, we only need to verify that $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}\leq 0$. To that end, we show that $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}$ is decreasing in θ. Then by the fact that $\frac{1}{\alpha(\theta_L)}+\frac{g'(\theta_L)}{g(\theta_L)}=0$, we have $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}\leq 0$ for any $\theta \in [\theta_L,\theta_H]$.

The derivative of $\frac{1}{\alpha(\theta)}+\frac{g'(\theta)}{g(\theta)}$ w.r.t. θ is given by

$$\frac{-\alpha'(\theta)}{\alpha(\theta)^2}+(\frac{g'(\theta)}{g(\theta)})' \;\leq\; 0\ .$$

The inequality holds because by Assumption 2, we have $\alpha'(\theta)\geq 0$, and by Assumption 1, $\frac{g'(\theta)}{g(\theta)}$ is decreasing in θ.

G.2. Proof of Lemma 5

To show the result, we will verify that $\lambda_g(\theta)\geq 0$ for any $\theta \leq \theta_L$.

By Eq. (19), for any $\theta \leq \theta_L$,

$$\lambda_g(\theta) \;=\; f'(\theta)\left(\frac{g(\theta)}{g'(\theta)}+\alpha(\theta))\right)+f(\theta)\left(\left(\frac{g(\theta)}{g'(\theta)}\right)'+\alpha'(\theta)\right)\ .$$

We consider the following cases:

1- $f'(\theta) \le 0$: Observe that the first term of $\lambda_g(\theta)$ is nonnegative. This is the case because by Assumption 1, $\frac{g(\theta)}{g'(\theta)} + \alpha(\theta) \le 0$ for any $\theta \le \theta_L$. To see why note that $\frac{g(\theta)}{g'(\theta)} + \alpha(\theta)$ is increasing in θ, and $\frac{g(\theta_L)}{g'(\theta_L)} + \alpha(\theta_L) = 0$. In addition, note that the second term of $\lambda_g(\theta)$, i.e., $f(\theta)\big(\big(\frac{g(\theta)}{g'(\theta)}\big)' + \alpha'(\theta)\big)$, is greater than or equal to zero. This holds because by Assumption 1, $\left(\frac{g(\theta)}{g'(\theta)}\right)' \ge 0$.

1- $f'(\theta) \ge 0$: Since $g'(\theta) \ge 0$, we have

$$\lambda_g(\theta) \;\ge\; f'(\theta)\alpha(\theta) + f(\theta)\alpha'(\theta) \;=\; (f(\theta)\alpha(\theta))' \;\ge\; 0\ .$$

The last inequality holds because $f(\theta)\alpha(\theta) = F(\theta) - 1$ is increasing in θ.

G.3. Proof of Lemma 6

Here, we show that for any $\theta \ge \theta_L$, $\arg\max_{t\ge 0}\{R(\theta,t)\} = \mathfrak{t}_g(\theta)$. To this end, we show that the objective function, i.e., $R(\theta,t)$, has an inverted u-shape in t. Then, we show that for $\theta \ge \theta_H$, $R(\theta,t)$ achieves its maximum at $t = 0$, and for $\theta \in [\theta_L, \theta_H]$, $R(\theta,t)$ gets maximized at $\mathfrak{t}_g(\theta)$, where $\mathfrak{t}_g(\theta)$ solves the FOC; that is, $\mathfrak{t}_g(\theta) = \arg\max_t R(\theta,t)$ for $\theta \in [\theta_L, \theta_H]$.

$R(\theta,t)$ Has an Inverted U-shape: Let $t_0 = \frac{g(\theta)\theta + 2\alpha(\theta)g'(\theta)\theta + \alpha(\theta)g(\theta)}{\alpha(\theta)g(\theta)g'(\theta)\theta}$. We will show that $R(\theta,t)$ is concave for any $t \le t_0$ and is convex otherwise. We further show that $R(\theta,t)$ is decreasing for any $t \ge t_0$ and is increasing at $t = -\infty$. This implies that $R(\theta,t)$ has an inverted u-shape and $\arg\max_t R(\theta,t) < t_0$.

The second derivative of $R(\theta,t)$ w.r.t. t is given by

$$\frac{\partial^2 R(\theta,t)}{\partial^2 t} \;=\; -g(\theta)e^{-g(\theta)t}(-g(\theta)\theta - \alpha(\theta)g(\theta)(1 - g'(\theta)\theta t) - 2\alpha(\theta)g'(\theta)\theta)$$

It is easy to observe that the second derivative is negative for any $t \le t_0$, and positive otherwise. This implies that the objective function is concave for any $t \le t_0$.

Next, we discuss the first derivative of $R(\theta,t)$ w.r.t. t. By definition,

$$\frac{\partial R(\theta,t)}{\partial t} \;=\; e^{-g(\theta)t}\big(-g(\theta)\theta - \alpha(\theta)g(\theta)(1 - g'(\theta)\theta t) - \alpha(\theta)g'(\theta)\theta\big). \tag{48}$$

This leads to

$$\frac{\partial R(\theta,t)}{\partial t}\Big|_{t=t_0} \;=\; e^{-g(\theta)t_0}\big(\alpha(\theta)g'(\theta)\theta\big) \le 0$$

Then, considering the fact that $\lim_{t\to\infty}\frac{\partial R(\theta,t)}{\partial t} = 0$ and $R(\theta,t)$ is convex for any $t \ge t_0$, we can conclude that $\frac{\partial R(\theta,t)}{\partial t} \le 0$ for any $t \ge t_0$. The proof of this part is completed by observing $\lim_{t\to-\infty}\frac{\partial R(\theta,t)}{\partial t} > 0$.

So far, we established that $R(\theta,t)$ has an inverted u-shape in t. This implies that the unique maximizer of the $R(\theta,t)$, i.e., $\arg\max_t R(\theta,t)$ solves the FOC. Let us call the unique maximizer, the FOC solution and denote it by $\mathfrak{t}_f(\theta)$. We will show that for any $\theta \ge \theta_H$, the FOC solution is negative. Then, by the fact that $R(\theta,t)$ has an inverted u-shape, we can conclude that for any $\theta \le \theta_H$, $R(\theta,t)$ gets maximized at $\mathfrak{t}_g(\theta) = 0$. Similarly, one can show that the FOC solution is positive for any $\theta \in [\theta_L, \theta_H]$.

By Eq. (48), the FOC solution, which solves $\frac{\partial R(\theta,t)}{\partial t} = 0$, is equal to $\mathfrak{t}_f(\theta) = \frac{\alpha(\theta)g(\theta) + g(\theta)\theta + \alpha(\theta)g'(\theta)\theta}{\alpha(\theta)g(\theta)g'(\theta)\theta}$. Since $g'(\theta) \ge 0$, the FOC solution is negative if

$$g'(\theta)\mathfrak{t}_f(\theta) \;=\; \frac{1}{\alpha(\theta)} + \frac{g'(\theta)}{g(\theta)} + \frac{1}{\theta} \;\le\; 0\ .$$

By Assumptions 1 and 2, $\frac{1}{\alpha(\theta)} + \frac{g'(\theta)}{g(\theta)} + \frac{1}{\theta}$ is decreasing in θ. Then, considering this and the fact that $g'(\theta_H)\mathfrak{t}_f(\theta_H) = 0$, we have $g'(\theta)\mathfrak{t}_f(\theta) \le 0$ for any $\theta \ge \theta_H$. This implies that $\mathfrak{t}_f(\theta) \le 0$ for any $\theta \ge \theta_H$.

G.4. Proof of Lemma 7

First of all, it is easy to observe that $\mathsf{t}_g(\theta)$ is continuous and by definition of $\mathsf{t}_g(\cdot)$ and Assumption 1, $\mathsf{t}_g(\theta)$ is decreasing for $\theta \geq \theta_H$ and $\theta < \theta_L$. Considering this, in the following we will show that $\mathsf{t}_f(\theta)$ is decreasing when $\theta \in [\theta_L, \theta_H]$. Recall that for this range of θ, $\mathsf{t}_f(\theta)$ is the FOC solution.

By Eq. (48), the FOC solution, $\mathsf{t}_f(\theta)$ solves $Q(\theta, \mathsf{t}_f(\theta)) = 0$, where

$$Q(\theta, t) \;=\; \theta + \alpha(\theta)\Big(1 - g'(\theta)\theta t + \frac{g'(\theta)}{g(\theta)}\theta\Big). \tag{49}$$

This implies that $\partial_1 Q(\theta, \mathsf{t}_f(\theta)) + \partial_2 Q(\theta, \mathsf{t}_f(\theta))\frac{d\mathsf{t}_f(\theta)}{d\theta} = 0$, where $\partial_i Q(\theta, \mathsf{t}_f(\theta))$, $i = 1, 2$, is the derivative of Q w.r.t. to its i^{th} argument. This leads to

$$\frac{d\mathsf{t}_f(\theta)}{d\theta} \;=\; -\frac{\partial_1 Q(\theta, \mathsf{t}_f(\theta))}{\partial_2 Q(\theta, \mathsf{t}_f(\theta))} \;=\; \frac{1 + \alpha'(\theta)\Big(1 - g'(\theta)\theta\mathsf{t}_f(\theta) + \frac{g'(\theta)}{g(\theta)}\theta\Big) + \alpha(\theta)\partial_1 H(\theta, \mathsf{t}_f(\theta))}{g'(\theta)\theta\alpha(\theta)} \tag{50}$$

where $H(\theta, t) = 1 - g'(\theta)\theta t + \frac{g'(\theta)}{g(\theta)}\theta$. Next, we will show that $\partial_1 H(\theta, \mathsf{t}_f(\theta)) \leq 0$. This confirms that $\frac{d\mathsf{t}_f(\theta)}{d\theta} \leq 0$. This is so because $\alpha'(\theta) \geq 0$ and $\big(1 - g'(\theta)\theta\mathsf{t}_f(\theta) + \frac{g'(\theta)}{g(\theta)}\theta\big) = -\frac{\theta}{\alpha(\theta)} \geq 0$.

We consider the following cases:

- $\theta + \alpha(\theta) \geq 0$: By definition,

$$\partial_1 H(\theta, \mathsf{t}_f(\theta)) = -(g'(\theta)\theta)'\mathsf{t}_f(\theta) + \left(\frac{g'(\theta)}{g(\theta)}\right)'\theta = -\left(\frac{g'(\theta)}{g(\theta)}\theta g(\theta)\right)'\mathsf{t}_f(\theta) + \left(\frac{g'(\theta)}{g(\theta)}\right)'\theta$$

$$= -\left(\left(\frac{g'(\theta)}{g(\theta)}\right)'\theta g(\theta) + (\theta g(\theta))'\left(\frac{g'(\theta)}{g(\theta)}\right)\right)\mathsf{t}_f(\theta) + \left(\frac{g'(\theta)}{g(\theta)}\right)'\theta$$

$$= -\left(\frac{g'(\theta)}{g(\theta)}\right)'\theta(\mathsf{t}_f(\theta)g(\theta) - 1) - \left((\theta g(\theta))'\left(\frac{g'(\theta)}{g(\theta)}\right)\right)\mathsf{t}_f(\theta)$$

To show $\partial_1 H(\theta, \mathsf{t}_f(\theta)) \leq 0$, it suffices to verify that $(\mathsf{t}_f(\theta)g(\theta) - 1) \leq 0$. To see why note that $\frac{g'(\theta)}{g(\theta)}$ is decreasing and $g(\theta)$ is increasing in θ. By Eq. (5), we have

$$\mathsf{t}_f(\theta)g(\theta) - 1 \;=\; \frac{g'(\theta)}{g(\theta)}\left(\frac{\theta + \alpha(\theta)}{\alpha(\theta)\theta}\right) \;\leq\; 0$$

The inequality, which is the desired result, holds because $\theta + \alpha(\theta) \geq 0$.

- $\theta + \alpha(\theta) < 0$: By definition, $H(\theta, t) = 1 - g'(\theta)\theta\big(t - \frac{1}{g(\theta)}\big)$. Then, by taking derivative w.r.t. θ, we have

$$\partial_1 H(\theta, \mathsf{t}_f(\theta)) = -(g'(\theta)\theta)'\big(\mathsf{t}_f(\theta) - \frac{1}{g(\theta)}\big) - \frac{(g'(\theta))^2\theta}{g^2(\theta)}$$

To show $\partial_1 H(\theta, \mathsf{t}_f(\theta)) \leq 0$, it suffices to show the first term, i.e., $-(g'(\theta)\theta)'\big(\mathsf{t}_f(\theta) - \frac{1}{g(\theta)}\big)$, is negative. By Assumption 1, $(g'(\theta)\theta)$ is increasing. Thus, we only need to verify $(\mathsf{t}_f(\theta) - \frac{1}{g(\theta)}) \geq 0$. By Eq. (5),

$$\mathsf{t}_f(\theta) - \frac{1}{g(\theta)} \;=\; \frac{1}{g'(\theta)}\left(\frac{\theta + \alpha(\theta)}{\alpha(\theta)\theta}\right) \;\geq\; 0,$$

where the inequality holds because $\theta + \alpha(\theta) < 0$.

Appendix H: Proof of Supporting Results of Appendix F

H.1. Proof of Lemma 14

- **$\mathsf{t}_f(\theta)$ is decreasing in θ:** Since $\mathsf{t}_f(\theta)$ is the FOC solution, we have

$$\frac{\partial(R(\theta,t)-ht)}{\partial t}\Big|_{t=\mathsf{t}_f(\theta)} = -\theta e^{-\mathsf{t}_f(\theta)\theta}(\theta+\alpha(\theta)(2-\theta\mathsf{t}_f(\theta)))-h \ = \ 0\ .$$

Define $W(\theta,t)\ :=\ \frac{\partial(R(\theta,t)-ht)}{\partial t}\ =\ -\theta e^{-t\theta}(\theta+\alpha(\theta)(2-\theta t))-h$. Then, the FOC implies that $W(\theta,\mathsf{t}_f(\theta))=0$. Thus,

$$\frac{\partial \mathsf{t}_f(\theta)}{\partial\theta}=-\frac{W_\theta(\theta,\mathsf{t}_f(\theta))}{W_t(\theta,\mathsf{t}_f(\theta))}\ ,$$

where $W_\theta(\theta,\mathsf{t}_f(\theta))=\frac{\partial W(\theta,t)}{\partial\theta}\big|_{t=\mathsf{t}_f(\theta)}$ and $W_t(\theta,\mathsf{t}_f(\theta))=\frac{\partial W(\theta,t)}{\partial t}\big|_{t=\mathsf{t}_f(\theta)}$. Throughout the proof, for simplicity, we denote $\mathsf{t}_f(\theta)$ by t. In the following, we will show that both $W_\theta(\theta,t)$ and $W_t(\theta,t)$ are non-positive. This implies that $\frac{\partial \mathsf{t}_f(\theta)}{\partial\theta}\le 0$.

By definition, we get

$$\begin{aligned}W_\theta(\theta,t)&=-(1-t\theta)e^{-t\theta}\big(\theta+\alpha(\theta)(2-t\theta)\big)-\theta e^{-t\theta}(1+\alpha'(\theta)(2-t\theta)-t\alpha(\theta))\\&=(1-t\theta)\frac{h}{\theta}-\theta e^{-t\theta}(1+\alpha'(\theta)(2-t\theta)-t\alpha(\theta))\ ,\end{aligned}$$

where the second equality follows because $W(\theta,t)\ =\ 0$. Again, by the fact that $W(\theta,t)=0$, we can replace $-\theta e^{-t\theta}$ by $\frac{h}{(\theta+\alpha(\theta)(2-\theta t))}$. Then,

$$\begin{aligned}W_\theta&=(1-t\theta)\frac{h}{\theta}+\frac{h}{\theta+\alpha(\theta)(2-t\theta)}-\theta e^{-t\theta}(\alpha'(\theta)(2-t\theta)-t\alpha(\theta))\\&=h(2-t\theta)\frac{\theta+\alpha(\theta)(1-t\theta)}{\theta\big(\theta+\alpha(\theta)(2-t\theta)\big)}-\theta e^{-t\theta}(\alpha'(\theta)(2-t\theta)-t\alpha(\theta))\ \le\ 0\ .\end{aligned}\tag{51}$$

The inequality holds because by the FOC condition, i.e., $W(\theta,t)\ =\ 0$, we have $2-t\ge 0$ and $\big(\theta+\alpha(\theta)(2-t\theta)\big)\ \le\ 0$, and by our assumption that $R(\theta,t)-ht\ge 0$, we have $\theta+\alpha(\theta)(1-t\theta)\ge 0$. Note that since $R(\theta,t)-ht\ =\ e^{-t\theta}\big(\theta+\alpha(\theta)(1-t\theta)\big)-ht\ge 0$, we get $\theta+\alpha(\theta)(1-t\theta)\ge 0$.

Next, we show that $W_t(\theta,t)\le 0$. By definition,

$$W_t(\theta,t)=^2\theta^2e^{-t\theta}\big(\theta+\alpha(\theta)(2-t\theta)\big)+^2\theta^2\alpha(\theta)e^{-t\theta}\ \le\ 0\ ,$$

where the inequality holds because by the FOC condition, $\big(\theta+\alpha(\theta)(2-t\theta)\big)\le 0$. The above equation along with Eq. (51) imply that $\mathsf{t}_f(\cdot)$ is decreasing.

- **$A(\theta,\mathsf{t}_h(\theta))\in[0,1]$:** Note that $A(\theta,\mathsf{t}_h(\theta))=0$ for $\theta\le\theta_L^h$ and is 1 for $\theta\ge\theta_H^h$. Thus, it suffices to show that $A(\theta,\mathsf{t}_h(\theta))\in[0,1]$ for any $\theta\in[\theta_L^h,\theta_H^h]$; see Lemma 22.

LEMMA 22. *When $h\le H_l$, then $(1-\mathsf{t}_h(\theta)\theta)\ \ge\ 0$ for any $\theta\in[\theta_L^h,\theta_H^h]$.*

H.1.1. Proof of Lemma 22 Here, we show that set $\{\theta : \theta > \theta_L^h, \text{ and } 1 - \mathsf{t}_f(\theta)\theta \;=\; 0\}$ is empty. That is, there does not exists any $\theta > \theta_L^h$ with $1 - \mathsf{t}_f(\theta)\theta \;=\; 0$. Then, by the fact that $1 - \mathsf{t}_f(\theta_H^h)\theta_H^h \;=1$ and $1 - \mathsf{t}_f(\theta_L^h)\theta_L^h \;=\; 0$, we have $1 - \theta\mathsf{t}_f(\theta) \;\geq\; 0$ for any $\theta \in [\theta_L^h, \theta_H^h]$.

Assume, contrary to our result, that there exists $\theta^* > \theta_L^h$ that solves $1 - \mathsf{t}_f(\theta^*)\theta^* \;=\; 0$. Then, we show that this cannot happen.

Let $\theta \in \{\theta_L^h, \theta^*\}$. Since $\mathsf{t}_f(\theta)$ is the FOC solution, we have $\frac{\partial(R(\theta,t)-ht)}{\partial t}\Big|_{t=\mathsf{t}_f(\theta)} \;=\; 0$. This condition can be rewritten as

$$W(\theta,\zeta,h) \;:=\; -\theta e^{-\zeta}(\theta + \alpha(\theta)(2-\zeta)) - h \;=\; 0 \;,$$

where $\zeta = \theta\mathsf{t}_f(\theta)$. In the following, we will show that for $\theta \in \{\theta_L^h, \theta^*\}$, we have $\frac{\partial\zeta}{\partial\theta} \;=\; -\frac{W_\theta}{W_\zeta} \;\leq 0$, where $W_\theta := \frac{\partial W(\theta,\zeta,h)}{\partial\theta}$ and $W_\zeta := \frac{\partial W(\theta,\zeta,h)}{\partial\zeta}$. This implies that there does not exists $\theta^* > \theta_L^h$ that solves $1 - \mathsf{t}_f(\theta^*)\theta^* \;=\; 0$.

To show $\frac{\partial\zeta}{\partial\theta} \leq 0$, we will verify that $W_\theta \leq 0$ and $W_\zeta \leq 0$. By definition,

$$W_\zeta \;=\; \theta e^{-\zeta}\big(\theta + \alpha(\theta)(2-\zeta)\big) + \theta\alpha(\theta)e^{-\zeta} \;\leq\; 0 \;,$$

where the inequality follows from the FOC, i.e., the fact that $W(\theta,\zeta,h) = 0$. To make it more clear, by the FOC, $(\theta + \alpha(\theta)(2-\zeta)) < 0$ and as a result, $W_\zeta \leq 0$.

Next, we show that $W_\theta \leq 0$ for $\theta \in \{\theta_L^h, \theta^*\}$. By definition,

$$W_\theta \;=\; -e^{-\zeta}\left(2\theta + \big(\alpha(\theta) + \theta\alpha'(\theta)\big)(2-\zeta)\right).$$

By the fact that for $\theta \in \{\theta_L^h, \theta^*\}$, we have $1 - \theta\mathsf{t}_f(\theta) \;=\; 0$, and thus $\zeta = 1$. This shows that

$$W_\theta \;=\; -e^{-1}\left(2\theta + \alpha(\theta) + \theta\alpha'(\theta)\right) \;\leq 0 \;,$$

where the inequality holds because $\theta_L^h \geq \tilde{\theta}$ and as a result $2\theta + \alpha(\theta) \geq 0$ for $\theta \in \{\theta_L^h, \theta^*\}$. Recall that $\theta^* > \theta_L^h$.

H.2. Proof of Lemma 15

We show the result for $h \leq H_l$ where H_l is defined in Eq. (11). A similar argument holds for $h > H_l$.

By definition, for any $h \leq H_l$, we have

$$R(\theta,\mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta) = \begin{cases} \theta + \alpha(\theta) & \text{if } \theta \geq \theta_H^h; \\ R(\theta,\mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta) & \text{if } \theta \in [\theta_L^h, \theta_H^h]; \\ e^{-1}\theta - \frac{h}{\theta} & \text{if } \theta \in [\underline{\theta}_L, \theta_L^h]; \end{cases}$$

$R(\theta,\mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)$ is obviously increasing when and $\theta \geq \theta_H^h$ and $\theta \leq \theta_L^h$. Furthermore, $R(\theta,\mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)$ is a continuous function of θ because $\mathsf{t}_h(\theta)$ is continuous. Thus, it suffices to show that $R(\theta,\mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta)$ is increasing in $\theta \in [\theta_L^h, \theta_H^h]$.

Recall that $\mathsf{t}_h(\theta) = \mathsf{t}_f(\theta)$ for $\theta \in [\theta_L^h, \theta_H^h]$. That is, $\mathsf{t}_h(\theta)$ is the FOC solution. Thus, by the Envelope theorem, the derivative of $R(\theta,\mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta)$ w.r.t. θ is given by

$$\begin{aligned} \frac{\partial\big(R(\theta,\mathsf{t}_f(\theta)) - h\mathsf{t}_f(\theta)\big)}{\partial\theta} \;&=\; -\mathsf{t}_f(\theta)e^{-\mathsf{t}_f(\theta)\theta}(\theta + \alpha(\theta)(2 - \mathsf{t}_f(\theta)\theta)) \\ &+\; e^{-\mathsf{t}_f(\theta)\theta}(1 + \alpha'(\theta)(1 - \mathsf{t}_f(\theta)\theta)) \\ &=\; h\frac{\mathsf{t}_f(\theta)}{\theta} + e^{-\mathsf{t}_f(\theta)\theta}(1 + \alpha'(\theta)(1 - \mathsf{t}_f(\theta)\theta)) \;\geq\; 0 \;, \end{aligned}$$

where the inequality holds because, as we show in Lemma 14, $1-\mathsf{t}_f(\theta)\theta \geq 0$ for any $\theta \in [\theta_L^h, \theta_H^h]$, and the second equality follows from the FOC, i.e., by the fact that

$$\frac{\partial R(\theta,t)}{\partial t}\Big|_{t=\mathsf{t}_f(\theta)} - h \;=\; -\theta e^{-\mathsf{t}_f(\theta)\theta}(\theta+\alpha(\theta)(2-\theta\mathsf{t}_f(\theta))) - h \;=\; 0 \;.$$

Finally, since $R(\underline{\theta}_L, \mathsf{t}_h(\underline{\theta}_L)) - h\mathsf{t}_h(\underline{\theta}_L) \;=\; 0$, we have $R(\theta, \mathsf{t}_h(\theta)) - h\mathsf{t}_h(\theta) \geq 0$ for $\theta \geq \underline{\theta}_L$; see definition of $\underline{\theta}_L$ in Eq. (11).

H.3. Proof of Lemma 16

The proof is naturally divided into two parts. In the first part, we show that $\lambda_h(\theta) \geq 0$ for any $\theta < \underline{\theta}_L$ and in the second part, we show that $\lambda_h(\theta) \geq 0$ for any $\theta \in [\underline{\theta}_L, \theta_L^h]$.

First Part: By Eq. (35), for any $\theta \leq \underline{\theta}_L$, we have

$$\lambda_h(\theta) \;=\; f'(\theta)\,(2\theta+\alpha(\theta)) + f(\theta)\,(2+\alpha'(\theta)) \;. \tag{52}$$

We note that by definition, we have $e^{-1}(\underline{\theta}_L)^2 = h$. Thus, given that $h \leq H_l = \tilde{\theta}^2 e^{-1}$, we have $\underline{\theta}_L \leq \tilde{\theta}$. This implies that for any $\theta \leq \underline{\theta}_L$, we have $2\theta + \alpha(\theta) \leq 0$. Then, if $f'(\theta) \leq 0$, we have $\lambda_h(\theta) \geq 0$. Now, assume that $f'(\theta) > 0$. Then,

$$\lambda_h(\theta) \;\geq\; f'(\theta)\alpha(\theta) + f(\theta)\alpha'(\theta) \;=\; (f(\theta)\alpha(\theta))' = (F(\theta)-1)' \geq 0\,, \tag{53}$$

where the first inequality holds because $f'(\theta) \geq 0$.

Second Part: By definition, for any $\theta \in [\underline{\theta}_L, \theta_L^h]$, we have

$$\begin{aligned}
\lambda_h(\theta) \;&=\; f'(\theta)\left(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}}\right) + f(\theta)\left(1+\alpha'(\theta)-\frac{h}{\theta^2 e^{-1}}\right)\\
&\geq\; f'(\theta)\left(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}}\right) + f(\theta)\left(1+\alpha'(\theta)-\frac{h}{(\underline{\theta}_L)^2 e^{-1}}\right)\\
&=\; f'(\theta)\left(\theta+\alpha(\theta)+\frac{h}{\theta e^{-1}}\right) + f(\theta)\alpha'(\theta)\;,
\end{aligned}$$

where the inequality holds because $\theta \geq \underline{\theta}_L$, and the last equation follows from definition of $\underline{\theta}_L$. We consider the following two cases.

Case i: $f'(\theta) \leq 0$: To show $\lambda_h(\theta) \geq 0$, we use the fact that for $\theta \geq \underline{\theta}_L$, function $\theta \mapsto \theta + \alpha(\theta) + \frac{h}{\theta e^{-1}}$ is increasing in θ. Then, considering the fact that $\theta_L^h + \alpha(\theta_L^h) + \frac{h}{\theta_L^h e^{-1}} \;=\; 0$, we have $\left(\theta + \alpha(\theta) + \frac{h}{\theta e^{-1}}\right) \leq 0$ for $\theta \in [\underline{\theta}_L, \theta_L^h]$. This implies that $\lambda_h(\theta) \geq 0$ when $f'(\theta) \leq 0$.

The derivative of $\theta + \alpha(\theta) + \frac{h}{\theta e^{-1}}$ w.r.t. θ is given by

$$1+\alpha'(\theta)-\frac{h}{\theta^2 e^{-1}} \;\geq\; 1+\alpha'(\theta)-\frac{h}{(\underline{\theta}_L)^2 e^{-1}} \;=\; \alpha'(\theta) \;\geq\; 0\;,$$

where the first inequality holds because $\theta \geq \underline{\theta}_L$, and the second inequality follows from the definition of $\underline{\theta}_L$.

Case ii: $f'(\theta) > 0$: In this case, we have

$$\lambda_h(\theta) \;\geq\; f'(\theta)\alpha(\theta) + f(\theta)\alpha'(\theta) \;=\; (f(\theta)\alpha(\theta))' \;=\; (F(\theta)-1)' \;\geq\; 0\;.$$

The last inequality completes the proof.

H.4. Proof of Lemma 17

Here, we will show that for any θ, the objective function, $R(\theta,t)-ht$ is a unimodular function of t and achieves its maximum at the FOC solution, denoted by $\mathfrak{t}_f(\cdot)$. Then, we show that $\arg\max_{t\geq 0}\{R(\theta,t)-ht\} = \max\{\mathfrak{t}_f(\theta),0\} = \mathfrak{t}_h(\theta)$.

To show that the objective function is unimodular, we will make the following observations: 1- The derivative of the objective function w.r.t. t at $t=\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$ is negative, at $t=-\infty$ is ∞, and at $t=\infty$ is negative. 2- For any $t\leq\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$, the objective function is a concave function of t, and for any $t>\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$, the objective function is a convex function of t. These two observations imply that for any given θ, $R(\theta,t)$ is a unimodular function of t, and achieves its maximum at $t<\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$.

First Part: The derivative of the objective function with respect to t is given by

$$\frac{\partial R(\theta,t)}{\partial t}-h \;=\; -\theta e^{-t\theta}(\theta+\alpha(\theta)(2-\theta t))-h\;. \tag{54}$$

Note that as t approaches $-\infty$, the derivative of the objective function with respect to t converges to ∞. Furthermore, as t converges to ∞, the derivative goes to $-h$. In addition, one can easily show that the derivative is negative at $t=\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$.

Second Part: The second derivative of the objective function with respect to t is given by

$$(\theta)^2 e^{-t\theta}(\theta+\alpha(\theta)(3-\theta t))\;. \tag{55}$$

It is easy to observe that the second derivative is negative for any $t<\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$, and is nonnegative otherwise. This implies that the objective function is concave for any $t\leq\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$ and it is convex for any $t>\frac{\theta+3\alpha(\theta)}{\theta\alpha(\theta)}$.

So far, we established that $R(\theta,t)-ht$ is a unimodular function of t and achieves its maximum at the FOC solution, denoted by $\mathfrak{t}_f(\cdot)$. By Lemma 14, the FOC solution is decreasing in θ. This and the fact that $\mathfrak{t}_f(\theta_H^h)=0$ lead to $\max\{\mathfrak{t}_f(\theta),0\}=0$ for any $\theta\geq\theta_H^h$ and $\max\{\mathfrak{t}_f(\theta),0\}=\mathfrak{t}_f(\theta)=\mathfrak{t}_h(\theta)$ for any $\theta\in[\theta_L^h,\theta_H^h]$.

H.5. Proof of Lemma 18

Let $G(z,t):=ze^{-tz}(-1+2tz)-ht$. We show that for any $z\leq\underline{\theta}_L$, we have $\max_{t\geq 0}\{G(z,t)\}\;\leq\;0$. First observe that $G(z,t=0)=-z\leq 0$ and $G(z,t=\infty)=-\infty$. Then, to show that $\max_{t\geq 0}\{G(z,t)\}\leq 0$, we will verify that $G(z,t)\leq 0$ at the FOC solution, i.e., t that solves

$$\frac{\partial G(z,t)}{\partial t}\;=\;e^{-tz}z^2(3-2tz)-h\;=\;0\;.$$

We denote the FOC solution by $t_F(z)$, and we show that $G(z,t_F(z))\leq 0$.

To this aim, we show that i- $\frac{\partial G(z,t_F(z))}{\partial z}\geq 0$ when $(-1+2t_F(z)z)\geq 0$, ii- $zt_F(z)$ is increasing in z, and iii- $G(\underline{\theta}_L,t_F(\underline{\theta}_L))=0$. The fact that $zt_F(z)$ is increasing in z implies either $(-1+2t_F(z)z)\geq 0$ for any $z\leq\underline{\theta}_L$, $(-1+2t_F(z)z)\leq 0$ for any $z\leq\underline{\theta}_L$, or there exists $\hat{z}\in[\underline{\Theta},\underline{\theta}_L)$ such that $(-1+2t_F(z)z)>0$ for any $z>\hat{z}$ and $(-1+2t_F(z)z)\leq 0$. We will focus on the third case, as the proof for this case encompasses that of the other two cases.

First we show that $G(z,t_F(z))$ for any $z>\hat{z}$. Since $G(\underline{\theta}_L,t_F(\underline{\theta}_L))=0$ and $\frac{\partial G(z,t_F(z))}{\partial z}\geq 0$ when $(-1+2t_F(z)z)\geq 0$, for any $z>\hat{z}$, we have $G(z,t_F(z))\leq G(\underline{\theta}_L,t_F(\underline{\theta}_L))=0$, which is the desired result. Furthermore, for any $z\leq\hat{z}$, $G(z,t_F(z))\leq 0$ as for this range of z, we have $(-1+2t_F(z)z)\leq 0$.

- Claim i: $\frac{\partial G(z,t_F(z))}{\partial z} \le 0$ when $(-1+2t_F(z)z) > 0$. By the envelope theorem, we get

$$\frac{\partial G(z,t_F(z))}{\partial z} \;=\; e^{-t_F(z)z}\big(-1+t_F(z)z\big(5-2t_F(z)z\big)\big) \;\ge\; 0\;,$$

where the inequality holds because $(-1+2t_F(z)z) > 0$ and by the FOC $(3-2t_F(z)z) \ge 0$. To see why note that $x \mapsto -1+x\big(5-2x\big)$ is positive when $x \in [\frac{1}{2}, \frac{3}{2}]$.

- Claim ii: $z \mapsto (zt_F(z))$ is an increasing function. Define $\zeta = zt_F(z)$. By the FOC, we have $W(z,\zeta) := e^{-\zeta}z^2(3-2\zeta) - h \;=\; 0$. Then,

$$\frac{\partial \zeta}{\partial z} \;=\; -\frac{\frac{\partial W(z,\zeta)}{\partial z}}{\frac{\partial W(z,\zeta)}{\partial \zeta}} \;=\; \frac{e^{-\zeta}z^2(5-2\zeta)}{2ze^{-\zeta}(3-2\zeta)} \;\ge\; 0\;,$$

where the inequality holds because by the FOC $3-2\zeta \ge 0$.

- Claim iii: $G(\underline{\theta}_L, t_F(\underline{\theta}_L)) = 0$. Note that $t_F(\underline{\theta}_L) = \frac{1}{\underline{\theta}_L}$ and as a result,

$$G(\underline{\theta}_L, t_F(\underline{\theta}_L)) \;=\; \underline{\theta}_L e^{-1} - \frac{h}{\underline{\theta}_L} \;=\; 0\;,$$

where the last equation follows from definition of $\underline{\theta}_L$.

H.6. Proof of Lemma 20

The proof has two parts. In the first part, we show that when $h \ge H_l$ and $\theta \ge \tilde{\theta}$, we have $1 - \mathfrak{t}_f(\theta)\theta \ge 0$. Then, in the second part of the proof, we show that $\underline{\theta}_M \ge \tilde{\theta}$. This implies that $1 - \mathfrak{t}_f(\theta)\theta \ge 0$ for any $\theta \in [\underline{\theta}_M, \theta_H^h]$, which is the desired result.

First Part: Here, we show that any solution of $1 - \mathfrak{t}_f(\theta)\theta = 0$, denoted by θ^*, is less than equal to $\tilde{\theta}$. Let $\bar{\theta}^*$ be the maximum of such solution; that is $\bar{\theta}^* = \max\{\theta : 1 - \mathfrak{t}_f(\theta)\theta = 0\}$. Then, considering the fact that $\bar{\theta}^* \le \tilde{\theta}$, $1 - \theta_H^h \mathfrak{t}_f(\theta_H^h) = 1$, and $1 - \bar{\theta}^*\mathfrak{t}_f(\bar{\theta}^*) = 0$, we can conclude that $1 - \theta\mathfrak{t}_f(\theta) \;>\; 0$ for any $\theta \in [\tilde{\theta}, \theta_H^h]$.

Suppose, contrary to our claim, that there exists $\theta^* > \tilde{\theta}$ that solves $1 - \mathfrak{t}_f(\theta^*)\theta^* = 0$. By the FOC, we have

$$\frac{\partial R(\theta^*, t)}{\partial t}\Big|_{t=\mathfrak{t}_f(\theta^*)} \;=\; -\theta^* e^{-\theta^*\mathfrak{t}_f(\theta^*)}(\theta^* + \alpha(\theta^*)(2 - \theta^*\mathfrak{t}_f(\theta^*))) - h \;=\; 0$$

Since θ^* solves $1 - \mathfrak{t}_f(\theta^*)\theta^* = 0$, we get

$$\frac{\partial R(\theta^*, t)}{\partial t}\Big|_{t=\mathfrak{t}_f(\theta^*)} \;=\; -\theta^* e^{-1}(\theta^* + \alpha(\theta^*)) \;-\; h\;. \tag{56}$$

We note that $\theta^* \mapsto -\theta^* e^{-1}(\theta^* + \alpha(\theta^*))$ is decreasing in θ^*. This holds because

$$\frac{d(-\theta^*(\theta^* + \alpha(\theta^*)))}{d\theta^*} \;=\; -(2\theta^* + \alpha(\theta^*)) - \theta^*\alpha'(\theta^*) \;\le\; 0\;,$$

where the inequality follows because $\theta^* > \tilde{\theta}$. This implies that

$$\max_{\theta^* \ge \tilde{\theta}}\{-\theta^* e^{-1}(\theta^* + \alpha(\theta^*))\} \;=\; -\tilde{\theta}e^{-1}(\tilde{\theta} + \alpha(\tilde{\theta})) \;=\; \tilde{\theta}^2 e^{-1} \;=\; H_l\;.$$

Then, by Eq. (56), we can conclude that when $h > H_l$, there does not exists any $\theta^* > \tilde{\theta}$ such that $1 - \mathfrak{t}_f(\theta^*)\theta^* = 0$.

Second Part: Here, we show that $\underline{\theta}_M \ge \tilde{\theta}$. To this aim, we show that $\frac{\partial \underline{\theta}_M}{\partial h} \ge 0$ when $1 - \mathfrak{t}_f(\underline{\theta}_M)\underline{\theta}_M \ge 0$. This verifies that $\underline{\theta}_M$ increases as we increase h from H_l. The reason is that at $h = H_l$, we have $\underline{\theta}_M = \tilde{\theta}$ and $1 - \mathfrak{t}_f(\underline{\theta}_M)\underline{\theta}_M = 0$. This implies at $h = H_l$, when h is increased, we have $\underline{\theta}_M \ge \tilde{\theta}$. Then, by the first part of the lemma, we know that $1-\mathfrak{t}_f(\underline{\theta}_M)\underline{\theta}_M \geq 0$ when we increase h. This allows us to repeat this procedure to show that $\frac{\partial \underline{\theta}_M}{\partial h} \geq 0$ for any $h \geq H_l$.

Let $\theta = \underline{\theta}_M$ and $\zeta = \mathfrak{t}_f(\theta)\theta$. Then, by definition, we have

$$G(\theta,\zeta,h) := \theta e^{-\zeta}(\theta+\alpha(\theta)(1-\zeta)) - h\zeta \;=\; 0\;,$$
$$W(\theta,\zeta,h) := -\theta e^{-\zeta}(\theta+\alpha(\theta)(2-\zeta)) - h \;=\; 0\;.$$

The first equation follows from the fact that at $\theta = \underline{\theta}_M$, $R(\underline{\theta}_M, \mathfrak{t}_f(\underline{\theta}_M)) - h\mathfrak{t}_f(\underline{\theta}_M) = 0$ and the second equation follows from the FOC, i.e., $\frac{\partial(R(\theta,t)-ht)}{\partial t}\Big|_{t=\mathfrak{t}_f(\theta)} = 0$. In the following, we show that $\frac{\partial \theta}{\partial h} \geq 0$ when $1-\zeta \geq 0$.

The aforementioned equations imply that

$$\frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial h} + \frac{\partial G}{\partial \zeta}\frac{\partial \zeta}{\partial h} = \zeta\;,$$
$$\frac{\partial W}{\partial \theta}\frac{\partial \theta}{\partial h} + \frac{\partial W}{d\zeta}\frac{\partial \zeta}{\partial h} = 1\;.$$

This leads to

$$\frac{\partial \theta}{\partial h} \;=\; \frac{\begin{vmatrix} \zeta & \frac{\partial G}{\partial \zeta} \\ 1 & \frac{\partial W}{\partial \zeta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G}{d\theta} & \frac{\partial G}{d\zeta} \\ \frac{\partial W}{d\theta} & \frac{\partial W}{d\zeta} \end{vmatrix}} = \frac{\zeta\frac{\partial W}{\partial \zeta} - \frac{\partial G}{\partial \zeta}}{\frac{\partial G}{\partial \theta}\frac{\partial W}{\partial \zeta} - \frac{\partial G}{\partial \zeta}\frac{\partial W}{\partial \theta}}\;,$$

It is easy to observe that $\frac{\partial G}{\partial \zeta} \;=\; W(\theta,\zeta,h) \;=\; 0$. Thus, $\frac{\partial \theta}{\partial h} \;=\; \frac{\zeta}{\frac{\partial G}{\partial \theta}}$. In the following, we will show that $\frac{\partial \theta}{\partial h} \geq 0$ by verifying $\frac{\partial G}{\partial \theta} \geq 0$. By definition,

$$\frac{\partial G}{\partial \theta} \;=\; e^{-\zeta}(\theta+\alpha(\theta)(1-\zeta)) + \theta e^{-\zeta}(1+\alpha'(\theta)(1-\zeta))\;.$$

We note that the first term, i.e., $(\theta+\alpha(\theta)(1-\zeta))$, is nonnegative because $G(\theta,\zeta,h) = \theta e^{-\zeta}(\theta+\alpha(\theta)(1-\zeta)) - h\zeta \;=\; 0$. In addition, the second term is positive as $1-\zeta \geq 0$. This gives us $\frac{\partial G}{\partial \theta} \geq 0$ and thus $\frac{\partial \theta}{\partial h} \geq 0$.

References

[1] Paul Milgrom, and Ilya Segal. Envelope theorems for arbitrary choice sets. *Econometrica, 70(2):583–601, 2002.*