

Appendix. Proofs

Proof of Proposition 1. By comparing (3) to (4), one can obtain the desired result. \square

Proof of Lemma 1. For given levels of innovation T_1 and T_2 , we obtain the supplier's optimal prices for a duopoly and a monopoly by solving the first order condition of (3). Then, the supplier's optimal profits can be expressed as follows:

$$\pi_s^*(T_1, T_2) = \begin{cases} \frac{1}{24}(2 + (T_1 + T_2)(1 + \beta))^2, & \text{for a duopoly,} \\ \frac{1}{8}(1 + \beta T_1 + T_2)^2, & \text{for a monopoly.} \end{cases}$$

Next, we compare the supplier's optimal profits to derive the condition for the optimal downstream structure. The supplier will induce a duopoly if $\frac{1}{\sqrt{24}}(2 + (T_1 + T_2)(1 + \beta)) \geq \frac{1}{\sqrt{8}}(1 + \beta T_1 + T_2)$, which is equivalent to $\sqrt{3} - 2 + T_1(\beta(\sqrt{3} - 1) - 1) \leq (1 + \beta - \sqrt{3})T_2$. The left-hand side is negative for $\beta \in [0, 1]$. If $\beta \geq \sqrt{3} - 1$, then the right-hand side is positive and the inequality holds for all T_1 and T_2 . Otherwise, the inequality holds for $T_2 \leq \frac{\sqrt{3}-2+T_1(\beta(\sqrt{3}-1)-1)}{(1+\beta-\sqrt{3})}$. \square

Proof of Proposition 2. We first prove that the monopoly is not feasible. Let T_2^m be M_2 's innovation investment under its monopoly, i.e., $r = r_m$, $T_1 = 0$, and $q_1 = 0$. By optimizing M_2 's profit function with respect to T_2 , one can obtain $T_2^m = \frac{1}{16k_2 - 1}$. According to Lemma 1, $T_2^m > \hat{T}_2(0|\beta)$ and $\beta < \sqrt{3} - 1$ for the supplier to choose a monopoly. That is,

$$\begin{aligned} \frac{1}{16k_2 - 1} &> \frac{2 - \sqrt{3}}{\sqrt{3} - 1 - \beta} \\ &\equiv k_2 < \frac{1}{2 - \sqrt{3}} \frac{1}{16}. \end{aligned}$$

However, there is no such k_2 for $\frac{49}{144} < k_2$, resulting in no monopoly.

Next, we derive the equilibrium outcomes in a duopoly. Let $T_i(T_j)$ be M_i 's best innovation investment response to M_j 's investment T_j . We define T_i^I as M_i 's innovation investment under a duopoly by solving both manufacturers' first order conditions simultaneously, which can be expressed as follows:

$$T_i^I = \frac{(5\beta - 7)(-5\beta^2 + 12\beta + 12k_j - 7)}{6k_j((7 - 5\beta)^2 - 144k_i) + (7 - 5\beta)^2(\beta^2 + 6k_i - 1)}.$$

By substituting T_i with T_i^* , we derive the supplier's profit as $\pi_s^I(\beta) = \frac{216((\beta-1)(5\beta-7)k_2+k_1((\beta-1)(5\beta-7)-24k_2))^2}{((7-5\beta)^2(\beta^2+6k_2-1)+6k_1((7-5\beta)^2-144k_2))^2}$.

If $T_2^I < \hat{T}_2(T_1^I|\beta)$, then $T_2^* = T_2^I$. Otherwise, M_2 will invest $T_2^B = \hat{T}_2(T_1(T_2)|\beta)$, which is

$$T_2^B = \frac{48(2\sqrt{3} - 3)k_1 - (4\sqrt{3} - 7)(\beta - 1)(5\beta - 7)}{(4\sqrt{3} - 7)(\beta - 1)(\beta + 1)(5\beta - 7) - 48(\sqrt{3}\beta + \sqrt{3} - 3)k_1}.$$

Then, M_1 's investment becomes

$$\begin{aligned} T_1^B &= T_1(T_2^B) \\ &= -\frac{(4\sqrt{3} - 7)(\beta - 1)(5\beta - 7)}{(4\sqrt{3} - 7)(\beta - 1)(\beta + 1)(5\beta - 7) - 48(\sqrt{3}\beta + \sqrt{3} - 3)k_1}. \end{aligned}$$

Thus, the supplier's profit in this case is $\pi_s^B(\beta) = \frac{864(\beta-1)^2k_1^2}{((4\sqrt{3}-7)(\beta-1)(\beta+1)(5\beta-7)-48(\sqrt{3}\beta+\sqrt{3}-3)k_1)^2}$.

Next, by solving $T_2^I = T_2^B$, we obtain

$$\hat{k}_2(k_1|\beta) = -\frac{(4\sqrt{3}-5)(\beta-1)(5\beta-7)k_1}{(4\sqrt{3}-7)(\beta-1)(5\beta-7)+48(3-2\sqrt{3})k_1}.$$

If $k_2 < \hat{k}_2(k_1|\beta)$, then $T_i^* = T_i^B$. Otherwise, $T_i^* = T_i^I$.

As we assume $\frac{49}{144} < k_2$, the condition for the boundary duopoly is $\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$. We verify that $\frac{\partial \hat{k}_2(k_1|\beta)}{\partial \beta} < 0$ and $\hat{k}_2(k_1|1) = 0$, leading $\hat{k}_2(k_1|\beta)$ to decrease in β to zero. Therefore, if β is too large, $\hat{k}_2(k_1|\beta)$ cannot be greater than $\frac{49}{144}$, violating the condition. Solving the inequality $k_2 < \hat{k}_2(k_1|\beta)$ for β , we obtain $\beta < \hat{\beta}(k_1, k_2)$, where $\hat{\beta}(k_1, k_2) = \frac{6}{5} - \frac{1}{5} \sqrt{\frac{720(\sqrt{3}-2)k_1k_2 + (5\sqrt{3}-12)k_1 + (7\sqrt{3}-12)k_2}{(5\sqrt{3}-12)k_1 + (7\sqrt{3}-12)k_2}}$. \square

Proof for Proposition 3. The results follow by substituting $\beta = 0$ into the results in Proposition 2. \square

Proof of Proposition 4. Recall that the supplier's profit in equilibrium committing not to enforcing cross-licensing can be derived from Proposition 2 by setting β to 0. We are interested in seeking conditions under which the supplier is better off with cross-licensing compared to that without cross-licensing, i.e., $\pi_s^*(0) < \pi_s^*(\beta)$. Recall that $\hat{k}_2(k_1|\beta)$ is decreasing in β . Thus, there are three cases; (1) $k_2 < \hat{k}_2(k_1|\beta)$, (2) $\hat{k}_2(k_1|\beta) \leq k_2 < \hat{k}_2(k_1|0)$, and (3) $\hat{k}_2(k_1|0) \leq k_2$. We analyze each case sequentially.

(i) Consider that $\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$. In this region, the equilibrium outcome is always boundary duopoly, i.e., $\pi_s = \pi_s^B$ regardless of the cross-licensing decision. We can verify that $\pi_s^B(\beta) \geq \pi_s^B(0)$ in this region. That is, cross-licensing is always better for the supplier.

Since $\hat{k}_2(k_1|\beta)$ is decreasing in β to zero, $\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$ does not hold for a large enough β .

(ii) Next, suppose that $\hat{k}_2(k_1|\beta) \leq k_2 < \hat{k}_2(k_1|0)$. Then, the supplier's profit is π_s^B without cross-licensing but π_s^I with cross-licensing. Although the profit functions are quite complicated, we note that they have a positive square root. We compare the positive square roots of $\pi_s^I(\beta)$ to $\pi_s^B(0)$ to obtain a threshold value for k_2 .

First, we note that while $\pi_s^I(\beta)$ is decreasing in k_1 and k_2 , $\pi_s^B(0)$ is decreasing in k_1 but constant in k_2 . This implies that if $\pi_s^B(0) < \pi_s^I(\beta)|_{k_2=k_1}$, then $\pi_s^B(0) < \pi_s^I(\beta)$ for $k_2 < k_1$. As both profit functions are monotone decreasing in k_1 , one can verify whether they cross or not by solving $\sqrt{\pi_s^I(\beta)|_{k_2=k_1}} = \sqrt{\pi_s^B(0)}$. We obtain $\tilde{k}_1(\beta) = \frac{1}{72} (42 - (2\sqrt{3}+3)\beta(5\beta-2))$ such that $\pi_s^B(0) < \pi_s^I(\beta)$ for $k_1 < \tilde{k}_1$.

Next, if $\tilde{k}_1 \geq k_1$, there should exist a threshold \tilde{k}_2 such that $\pi_s^I(\beta) > \pi_s^B(0)$ for $k_2 < \tilde{k}_2$. We equate the positive roots of $\pi_s^I(\beta)$ and $\pi_s^B(0)$:

$$\Delta \triangleq \sqrt{\pi_s^I(\beta)} - \sqrt{\pi_s^B(0)} = \frac{6\sqrt{6}((\beta-1)(5\beta-7)k_2 + k_1((\beta-1)(5\beta-7) - 24k_2))}{(7-5\beta)^2(\beta^2 + 6k_2 - 1) + 6k_1((7-5\beta)^2 - 144k_2)} - \frac{12\sqrt{6}k_1}{7(4\sqrt{3}-7) - 48(\sqrt{3}-3)k_1}.$$

We notice that the first term is monotonically decreasing in k_2 . So is Δ . Then, there exists a unique solution $\tilde{k}_2(k_1|\beta)$ such that $\pi_s^*(0) \leq \pi_s^*(\beta)$ for $k_2 \leq \tilde{k}_2(k_1|\beta)$. Combining with the upper bound of Case 2, the supplier is better off with cross-licensing for $k_2 < \min\{\tilde{k}_2(k_1|\beta), \hat{k}_2(k_1|0)\}$.

(iii) Finally, suppose that $\hat{k}_2(k_1|0) \leq k_2 < k_1$. This region is nonempty if and only if $\hat{k}_2(k_1|0) < k_1$, or equivalently $\frac{7}{12} < k_1$. The corresponding supplier's profit function is π_s^I regardless of the cross-licensing decision. Although π_s^I is complicated, we note that it has a positive square root. We compare the positive square roots of $\pi_s^I(\beta)$ to $\pi_s^I(0)$. Specifically, we solve the following to obtain the threshold value for k_2 :

$$\begin{aligned} \sqrt{\pi_s^I(\beta)} - \sqrt{\pi_s^I(0)} &= \frac{6\sqrt{6}((\beta-1)(5\beta-7)k_2 + k_1((\beta-1)(5\beta-7) - 24k_2))}{(7-5\beta)^2(\beta^2+6k_2-1) + 6k_1((7-5\beta)^2 - 144k_2)} \\ &\quad - \frac{6\sqrt{6}(7k_2 + k_1(7-24k_2))}{49(6k_2-1) + 6k_1(49-144k_2)}. \end{aligned}$$

While the above equation is involved, it can be reduced to a quadratic equation with respect to k_2 with a positive coefficient for k_2^2 . Moreover, one can verify that $(\sqrt{\pi_s^I(\beta)} - \sqrt{\pi_s^I(0)})|_{k_2=0} < 0$. So there exists a root $\underline{k}_2(k_1|\beta)$ such that $\pi_s^I(\beta) > \pi_s^I(0)$ for $k_2 > \underline{k}_2(k_1|\beta)$.

$$\underline{k}_2(k_1|\beta) = \frac{25\beta^3(24k_1-7) + 70\beta^2(7-24k_1) - \sqrt{(7(\beta-1)(5\beta-7)(5\beta-2) - A - B + C}}{24(-35\beta + 12(5\beta-2)k_1 + 49)},$$

where $A = 24k_1(\beta(5\beta(5\beta-14) + 59) + 6(2-5\beta)k_1 + 21)^2$, $B = 336(5\beta-7)k_1(5\beta^2 - 7\beta - 12k_1 + 2)(-35\beta + 12(5\beta-2)k_1 + 49)$, and $C = \beta(24(59 - 30k_1k_1 - 413) + 72k_1(4k_1 + 7) + 98)$. In addition, if $\beta \geq \frac{2}{5}$, then $\underline{k}_2(k_1|\beta) \geq k_1$. In such a case, $k_2 < k_1 \leq \underline{k}_2(k_1|\beta)$, which means that $\pi_s^I(0) \geq \pi_s^B(\beta)$ and the supplier is worse off with cross-licensing. \square

Proof of Corollary 2. Suppose that $k_2 \leq \hat{k}_2(k_1|\beta)$. Then, the derivative of π_s^B around $\beta = 0$ is

$$\frac{\partial \pi_s^B}{\partial \beta} \Big|_{\beta=0} = -\frac{3456k_1^2(24(2\sqrt{3}-3)k_1 - 4\sqrt{3} + 7)}{(48(\sqrt{3}-3)k_1 - 28\sqrt{3} + 49)^3}.$$

One can show the above equation is positive. Next, if $\hat{k}_2(k_1|\beta) < k_2$, $\pi_s = \pi_s^I$. Similarly, for $k_1 \approx k_2$,

$$\lim_{k_2 \rightarrow k_1} \frac{\partial \pi_s^I}{\partial \beta} \Big|_{\beta=0} = \frac{3456k_1^2}{(72k_1-7)^3} > 0.$$

Next, consider $\beta = 1$ and recall that M_2 can be better off with cross-licensing for $\hat{k}_2(k_1|\beta) < k_2$ and $k_1 < k_1(k_2|\beta)$. It implies that M_2 is better off with cross licensing for some large $k_1 \approx k_2 > \hat{k}_2(k_1|\beta)$. We will show that the supplier is not under the same condition. If $k_1 = k_2$, then the supplier's profit can be greatly simplified to

$$\pi_s^I(\beta)_{k_1=k_2} = \frac{864k_1^2}{((\beta+1)(5\beta-7) + 72k_1)^2}.$$

Then, $\pi_s^I(0)|_{k_1=k_2} = \frac{864k_1^2}{(72k_1-7)^2}$ and $\pi_s^I(1)|_{k_1=k_2} = \frac{864k_1^2}{(72k_1-4)^2}$, showing that $\pi_s^I(0)|_{k_1=k_2} > \pi_s^I(1)|_{k_1=k_2}$. \square

Proof of Proposition 5. Consider T_1 with respect to β . We examine its derivatives for $k_2 \leq \hat{k}_2(k_1|\beta)$ and $\hat{k}_2(k_1|\beta) < k_2$ sequentially.

1. $k_2 \leq \hat{k}_2(k_1|\beta)$ or equivalently $\beta \leq \hat{\beta}(k_1, k_2)$

Recall that $T_1^* = T_1^B$. Then,

$$\frac{\partial T_1^B}{\partial \beta} = -(7-4\sqrt{3}) \frac{48(5\beta(\sqrt{3}\beta + 2\sqrt{3} - 6) - 19\sqrt{3} + 36)k_1 - (7-4\sqrt{3})(\beta-1)^2(7-5\beta)^2}{((4\sqrt{3}-7)(\beta-1)(\beta+1)(5\beta-7) - 48(\sqrt{3}\beta + \sqrt{3}-3)k_1)^2}.$$

Notice that if $\beta < \frac{1}{5}(5\sqrt{3} - \sqrt{5(39 - 22\sqrt{3})} - 5)$, the numerator is an increasing linear function in k_1 . Since $\hat{\beta}(k_1, k_2)$ is increasing in k_1 and $\lim_{k_1 \rightarrow \infty} \hat{\beta}(k_1, k_2) = \frac{6}{5} - \frac{\sqrt{240(39 - 22\sqrt{3})k_2 - 40\sqrt{3} + 73}}{5(4\sqrt{3} - 5)} < \frac{1}{5}(5\sqrt{3} - \sqrt{5(39 - 22\sqrt{3})} - 5)$ for $\frac{49}{144} < k_2$, the numerator is indeed an increasing linear function for $\beta < \hat{\beta}(k_1, k_2)$. This also implies that the numerator is positive for $k_1 > \frac{(7-4\sqrt{3})(7-5\beta)^2(\beta-1)^2}{48(5\beta(\sqrt{3}\beta+2\sqrt{3}-6)-19\sqrt{3}+36)} \triangleq \underline{k}_1$. Since we assume that $\frac{49}{144} < k_2 < k_1$, if we show $\underline{k}_1 \leq \frac{49}{144}$, then we prove that the numerator is positive and T_1^B is decreasing in β . By differentiating \underline{k}_1 with respect to β , one can show that \underline{k}_1 is increasing in β . Also, $\lim_{k_1 \rightarrow \infty} \hat{\beta}(k_1, k_2)$ has the maximum at $k_2 = \frac{49}{144}$. Thus, \underline{k}_1 has the maximum at $\beta = \lim_{k_1 \rightarrow \infty} \hat{\beta}(k_1, 49/144)$, which is approximately evaluated as $0.172 < \frac{49}{144}$, showing that T_1^B is decreasing in $\beta \in [0, \hat{\beta}(k_1, k_2)]$.

2. $\hat{k}_2(k_1|\beta) < k_2 < k_1$ or equivalently $\hat{\beta}(k_1, k_2) < \beta$

We note that

$$\hat{\beta}(k_1, k_2) \begin{cases} \leq 0 & \text{if } \hat{k}_2(k_1|0) \leq k_2, \\ > 0 & \text{otherwise.} \end{cases}$$

Thus, the lower bound of β depends on k_1 and k_2 . Given this setup, we examine the derivative of $T_1^* = T_1^I$ with respect to β .

$$\frac{\partial T_1^I}{\partial \beta} = \frac{f(\beta)}{((7-5\beta)^2(\beta^2+6k_2-1)+6k_1((7-5\beta)^2-144k_2))^2},$$

where $f(\beta) = (\beta-1)^2(7-5\beta)^4 - 6(7-5\beta)^2k_2(\beta(55\beta-98)+60k_2+39) - 6k_1((7-5\beta)^4 - 12(5\beta-7)(155\beta-169)k_2+8640k_2^2)$. We want to show that f is decreasing in β by evaluating f and $f_n \triangleq \frac{\partial^n f(\beta)}{\partial \beta^n}$ at the boundaries of β . In what follows, we omit complicated evaluated expressions if they can be verified.

- (a) Evaluation of f_4

Notice that f_4 is a convex quadratic equation for β . Although $f_4(0)$ and $f_4(\hat{\beta}(k_1, k_2))$ can be either negative or positive, $f_4(1) < 0$. Then, f_3 is either decreasing or increasing and then decreasing in β .

- (b) Evaluation of f_3

As we verify that $f_3(0) \geq 0$, $f_3(\hat{\beta}(k_1, k_2)) \geq 0$, and $f_3(1) \geq 0$, $f_3 \geq 0$ and f_2 is increasing in β .

- (c) Evaluation of f_2

Similar to f_4 , $f_2(0)$ and $f_2(\hat{\beta}(k_1, k_2))$ can be either negative or positive, but we verify $f_2(1) > 0$. Then, f_1 is either increasing or decreasing and then increasing in β .

- (d) Evaluation of f_1

We verify that $f_1(0) < 0$, $f_1(\hat{\beta}(k_1, k_2)) < 0$, and $f_1(1) < 0$. Hence, $f_1 < 0$ and f is decreasing in β .

- (e) Evaluation of f

We verify $f(1) < 0$. We can show that $\hat{k}_2(k_1|0) < k_1$ if and only if $k_1 > \frac{7}{12}$. So if $k_1 < \frac{7}{12}$, then $k_2 < k_1 < \hat{k}_2(k_1|0)$. If $k_2 \leq \hat{k}_2(k_1|0)$, the lower bound of $f(\cdot)$ is $\hat{\beta}(k_1, k_2)$. We further verify that $f(\hat{\beta}(k_1, k_2)) > 0$ for $k_2 \leq \hat{k}_2(k_1|0)$. In addition, if $k_2 > \hat{k}_2(k_1|0)$, which is only possible if $k_1 > \frac{7}{12}$,

then the lower bound of β is 0. However, $f(0) = -51840k_1k_2^2 - 17640k_2^2 + 85176k_1k_2 - 11466k_2 - 14406k_1 + 2401$ is a concave quadratic function for k_2 and may or may not be less than 0. we verify that $f(0)|_{k_2=\hat{k}_2(k_1|0)} > 0$ if and only if $k_1 > \frac{7}{12}$. So in this case, there exists a solution of $f(0) = 0$ for k_2 , which is greater than $\hat{k}_2(k_1|0)$, denoted as $\bar{k}_2 = \frac{7(\sqrt{24k_1(100806k_1-58387)+170569+2028k_1-273})}{120(144k_1+49)} > \hat{k}_2(k_1, 0)$, such that $f(0) > 0$ if and only if $k_2 < \hat{k}_2(k_1|0)$. To summarize, the sign of $f(\cdot)$ at the low bound of β is

$$f(\underline{\beta}) = \begin{cases} f(\hat{\beta}(k_1, k_2)) > 0 & \text{if } \hat{k}_2(k_1|\beta) < k_2 < \hat{k}_2(k_1|0), \\ f(0) > 0 & \text{if } \hat{k}_2(k_1|0) \leq k_2 \leq \bar{k}_2, \\ f(0) \leq 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} > 0 & \text{if } \hat{k}_2(k_1|\beta) \leq k_2 \leq \bar{k}_2, \\ \leq 0 & \text{otherwise.,} \end{cases}$$

Then considering that f decreases in β and $f(1) < 0$, the result follows.

The proof of how T_2 changes in β is similar and thus omitted. \square

Proof for Proposition 6. Taking derivative of $(T_1^*(\beta) + T_2^*(\beta)) - (T_1^*(0) + T_2^*(0))$ with respect to k_2 , we obtain the result as stated in (i).

Comparing the total level of innovation $T_1 + T_2$, we get that cross licensing increases the total innovation when $k_2 < \hat{k}_2(k_1|\beta)$, and lowers the total innovation when $k_2 \geq \hat{k}_2(k_1|0)$. For $\hat{k}_2(k_1|\beta) \leq k_2 < \hat{k}_2(k_1|0)$, because $(T_1^*(\beta) + T_2^*(\beta)) - (T_1^*(0) + T_2^*(0))$ is decreasing in k_2 , there must exists a threshold, which we denote as $\kappa(k_1|\beta)$, where $\hat{k}_2(k_1|\beta) \leq \kappa(k_1|\beta) < \hat{k}_2(k_1|0)$, such that cross-licensing lowers the total innovation $T_1 + T_2$ if and only if $k_2 > \kappa(k_1|\beta)$.

Finally, We verify that when $\beta = 1$ the total level of innovation $T_1 + T_2$ is always lower with cross-licensing, by comparing the total level of innovation, and get the result in (c). \square

Proof of Proposition 7. We prove the profits of M_1 and M_2 sequentially.

(i) Consider M_1 's profit first. If $k_2 < \hat{k}_2(k_1|\beta)$, we verify that $\pi_1^B(\beta) > \pi_1^B(0)$. We next consider the case when $k_2 \geq \hat{k}_2(k_1|\beta)$. First, we can show that $T_2^I(\beta)$ decreases in β , so $T_2^I(\beta) < T_2^I(0)$ for any $\beta > 0$. We next verify that without cross-licensing, i.e., $\beta = 0$, manufacturer 1's profit decreases in T_2 . Therefore, $\pi_1(T_1^*(0), T_2^I(\beta)|0) > \pi_1(T_1^*(0), T_2^I(0)|0)$. Then, given fixed T_1 and T_2 , we can show that manufacturer 1's profit is always higher with cross-licensing. So $\pi_1(T_1^*(0), T_2^I(\beta)|\beta) > \pi_1(T_1^*(0), T_2^I(\beta)|0)$. Then we have, $\pi_1(T_1^*(\beta), T_2^I(\beta)|\beta) \geq \pi_1(T_1^*(0), T_2^I(\beta)|\beta) > \pi_1(T_1^*(0), T_2^I(\beta)|0) > \pi_1(T_1^*(0), T_2^I(0)|0)$.

Next, consider the derivative of π_1 with respect to β . One can show that the derivative evaluated around $\beta = 1$ is negative.

$$\frac{\partial}{\partial \beta} \pi_1(\beta)|_{\beta=1} = -\frac{k_1k_2(-36k_1^2 + k_1 + 15(1 - 36k_1)k_2^2 + (9k_1 + 1)(24k_1 - 1)k_2)}{3(k_2 + k_1(1 - 36k_2))^3}$$

(ii) Now, we focus on M_2 's profit. Recall that $\pi_2^*(T_1, T_2|\beta) = \Pi_2^*(T_1, T_2|\beta) - k_2T_2^2 = \frac{1}{144}((5\beta - 7)T_2 + (5 - 7\beta)T_1 - 2)^2 - k_2T_2^2$, of which the first term is revenue and the second term is innovation investment. First, assume that $k_2 < \hat{k}_2(k_1|\beta)$. If the second term decrease faster than the first term, then π_2 is decreasing. We compare the derivatives of their squares, $\frac{\partial}{\partial \beta} \sqrt{\Pi_2(T_1^B, T_2^B|\beta)} - \frac{\partial}{\partial \beta} \sqrt{k_2T_2^{B2}}$, of which sign

is equivalent to $f(k_1) \triangleq 24k_1((936 - 528\sqrt{3})k_1 - (48\sqrt{3} - 83)(\beta - 1)^2(5\beta - 1)) + \sqrt{k_2}((56\sqrt{3} - 97)(\beta - 1)^2(7 - 5\beta)^2 + 48k_1(\beta((615 - 355\sqrt{3})\beta + 554\sqrt{3} - 960) + 144(\sqrt{3} - 2)k_1 - 147\sqrt{3} + 255))$. While f is complicated, it is a concave quadratic function in k_1 . We notice that its maximum exists between 0 and $\frac{49}{144}$, implying that f is decreasing in $\frac{49}{144} < k_1$. Since $\frac{49}{144} < k_2$, $f(k_2)|_{k_2=49/144} < 0$ is sufficient for $f < 0$ for $k_2 < k_1$. One can verify that $f(k_2)|_{k_2=49/144}$ is decreasing in $\beta \in [0, 1]$ and $f(k_2)|_{k_2=49/144, \beta=0} < 0$.

Then, suppose that $k_2 > \hat{k}_2(k_1|\beta)$. Due to the analytical complexity, the general comparative statistics of π_2 with respect to β cannot be shown. For $k_2 > \hat{k}_2(k_1|0) > \hat{k}_2(k_1|\beta)$, however, we can show that there exists a threshold k_1 under which M_2 can be better off with cross-licensing.

When $k_2 > \hat{k}_2(k_1|0) > \hat{k}_2(k_1|\beta)$, both manufacturers invest T_i^I regardless of β , and M_2 's profit becomes $\pi_2^I(\beta)$. We notice that $\pi_2^I(\beta)$ has a square fraction term with respect to k_1 , which makes $\sqrt{\pi_2^I(0)} - \sqrt{\pi_2^I(\beta)}$ a quadratic equation in k_1 . Thus, there exists a positive solution $\bar{k}_1(k_2|\beta)$ under which $\sqrt{\pi_2^I(0)} < \sqrt{\pi_2^I(\beta)}$. \square

Proof of Corollary 3. Equation (5) shows that consumer surplus is proportional to the supplier's profit under a duopoly as given in Lemma 1. Therefore, the impact of cross-licensing on consumer surplus is the same as that on the supplier's profit, which is stated in Proposition 4. \square

Proof of Proposition 8. Let us define $\Delta_f(\beta)$ as the profit difference with and without cross-licensing of which effectiveness is β for the supply chain entity f . Then, social welfare difference is the sum of the differences of consumer surplus and all firms' profits, which can be expressed as

$$\Delta_{SW}(\beta) = \Delta_1(\beta) + \Delta_2(\beta) + \Delta_s(\beta) + \Delta_{CS}(\beta).$$

Recall that Lemma 1 shows that each firm's profit function is a square of a linear combination T_1 and T_2 . If the downstream is either a boundary duopoly ($\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$) or an interior duopoly ($\hat{k}_2(k_1|0) \leq k_2 < k_1$) regardless of cross-licensing, Proposition 2 shows that both T_1 and T_2 are fractions with the same denominator. It implies that although Δ_{SW} is algebraically complicated, it is a difference of two complicated fractions with positive denominators. However, if the downstream structure depends on cross-licensing ($\hat{k}_2(k_1|\beta) \leq k_2 < \hat{k}_2(k_1|0)$), namely an interior duopoly with cross-licensing but a boundary one without it, then Δ_{SW} is a linear combination of four complicated fractions, hindering us from analyzing it. As such, we examine when $\Delta_{SW}(\beta) > 0$ for $\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$ and $\hat{k}_2(k_1|0) \leq k_2 < k_1$, sequentially.

- (i) $\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$ In this case, the downstream is a boundary duopoly regardless of cross-licensing. Notice that the denominator of T_1^B and T_2^B is identical to $d^B(\beta) \triangleq (4\sqrt{3} - 7)(\beta - 1)(\beta + 1)(5\beta - 7) - 48(\sqrt{3}\beta + \sqrt{3} - 3)k_1$. Thus, each entity's profit function with cross-licensing can be expressed as a fraction with the same denominator, i.e., $(d^B(\beta))^2$. This applies to the no cross-licensing case by replacing β with 0. In short, $\Delta_{SW}(\beta)$ is a difference of two complicated fractions with positive denominators $(d^B(\beta))^2$ and $(d^B(0))^2$.

After multiplying both denominators to $\Delta_{SW}(\beta)$, we have $g \triangleq (48(\sqrt{3} - 3)k_1 - 28\sqrt{3} + 49)^2((56\sqrt{3} - 97)(\beta - 1)^2(7 - 5\beta)^2k_2 - 288k_1^2((48\sqrt{3} - 89)(\beta - 1)^2 + 24(7 - 4\sqrt{3})k_2) + (\beta - 1)(5\beta - 7)k_1((56\sqrt{3} - 97)(\beta - 1)(5\beta - 7) + 96(45 - 26\sqrt{3})k_2)) - (288(24(4\sqrt{3} - 7)k_2 - 48\sqrt{3} + 89)k_1^2 - 7(96(26\sqrt{3} - 45)k_2 -$

$392\sqrt{3}+679)k_1+49(56\sqrt{3}-97)k_2)((4\sqrt{3}-7)(\beta-1)(\beta+1)(5\beta-7)-48(\sqrt{3}\beta+\sqrt{3}-3)k_1)^2$. Although g is algebraically complicated, one can verify that g is decreasing in k_2 and $g > 0$ for $\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$, proving the desired result.

(ii) $\hat{k}_2(k_1|0) \leq k_2 < k_1$

As noted, the downstream is an interior duopoly regardless of cross-licensing. We use an approach similar to the first case. Since the denominator of T_1^I and T_2^I are identical and equals $d^I(\beta) \triangleq (7-5\beta)^2(\beta^2+6k_1+6k_2-1)-864k_1k_2$, each entity's profit function with cross-licensing can be expressed as a fraction with the same denominator, i.e., $(d^I(\beta))^2$. Similar to the previous case, $\Delta_{SW}(\beta)$ is a difference of two complicated fractions with positive denominators, $(d^I(\beta))^2$ and $(d^I(0))^2$. After multiplying both denominators to $\Delta_{SW}(\beta)$, we have the numerator $h \triangleq (-294k_2+6k_1(144k_2-49)+49)^2((\beta-1)^2-(7-5\beta)^2)k_2((7-5\beta)^2-432k_2)-144k_1^2(-3(\beta-1)^2(7-5\beta)^2+(5\beta-7)(125\beta-127)k_2-1440k_2^2)-(5\beta-7)k_1((\beta-1)^2(5\beta-7)^3+48k_2((375\beta-381)k_2-(\beta-1)(5\beta-7)(17\beta-19)))-(144(k_2(1440k_2-889)+147)k_1^2-7(48k_2(381k_2-133)+343)k_1+49k_2(432k_2-49))((7-5\beta)^2(\beta^2+6k_2-1)+6k_1((7-5\beta)^2-144k_2))^2$.

We verify that $h|_{k_2=\hat{k}_2(k_1|0)} < 0$ and $h|_{k_2=k_1} \geq 0$, suggesting the existence of thresholds for $\Delta_{SW} > 0$. However, it is unclear whether $h = 0$ has a unique solution. To show the uniqueness, we focus on the key result by omitting some tedious algebras, as h is algebraically complicated. Let us define h_n as the n^{th} derivative of h with respect to k_2 . That is,

$$h_4(k_1) \triangleq \frac{\partial^4 h}{\partial k_2^4} = 124416\beta(588(6\beta-7)(7-5\beta)^2+k_1 \cdot m(k_1, \beta)),$$

where $m(k_1, \beta) \triangleq 288k_1\beta(5\beta(455\beta-1684)+17874)+360(22-25\beta)k_1-13748-7(5\beta-7)(\beta(14365\beta-36141)+34202)$. Suppose that $h_4 \geq 0$, implying that h_3 is weakly increasing. One can show that $h_3|_{k_2=\hat{k}_2(k_1|0)} \geq 0$, implying that h_2 is also weakly increasing. Although it can be shown that $h_2|_{k_2=k_1} \geq 0$, we lose tractability for the sign of $h_2|_{k_2=\hat{k}_2(k_1|0)}$. Then, there are two cases:

- (a) Assume that $h_2|_{k_2=\hat{k}_2(k_1|0)} < 0$, implying that h_1 is decreasing and then increasing. Moreover, it can be shown that $h_1|_{k_2=\hat{k}_2(k_1|0)} < 0$, showing that h_1 's sign changes once from negative to positive. This means h is decreasing from negative then increasing to positive, showing the unique solution of $h = 0$.
- (b) Assume that $h_2|_{k_2=\hat{k}_2(k_1|0)} \geq 0$, implying that h_1 is weakly increasing. h_1 's sign may remain as positive or change once from negative to positive. If h_1 's sign is positive, h is increasing from negative to positive, proving the unique solution. Even if h_1 's sign changes from negative to positive, there exists a unique solution for $h = 0$, as it was shown in the previous case.

Therefore, there must be a threshold for k_2 under which $\Delta_{SW}(\beta) < 0$ and over which $\Delta_{SW}(\beta) > 0$.

Next, we analyze when $h_4 \geq 0$. We note that $\hat{k}_2(k_1|0) < k_1$ holds for $\frac{7}{12} < k_1$. If $\beta < \frac{22}{25}$, m is a convex function and its inflection point is greater than $\frac{7}{12}$. This implies that $k_1 \cdot m(k_1, \beta)$ is decreasing and then increasing to positive in k_1 . Although $588(6\beta-7)(7-5\beta)^2 < 0$, $k_1 \cdot m(k_1, \beta)$ becomes large enough to make $h_4 > 0$ for sufficiently high k_1 . However, it is unclear if $h_4 > 0$ for small k_1 .

$$h_4\left(\frac{7}{12}\right) = 12700800\beta(5\beta(\beta(35\beta-52)-65)-1162),$$

which is less than 0 for $0 < \beta < 22/25$. Thus, h_4 is increasing from negative to positive in k_1 , proving that there exists a threshold $\underline{k}_1^{SW}(\beta)$ over which $h_4 > 0$.

If $h_4 < 0$, we lose tractability due to the complexity of the polynomial. Although we cannot show a general result, we can show that $\Delta_{SW}(\beta)$ is negative for a special case. Assume $\beta = 1$ and k_1 is large enough, assuring $h_4 < 0$. Then,

$$\lim_{k_1 \rightarrow \infty} \Delta_{SW}(1) = -\frac{3(k_2(72k_2(360k_2 + 673) - 14497) + 196)}{(5184k_2^2 - 1908k_2 + 49)^2} < 0.$$

As the next proof show, $\Delta_{SW}(1)$ is indeed negative.

(iii) $\beta = 1$

We note that $\hat{k}_2(k_1|1) = 0$ implies that the downstream is an interior duopoly with cross-licensing regardless of k_2 . However, it may or may not be an interior duopoly without cross-licensing, contingent on k_2 .

First, consider that $\hat{k}_2(k_1|0) \leq k_2$ such that the downstream is an interior duopoly regardless of cross-licensing. Then, $\Delta_{SW}(1)$ is a fraction, where the numerator is a polynomial of degree 4 in k_1 and k_2 , $-108(k_2(72k_2(360k_2 + 673) - 14497) + 196)k_1^4 + (2401 - 36k_2(9k_2(48k_2(180k_2 - 1091) + 7651) + 6419))k_1^3 + 2k_2(2401 - 162k_2(k_2(16152k_2 + 7651) - 1372))k_1^2 + 14k_2^2(18k_2(6213k_2 - 917) + 343)k_1 + 49k_2^3(49 - 432k_2)$, and the denominator is the product of two square terms, $(k_2 + k_1(1 - 36k_2))^2(-294k_2 + 6k_1(144k_2 - 49) + 49)^2$. Thus, the numerator's sign determines the sign of $\Delta_{SW}(1)$. Although it is tedious, one can show that the numerator is decreasing in k_1 , where its maximum at $k_1 = k_2$ is negative for $\hat{k}_2(k_1|0) \leq k_2$.

Next, suppose that $k_2 < \hat{k}_2(k_1|0)$. The downstream is a boundary duopoly without cross-licensing. Similar to the previous case, $\Delta_{SW}(1)$ is a fraction, where the numerator is a polynomial of degree 4 in k_2 and degree 3 in k_1 , and the denominator is the product of two square terms. It can be also verified that the numerator is increasing in k_2 , where its maximum at $k_2 = k_1$ is negative for $k_2 < \hat{k}_2(k_1|0)$. \square

Proof of Corollary 4. We synthesize Propositions 4 and 8 to derive the desired results. For $\frac{49}{144} < k_2 < \hat{k}_2(k_1|\beta)$, Proposition 4(i) and Proposition 8(i) show that cross-licensing always benefits both the supplier and the social welfare. For $\hat{k}_2(k_1|0) \leq k_2$, Proposition 4(iii) shows that cross-licensing benefits the supplier if $\underline{k}_2(k_1|\beta) \leq k_2$. It also implies that if $\hat{k}_2(k_1|0) \leq k_2 < \underline{k}_2(k_1|\beta)$, the supplier will not impose cross-licensing. According to Proposition 8(ii), enforcing cross-licensing increases social welfare if $\underline{k}_2^{SW}(k_1|\beta) \leq k_2 < \underline{k}_2(k_1|\beta)$ and $h_4(k_1) > 0$. \square

Proof of Lemma 2. It follows the same proof of Lemma 1. \square

Proof of Proposition 9. As the proof closely follows the same proof of Proposition 2 with Q_{is} and r_s^* , we outline the key differences. First, define T_{is}^I and T_{is}^B which correspond to T_i^I and T_i^B in Proposition 2. It is straightforward to show both $T_{is}^I = (1 + T_s)T_i^I$ and $T_{is}^B = (1 + T_s)T_i^B$. Then, solving $T_{is}^I = T_{is}^B$ results in the same threshold $\hat{k}_2(k_1|\beta)$. Hence, each manufacturer increases its investment by the same factor $(1 + T_s)$, i.e., $T_{is}^* = (1 + T_s)T_i^*$. Indeed, even the monopoly investment T_{2s}^m is $(1 + T_s)$ times more than T_2^m . One can also show the monopoly is infeasible. \square

Proof of Proposition 10. Let us define the supplier's profit excluding its investment as $A(\beta) \triangleq \frac{1}{24}(2 + (T_1^* + T_2^*)(1 + \beta))^2$. Notice that (6) is a quadratic equation in T_s and $A(\beta)$ is a constant in T_s . By solving the first order condition, one can obtain $T_s^*(\beta) = \frac{A(\beta) + \sqrt{A(\beta)^2 + 48A(\beta)k_s}}{24k_s}$. Then, the supplier's optimal profit is

$$\pi_s^*(A(\beta)) = \frac{A(\beta) \left(A(\beta)^{3/2} \sqrt{A(\beta) + 48k_s} + A(\beta)^2 + 24A(\beta)k_s - 288k_s^2 \right)}{6912k_s^2},$$

which is increasing in $A(\beta)$. Therefore, if $A(\beta) \geq A(0)$, then the supplier is better off with cross-licensing with $T_s^*(\beta)$.

We note that $A(\beta)$ is indeed the supplier's optimal profit without the supplier's investment in Lemma 1 with the optimal manufacturer's investment. Recall that Proposition 4 is the result of $A(\beta) \geq A(0)$. Thus, the supplier's innovation investment does not alter the cross-licensing decision. \square

Proof of Proposition 11 We are interested in whether cross-licensing can achieve higher total innovation with the supplier's investment under wider conditions. According to Proposition 6(ii), if the supplier does not invest, i.e., $T_s = 0$, $\Delta T = (T_1^*(\beta) + T_2^*(\beta)) - (T_1^*(0) + T_2^*(0)) \leq 0$ at $k_2 \geq \kappa(k_1|\beta)$. Now consider that $T_s(\beta) > 0$. According to Proposition 9, the optimal investment of a manufacturer i becomes $T_{i_s}^* = (1 + T_s)T_i^*(\beta)$. Define the total innovation with the supplier's investment T_s is $T_T(\beta) \triangleq T_s(\beta) + (1 + T_s(\beta))(T_1^*(\beta) + T_2^*(\beta))$. For $k_2 \geq \kappa(k_1|\beta)$ we analyze the following total innovation difference:

$$\begin{aligned} T_T(\beta) - T_T(0) &= T_s(\beta) + (1 + T_s(\beta))(T_1^*(0) + T_2^*(0) + \Delta T) - (T_s(0) + (1 + T_s(0))(T_1^*(0) + T_2^*(0))) \\ &= (T_s(\beta) - T_s(0))(1 + T_1^*(0) + T_2^*(0)) + (1 + T_s(\beta))\Delta T. \end{aligned}$$

Remind that $k_2 = \kappa(k_1|\beta)$ implies $\Delta T = 0$ and $T_s(\beta) > T_s(0)$ is sufficient for $T_T(\beta) > T_T(0)$. If $k_2 > \kappa(k_1|\beta)$, $\Delta T < 0$. Since ΔT is continuous, $T_s(\beta) > T_s(0)$ is still sufficient to find $\varepsilon > 0$ such that ΔT is large enough to make $T_T(\beta) > T_T(0)$ for $k_2 < \kappa(k_1|\beta) + \varepsilon$.

Next, we prove $T_s^*(\beta) > T_s^*(0)$ for $\kappa(k_1|\beta) \geq k_2 < \hat{k}_2(k_1|\beta)$. With the results from Propositions 2 and 9, we obtain the supplier's optimal investment for $k_2 < \hat{k}_2(k_1|\beta)$:

$$T_s^{B*}(\beta) = - \frac{1728(\beta - 1)^2 k_1^2}{1728(\beta - 1)^2 k_1^2 - 2k_s((4\sqrt{3} - 7)(\beta - 1)(\beta + 1)(5\beta - 7) - 48(\sqrt{3}\beta + \sqrt{3} - 3)k_1)^2}.$$

One can verify that $\frac{\partial T_s^{B*}(\beta)}{\partial \beta} |_{\beta=0} > 0$ for $k_2 < \hat{k}_2(k_1|\beta)$, showing that $T_s^*(\beta) > T_s^*(0)$. \square