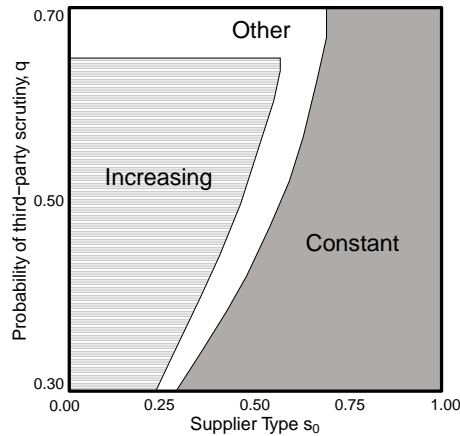


Appendix O.1: Examining the Expected Final SR Level

We discuss here in more detail the effect of visibility on the expected final SR level given the supplier's type s_0 , with the expectation taken over all possible signals that the firm may observe. We focus on the No Firm Disclosure setting, noting that the results in the Firm Disclosure setting are qualitatively the same. Figure O.1 illustrates the results with respect to the supplier's type (s_0) and the probability of third-party scrutiny (q). We identify three scenarios: (i) the expected final SR level is *increasing* in visibility; (ii) the expected final SR level is *constant* in visibility; and (iii) for at least some range of v , the expected final SR level is *decreasing* in visibility (denoted as "Other" in Figure O.1). We first observe from Figure O.1 that increased visibility primarily has a nonnegative effect on the expected final SR level (i.e., scenarios (i) and (ii)). For scenario (i), when the supplier's current SR level is low, greater visibility increases the expected final SR level for a wide range of q values. This is due to the positive effects of greater visibility on both the accuracy of the signal and the firm's investment. Specifically, greater visibility makes the firm more certain that the supplier's current practice are poor. Hence, the firm invests a high amount ($\bar{\beta}_H$). The firm's high investment increases with visibility (Proposition 1(i)) and results in an improved SR level by the supplier ($s^*(s_0, \bar{\beta}_H) = s_u^*(\bar{\beta}_H)$ for $s_0 \leq \tau$; see Proposition A.4).

Figure O.1 Effect of Visibility on the Expected Final SR Level



Note. Parameters: $\alpha = 0.4$, $r = 0.5$, $w = 0.15$, $c = 0$. Functions: $\rho(s) = e^{1.55s}$, $\gamma(s) = 0.5 - e^{-s}$, $\delta(\beta) = \frac{0.7\beta^2}{1-\beta}$. The supplier's type s_0 is distributed uniformly between $m = 0$ and $M = 1$.

Conversely, for scenario (ii) the supplier's current SR level is already high, and thus, visibility has no impact on the expected final SR level because the firm invests β_L and $s^*(s_0, \beta_L) = s_0$ for $s_0 > \tau$ (Proposition A.4). That is, the supplier does not change his SR level in response to the firm's low investment. As the probability of third-party scrutiny increases, the constant region shrinks because (a) the firm invests a high amount for a wider range of supplier types due to the increased chance of the consumers observing the supplier's final SR level (Proposition 2(ii)); and (b) the supplier increases his SR level in response to the firm's high investment.

For scenario (iii), Figure O.1 highlights two regions where an increase in visibility can lead to a decrease in the expected final SR level: (I) when s_0 takes intermediate values, and (II) when s_0 is low and q is high. Region (I) is discussed in §3.1. In region (II), the probability of third-party scrutiny is so high that the firm invests the highest possible amount $\hat{\beta}$ even when visibility is low. Consequently, an increase in v does not yield an increase in her high investment. It does, however, decrease her low investment β_L (Proposition 1(i)). As a result, the expected final SR level decreases with v .

Appendix O.2: The Effect of q on β_L and β_H

The key element that determines whether an increase in q has a positive or negative effect on β_L and β_H is how the rates of increase of the functions $\rho(s)$ and $\gamma(s)$ – captured by $R_\rho(s)$ and $R_\gamma(s)$ – change with s . If $\rho(s)$ increases extremely fast in s for high values of s and/or $\gamma(s)$ decreases extremely fast as s decreases for low values of s (e.g., if $R'_\rho(s) \ll 0$ and/or $R'_\gamma(s) \gg 0$), then the firm's optimal investment in the supplier's capabilities is likely to *decrease* with q . In this case, as q increases, the supplier already has a strong incentive to improve SR. Thus, attempting to further motivate the supplier to increase s is too costly. Conversely, if the rates of increase of $\rho(s)$ and $\gamma(s)$ are similar across s (e.g., if $R'_\rho(s) = R'_\gamma(s) = 0$; or equivalently, if $\rho(s)$ and $\gamma(s)$ are exponential in s), then β is more likely to *increase* with q . In this case, the firm complements an increase in q with a greater β , as the corresponding increase in s is worth the additional cost. For example, when s_0 is distributed uniformly between $m = 0$ and $M = 1$, we identify the following cases:

- **Example 1.** β_L and β_H are *increasing* in q if $\rho(s) = e^{1.75s}$ and $\gamma(s) = 0.5 - e^{-s}$ (additional functions and parameters: $\delta(\beta) = \frac{0.5\beta^2}{1-\beta}$, $\alpha = 0.4$, $q = \{0.45, 0.55\}$, $r = 0.5$, $w = 0.15$, $c = 0$, market size equals 100).
- **Example 2.** β_L and β_H are *decreasing* in q if $\rho(s) = 3000e^{0.01e^{0.01s}}$ and $\gamma(s) = 0.01 - e^{-1.25e^{1.5s}}$ (additional functions and parameters: $\delta(\beta) = \frac{0.1\beta^2}{1-\beta}$, $\alpha = 0.3$, $q = \{0.25, 0.35\}$, $r = 0.5$, $w = 0.15$, $c = 0$, market size equals 100).

Appendix O.3: Proofs

Note that the No Firm Disclosure setting is a special case of the Firm Disclosure setting, where (i) the value of s_D is fixed and equal to \hat{s} ; and (ii) there is no penalty associated with the disclosure of SR practices; i.e., $p = 0$. Therefore, we present proofs only for the Firm Disclosure setting and indicate how they can be used to prove the results in the No Firm Disclosure setting. We also note that Assumptions 2(i) and 2(ii) are equivalent to the following:

ASSUMPTION 2. *The functions $\gamma(s)$ and $\rho(s)$ satisfy $\rho'(m) < \alpha q(w - c)\gamma'(m)$.*

ASSUMPTION 3. *$\delta'(\beta_M)(1 - \beta_M) > (\alpha q(r - w)\gamma'(M) + pq)(R_\rho(M) - R_\gamma(M))^{-1}$, where β_M satisfies $s^*(m, \beta_M) = M$ assuming that the third party revealed s to the consumers regardless of how it compares to s_D .*

To simplify notation, define $R(s) \equiv (R_\rho(s) - R_\gamma(s))^{-1}$. Note that for any s , $R(s) > 0$ because $\rho(s)$ is strictly increasing and convex in s , and $\gamma(s)$ is strictly increasing and concave in s . Throughout the proofs, we utilize \check{s} defined in Lemma O.1 and the fact that $s^*(s_0, \beta_D, s_D) = \max\{s_0, \min\{s_D, \check{s}\}\}$ shown in Appendix O.3.1. Due to space constraints, we only include here the proofs of the main theorems in the paper. The remaining proofs are available from the authors.

O.3.1. The Supplier's Best Response

In this section, we demonstrate our results regarding the supplier's best response to the firm's choice of s_D and β_D . In Lemma O.1, we define a modified objective function for the supplier where: (i) the supplier can select any value of s regardless of s_0 ; and (ii) with probability q , the third party communicates any value of s that it observes. We then use this result to prove Theorem A.1 (and Theorem 1 in the No Firm Disclosure setting).

LEMMA O.1. *For any s_D , define $F(s) \equiv (1 - r + \alpha(1 - q)\gamma(s_D) + \alpha q\gamma(s))(w - c) - (1 - \beta_D)(\rho(s) - \rho(s_0))$. $F(s)$ is strictly concave in s and has a unique maximum at $\check{s} \equiv \arg \max_s \{F(s)\}$. Furthermore, $\frac{\partial \check{s}}{\partial \beta_D} > 0$, $\frac{\partial \check{s}}{\partial q} > 0$, and \check{s} is independent of v , p , and \tilde{s} .*

Proof of Theorem A.1. First, since the third party only communicates $\min\{s, s_D\}$, the supplier has no incentive to select $s > s_D$. Hence, the supplier's profit function is equal to $F(s)$ for any $s \leq s_D$, and equal to $(1 - r + \alpha\gamma(s_D))(w - c) - (1 - \beta_D)(\rho(s) - \rho(s_0))$ for any $s > s_D$. Therefore, it is strictly decreasing in s if $s > s_D$. As a result, by the concavity of $F(s)$, the unconstrained optimal solution to Equation (8) is equal to $s_{uD}^*(\beta_D, s_D) = \min\{s_D, \check{s}\} \leq s_D$. Similarly, by the concavity of $F(s)$ and due to the constraint $s \geq s_0$, if $s_{uD}^*(\beta_D, s_D) < s_0$, then the supplier's optimal decision is $s^* = s_0$. Thus, the supplier's optimal solution in the original constrained problem is $s^*(s_0, \beta_D, s_D) = \max\{s_0, s_{uD}^*(\beta_D, s_D)\} = \max\{s_0, \min\{s_D, \check{s}\}\}$. Furthermore, from Lemma O.1, it follows that $\frac{\partial s^*(s_0, \beta_D, s_D)}{\partial \beta_D} \geq 0$ and $\frac{\partial s^*(s_0, \beta_D, s_D)}{\partial q} \geq 0$.

REMARK: To prove Theorem 1 in the No Firm Disclosure setting, we replace s_D by \hat{s} and β_D by β in this proof. \square

O.3.2. The Firm's Optimal Strategy

To prove Theorems 2, 3, and A.2, we first summarize a set of useful lemmas. We use Lemmas O.3–O.5 to rewrite the firm's objective function and simplify our analysis.

LEMMA O.2. $\frac{d\tilde{s}}{d\beta_D} = \frac{R(\tilde{s})}{1 - \beta_D}$.

LEMMA O.3. *Define*

$$\begin{aligned} G_L(\beta_D, s_D, \tilde{s}) \equiv & \left[1 - r + \alpha(1 - q)\gamma(s_D) + \alpha q \left(v\gamma(\min\{\tilde{s}, s_D\}) \right. \right. \\ & \left. \left. + (1 - v) \left(\gamma(\tilde{s})\Phi(\tilde{s}) + \int_{\tilde{s}}^{s_D} \gamma(x)\phi(x) dx + \gamma(s_D)(1 - \Phi(s_D)) \right) \right] (r - w) - \delta(\beta_D) \\ & - pq \left(v(s_D - \min\{\tilde{s}, s_D\}) + (1 - v) \left((s_D - \tilde{s})\Phi(\tilde{s}) + \int_{\tilde{s}}^{s_D} (s_D - x)\phi(x) dx \right) \right). \end{aligned} \quad (\text{O.1})$$

For any s_D and \tilde{s} , there exists a β_{LD} that maximizes $G_L(\beta_D, s_D, \tilde{s})$ and satisfies the first order condition

$$\frac{\partial G_L(\beta_D, s_D, \tilde{s})}{\partial \beta_D} = (\alpha q(r - w)\gamma'(\tilde{s}) + pq) \frac{R(\tilde{s})}{1 - \beta_D} (1 - v)\Phi(\tilde{s}) - \delta'(\beta_D) = 0. \quad (\text{O.2})$$

Furthermore, $\frac{\partial G_L(\beta_D, s_D, \tilde{s})}{\partial \beta_D}$ and β_{LD} are independent of s_D and \tilde{s} .

LEMMA O.4. *Define*

$$\begin{aligned} G_H(\beta_D, s_D) \equiv & \left[1 - r + \alpha(1 - q)\gamma(s_D) + \alpha q \left(v\gamma(\tilde{s}) \right. \right. \\ & \left. \left. + (1 - v) \left(\gamma(\tilde{s})\Phi(\tilde{s}) + \int_{\tilde{s}}^{s_D} \gamma(x)\phi(x) dx + \gamma(s_D)(1 - \Phi(s_D)) \right) \right] (r - w) - \delta(\beta_D) \\ & - pq \left(v(s_D - \tilde{s}) + (1 - v) \left((s_D - \tilde{s})\Phi(\tilde{s}) + \int_{\tilde{s}}^{s_D} (s_D - x)\phi(x) dx \right) \right), \end{aligned} \quad (\text{O.3})$$

which is independent of \tilde{s} . For any s_D , there exists a β_{HD} that maximizes $G_H(\beta_D, s_D)$ and satisfies the F.O.C.

$$\frac{\partial G_H(\beta_D, s_D)}{\partial \beta_D} = (\alpha q(r - w)\gamma'(\tilde{s}) + pq) \frac{R(\tilde{s})}{1 - \beta_D} (v + (1 - v)\Phi(\tilde{s})) - \delta'(\beta_D) = 0. \quad (\text{O.4})$$

Furthermore, $\frac{\partial G_H(\beta_D, s_D)}{\partial \beta_D}$ and β_{HD} are independent of s_D .

LEMMA O.5. *Given a signal \tilde{s} , the firm's expected profit function as a function of β_D and s_D can be written as*

$$\begin{aligned} \mathbb{E}_{s_0} [\Pi_F(\beta_D, s_D) | \tilde{s}] = & \left(G_L(\beta_D, s_D, \tilde{s}) \mathbb{1}_{\tilde{s} \leq \tilde{s}} + G_H(\beta_D, s_D) \mathbb{1}_{\tilde{s} > \tilde{s}} \right) \mathbb{1}_{\tilde{s} \leq s_D} \\ & + \left((1 - r + \alpha\gamma(s_D))(r - w) - \delta(\beta_D) \right) \mathbb{1}_{\tilde{s} > s_D}. \end{aligned} \quad (\text{O.5})$$

As we show below, the first order condition of the firm's problem with respect to β_D (s_D) is independent of s_D (β_D), except when $\tilde{s}(\beta_D) = s_D$. In our analysis, we always compare any local maximum with the local maximum that satisfies $\tilde{s}(\beta_D) = s_D$ (which is unique by Lemma O.6). Therefore, we can swap the order of maximization for the firm's problem in Equation (9). To simplify our analysis, Theorem O.1 first characterizes the optimal choice of β_D given s_D .

THEOREM O.1. *Define $\hat{\beta}_{s_D}$ such that $\tilde{s}(\hat{\beta}_{s_D}) = s_D$. Given v and s_D , there exist β_{LD} and β_{HD} such that $0 \leq \beta_{LD} \leq \beta_{HD} < 1$ and the firm's optimal choice of β_D , $\beta_D^*(v, \tilde{s}, s_D)$, can be characterized as follows:*

- (a) *If $\beta_{LD} \geq \hat{\beta}_{s_D}$, then $\beta_D^*(v, \tilde{s}, s_D) = \hat{\beta}_{s_D}$ for all \tilde{s} .*
- (b) *If $\beta_{LD} < \hat{\beta}_{s_D}$, then there exists a threshold $\tau_{s_D} \in (m, \hat{s})$ such that:*
 - (i) *If $\tilde{s} \leq \tau_{s_D}$, then $\beta_D^*(v, \tilde{s}, s_D) = \min\{\beta_{HD}, \hat{\beta}_{s_D}\}$;*
 - (ii) *If $\tilde{s} > \tau_{s_D}$, then $\beta_D^*(v, \tilde{s}, s_D) = \beta_{LD}$.*

Here, $\beta_{LD} = \arg \max_{\beta_D} G_L(\beta_D, s_D, \tilde{s})$ in Lemma O.3 and $\beta_{HD} = \arg \max_{\beta_D} G_H(\beta_D, s_D)$ in Lemma O.4. Furthermore, $\beta_{LD} = \beta_{HD}$ if and only if $v = 0$; $\beta_{LD} = 0$ if and only if $v = 1$.

Proof. We first show that $\beta_{LD} = 0$ if and only if $v = 1$. Note that β_{LD} satisfies the first order condition characterized by Equation (O.2) in Lemma O.3. If $v = 1$, this condition simplifies to $\delta'(\beta_{LD}) = 0$. By definition of the function $\delta(\cdot)$, β_{LD} must be equal to 0. Conversely, for $\beta_{LD} = 0$ to satisfy Equation (O.2), we must have $(\alpha q(r-w)\gamma'(\tilde{s}) + pq)R(\tilde{s})(1-v)\Phi(\tilde{s}) = 0$. Therefore, v must be equal to 1.

Next, we show $\beta_{LD} \leq \beta_{HD}$. To simplify notation, we write $\frac{\partial G_H(\beta_D, s_D)}{\partial \beta_D}$ as $G'_H(\beta_D)$ and $\frac{\partial G_L(\beta_D, s_D, \tilde{s})}{\partial \beta_D}$ as $G'_L(\beta_D)$. Based on Equations (O.2) and (O.4), we have for any β_D :

$$\begin{aligned} G'_H(\beta_D) &= (\alpha q(r-w)\gamma'(\tilde{s}) + pq) \frac{R(\tilde{s})}{1-\beta_D} (v + (1-v)\Phi(\tilde{s})) - \delta'(\beta_D) \\ &\geq (\alpha q(r-w)\gamma'(\tilde{s}) + pq) \frac{R(\tilde{s})}{1-\beta_D} (1-v)\Phi(\tilde{s}) - \delta'(\beta_D) \\ &= G'_L(\beta_D). \end{aligned} \tag{O.6}$$

In particular, Equation (O.6) holds for $\beta_D = \beta_{LD}$; i.e., $G'_H(\beta_{LD}) \geq G'_L(\beta_{LD}) = 0$. Since $G_H(\beta_D, s_D)$ is unimodal in β_D and $G'_H(\beta_{HD}) = 0$, it must be that $\beta_{HD} \geq \beta_{LD}$. From this analysis, it also follows that $\beta_{HD} = \beta_{LD}$ if and only if $v = 0$, as it is the only case in which $G'_H(\beta_D) = G'_L(\beta_D)$. Similarly, since \tilde{s} is strictly increasing in β_D , we also have $\tilde{s}(\beta_{HD}) \geq \tilde{s}(\beta_{LD})$, with the inequality being strict if $v > 0$.

Define $s_L = \tilde{s}(\beta_{LD})$, $s_H = \tilde{s}(\beta_{HD})$, and $\tilde{\beta}$ as the value of β_D such that $\tilde{s}(\beta_D = \tilde{\beta}) = \tilde{s}$. Also simplify $\mathbb{E}_{s_0}[\Pi_F(\beta_D, s_D) | \tilde{s}]$, $G_L(\beta_D, s_D, \tilde{s})$, and $G_H(\beta_D, s_D)$ to $\Pi_F(\beta_D)$, $G_L(\beta_D)$, and $G_H(\beta_D)$. We note from Lemma O.5 and Equations (O.1) and (O.3) that $\Pi_F(\beta_D)$ is continuous in β_D . However, its derivative has two discontinuity points at $\beta_D = \tilde{\beta}$ and $\beta_D = \hat{\beta}_{s_D}$. Also note that for any $\tilde{s} > s_D$, equivalently $\beta_D > \hat{\beta}_{s_D}$, $\Pi_F(\beta_D)$ is strictly decreasing in β_D . Since β_{LD} (β_{HD}) is the unique local maximum for $G_L(\beta_D)$ ($G_H(\beta_D)$), the firm's optimal investment given s_D and \tilde{s} can only take one of four possible values: (i) $\beta_D^*(v, \tilde{s}) = \beta_{LD}$; (ii) $\beta_D^*(v, \tilde{s}) = \tilde{\beta}$; (iii) $\beta_D^*(v, \tilde{s}) = \beta_{HD}$; or (iv) $\beta_D^*(v, \tilde{s}) = \hat{\beta}_{s_D}$.

We first rule out $\tilde{\beta}$ as the optimal solution. Note from Lemma O.5 that \tilde{s} and hence $\tilde{\beta}$ is relevant only if $\tilde{s} \leq s_D$. In this case, $\Pi_F(\beta_D) = G_L(\beta_D)$ if $\tilde{s} \leq \tilde{s}$ and $\Pi_F(\beta_D) = G_H(\beta_D)$ if $\tilde{s} > \tilde{s}$. Therefore, $\lim_{\beta_D \downarrow \tilde{\beta}} \Pi'_F(\beta_D) = \lim_{\beta_D \downarrow \tilde{\beta}} G'_H(\beta_D)$ and $\lim_{\beta_D \uparrow \tilde{\beta}} \Pi'_F(\beta_D) = \lim_{\beta_D \uparrow \tilde{\beta}} G'_L(\beta_D)$. From Equation (O.6) we know that $G'_H(\tilde{\beta}) \geq G'_L(\tilde{\beta})$. Thus, $\lim_{\beta_D \downarrow \tilde{\beta}} \Pi'_F(\beta_D) \geq \lim_{\beta_D \uparrow \tilde{\beta}} \Pi'_F(\beta_D)$ and $\beta_D = \tilde{\beta}$ cannot be a local maximum.

Regarding the other possible optimal solutions, we note from Equation (O.5) and the unimodality of $G_L(\beta_D)$ and $G_H(\beta_D)$ that: (i) β_{LD} is a local maximum if and only if $s_L \leq \tilde{s}$ and $s_L \leq s_D$; (ii) β_{HD} is a local maximum if and only if $\tilde{s} < s_H \leq s_D$; and (iii) $\hat{\beta}_{s_D}$ is a local maximum if and only if $\tilde{s} < s_D < s_H$, or $\tilde{s} \geq s_D$ and $s_L > s_D$. Case (iii) can be seen from the following. If $\tilde{s} < s_D < s_H$, then $\beta_{HD} > \hat{\beta}_{s_D}$ and $G_H(\beta_D)$ is strictly increasing in β_D for $\beta_D \leq \hat{\beta}_{s_D}$. Therefore, since in this case $\Pi_F(\hat{\beta}_{s_D}) = G_H(\hat{\beta}_{s_D})$ and $\Pi_F(\beta_D)$ is strictly decreasing in β_D for $\beta_D > \hat{\beta}_{s_D}$, it follows that $\hat{\beta}_{s_D}$ is a local maximum. Similarly, if $\tilde{s} \geq s_D$ and $s_L > s_D$, then $\beta_{LD} > \hat{\beta}_{s_D}$ and $G_L(\beta_D)$ is strictly increasing in β_D for $\beta_D \leq \hat{\beta}_{s_D}$. Therefore, since in this case $\Pi_F(\hat{\beta}_{s_D}) = G_L(\hat{\beta}_{s_D})$, $\hat{\beta}_{s_D}$ is a local maximum. Given the above, we have the following cases:

- (i) If $s_L > s_D$, then $s_H > s_D$ because $s_L \leq s_H$. The only local (and hence global) maximum is $\beta_D^*(v, \tilde{s}) = \hat{\beta}_{s_D}$ for all \tilde{s} .
- (ii) If $\tilde{s} < s_L \leq s_D$, then the only local maximum when $s_H \leq s_D$ is β_{HD} ; and when $s_H > s_D$ it is $\hat{\beta}_{s_D}$. Therefore, since \tilde{s} is increasing in β_D , $\beta_D^*(v, \tilde{s}) = \min\{\beta_{HD}, \hat{\beta}_{s_D}\}$.
- (iii) If $s_L \leq s_D$ and $\tilde{s} \geq \min\{s_H, s_D\}$, then the only local (and hence global) maximum is $\beta_D^*(v, \tilde{s}) = \beta_{LD}$ for all \tilde{s} .
- (iv) If $s_L \leq s_D$ and $s_L \leq \tilde{s} < \min\{s_H, s_D\}$, then both β_{LD} and $\min\{\beta_{HD}, \hat{\beta}_{s_D}\}$ are local maxima.

It only remains to find the global maximum for case (iv). Define $\bar{\beta}_{HD} \equiv \min\{\beta_{HD}, \hat{\beta}_{s_D}\}$. First note from Lemma O.5 that in this case, $\Pi_F(\beta_{LD}) = G_L(\beta_{LD})$ and $\Pi_F(\bar{\beta}_{HD}) = G_H(\bar{\beta}_{HD})$. In addition, if $s_L \leq s_D$, then $\partial G_L(\beta_{LD}, s_D, \tilde{s}) / \partial \tilde{s} = \alpha q v \gamma'(\tilde{s})(r-w) + pqv > 0$; i.e., $G_L(\beta_{LD}, s_D, \tilde{s})$ is strictly increasing in \tilde{s} . Conversely, by Lemma O.4, $G_H(\bar{\beta}_{HD})$ is constant in \tilde{s} . Thus, for $\tilde{s} \in [s_L, \min\{s_H, s_D\})$, $\Pi_F(\beta_{LD})$ is continuous and strictly increasing in \tilde{s} and $\Pi_F(\bar{\beta}_{HD})$ is constant in \tilde{s} . In addition, by case (ii) above, as $\tilde{s} \uparrow s_L$, the firm strictly prefers $\bar{\beta}_{HD}$ over β_{LD} ; and by case (iii) above, if $\tilde{s} = \min\{s_H, s_D\}$, then the firm strictly prefers β_{LD} over β_{HD} . Hence, there must exist a unique threshold $\tau_{s_D} \in [s_L, \min\{s_H, s_D\})$ such

that (a) if $\tilde{s} < \tau_{s_D}$, then $\Pi_F(\beta_{HD}) > \Pi_F(\beta_{LD})$; (b) if $\tilde{s} = \tau_{s_D}$, then $\Pi_F(\beta_{HD}) = \Pi_F(\beta_{LD})$; and (c) if $\tilde{s} > \tau_{s_D}$, then $\Pi_F(\beta_{HD}) < \Pi_F(\beta_{LD})$.

Finally, note that $\beta_{LD} \geq (<)\hat{\beta}_{s_D}$ is equivalent to $s_L \geq (<)s_D$. Summarizing the above and assuming that the firm chooses $\beta_D = \bar{\beta}_{HD}$ when she is indifferent between β_{LD} and $\bar{\beta}_{HD}$, we have the following scenarios as in the theorem: (a) If $\beta_{LD} \geq \hat{\beta}_{s_D}$, then $\beta_D^*(v, \tilde{s}) = \hat{\beta}_{s_D}$ for all \tilde{s} ; (b) If $\beta_{LD} < \hat{\beta}_{s_D}$, then (i) if $\tilde{s} \leq \tau_{s_D}$, then $\beta_D^*(v, \tilde{s}) = \beta_{LD}$, and (ii) if $\tilde{s} > \tau_{s_D}$, then $\beta_D^*(v, \tilde{s}) = \bar{\beta}_{HD}$.

REMARK: To prove Theorem 2 in the No Firm Disclosure setting, we simply replace s_D by \hat{s} , β_D by β , $\hat{\beta}_{s_D}$ by $\hat{\beta}$, and let $p = 0$ in this proof. \square

The next four results are intermediate steps to eventually prove Theorems 3 and A.2.

THEOREM O.2. *Define $G(\beta_D, s_D, \tilde{s}) \equiv G_L(\beta_D, s_D, \tilde{s}) \mathbb{1}_{\tilde{s} \leq \tilde{s}} + G_H(\beta_D, s_D) \mathbb{1}_{\tilde{s} > \tilde{s}}$, which is equal to the firm's expected profit when $\tilde{s} \leq s_D$. The function $G(\beta_D, s_D, \tilde{s})$ is unimodal in s_D . There exist $\hat{\tau}_L$ and $\hat{\tau}_H$ such that $m < \hat{\tau}_L < \hat{\tau}_H \leq M$ and $z(\tilde{s}) \equiv \arg \max_{s_D} G(\beta_D, s_D, \tilde{s})$ can be characterized as follows:*

- (i) *If $\tilde{s} < \hat{\tau}_L$, then $z(\tilde{s}) = \hat{\tau}_L$;*
- (ii) *If $\hat{\tau}_L \leq \tilde{s} < \hat{\tau}_H$, then $z(\tilde{s}) = \tilde{s}$;*
- (iii) *If $\tilde{s} \geq \hat{\tau}_H$, then $z(\tilde{s}) = \hat{\tau}_H$.*

$z(\tilde{s})$ is independent of β_D , and $\hat{\tau}_H$ and $\hat{\tau}_L$ are the unique solutions to

$$V_1(s_D = \hat{\tau}_H) \equiv \alpha(r - w) \gamma'(\hat{\tau}_H) (1 - q(1 - v) \Phi(\hat{\tau}_H)) - pq(1 - v) \Phi(\hat{\tau}_H) = 0 \quad (\text{O.7})$$

$$V_2(s_D = \hat{\tau}_L) \equiv \alpha(r - w) \gamma'(\hat{\tau}_L) (1 - q \Phi(\hat{\tau}_L)) - pq(v + (1 - v) \Phi(\hat{\tau}_L)) = 0. \quad (\text{O.8})$$

LEMMA O.6. *There exists at most one pair of values (β_D, s_D) that satisfy $s_D = \check{s}(\beta_D)$ and are a local maximum of the firm's expected profit. That is, they satisfy the first order condition*

$$\alpha \gamma'(\check{s}) (r - w) \frac{R(\check{s})}{1 - \beta_D} - \delta'(\beta_D) = 0. \quad (\text{O.9})$$

Furthermore, both of these values are independent of v , \tilde{s} , and p . We define $(\hat{\beta}_D, \hat{s}_D)$ as this pair of values if they exist; otherwise, let $(\hat{\beta}_D, \hat{s}_D) = (\beta_M, M)$.

LEMMA O.7. *There are at most three local maxima to the firm's optimization problem:*

- (i) *If $\check{s}(\beta_{LD}) \leq \tilde{s}$ and $\check{s}(\beta_{LD}) < z(\tilde{s})$, then $(\beta_D, s_D) = (\beta_{LD}, \min\{z(\tilde{s}), M\})$ is a local maximum;*
- (ii) *If $\check{s} < \check{s}(\beta_{HD}) < z(\tilde{s})$, then $(\beta_D, s_D) = (\beta_{HD}, \min\{z(\tilde{s}), M\})$ is a local maximum;*
- (iii) *If (a) $z(\tilde{s}) \leq \check{s}(\beta_{LD}) \leq \tilde{s}$, or (b) $\check{s}(\beta_{HD}) > \tilde{s}$ and $\check{s}(\beta_{HD}) \geq z(\tilde{s})$, then $(\beta_D, s_D) = (\hat{\beta}_D, \hat{s}_D)$ is a local maximum.*

Here β_{LD} , β_{HD} , $z(\tilde{s})$, and $(\hat{\beta}_D, \hat{s}_D)$ are defined in Lemmas O.3, O.4, Theorem O.2, and Lemma O.6.

LEMMA O.8. *Given \tilde{s} , any local maximum (β_D, s_D) to the firm's optimization problem must satisfy $\beta_D \leq \hat{\beta}_D$, where $\hat{\beta}_D$ is defined in Lemma O.6. Furthermore, if a local maximum satisfies $\beta_D = \hat{\beta}_D$, then it must also satisfy $s_D = \hat{s}_D$.*

COROLLARY O.1. *If $\tilde{s} = \tau_D$, then the firm is indifferent between the strategies $(\bar{\beta}_{HD}, s_{D_H})$ and (β_{LD}, s_{D_L}) , where s_{D_H} (s_{D_L}) is equal to the optimal value of s_D given a signal immediately to the left (immediately to the right) of $\tilde{s} = \tau_D$. Furthermore, τ_D satisfies $\check{s}(\beta_{LD}) < \tau_D < \check{s}(\bar{\beta}_{HD})$.*

Proof of Theorems 3 and A.2. First note that disclosing a value of s_D that is outside the range $[m, M]$ is not credible. Thus, we solve for the optimal s_D that satisfies $s_D \in [m, M]$. Define the threshold $\tau_H \equiv \min\{\hat{\tau}_H, M\}$, where $\hat{\tau}_H$ is defined in Theorem O.2. First, from Lemmas O.7 and O.8, we know that if $\beta_{LD} \geq \hat{\beta}_D$, then $\beta_{HD} > \hat{\beta}_D$ and thus, $\beta_D^* = \hat{\beta}_D$ for any signal. This in turn means that $s_D^* = \hat{s}_D = \check{s}(\hat{\beta}_D)$ for any signal. From Assumption 2(i) and the definition of $\check{s}(\beta_D)$ in Lemma O.1, it follows that $s^*(s_0 = m, \beta_D = \hat{\beta}_D) = \check{s}(\hat{\beta}_D)$, proving case (a) in Theorems 3 and A.2.

Second, by Theorem O.2, if $\beta_{LD} < \hat{\beta}_D$ and $\hat{\tau}_L \geq M$, then for any signal \tilde{s} , we know that $z(\tilde{s}) \geq M$. From Assumption 2(ii), we know that any β_D that is optimal for the firm satisfies $\check{s}(\beta_D) < M$ and hence $\check{s}(\beta_{LD}) \leq \check{s}(\beta_{HD}) < M$. Thus, by Lemma O.7 and since $\hat{\tau}_H \geq \hat{\tau}_L$, the only local maximum for the firm's disclosure decision is $s_D = \min\{z(\tilde{s}), M\} = M$, regardless of the signal \tilde{s} . In addition, by Theorem O.1, for any s_D that satisfies $s_D \geq \check{s}(\beta_{HD})$ (or equivalently, $\hat{\beta}_{s_D} \geq \beta_{HD}$), we know that the structure of the optimal β_D is that $\beta_D^* = \beta_{HD}$ for $\tilde{s} \leq \tau_D$, and $\beta_D^* = \beta_{LD}$ thereafter. By the definition of $\hat{\tau}_L$ in Equation (O.8) and that the function $G(\beta_D, s_D, \tilde{s})$ in Theorem O.2 is unimodal in s_D , it follows that $\hat{\tau}_L \geq M$ if and only if $\alpha(1-q)(r-w)\gamma'(M) \geq pq$, proving case (b) in Theorem 3.

Third, if $\beta_{LD} < \hat{\beta}_D$ and $\hat{\tau}_L < M$ (or equivalently, if $\alpha(1-q)(r-w)\gamma'(M) < pq$), by Theorem O.2 we identify two additional cases. First, if $\check{s}(\beta_{HD}) < \hat{\tau}_L$, then we also have $\check{s}(\beta_{LD}) < \hat{\tau}_L$. In addition, since $z(\tilde{s})$ is non-decreasing in \tilde{s} (Theorem O.2), it follows that $\check{s}(\beta_{LD}) \leq \check{s}(\beta_{HD}) < z(\tilde{s})$ for all \tilde{s} . Thus, by Lemma O.7, the optimal disclosure decision is $s_D^* = z(\tilde{s})$. In particular, s_D^* is constant and equal to $\hat{\tau}_L$ for $\tilde{s} \leq \hat{\tau}_L$. Again by Theorem O.1, for a given s_D such that $s_D \geq \check{s}(\beta_{HD})$, the firm's optimal choice of β_D is $\beta_D^* = \beta_{HD}$ for $\tilde{s} \leq \tau_D$, and $\beta_D^* = \beta_{LD}$ for $\tilde{s} > \tau_D$. At $\tilde{s} = \tau_D$, the firm is indifferent between the strategies $(\beta_{HD}, z(\tau_D))$ and $(\beta_{LD}, z(\tau_D))$. We break the tie by assuming that she selects the former. We also note that at $\tilde{s} = \hat{\tau}_L$, we cannot simultaneously have $\check{s}(\beta_{HD}) < \hat{\tau}_L = \tilde{s}$ and $\check{s}(\beta_{HD}) > \tilde{s}$. Thus, by Lemma O.7, $(\beta_{HD}, z(\tilde{s}) = \hat{\tau}_L)$ cannot be a local maximum and hence at $\tilde{s} = \hat{\tau}_L$, $(\beta_D^*, s_D^*) = (\beta_{LD}, \hat{\tau}_L)$. We can further define the threshold τ_L as $\tau_L \equiv \hat{\tau}_L$ such that $\tau_D < \tau_L$.

Next, we study the case where $\beta_{LD} < \hat{\beta}_D$, $\hat{\tau}_L < M$, and $\check{s}(\beta_{HD}) \geq \hat{\tau}_L$. By Lemma O.7, we know that the last inequality implies that $(\beta_{HD}, z(\tilde{s}))$ is not a local maximum in this case, as $\tilde{s} < z(\tilde{s})$ only if $\tilde{s} < \tau_L$ (Theorem O.2). In addition, if $\tilde{s} < \check{s}(\beta_{LD})$, then the only local (and hence, global) maximum is $(\hat{\beta}_D, \hat{s}_D)$. Conversely, for $\tilde{s} \geq \check{s}(\beta_{LD})$, we know that if $\check{s}(\beta_{LD}) \geq z(\tilde{s})$, then $(\hat{\beta}_D, \hat{s}_D)$ is the unique maximum; however, if $\check{s}(\beta_{LD}) < z(\tilde{s})$, then we need to compare between $(\hat{\beta}_D, \hat{s}_D)$ and $(\beta_{LD}, z(\tilde{s}))$ as follows.

First, by Lemmas O.5 and O.6, we have $\mathbb{E}_{s_0}[\Pi_F(\hat{\beta}_D, \hat{s}_D)|\tilde{s}] = (1-r+\alpha\gamma(\hat{s}_D))(r-w) - \delta(\hat{\beta}_D)$, which is independent of \tilde{s} . Conversely, we can define the strategy $(\beta_D = \beta_{LD}, s_D = \tilde{s})$ such that for any signal $\tilde{s} \geq \check{s}(\beta_{LD})$, $\mathbb{E}_{s_0}[\Pi_F(\beta_D = \beta_{LD}, s_D = \tilde{s})|\tilde{s}] = G_L(\beta_D = \beta_{LD}, s_D = \tilde{s}, \tilde{s})$, which by Lemma O.3 is continuous in \tilde{s} and satisfies $\frac{dG_L(\beta_{LD}, \tilde{s}, \tilde{s})}{d\tilde{s}} = \alpha q v (r-w)\gamma'(\tilde{s}) + p q v > 0$. That is, the firm's expected profits are constant in \tilde{s} when she selects $(\hat{\beta}_D, \hat{s}_D)$, and strictly increasing in \tilde{s} when she selects instead (β_{LD}, \tilde{s}) . Furthermore, by Lemma O.5 we know that at $\tilde{s} = \hat{s}_D$ the firm's expected profit is equal to $G_L(\beta_{LD}, s_D = \hat{s}_D, \tilde{s} = \hat{s}_D)$ for any β_D that satisfies $\check{s}(\beta_D) \leq s_D$, including $\check{s}(\hat{\beta}_D) = \hat{s}_D = \tilde{s}$. Since β_{LD} is the unique maximizer of $G_L(\beta_D, s_D, \tilde{s})$ for any s_D and \tilde{s} , it follows that $G_L(\beta_D = \beta_{LD}, s_D = \hat{s}_D, \tilde{s} = \hat{s}_D) > (1-r+\alpha\gamma(\hat{s}_D))(r-w) - \delta(\hat{\beta}_D) = \mathbb{E}_{s_0}[\Pi_F(\hat{\beta}_D, \hat{s}_D)|\tilde{s}]$. As a result, at $\tilde{s} = \hat{s}_D$, the optimal solution must be $(\beta_D^*, s_D^*) = (\beta_{LD}, z(\tilde{s}))$, as we already ruled out $(\beta_{HD}, z(\tilde{s}))$ as a local maximum. Since the firm's expected profit is continuous and increasing in \tilde{s} when she selects $(\beta_{LD}, z(\tilde{s}))$, there must exist a threshold signal, $\tau_D < \hat{s}_D$, such that the firm prefers $(\hat{\beta}_D, \hat{s}_D)$ for $\tilde{s} \leq \tau_D$ and prefers $(\beta_{LD}, z(\tilde{s}))$ for $\tilde{s} > \tau_D$. At $\tilde{s} = \tau_D$, the firm is indifferent between the two alternatives. We break the tie by assuming that she selects $(\hat{\beta}_D, \hat{s}_D)$.

To summarize, if $\beta_{LD} < \hat{\beta}_D$, $\hat{\tau}_L < M$, and $\check{s}(\beta_{HD}) \geq \hat{\tau}_L$, we can define $\tau_L \equiv \max\{\tau_D, \hat{\tau}_L\}$, such that (i) if $\tau_D < \hat{\tau}_L$, then $\tau_L = \hat{\tau}_L$ and the firm's optimal strategy is $(\hat{\beta}_D, \hat{s}_D)$ for all $\tilde{s} \leq \tau_D$ and $(\beta_{LD}, z(\tilde{s}) = \tau_L)$ for all $\tilde{s} \in (\tau_D, \tau_L]$; and (ii) if $\tau_D \geq \hat{\tau}_L$, then $\tau_L = \tau_D$ and the firm's optimal strategy is $(\hat{\beta}_D, \hat{s}_D)$ for all $\tilde{s} \leq \tau_L$. This proves Theorem 3(c) Region (i).

Finally, by Theorem O.2, we note that for $\tilde{s} \geq \hat{\tau}_H$, $z(\tilde{s}) = \hat{\tau}_H$ and the firm's expected profit is equal to $G_L(\beta_D, s_D = \hat{\tau}_H, \tilde{s})$, which is constant in \tilde{s} . Therefore, τ_D must satisfy $\tau_D < \hat{\tau}_H$. Also by Theorem O.2, $\hat{\tau}_L < \hat{\tau}_H$. Since in case (c) of Theorem 3 we assume that $\tau_L < M$, we must have $\tau_L < \tau_H \equiv \min\{\hat{\tau}_H, M\}$. In addition, by Equation (O.7) and the fact that $G(\beta_D, s_D, \tilde{s})$ is unimodal in s_D (Theorem O.2), it follows that $\hat{\tau}_H < M$ (and hence $\tau_H < M$) if and only if $\alpha(r-w)\gamma'(M)(1-q(1-v)) < pq(1-v)$.

Summarizing the above, we have shown that if $\beta_{LD} < \hat{\beta}_D$, then (i) the optimal investment is $\beta^* = \bar{\beta}_{HD} = \min\{\beta_{HD}, \hat{\beta}_D\}$ for $\tilde{s} \leq \tau_D$ and $\beta^* = \beta_{LD}$ for $\tilde{s} > \tau_D$; (ii) if further $\alpha(1-q)(r-w)\gamma'(M) < pq$, then the optimal disclosure is piece-wise

constant and strictly less than M for $\tilde{s} \leq \tau_L$ (with at most one discontinuous drop at $\tilde{s} = \tau_D$), and $s_D^* = z(\tilde{s})$ for $\tilde{s} > \tau_L$. By Theorem O.2, the latter means that if $\tilde{s} > \tau_L$, then $s_D^* = \tilde{s}$ for $\tilde{s} \leq \tau_H$; and $s_D^* = \tau_H$ for $\tilde{s} \in (\tau_H, M]$.

We finally compare s_D^* to the firm's best estimate of the supplier's final SR level, $s^*(\tilde{s}, \beta_D^*)$. First, by Proposition A.4, we know that $s^*(\tilde{s}, \beta_D^*) = \tilde{s}(\bar{\beta}_{HD})$ for $\tilde{s} \leq \tau_D$ and $s^*(\tilde{s}, \beta_D^*) = \tilde{s} \leq \tau_L$ for $\tilde{s} \in (\tau_D, \tau_L]$. For $\tilde{s} > \tau_L$, since $\tau_L \geq \tau_D$, by Proposition A.4 we know that $s^*(\tilde{s}, \beta_D^*) = \tilde{s}$. From the above proof, we see that in Region (i) either (a) $s_D^* = \hat{s}_D = \tilde{s}(\bar{\beta}_{HD})$ for $\tilde{s} \leq \tau_D$ and $s_D^* = \tau_L$ for $\tilde{s} \in (\tau_D, \tau_L]$; or (b) $s_D^* = \tau_L > \tilde{s}(\bar{\beta}_{HD}) > \tilde{s}(\beta_{LD})$ for $\tilde{s} \leq \tau_L$. Thus, $s_D^* \geq s^*(\tilde{s}, \beta_D^*)$ in both cases. In Region (ii), the firm discloses $s_D^* = \tilde{s} = s^*(\tilde{s}, \beta_D^*)$. Finally, in Region (iii) (if it exists, i.e. if $\tau_H < M$), the firm discloses $s_D^* = \tau_H$ for all $\tilde{s} \in (\tau_H, M]$. Thus, in this region $s_D^* = \tau_H < \tilde{s} = s^*(\tilde{s}, \beta_D^*)$. \square

O.3.3. No Firm Disclosure versus Firm Disclosure

The following lemma is useful in the proof of Proposition A.1.

LEMMA O.9. *Define*

$$\Delta(\tilde{s}, x) \equiv \mathbb{E}_{s_0} [\Pi_F(\bar{\beta}_{HD}(v), s_{DH}) | \tilde{s}] - \mathbb{E}_{s_0} [\Pi_F(\beta_{LD}(v), s_{DL}) | \tilde{s}], \quad (\text{O.10})$$

where x represents any parameter of the firm's expected profit function and s_{DH} (s_{DL}) is equal to the optimal value of s_D given a signal immediately to the left (right) of \tilde{s} . The threshold $\tau_D(v)$ is strictly increasing in x if and only if $\frac{\partial \Delta(\tilde{s} = \tau_D(v), x)}{\partial x} > 0$.

Proof of Proposition A.1. We first show $\beta_{LD} > \beta_L$ and $\beta_{HD} > \beta_H$ for all v . By Lemmas O.3 and O.4, β_{LD} and β_{HD} are independent of s_D . Also recall that the No Firm Disclosure setting is a special case of the Firm Disclosure setting with $s_D = \hat{s}$ and $p = 0$. Thus, it suffices to show that β_{LD} and β_{HD} are strictly increasing in p .

Based on Equations (O.2) and (O.4), we have

$$\frac{\partial^2 G_L}{\partial \beta_D \partial p} = R(\tilde{s})(1-v)\Phi(\tilde{s}), \quad \frac{\partial^2 G_H}{\partial \beta_D \partial p} = R(\tilde{s})(v+(1-v)\Phi(\tilde{s})),$$

where $R(s) > 0$ for all s . By Assumption 2(i), we also have $\Phi(\tilde{s}) > 0$. Then by Topkis's theorem, β_{LD} and β_{HD} are strictly increasing in p .

Next, to show $\bar{\beta}_{HD} = \min\{\beta_{HD}, \hat{\beta}_D\} > \bar{\beta}_H = \min\{\beta_H, \hat{\beta}\}$, we also need to show $\hat{\beta}_D > \beta_H$. To show this, we note that if $\beta_D^*(v, \tilde{s}) = \hat{\beta}_D$, then by Lemma O.7 we have $s_D = \tilde{s}(\hat{\beta}_D) = \hat{s}_D$. By Lemma O.6, $\hat{\beta}_D$ satisfies the first order condition

$$\Pi'_F(\hat{\beta}_D) = \alpha \gamma'(\tilde{s}(\hat{\beta}_D))(r-w) \frac{R(\tilde{s}(\hat{\beta}_D))}{1-\hat{\beta}_D} - \delta'(\hat{\beta}_D) = 0,$$

where $\Pi_F(\hat{\beta}_D)$ is the firm's expected profit if $s_D = \tilde{s}(\hat{\beta}_D)$. Replacing $\hat{\beta}_D$ with β_H in the above expression, we have

$$\begin{aligned} \Pi'_F(\beta_H) &= \alpha \gamma'(\tilde{s}(\beta_H))(r-w) \frac{R(\tilde{s}(\beta_H))}{1-\beta_H} - \delta'(\beta_H) \\ &> \left(\alpha q \gamma'(\tilde{s}(\beta_H))(r-w) \right) (v+(1-v)\Phi(\tilde{s}(\beta_H))) \frac{R(\tilde{s}(\beta_H))}{1-\beta_H} - \delta'(\beta_H) = 0. \end{aligned}$$

The inequality follows from $q \leq 1$ and $\Phi(\tilde{s}) < 1$ (Assumption 2(ii)), and the equality follows from Equation (O.4). Since $\Pi_F(\beta_D)$ is unimodal, we have $\hat{\beta}_D > \beta_H$ and hence $\bar{\beta}_{HD} = \min\{\beta_{HD}, \hat{\beta}_D\} > \bar{\beta}_H = \min\{\beta_H, \hat{\beta}\}$.

Finally, we show $\tau_D > \tau$. Similar to the above, we need to show that τ_D is increasing in p , as the value of $\Delta(\tau_D, p)$ when $p = 0$ is exactly equal to the difference in profits between selecting $\bar{\beta}_H$ and β_L in the No Firm Disclosure setting. By Lemma O.9, it suffices to show $\partial \Delta(\tau_D, p) / \partial p > 0$, where $\Delta(\tau_D, x)$ is defined in Equation (O.10) for any model parameter x . By Equation (9), we know that

$$\frac{\partial \Delta(\tau_D, p)}{\partial p} = q \left(E_{s_0} \left[(s_{DL} - s^*(s_0, \beta_{LD}))^+ | \tau_D \right] - E_{s_0} \left[(s_{DH} - s^*(s_0, \bar{\beta}_{HD}))^+ | \tau_D \right] \right), \quad (\text{O.11})$$

where $x^+ \equiv \max\{x, 0\}$. Since $q > 0$, it suffices to show that the difference in the parentheses, i.e., the difference in expected penalty between the alternative strategies $(\bar{\beta}_{HD}, s_{DH})$ and (β_{LD}, s_{DL}) evaluated at $\tilde{s} = \tau_D$, is strictly positive.

We first note that at τ_D , both of the above strategies are local maxima (see the proof of Theorems 3 and A.2). Furthermore, we can identify two cases. First, if $\bar{\beta}_{HD} = \min\{\beta_{HD}, \hat{\beta}_D\} = \hat{\beta}_D$, then we must have $s_{DH} = \check{s}(\hat{\beta}_D)$ (see Lemma O.7). As a result, $s^*(s_0, \hat{\beta}_D) = \max\{s_0, \min\{s_{DH}, \check{s}(\hat{\beta}_D)\}\} \geq s_{DH}$ and hence $E_{s_0} \left[(s_{DH} - s^*(s_0, \bar{\beta}_{HD}))^+ | \tau_D \right] = 0$. Similarly, we know that β_{LD} must satisfy $\beta_{LD} < \hat{\beta}_D$ for τ_D to exist. Since s_{DL} is the disclosed SR level when selecting $\beta_D = \beta_{LD}$, by Lemma O.6 and the fact that any optimal solution must satisfy $\check{s} \leq s_D$, it follows that $\check{s}(\beta_{LD}) < s_{DL}$. Therefore, for any $s_0 \leq \check{s}(\beta_{LD})$, the supplier's optimal response is $s^*(s_0, \beta_{LD}) = \max\{s_0, \check{s}(\beta_{LD})\} = \check{s}(\beta_{LD}) < s_{DL}$, and hence, $E_{s_0} \left[(s_{DL} - s^*(s_0, \beta_{LD}))^+ | \tau_D \right] > 0$. Taken together, these results imply that if $\bar{\beta}_{HD} = \hat{\beta}_D$, then the difference in expected penalty is positive.

Second, if $\bar{\beta}_{HD} < \hat{\beta}_D$, then $\beta_H < \hat{\beta}_D$ and $\bar{\beta}_{HD} = \beta_{HD}$. Since at $\tilde{s} = \tau_D$ both strategies (β_{HD}, s_{DH}) and (β_{LD}, s_{DL}) are local maxima, by Lemma O.7 (cases (i) and (ii)) it follows that $s_{DH} = s_{DL} = \min\{z(\tau_D), M\}$, $\check{s}(\beta_{LD}) < z(\tau_D)$, and $\check{s}(\beta_{HD}) < z(\tau_D)$. In addition, since $\beta_{LD} < \beta_{HD}$ and \check{s} is strictly increasing in β_D , we have $\check{s}(\beta_{LD}) < \check{s}(\beta_{HD})$. Thus, $s^*(s_0, \beta_{LD}) \leq s^*(s_0, \bar{\beta}_{HD})$ for all s_0 . In particular, given the supplier's best response in Theorem A.1, the last inequality is strict for any $s_0 \in [\check{s}(\beta_{LD}), \check{s}(\beta_{HD})]$. Combining these results, it follows that $E_{s_0} \left[(s_{DL} - s^*(s_0, \beta_{LD}))^+ | \tau_D \right]$ is strictly positive and strictly greater than $E_{s_0} \left[(s_{DH} - s^*(s_0, \bar{\beta}_{HD}))^+ | \tau_D \right]$. As a result, the difference in expected penalty is also positive in this case. \square

Proof of Theorem 4. First note by Corollary O.1 that $\check{s}(\beta_L) < \tau < \check{s}(\bar{\beta}_H)$ and $\check{s}(\beta_{LD}) < \tau_D < \check{s}(\bar{\beta}_{HD})$. In addition, we know from Proposition A.1 that $\tau_D > \tau$. Since the No Firm Disclosure setting is a special case of the Firm Disclosure setting with $p=0$ and $s_D = \hat{s}$, by Lemma O.5, Theorems 2, 3, and A.2, we have the following:

- (i) If $\tilde{s} \in [m, \tau)$, then $\Pi_{FD}^*(v, \tilde{s}) = G_H(\bar{\beta}_{HD}, s_D)$ and $\Pi_F^*(v, \tilde{s}) = G_H(\bar{\beta}_H, s_D = \hat{s}, p=0)$. Since G_H is independent of \tilde{s} , it follows that $d\Delta_{\Pi}(v, \tilde{s})/d\tilde{s} = 0$.
- (ii) If $\tilde{s} \in [\tau, \min\{\tau_D, \hat{s}\})$, then $\Pi_{FD}^*(v, \tilde{s}) = G_H(\bar{\beta}_{HD}, s_D)$ and $\Pi_F^*(v, \tilde{s}) = G_L(\beta_L, s_D = \hat{s}, \tilde{s}, p=0)$. Since G_H is independent of \tilde{s} , it follows that $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} = 0 - \frac{dG_L(\beta_L, s_D = \hat{s}, \tilde{s}, p=0)}{d\tilde{s}} = -\alpha q v \gamma'(\tilde{s}) < 0$.
- (iii) If $\hat{s} < \tau_D$ and $\tilde{s} \in [\hat{s}, \tau_D)$, then $\Pi_{FD}^*(v, \tilde{s}) = G_H(\bar{\beta}_{HD}, s_D)$ and $\Pi_F^*(v, \tilde{s}) = G_L(\beta_L, s_D = \hat{s}, \tilde{s}, p=0)$. The second function is independent of \tilde{s} because in No Firm Disclosure, the third party does not disclose $s > \hat{s}$. Thus, $d\Delta_{\Pi}(v, \tilde{s})/d\tilde{s} = 0$.

Case 1: $\hat{s} \leq \tau_L$. In this case, we obtain the following results:

- (iv) If $\tilde{s} \geq \tau_D$ and $\tilde{s} \in [\tau_D, \hat{s})$, then $\Pi_{FD}^*(v, \tilde{s}) = G_L(\beta_{LD}, s_D \geq \tilde{s}, \tilde{s})$ and $\Pi_F^*(v, \tilde{s}) = G_L(\beta_L, s_D = \hat{s}, \tilde{s}, p=0)$. We know that under Firm Disclosure $s_D \geq \tilde{s}$ because by Theorem 3, $s_D \geq s^*(\tilde{s}) \geq \tilde{s}$ in Region (i). Therefore, we have $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} = \frac{dG_L(\beta_{LD}, s_D \geq \tilde{s}, \tilde{s})}{d\tilde{s}} - \frac{dG_L(\beta_L, s_D = \hat{s}, \tilde{s}, p=0)}{d\tilde{s}} = p q v > 0$.
- (v) If $\tilde{s} \in [\max\{\tau_D, \hat{s}\}, \tau_L)$, then $\Pi_{FD}^*(v, \tilde{s}) = G_L(\beta_{LD}, s_D \geq \tilde{s}, \tilde{s})$ and $\Pi_F^*(v, \tilde{s}) = G_L(\beta_L, s_D = \hat{s}, \tilde{s}, p=0)$. Thus, $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} = \frac{dG_L(\beta_{LD}, s_D \geq \tilde{s}, \tilde{s})}{d\tilde{s}} - 0 = \alpha q v \gamma'(\tilde{s}) + p q v > 0$.
- (vi) If $\tilde{s} \in [\tau_L, \tau_H)$, by Theorem 3, $s_D^* = \tilde{s}$. We can show that $\frac{dG_L(\beta_{LD}, s_D = \tilde{s}, \tilde{s})}{d\tilde{s}} = \alpha(r-w)\gamma'(\tilde{s})(1-q(1-v)\Phi(\tilde{s})) - p q (1-v)\Phi(\tilde{s}) = V_1(\tilde{s})$, with $V_1(\cdot)$ defined in Theorem O.2. Since $V_1(\cdot)$ is decreasing and $\tilde{s} < \tau_H$, $\frac{dG_L(\beta_{LD}, s_D = \tilde{s}, \tilde{s})}{d\tilde{s}} > 0$. Finally, $\Pi_F^*(v, \tilde{s})$ is constant in \tilde{s} because $\hat{s} \leq \tau_L$, and thus $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} = \frac{dG_L(\beta_{LD}, s_D = \tilde{s}, \tilde{s})}{d\tilde{s}} > 0$.
- (vii) Last, if $\tilde{s} \in [\tau_H, M]$, then both $\Pi_{FD}^*(v, \tilde{s})$ and $\Pi_F^*(v, \tilde{s})$ are independent of \tilde{s} and thus, $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} = 0$.

Case 2: $\hat{s} > \tau_L$. In this case, we can instead identify the following situations:

- (iv) If $\tilde{s} \in [\tau_D, \tau_L)$, then the same results as in Case 1(iv) above hold and $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} = p q v > 0$.
- (v) If $\tilde{s} \in [\tau_L, \hat{s})$, then by Theorem 3 in the Firm Disclosure setting, $s_D^* \leq \tilde{s} < \hat{s}$. Thus, the firm would be communicating *negative* SR information, which leads to the firm preferring the No Firm Disclosure setting (for a formal proof, please see Lemma O.12 and its proof in the Supplemental Appendix). In other words, in this case, $\Delta_{\Pi}(v, \tilde{s}) < 0$.
- (vi) If $\hat{s} < \tau_H$ and $\tilde{s} \in [\hat{s}, \tau_H)$, then the same results as in Case 1(vi) above hold and $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} > 0$.
- (vii) Finally, if $\tilde{s} \in [\max\{\tau_H, \hat{s}\}, M]$, then both $\Pi_{FD}^*(v, \tilde{s})$ and $\Pi_F^*(v, \tilde{s})$ are independent of \tilde{s} and thus, $\frac{d\Delta_{\Pi}(v, \tilde{s})}{d\tilde{s}} = 0$.

Thus, $\Delta_{\Pi}(v, \tilde{s})$ is constant in \tilde{s} for $\tilde{s} \in [m, \tau)$; and non-increasing in \tilde{s} for $\tilde{s} \in [\tau, \tau_D)$. Then, Case 1 means that if $\hat{s} \leq \tau_L$, then $\Delta_{\Pi}(v, \tilde{s})$ is non-decreasing in \tilde{s} for $\tilde{s} \in [\tau_D, M]$; and Case 2 means that, if $\hat{s} > \tau_L$, then $\Delta_{\Pi}(v, \tilde{s})$ is negative for $\tilde{s} \in [\tau_D, \hat{s})$ and non-decreasing in \tilde{s} for $\tilde{s} \in [\hat{s}, M]$. \square