Online Appendix

A. Auxiliary Results

In this section, we first establish that given commissions/subscriptions the corresponding competitive equilibrium can be obtained through the optimal primal/dual solutions of a convex optimization problem (see (1)). Then, we show that optimal solutions of both this problem and the revenue optimization problem (7) can be characterized by maximizing a concave function over a polymatroid. The solutions of such problems admit an interesting structure, which we leverage in the subsequent sections to establish our key findings.

We start by presenting our *equilibrium problem*. Given $(\boldsymbol{\gamma},\boldsymbol{\mu})\in\Gamma\times\mathcal{U}$ such that $\gamma_i^s<1$ for all $i\in\mathcal{S}$, define functions $\tilde{F}_{s_i}^{-1}:[0,1]\to\left[\frac{\mu_i^s}{1-\gamma_i^s},\frac{\bar{v}_{s_i}+\mu_i^s}{1-\gamma_i^s}\right]$ and $\tilde{F}_{b_j}^{-1}:[0,1]\to\left[\frac{-\mu_j^b}{1+\gamma_j^b},\frac{-\mu_j^b+\bar{v}_{b_j}}{1+\gamma_j^b}\right]$ such that $\tilde{F}_{s_i}^{-1}(x):=\frac{F_{s_i}^{-1}(x)+\mu_i^s}{1-\gamma_i^s}$ and $\tilde{F}_{b_j}^{-1}(x):=\frac{F_{b_j}^{-1}(x)-\mu_j^b}{1+\gamma_j^b}$. Consider the following problem:[1]

$$\max_{\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b}\quad \sum_{j\in\mathcal{B}}\int_0^{q_j^b}\tilde{F}_{b_j}^{-1}\left(1-\frac{z}{b_j}\right)dz-\sum_{i\in\mathcal{S}}\int_0^{q_i^s}\tilde{F}_{s_i}^{-1}\left(\frac{z}{s_i}\right)dz-\sum_{(i,j)\in E}\frac{c_i}{1+\gamma_j^b}x_{ij} \tag{1a}$$

$$\text{s.t.}\quad \sum_{i:(i,j)\in E}x_{ij}\;=\;q_j^b,\quad \forall j\in\mathcal{B}, \tag{1b}$$

$$\sum_{j:(i,j)\in E}x_{ij}\;=\;q_i^s,\quad \forall i\in\mathcal{S}, \tag{1c}$$

$$q_j^b\;\le\;b_j\quad \forall j\in\mathcal{B}, \tag{1d}$$

$$q_i^s\;\le\;s_i,\quad \forall i\in\mathcal{S}, \tag{1e}$$

$$x_{ij}\;\ge\;0,\quad \forall (i,j)\in E. \tag{1f}$$

In Section EC.1.1 of Birge et al. (2018), we prove that for any optimal solution $(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ to problem (1), vector $(\boldsymbol{q}^s,\boldsymbol{q}^b)$ is unique. Moreover, among all possible dual optimal solutions $(\boldsymbol{\theta}^b,\boldsymbol{\theta}^s,\boldsymbol{\eta}^b,\boldsymbol{\eta}^s,\boldsymbol{\pi})$ corresponding to constraints (1b)–(1f), we consider the one that maximizes $\sum_{j\in\mathcal{B}}[sgn(q_j^b)-sgn(b_j-q_j^b)](\theta_j^b+\eta_j^b)+\sum_{i\in\mathcal{S}}[-sgn(q_i^s)+sgn(s_i-q_i^s)](\theta_i^s-\eta_i^s)$. Lemma EC.2 in Section EC.1.1 of Birge et al. (2018) establishes that such a dual optimal vector $(\boldsymbol{\theta}^b,\boldsymbol{\theta}^s,\boldsymbol{\eta}^b,\boldsymbol{\eta}^s,\boldsymbol{\pi})$ is unique. We refer to problem (1) as the *equilibrium problem*, since its solutions correspond to competitive equilibria.

Proposition A.1. *For any $(\boldsymbol{\gamma},\boldsymbol{\mu})$, the tuple $(\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ constitutes a competitive equilibrium if and only if*

(i) *$(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ is an optimal solution to the optimization problem (1);*

(ii) *the price vector $\boldsymbol{p}$ satisfies $p_i=\theta_i^s$ for all i such that $q_i^s>0$ and $\max_{j:(i,j)\in E}\{\theta_j^b-\frac{c_i}{1+\gamma_j^b}\}\le p_i\le\theta_i^s$ for all i such that $q_i^s=0$, where $(\boldsymbol{\theta}^b,\boldsymbol{\theta}^s,\boldsymbol{\eta}^b,\boldsymbol{\eta}^s,\boldsymbol{\pi})$ is the unique special dual multiplier specified above.*

Next we introduce a class of optimization problems involving polymatroids. Let $g,h:[0,1]\to\mathbb{R}\cup\{-\infty,+\infty\}$ be functions such that $|g(r)|,|h(r)|<\infty$ for all $r\in[0,1)$. Suppose that these functions satisfy the following assumptions:

(A1) $g(r)$ is continuously differentiable, strictly concave in $r\in(0,1)$, continuous at $r=0$, and continuous at $r=1$ if $g(1)>-\infty$.

(A2) $h(r)$ is continuously differentiable, strictly convex in $r\in(0,1)$, continuous at $r=0$, and continuous at $r=1$ if $h(1)<\infty$. [2]

(A3) $th(\frac{r}{t})$ is strictly decreasing in t for $r>0$ and jointly convex in $(r,t)\in\left\{(r',t'):\ t'>0,\frac{r'}{t'}\in[0,1],\ h(\frac{r'}{t'})<\infty\right\}$.

[1] Observe that under Assumption 1, we have $\int_0^0\tilde{F}_{b_j}^{-1}(1-z/b_j)dz=\lim_{q\to0}\int_0^q\tilde{F}_{b_j}^{-1}(1-z/b_j)dz=0$ and $\int_0^{b_j}\tilde{F}_{b_j}^{-1}(1-z/b_j)dz=\lim_{q\to b_j}\int_0^q\tilde{F}_{b_j}^{-1}(1-z/b_j)dz<\infty$.

[2] Given Assumptions (A1) - (A2), we let $g'(r)$ and $h'(r)$ be the derivative of $g(r)$ and $h(r)$ evaluated at $r\in(0,1)$. With some abuse of notation, we let $g'(0):=\lim_{r\downarrow0}g'(r)$, $g'(1):=\lim_{r\uparrow1}g'(r)$, $h'(0):=\lim_{r\downarrow0}h'(r)$, and $h'(1):=\lim_{r\uparrow1}h'(r)$. Note that when $g'(0)$ and $h'(0)$ are finite, we can leverage the mean value theorem to show that they respectively correspond to the right derivative $g(r)$ and $h(r)$ at $r=0$. Similarly, when $g'(1)$ and $h'(1)$ are finite, they correspond to the left derivative of $g(r)$ and $h(r)$ at $r=1$.

(A4) $g'(0) > h'(0) \geq 0$ and $g'(1) \leq 0$.

Define a function $f:(0,\infty)\to\mathbb{R}$, and correspondence $\rho:(0,\infty)\rightrightarrows\mathbb{R}$, which (for $t>0$) are given by

$$f(t) := \max_{r\in[0,\min\{1,t\}]} g(r) - th\left(\frac{r}{t}\right), \tag{2}$$

$$\rho(t) := \arg\max_{r\in[0,\min\{1,t\}]} g(r) - th\left(\frac{r}{t}\right). \tag{3}$$

We have the following properties for $f(\cdot),\rho(\cdot)$:

Lemma A.1. *Suppose that Assumptions* (A1)–(A4) *hold. Then:*

(i) $\rho(t)$ is a singleton for $t>0$, and hence $\rho(\cdot)$ is a function. Moreover, $\rho(t)$ is strictly increasing in t. Define $t_0 = [g']^{-1}(h'(1))$ with the convention $[g']^{-1}(x) = 0$ for $x \geq \sup_{x'\in(0,1)} g'(x')$ and $[g']^{-1}(x) = 1$ for $x \leq \inf_{x'\in(0,1)} g'(x')$. We have that $\rho(t)/t = 1$ for $t\in(0,t_0]$ and that $\rho(t)/t$ is strictly decreasing in t for $t>t_0$.

(ii) $f(t)$ is continuous, strictly increasing, and strictly concave in $(0,\infty)$. Furthermore, $\lim_{t\downarrow 0} f(t) = g(0)$.

Using the first part of this lemma, when Assumptions (A1)–(A4) hold, we extend the domain of $f(\cdot)$ to include 0 and, in particular, we let $f(0) = \lim_{t\downarrow 0} f(t)$. Note that this extension ensures that $f(t)$ is still continuous, strictly increasing, and strictly concave for $t \geq 0$. In the remainder of the paper, in settings where Assumptions (A1)–(A4) hold, we always focus on $f(\cdot)$ with the extended domain, which enjoys these properties. Thus, with some abuse, when we invoke Lemma A.1(ii), we conclude that $f(t)$ has the aforementioned properties for $t\geq 0$. Similar to $f(\cdot)$, we extend the domain of $\rho(\cdot)$ and, in particular, we let $\rho(0) = \lim_{t\downarrow 0}\rho(t) = 0$. We consider the following optimization problem:

$$\max_{\boldsymbol{y}} \quad \sum_{j\in\mathcal{B}} b_j f\left(\frac{y_j}{b_j}\right) \tag{4a}$$

$$\text{s.t.} \quad \sum_{j\in B} y_j \leq \sum_{i\in N_E(B)} s_i, \quad \forall B\subset\mathcal{B}, \tag{4b}$$

$$y_j \geq 0, \quad \forall j\in\mathcal{B}. \tag{4c}$$

Observe that under Assumptions (A1)–(A4) this problem has a strictly concave objective (by Lemma A.1) and its feasible set is the polymatroid $\mathcal{P}$ given in (9). Optimal solutions of this problem admit a special structure, which we exploit in our subsequent analysis:

Lemma A.2. *The optimal solution to problem* (4) *is unique. Let $\boldsymbol{y}^\star$ be this solution. Then $\boldsymbol{y}^\star$ is the lexicographically optimal base for polymatroid $\mathcal{P} = \{\boldsymbol{y}\geq\boldsymbol{0} : \sum_{j\in B} y_j \leq \sum_{i\in N_E(B)} s_i,\ \forall B\subset\mathcal{B}\}$ with respect to weight vector $\boldsymbol{b}$. Moreover, $y_j^\star > 0$ for all $j\in\mathcal{B}$.*

Furthermore, the solution to this problem (for appropriately chosen $g(\cdot),h(\cdot)$) sheds light on the optimal solutions of the revenue optimization and equilibrium problems.

Proposition A.2. *Suppose that Assumptions 1, 2, and 3 hold and consider the associated revenue optimization problem* (7). *Let $f(\cdot)$ be given as in* (2), *where $g(r) := F_b^{-1}(1-r)r$ and $h(r) := F_s^{-1}(r)r$ for $r\in[0,1]$, and consider the associated problem* (4).

(i) Assumptions (A1)–(A4) *hold.*

(ii) Let $\boldsymbol{y}^\star$ be the optimal solution to (4), *and $\boldsymbol{r}^\star = \{r_j^\star\}_{j\in\mathcal{B}}$ be such that $r_j^\star = \rho(y_j^\star/b_j)$. Any optimal solution $(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ of the revenue optimization problem* (7) *is such that (i) $q_j^b = r_j^\star b_j$ for all $j\in\mathcal{B}$, and (ii) $q_i^s = \frac{r_j^\star b_j}{y_j^\star} s_i$ for all $i\in\mathcal{S}$ and $j\in\mathcal{B}$ such that $x_{ij}>0$.*

(iii) The optimal objective value of (4) *is the optimal revenue in* (7), *i.e., $V_{opt} = \sum_{j\in\mathcal{B}} b_j f\left(\frac{y_j^\star}{b_j}\right)$.*

Proposition A.3. *Suppose that Assumptions 1, 2, and 3 hold. Suppose further that commissions/subscriptions $(\boldsymbol{\gamma},\boldsymbol{\mu})$ are homogeneous, i.e., $\gamma_j^b = \gamma^b$, $\mu_j^b = \mu^b$, $\gamma_i^s = \gamma^s$, $\mu_i^s = \mu^s$ for all $i\in\mathcal{S}$ and $j\in\mathcal{B}$, and consider the associated equilibrium problem* (1). *Let $f(\cdot)$ be given as in* (2), *where $g(r) := \int_0^r \frac{1}{1+\gamma^b}F_b^{-1}(1-x) - \frac{\mu^b}{1+\gamma^b}dx$ and $h(r) := \int_0^r \frac{1}{1-\gamma^s}F_s^{-1}(x) + \frac{\mu^s}{1-\gamma^s}dx$ for $r\in[0,1]$, and consider the associated formulation* (4).

(i) Assumptions (A1)–(A3) *hold. If $\mu^b + \frac{1+\gamma^b}{1-\gamma^s}\mu^s < F_b^{-1}(1)$, then Assumption* (A4) *also holds.*[3]

[3] Note that $\mu^b + \frac{1+\gamma^b}{1-\gamma^s}\mu^s \geq F_b^{-1}(1)$ yields the trivial equilibrium where no one trades in the system.

(ii) *Let $\boldsymbol{y}^\star$ be the optimal solution to* (4)*, and $\boldsymbol{r}^\star = \{r_j^\star\}_{j\in\mathcal{B}}$ be such that $r_j^\star = \rho(y_j^\star/b_j)$. Any optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ of the equilibrium problem* (1) *is such that (i) $q_j^b = r_j^\star b_j$ for all $j \in \mathcal{B}$, and (ii) $q_i^s = \frac{r_j^\star b_j}{y_j^\star} s_i$ for all $i \in \mathcal{S}$ and $j \in \mathcal{B}$ such that $x_{ij} > 0$.*

(iii) *The optimal objective value of* (4) *is the same as the optimal objective value of problem* (1)*.*

B. Proofs of Results in Section 3

Proof of Theorem 2. As Assumptions 1, 2, and 3 hold, Proposition A.2 implies that the solution of the revenue optimization problem (7) can be characterized in terms of the solution of (4), where $f(\cdot)$ is as in (2), and $g(r) := F_b^{-1}(1-r)r$ and $h(r) := F_s^{-1}(r)r$ for $r \in [0,1]$. That is, by Proposition A.2, let $\boldsymbol{y}^\star > 0$ be the unique optimal solution to this problem[4] and let $\boldsymbol{r}^\star = \{r_j^\star\}_{j\in\mathcal{B}}$ be such that $r_j^\star = \rho(y_j^\star/b_j)$ (where $\rho(\cdot)$ is as in (3)). By Lemma A.2, it follows that $\boldsymbol{y}^\star$ is the lexicographically optimal base of the feasible set of (4) (which we denote by $\mathcal{P}$) with respect to weight vector $\boldsymbol{b}$.

To establish the claim, we proceed in two steps. First, we show that the lexicographically optimal base $\boldsymbol{y}^\star$ can be characterized in terms of the sets $\{\mathcal{S}_\tau, \mathcal{B}_\tau\}$ given in the theorem statement. Then, we relate this characterization to the the ranking of the marginal agents.

Step 1: Characterization of the lexicographically optimal base. The construction in the theorem statement identifies a nonempty set $\mathcal{B}_\tau$ at each step and removes the elements of this set from $\mathcal{B}^{(\tau-1)}$. Since $\mathcal{B}$ is a finite set, it follows that in $\ell \le |\mathcal{B}|$ iterations sets $\{\mathcal{B}_\tau\}_{\tau=1}^\ell$ and $\{\mathcal{S}_\tau\}_{\tau=1}^\ell$ are constructed. Let the vector $\boldsymbol{t}' = \{t_j'\}_{j\in\mathcal{B}}$ be defined as $t_j' := \big(\sum_{i\in N_{E^{(\tau-1)}}(\mathcal{B}_\tau)} s_i\big)/\big(\sum_{k\in\mathcal{B}_\tau} b_k\big)$ for $j \in \mathcal{B}_\tau$ and $\tau = 1,\dots,\ell$. We claim that the vector $\boldsymbol{y}' := \{t_j' b_j\}_{j\in\mathcal{B}}$ is the unique lexicographically optimal base of polymatroid $\mathcal{P}$ under weight vector $\boldsymbol{b}$, i.e., $\boldsymbol{y}' = \boldsymbol{y}^\star$.

In order to establish this claim, we employ Theorem 3.1 in Fujishige (1980), which establishes the connection between vector $\boldsymbol{y}'$ and the lexicographically optimal base of polymatroid $\mathcal{P}$ with respect to weight vector $\boldsymbol{b}$. (For the reader's convenience, the statement of Theorem 3.1 in Fujishige (1980) is replicated in Theorem EC.1 in Section (EC.1.1.1) of Birge et al. (2018).) To this end, first define the vector $\boldsymbol{v}^a := (ab_j)_{j\in\mathcal{B}}$ for any $a \ge 0$. Second, let the vector $\boldsymbol{u}^a$ be a vector such that: (i) $u_j^a := ab_j$ if $0 \le a \le t_j'$, and (ii) $u_j^a := t_j' b_j$ if $a \ge t_j'$. It can be readily seen that $\boldsymbol{u}^a \le \boldsymbol{u}^{a'}$ (where the inequality is entrywise) for $0 \le a \le a'$. We claim that for any $a \ge 0$, $\boldsymbol{u}^a$ is a base of the polymatroid $\mathcal{P}_a = \{\boldsymbol{z} \in \mathcal{P} | \boldsymbol{z} \le \boldsymbol{v}^a\}$ (where once again the inequality is entrywise). Note that if this claim holds, then the above observations imply that the weight vector $\boldsymbol{b}$ and vector $\boldsymbol{t}'$ satisfy Conditions (3.1)–(3.5) of Fujishige (1980). Hence, Theorem 3.1 of Fujishige (1980) applies, and it implies that $\boldsymbol{y}'$ is the unique lexicographically optimal base of polymatroid $\mathcal{P}$ under weight vector $\boldsymbol{b}$.

Fix some $a \ge 0$. We complete the proof of Step 1 by establishing that $\boldsymbol{u}^a$ is a base of $\mathcal{P}_a$. Note that for any $B \subset \mathcal{B}$ we have that

$$\begin{aligned}\sum_{j\in B} u_j^a &\overset{(1)}{\le} \sum_{j\in B} t_j' b_j \overset{(2)}{=} \sum_{\tau=1}^{\ell} \sum_{j\in B\cap\mathcal{B}_\tau} t_j' b_j \overset{(3)}{\le} \sum_{\tau=1}^{\ell} \Big(\sum_{i\in N_{E^{(\tau-1)}}(\mathcal{B}_\tau)} s_i \Big) \frac{\sum_{j\in B\cap\mathcal{B}_\tau} b_j}{\sum_{j\in\mathcal{B}_\tau} b_j} \\ &\overset{(4)}{\le} \sum_{\tau=1}^{\ell} \sum_{i\in N_{E^{(\tau-1)}}(B\cap\mathcal{B}_\tau)} s_i = \sum_{i\in N_E(B)} s_i.\end{aligned} \tag{5}$$

Here, (1) uses the definition of $\boldsymbol{u}^a$, (2) follows since $\cup_{\tau=1}^\ell \mathcal{B}_\tau = \mathcal{B}$ and the sets $\{\mathcal{B}_\tau\}$ are disjoint, and (3) uses the definition of t_j'. Finally, (4) follows since the definition of $\mathcal{B}_\tau$ implies that $(\sum_{i\in N_{E^{(\tau-1)}}(\mathcal{B}_\tau)} s_i)/(\sum_{j\in\mathcal{B}_\tau} b_j) \le (\sum_{i\in N_{E^{(\tau-1)}}(B')} s_i)/(\sum_{j\in B'} b_j)$ for any $B' \subset \mathcal{B}_\tau$. The inequality (5) together with the fact that $\boldsymbol{u}^a \le \boldsymbol{v}^a$ (which holds by the construction of $\boldsymbol{u}^a, \boldsymbol{v}^a$) implies that $\boldsymbol{u}^a \in \mathcal{P}_a$.

Let $\bar{B}_1 = \{j |\ a \le t_j'\}$ and $\bar{B}_2 = \mathcal{B} \setminus \bar{B}_1$. Note that by the definition of $\{\mathcal{B}_\tau\}_{\tau=1}^\ell$ it follows that $t_{j_1}' < t_{j_2}'$ for $j_1 \in \mathcal{B}_{\tau_1}$ and $j_2 \in \mathcal{B}_{\tau_2}$ and $\tau_1, \tau_2 \in \{1,\dots,\ell\}$ such that $\tau_1 < \tau_2$. Moreover, $t_{j_1}' = t_{j_2}'$ for $j_1, j_2 \in \mathcal{B}_\tau$ for $\tau \in \{1,\dots,\ell\}$. These observations imply that $\bar{B}_1 = \cup_{\tau=\ell'+1}^\ell \mathcal{B}_\tau$ and $\bar{B}_2 = \cup_{\tau=1}^{\ell'} \mathcal{B}_\tau$ for some $\ell' \in \{1,\dots,\ell\}$. By the definition of $\boldsymbol{u}^a$, we have that $u_j^a = v_j^a$ for $j \in \bar{B}_1$. Moreover, for $\tau \le \ell'$ we have that $\mathcal{B}_\tau \subset \bar{B}_2$, and hence

[4] The problem of maximizing (Schur) concave functions over a polymatroid is also studied in Zhang (2008). In that paper, the author provides analytical solutions to a class of polymatroid optimization problems, and establishes certain monotonicity properties of the optimal solutions.

the definition of $\boldsymbol{u}^a$ and t'_j imply that $\sum_{k\in\mathcal{B}_\tau} u^a_k = \sum_{k\in\mathcal{B}_\tau} t'_k b_k = \sum_{i\in N_{E(\tau-1)}(\mathcal{B}_\tau)} s_i = \sum_{i\in\mathcal{S}_\tau} s_i$. Hence, we conclude that

$$\sum_{k\in\mathcal{B}} u^a_k = \sum_{k\in\bar{B}_1} v^a_k + \sum_{\tau=1}^{\ell'} \sum_{k\in\mathcal{S}_\tau} s_k. \tag{6}$$

Consider an arbitrary $z\in\mathcal{P}_a$, and observe that

$$\sum_{k\in\mathcal{B}} z_k = \sum_{k\in\bar{B}_1} z_k + \sum_{\tau=1}^{\ell'} \sum_{k\in\mathcal{B}_\tau} z_k \le \sum_{k\in\bar{B}_1} v^a_k + \sum_{\tau=1}^{\ell'} \sum_{k\in N_E(\mathcal{B}_\tau)} s_k,$$

where we use $z_k \le v^a_k$ for $k\in\bar{B}_1$ and the fact that $\sum_{k\in\mathcal{B}_\tau} z_k \le \sum_{k\in N_E(\mathcal{B}_\tau)} s_k$ for any $z\in\mathcal{P}_a\subset\mathcal{P}$. Together with (6), this implies that $\boldsymbol{u}^a$ is a base of $\mathcal{P}_a$. Hence, we conclude that $\boldsymbol{y}' = \boldsymbol{y}^\star$.

Step 2: Characterization of marginal agents. By Proposition A.2, we have that $\frac{q^b_j}{b_j} = r^\star_j = \rho(y^\star_j/b_j)$ for all $j\in\mathcal{B}$, which implies that the valuation of the marginal agent of type $j\in\mathcal{B}$ is given by $v^m_{b_j} = F_b^{-1}(1-\frac{q^b_j}{b_j}) = F_b^{-1}(1-\rho(y^\star_j/b_j))$. By Lemma A.1, $\rho(\cdot)$ is a strictly increasing function, and hence the entries of the vector $\{\rho(y^\star_j/b_j)\}_{j\in\mathcal{B}}$ admit the same ranking as the entries of the vector $\{\frac{y^\star_j}{b_j}\}_{j\in\mathcal{B}}$ (note that, by Proposition A.2, the assumptions in Lemma A.1 are satisfied). Since $\boldsymbol{y}^\star = \boldsymbol{y}'$ by the definition of $\boldsymbol{y}'$, these observations imply that

$$v^m_{b_j} = v^m_{b_k} \quad \text{for } j,k\in\mathcal{B}_\tau \quad \text{with} \quad \tau\in\{1,\dots,\ell\}, \tag{7}$$

$$v^m_{b_j} > v^m_{b_k} \quad \text{for } j\in\mathcal{B}_{\tau_1}, k\in\mathcal{B}_{\tau_2}, \quad \text{with} \quad \tau_1,\tau_2\in\{1,\dots,\ell\} \quad \text{and} \quad \tau_1<\tau_2. \tag{8}$$

Thus, we have established that $j\in\mathcal{B}_\tau$ if and only if type j's marginal agent has the τth-highest value among the values of marginal agents of all buyer types.

We next establish the ranking result for the marginal agents of seller types. By Theorem 1, under any optimal commission-subscription pair $(\boldsymbol{\gamma},\boldsymbol{\mu})$, the mass of agents who trade $(\boldsymbol{q}^b,\boldsymbol{q}^s)$, and therefore the marginal agents, are the same. Furthermore, by Proposition 6(ii), and the fact that Assumption 3 holds, it follows that there exists an optimal commission-subscription pair where $\boldsymbol{\gamma}^b = \boldsymbol{\mu}^b = 0$. Consider such an optimal commission-subscription pair, and note that since $\boldsymbol{\gamma}^b = \boldsymbol{\mu}^b = 0$, under this solution the price trading buyers of type $j\in\mathcal{B}$ pay is equal to $v^m_{b_j} = F_b^{-1}\left(1-\frac{q^b_j}{b_j}\right)$.

We claim that buyers in $\mathcal{B}_\tau$ trade only with the sellers in $\mathcal{S}_\tau$, and vice versa. To see this, first focus on $\tau=1$ and observe that buyers in $\mathcal{B}_1$ can trade only with sellers in $\mathcal{S}_1 = N_{E(0)}(\mathcal{B}_1)$, as they are not adjacent to any other seller type. Since the price the trading buyers in $\mathcal{B}_\tau$ pay is given by $v^m_{b_j}$ for any $j\in\mathcal{B}_\tau$ (by (7)), it follows by (8) that type $\mathcal{B}_1$ buyers trade at the highest prices. Thus, if sellers of type $\mathcal{S}_1$ trade, then they trade with buyers whose types belong to $\mathcal{B}_1$. On the other hand, by the construction in Proposition A.2, it follows that all seller types involve in some trade at the revenue-maximizing solution. Hence, it follows that all seller types in $\mathcal{S}_1$ have nonzero trade with some buyer type in $\mathcal{B}_1$, and buyers in $\mathcal{B}_1$ trade only with sellers in $\mathcal{S}_1$ and vice versa. By induction, the same argument implies that buyers in $\mathcal{B}_\tau$ trade positive quantities with the sellers in $\mathcal{S}_\tau$. Thus, Proposition A.2 implies that for any $i\in\mathcal{S}_\tau$ we have that $v^m_{s_i} = F_s^{-1}\left(\frac{q^s_i}{s_i}\right) = F_s^{-1}\left(\frac{\rho(y^\star_j/b_j)}{y^\star_j/b_j}\right)$ for some buyer type $j\in\mathcal{B}_\tau$. Moreover, since $y^\star_j = y'_j = t'_j b_j$, it follows that $v^m_{s_i} = F_s^{-1}\left(\frac{\rho(t'_j)}{t'_j}\right)$.

By Lemma A.1, it follows that $\rho(t)/t = 1$ for $0<t\le t_0$, where t_0 is a constant that depends on the value distributions. Let $\mathcal{T} = \{\tau | t'_j \le t_0 \text{ for } j\in\mathcal{B}_\tau\}$, and observe that by construction of $\{t'_j\}$ we have that $t'_j>0$ for all $j\in\mathcal{B}$ and $\mathcal{T} = \{1,\dots,\bar\tau\}$ for some $\bar\tau\in\mathbb{Z}$. Consider some $\tau\le\bar\tau$ and $i\in\mathcal{S}_\tau$. It follows from the observations above that $v^m_{s_i} = F_s^{-1}(\rho(t'_j)/t'_j) = F_s^{-1}(1) = \bar{v}_{s_i}$. Next, consider $\tau>\bar\tau$ and $i\in S_\tau$. Observe that Lemma A.1 also implies that $\rho(t)/t$ is strictly decreasing for $t>t_0$. Hence, for such τ we have that $v^m_{s_i} = F_s^{-1}\left(\frac{\rho(t'_j)}{t'_j}\right) < \bar{v}_{s_i}$ where $j\in\mathcal{B}_\tau$. Moreover, $v^m_{s_i}$ is strictly decreasing in τ. These observations together imply that

$$v^m_{s_i} = \bar{v}_{s_i} \quad \text{for } i\in\mathcal{S}_\tau, \quad \tau\le\bar\tau, \tag{9}$$

$$v^m_{s_{i_1}} = v^m_{s_{i_2}} \quad \text{for } i_1,i_2\in\mathcal{S}_\tau \quad \text{with} \quad \tau\in\{1,\dots,\ell\}, \tag{10}$$

$$v^m_{s_{i_1}} > v^m_{s_{i_2}} \quad \text{for } i_1\in\mathcal{S}_{\tau_1}, i_2\in\mathcal{S}_{\tau_2}, \quad \text{with} \quad \tau_1,\tau_2\in\{1,\dots,\ell\}, \quad \tau_2>\bar\tau \quad \text{and} \quad \tau_1<\tau_2. \tag{11}$$

Hence the claim follows. Q.E.D.

Proof of Theorem 3. Proof of part (i). Let

$$f(t) := \max_{r\in[0,\min\{1,t\}]} \left[F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t}\right)\right] r \tag{12}$$

for $t>0$; note that $f(\cdot)$ is defined as in the statement of Proposition A.2. Let $\bar{V}(\boldsymbol{s},\boldsymbol{b})$ be defined as $\bar{V}(\boldsymbol{s},\boldsymbol{b}) := b_0 f\left(\frac{s_0}{b_0}\right)$ where s_0 and b_0 are defined as in the statement of the theorem. Therefore, we need to show that $V_{max}(\boldsymbol{s},\boldsymbol{b}) = \bar{V}(\boldsymbol{s},\boldsymbol{b})$.

We first prove that $V_{max}(\boldsymbol{s},\boldsymbol{b}) \le \bar{V}(\boldsymbol{s},\boldsymbol{b})$. Fix any arbitrary network $G(\mathcal{S}\cup\mathcal{B},E)$ and, by Proposition A.2, let $\boldsymbol{y}^\star > 0$ be the unique optimal solution to Problem (4) when f is defined as in (12). Then,

$$\begin{aligned} V_{opt}(E,\boldsymbol{s},\boldsymbol{b}) &\overset{(a)}{=} \left(\sum_{j\in\mathcal{B}} b_j\right)\sum_{j\in\mathcal{B}} \frac{b_j}{\sum_{j\in\mathcal{B}} b_j} f\left(\frac{y_j^\star}{b_j}\right) \overset{(b)}{\le} \left(\sum_{j\in\mathcal{B}} b_j\right) f\left(\frac{\sum_{j\in\mathcal{B}} y_j^\star}{\sum_{j\in\mathcal{B}} b_j}\right) \\ &\overset{(c)}{\le} \left(\sum_{j\in\mathcal{B}} b_j\right) f\left(\frac{\sum_{i\in\mathcal{S}} s_i}{\sum_{j\in\mathcal{B}} b_j}\right) \overset{(d)}{=} \bar{V}(\boldsymbol{s},\boldsymbol{b}), \end{aligned} \tag{13}$$

where equality (a) follows from part (iii) of Proposition A.2 (note that the assumptions in the statement of the theorem imply that the assumptions in Proposition A.2 hold). Inequality (b) follows from the fact that f is strictly concave (Proposition A.2 together with Lemma A.1). Inequality (c) follows from the fact that f is strictly increasing and that $\boldsymbol{y}^\star$ is feasible and thus must satisfy $\sum_{j\in\mathcal{B}} y_j^\star \le \sum_{i\in N(\mathcal{B})=\mathcal{S}} s_i$. Finally, equality (d) follows from the fact that, by definition, $\bar{V}(\boldsymbol{s},\boldsymbol{b}) = b_0 f(\frac{s_0}{b_0})$. Therefore, we have established that $V_{opt}(E,\boldsymbol{s},\boldsymbol{b}) \le \bar{V}(\boldsymbol{s},\boldsymbol{b})$ for any network $G(\mathcal{S}\cup\mathcal{B},E)$ and, thus, we must have that $V_{max}(\boldsymbol{s},\boldsymbol{b}) = \max_{E\subseteq\mathcal{B}\times\mathcal{S}} V_{opt}(E,\boldsymbol{s},\boldsymbol{b}) \le \bar{V}(\boldsymbol{s},\boldsymbol{b})$.

Next, let $\boldsymbol{y}$ be defined as $y_j := b_j \frac{\sum_{i\in\mathcal{S}} s_i}{\sum_{j\in\mathcal{B}} b_j}$ for all $j\in\mathcal{B}$. As the function f is strictly concave (Lemma A.1), inequality (b) in (13) holds by equality if and only if $\boldsymbol{y}^\star = \boldsymbol{y}$. Note that, in fact, such a y is feasible for Problem (4) with f given by (12) (i.e. it satisfies constraints (4b) and (4c)) if and only if the network satisfies the weighted Hall's marriage condition; this readily follows from Definition 2. Moreover, by the definition of $\boldsymbol{y}$, we have that $\sum_{j\in\mathcal{B}} y_j = \sum_{i\in\mathcal{S}} s_i$, and thus inequality (c) in Equation (13) holds with equality as well. Therefore, we have established that $V_{opt}(E,\boldsymbol{s},\boldsymbol{b}) = \bar{V}(\boldsymbol{s},\boldsymbol{b})$ if and only if network $G(\mathcal{S}\cup\mathcal{B},E)$ satisfies the weighted Hall's marriage condition. Because a complete bipartite network satisfies the weighted Hall's marriage condition, we obtain that $V_{max}(\boldsymbol{s},\boldsymbol{b}) = \max_{E\subseteq\mathcal{B}\times\mathcal{S}} V_{opt}(E,\boldsymbol{s},\boldsymbol{b}) \ge \bar{V}(\boldsymbol{s},\boldsymbol{b})$. This completes the proof of part (i).

Proof of part (ii): By Claim (ii) in Proposition 5 we have that, if a network $G(\mathcal{S}\cup\mathcal{B},E)$ satisfies the ε-marriage condition, then $V_h(E,\boldsymbol{s},\boldsymbol{b}) \ge (1-\varepsilon)V_{max}(\boldsymbol{s},\boldsymbol{b})$ where V_h is the revenue derived from the optimal homogeneous commissions/subscriptions. Hence, $V_{opt}(E,\boldsymbol{s},\boldsymbol{b}) \ge V_h(E,\boldsymbol{s},\boldsymbol{b}) \ge (1-\varepsilon)V_{max}(\boldsymbol{s},\boldsymbol{b})$, as desired. Q.E.D.

Proof of Proposition 3. Let $\boldsymbol{y}_1^\star, \boldsymbol{y}_2^\star$ be optimal solutions to problem (4) associated with networks $G(\mathcal{S}\cup\mathcal{B},E_1)$ and $G(\mathcal{S}\cup\mathcal{B},E_2)$, respectively, when f is defined as in the statement of Proposition A.2. As $\sum_{i\in N_{E_1}(B)} s_i \ge \sum_{i\in N_{E_2}(B)} s_i$ for all $B\subset\mathcal{B}$, we have that $\boldsymbol{y}_2^\star$ is a feasible solution for the problem associated with network $G(\mathcal{S}\cup\mathcal{B},E_1)$. Therefore, by Proposition A.2, we have that $\boldsymbol{y}_1^\star > 0$, $\boldsymbol{y}_2^\star > 0$ and

$$V_{opt}(E_1,\boldsymbol{s},\boldsymbol{b}) = \sum_{j\in\mathcal{B}} b_j f\left(\frac{(y_1^\star)_j}{b_j}\right) \ge \sum_{j\in\mathcal{B}} b_j f\left(\frac{(y_2^\star)_j}{b_j}\right) = V_{opt}(E_2,\boldsymbol{s},\boldsymbol{b}),$$

which completes the proof. Q.E.D.

C. Proofs of Results in Section 4

Proof of Proposition 5. Fix a network $G(\mathcal{S}\cup\mathcal{B},E)$. To ease exposition, throughout the rest of the proof we omit the dependence on the network in the notation.

With a slight abuse of notation, let $\bar{V}_h(\mu^b)$ denote the revenue under homogeneous commissions/commissions as a function of the buyers' subscription fee μ^b, when μ^s, γ^b, and γ^s are all set to zero, and define $\bar{V}_h = \max_{\mu^b\ge 0} \bar{V}_h(\mu^b)$. Note that $V_h \ge \bar{V}_h$, and therefore it suffices to establish the results in parts (i) and (ii) using either $\bar{V}_h$ or $\bar{V}_h(\mu^b)$ for some $\mu^b \geq 0$. As $\bar{v}_b < \infty$, it suffices to consider $\mu^b \in [0, \bar{v}_b]$ as otherwise no trade will take place and $\bar{V}_h(\mu^b) = 0$ when $\mu^b > \bar{v}_b$.

Before proceeding to the proofs of parts (i) and (ii) of this proposition, we establish a couple of intermediate results that will be exploited later on.

Claim 1: $\bar{V}_h(\mu^b) = \sum_{j\in\mathcal{B}} \mu^b b_j r_j^\star(\mu^b)$ for $\mu^b \in [0, \bar{v}_b]$, where $r_j^\star(\mu^b)$ is defined as in (14). As a first step in our proof, we provide an expression for $\bar{V}_h(\mu^b)$. We start by finding the optimal equilibrium demand as a function of μ^b, which we denote by $\boldsymbol{q}^b(\mu^b)$. (With a slight abuse of notation, in what follows we define the relevant quantities as functions of μ^b to make this dependency explicit.)

Fix $\mu^b \in [0, \bar{v}_b)$. By Proposition A.3(i), Assumptions (A1)–(A4) hold. Using Proposition A.3(ii), we know that the equilibrium demands are $q_j^\star(\mu^b) = b_j r_j^\star(\mu^b)$ where

$$r_j^\star(\mu^b) = \underset{r\in[0,\min\{1, y_j^\star(\mu^b)/b_j\}]}{\arg\max} \int_0^r F_b^{-1}(1-x) - \mu^b - F_s^{-1}\left(\frac{x}{y_j^\star(\mu^b)/b_j}\right) dx, \tag{14}$$

and $\boldsymbol{y}^\star(\mu^b) > 0$ is the optimal solution to (4). By Lemma A.2, $\boldsymbol{y}^\star$ is the lexicographically optimal base of polymatroid $\mathcal{P} = \{\boldsymbol{y} \geq \boldsymbol{0} : \sum_{j\in B} y_j \leq \sum_{i\in N_E(B)} s_i,\ \forall B \subset \mathcal{B}\}$, which is independent of μ^b. Therefore, we drop the dependency on μ^b in $y^\star(\mu^b)$, and define $t_j := \frac{y_j^\star}{b_j}$ for all $j \in \mathcal{B}$. Note that, as $y_j^\star > 0$, we have that $t_j > 0$. From expression (14), it follows that $r_j^\star(\bar{v}^b) = 0$ (corresponding to no trade in equilibrium). Therefore can use expression (14) for all $\mu^b \in [0, \bar{v}^b]$.

Notice that the value of $r_j^\star(\mu^b)$ can be found by solving a (strictly) concave maximization problem. Using the first-order optimality condition, we can express $r_j^\star(\mu^b)$ as

$$r_j^\star(\mu^b) = \max\left\{r : F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right) \geq \mu^b,\ 0 \leq r \leq \min\{1, t_j\}\right\}, \qquad \forall \mu^b \in [0, \bar{v}_b]. \tag{15}$$

Finally, the revenue can be expressed as

$$\bar{V}_h(\mu^b) = \sum_{j\in\mathcal{B}} \mu_j^b q_j^\star(\mu^b) = \sum_{j\in\mathcal{B}} \mu^b b_j r_j^\star(\mu^b) \qquad \text{for } \mu^b \in [0, \bar{v}_b]. \tag{16}$$

Claim 2: The functions $\{\mu^b r_j^\star(\mu^b)\}$ are continuous, concave, and satisfy $0 \leq \mu^b r_j^\star(\mu^b) \leq f(t_j)$, for all $\mu^b \in [0, \bar{v}_b]$, where $f(\cdot)$ is defined as in (17). Moreover, these bounds are tight. We now derive some properties of the functions $\mu^b r_j^\star(\mu^b)$ that we will later use to bound $\bar{V}_h(\mu^b)/V_{opt}$. In what follows, we show that the functions $\mu^b r_j^\star(\mu^b)$ are continuous and concave in $\mu^b \in [0, \bar{v}_b]$ for every $j \in \mathcal{B}$, and that $0 \leq r_j^\star(\mu^b) \leq f(t_j)$, where

$$f(t) := \max_{r\in[0,\min\{1,t\}]} \left[F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t}\right)\right] r \text{ for } t > 0. \tag{17}$$

Fix j. We first show that $r_j^\star(\mu^b)$ is continuous, weakly decreasing, and concave in $\mu^b \in [0, \bar{v}_b]$. Consider the expression in (15) and note that, the objective function r is jointly continuous in (r, μ^b) and that the function $F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{t_j}) - \mu^b$ is also jointly continuous in (r, μ^b). By the maximum theorem on page 116 of Berge (1963), the value function $r_j^\star(\mu^b)$ is thus continuous in $\mu^b \in [0, \bar{v}_b]$. Moreover, the function $F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right)$ is continuous and strictly decreasing in r (recall that $t_j > 0$ by definition). Thus, for any $\mu_1^b \geq \mu_2^b$ we have that $\{r : F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{t_j}) \geq \mu_1^b,\ 0 \leq r \leq \min\{1, t_j\}\} \subset \{r : F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{t_j}) \geq \mu_2^b,\ 0 \leq r \leq \min\{1, t_j\}\}$, which implies that $r_j^\star(\mu^b)$ is weakly decreasing in μ^b. Finally, for any $\mu_1^b, \mu_2^b \in [0, \bar{v}_b]$, let $\bar{r} := r_j^\star\left(\frac{1}{2}\mu_1^b + \frac{1}{2}\mu_2^b\right)$, $r_1 := r_j^\star\left(\mu_1^b\right)$, and $r_2 := r_j^\star\left(\mu_2^b\right)$. Given that functions $F_b^{-1}(1-r)$ and $-F_s^{-1}(r)$ are concave, we have that $F_b^{-1}\left(1 - \frac{(r_1+r_2)}{2}\right) - F_s^{-1}\left(\frac{r_1+r_2}{2t_j}\right) \geq \frac{1}{2}\left(F_b^{-1}(1-r_1) - F_s^{-1}(\frac{r_1}{t_j})\right) + \frac{1}{2}\left(F_b^{-1}(1-r_2) - F_s^{-1}(\frac{r_2}{t_j})\right) \geq \frac{1}{2}\mu_1^b + \frac{1}{2}\mu_2^b$. Moreover, by definition of r_1 and r_2, we have that $\frac{r_1+r_2}{2} \leq t_j$ and $\frac{r_1+r_2}{2} \leq 1$ because both $r_1, r_2 \leq \min\{1, t_j\}$. This implies that $\frac{r_1+r_2}{2} \in \{r : F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{t_j}) \geq \frac{\mu_1^b+\mu_2^b}{2},\ r \leq t_j\}$. Thus, we have $\bar{r} \geq \frac{r_1+r_2}{2}$, and hence $r_j^\star(\mu^b)$ is a concave function in $\mu^b \in [0, \bar{v}_b]$. Therefore, we have established that $r_j^\star(\mu^b)$ is continuous, weakly decreasing, and concave, which implies that the function $\mu^b r_j^\star(\mu^b)$ is continuous and concave in $\mu^b \in (0, \bar{v}_b]$.

Next, we obtain tight lower and upper bounds on $\mu^b r_j^\star(\mu^b)$. To that end, we start by showing that $\max_{\mu^b\in[0,\bar{v}_b]} \mu^b r_j^\star(\mu^b) = f(t_j)$ for all $j \in \mathcal{B}$, where f is defined as in (17). Recall that we have defined t_j as $t_j = y_j^\star/b_j$. Let $\bar{\mu}^b := \arg\max_{\mu^b\in[0,\bar{v}_b]} \mu^b r_j^\star(\mu^b)$. From (15), we have that $\bar{\mu}^b \leq F_b^{-1}(1 - r_j^\star(\bar{\mu}^b)) - F_s^{-1}\left(\frac{r_j^\star(\bar{\mu}^b)}{t_j}\right)$.

Therefore, $\bar{\mu}^b r_j^\star(\bar{\mu}^b) \le [F_b^{-1}(1-r_j^\star(\bar{\mu}^b)) - F_s^{-1}(\frac{r_j^\star(\bar{\mu}^b)}{t_j})] r_j^\star(\bar{\mu}^b) \le \max_{r\in[0,\min\{1,t_j\}]}[F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{t_j})]r = f(t_j)$. To show that this bound is tight, let $\bar{r} := \arg\max r \in [0,\min\{1,t_j\}] \left[F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right)\right] r$, and consider $\mu^b = F_b^{-1}(1-\bar{r}) - F_s^{-1}(\frac{\bar{r}}{t_j})$. Notice that $\mu^b \in [0,\bar{v}_b]$ and, by (15), we have that $r_j^\star(\mu^b) = \bar{r}$. Therefore, $\mu^b r_j^\star(\mu^b) = \mu^b \bar{r} = [F_b^{-1}(1-r) - F_s^{-1}(\frac{\bar{r}}{t_j})]\bar{r} = f(t_j)$. Finally, by (15), it is immediate to see that $r_j^\star(\mu^b)\mu^b \ge 0$ and that it is equal to zero if $\mu^b \in \{0,\bar{v}_b\}$. Therefore, we conclude that $0 \le \mu^b r_j^\star(\mu^b) \le f(t_j)$, and that both these bounds are tight.

Proof of part (i): $\frac{V_h}{V_{opt}} \ge \frac{1}{2}$. Define $\left(\tilde{b}_j\right)_{j\in\mathcal{B}}$ as $\tilde{b}_j := b_j f(t_j)$, and define $\tilde{r}_j^\star(\mu^b) := \frac{b_j}{\tilde{b}_j} r_j^\star(\mu^b)$ for $\mu^b \in [0,\bar{v}_b]$. Note that the function $\tilde{r}_j^\star(\mu^b)\mu^b$ is still concave in $\mu^b \in [0,\bar{v}_b]$ as $\tilde{b}_j$ is a positive constant. Moreover, by using the tight upper bound on $r_j^\star$, we conclude that $\max_{\mu^b\in[0,\bar{v}_b]} \tilde{r}_j^\star(\mu^b)\mu^b = 1$ for every $j \in \mathcal{B}$. Let the weight vector $\boldsymbol{w}$ be defined as $w_j := \frac{\tilde{b}_j}{\sum_{j'\in\mathcal{B}} \tilde{b}_{j'}}$ for all $j \in \mathcal{B}$. Then,

$$\frac{V_h}{V_{opt}} \overset{(a)}{\ge} \frac{\max_{\mu^b\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} b_j r_j^\star(\mu^b)\mu^b}{\sum_{j\in\mathcal{B}} b_j f(\frac{y_j^\star}{b_j})} \overset{(b)}{=} \frac{\max_{\mu^b\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} \tilde{b}_j \tilde{r}_j^\star(\mu^b)\mu^b}{\sum_{j\in\mathcal{B}} \tilde{b}_j}$$
$$\overset{(c)}{=} \max_{\mu^b\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} w_j \tilde{r}_j^\star(\mu^b)\mu^b \overset{(d)}{\ge} \max_{\mu^b\in[0,\bar{v}_b]} \min_{j\in\mathcal{B}} \tilde{r}_j^\star(\mu^b)\mu^b, \tag{18}$$

where inequality (a) follows from the fact that the numerator satisfies $V_h \ge \max_{\mu\in[0,\bar{v}_b]} \bar{V}_h(\mu) = \max_{\mu\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} b_j r_j^\star(\mu^b)\mu^b$, and in the denominator we used Proposition A.2(iii) to derive $V_{opt} = \sum_{j\in\mathcal{B}} b_j f(\frac{y_j^\star}{b_j})$. Equality (b) follows from the definition of $\tilde{\boldsymbol{b}}$ and, similarly, equality (c) follows from using the definition of $\tilde{\boldsymbol{w}}$. Finally, inequality (d) follows by noting that $\sum_{j\in\mathcal{B}} w_j = 1$, $w_j \ge 0$ for all $j \in \mathcal{B}$, which in turn implies that $\sum_{j\in\mathcal{B}} w_j \tilde{r}_j^\star(\mu^b)\mu^b \ge \min_{j\in\mathcal{B}} \tilde{r}_j^\star(\mu^b)\mu^b$ for all $\mu^b \in [0,\bar{v}_b]$.

Next, we provide a lower bound on $\max_{\mu^b\in[0,\bar{v}_b]} \min_{j\in\mathcal{B}} \tilde{r}_j^\star(\mu^b)\mu^b$. To ease notation, let $H_j(\mu^b) := \mu^b \tilde{r}_j^\star(\mu^b)$ for all $\mu^b \in [0,\bar{v}_b]$ and define two functions $H, G : [0,\bar{v}_b] \to [0,1]$ where $H(\mu^b) := \min_{j\in\mathcal{B}} H_j(\mu^b)$ and $G(\mu^b) := \min\{\frac{\mu^b}{\bar{v}_b}, 1 - \frac{\mu^b}{\bar{v}_b}\}$ for $\mu^b \in [0,\bar{v}_b]$. By Claim 2 above, we have that $H_j(\mu^b)$ is continuous and concave for every $\mu^b \in [0,\bar{v}_b]$. We conclude that $H(\mu^b)$ is continuous and concave in $\mu^b \in [0,\bar{v}_b]$ because it is the minimum of a finite set of continuous and concave functions. Moreover, as each H_j is continuous and concave, it is differentiable almost everywhere, and thus H is differentiable almost everywhere. By Claim 2 above, we also have that $H(0) = H(\bar{v}_b) = 0$. In addition, note that $G(\mu^b)$ is a piecewise linear concave function, consisting of two pieces, and it is symmetric around $\mu^b = \frac{1}{2}\bar{v}_b$ where it achieves its peak value of $\frac{1}{2}$.

We want to show that $H\left(\frac{1}{2}\bar{v}_b\right) \ge G\left(\frac{1}{2}\bar{v}_b\right) = 1/2$; this will imply the desired bound as $\max_{\mu^b\in[0,\bar{v}_b]} \min_{j\in\mathcal{B}} \tilde{r}_j^\star(\mu^b)\mu^b \ge \min_{j\in\mathcal{B}} \tilde{r}_j^\star(\frac{1}{2}\bar{v}_b)\frac{1}{2}\bar{v}_b = H(\frac{1}{2}\bar{v}_b)$. In fact, we show a stronger result: we show that $H(\mu^b) \ge G(\mu^b)$ for all $\mu^b \in [0,\frac{1}{2}\bar{v}_b]$. (This trivially holds at $\mu^b = 0$ as $H(0) = G(0)$.)

Suppose on the contrary that there exists $\bar{\mu} \in (0,\frac{1}{2}\bar{v}_b]$ such that $H(\bar{\mu}) < G(\bar{\mu})$, and suppose that H is differentiable at $\bar{\mu}$. The latter is without loss of generality; as both H and G are continuous in $[0,\bar{v}_b]$ and H is differentiable almost everywhere. Next, we consider different cases which cover different possible values of $H'(\bar{\mu})$, and obtain contradictions in each case:

(1) If $H'(\bar{\mu}) > \frac{1}{\bar{v}_b}$, then by the concavity of H, we have that $H(0) \le H(\bar{\mu}) + H'(\bar{\mu})(0-\bar{\mu})$. As $H'(\bar{\mu})(0-\bar{\mu}) \le -\frac{\bar{\mu}}{\bar{v}_b}$, we have that $H(\bar{\mu}) + H'(\bar{\mu})(0-\bar{\mu}) < G(\bar{\mu}) - \frac{\bar{\mu}}{\bar{v}_b} = 0$, and thus $0 = H(0) < 0$, which is a contradiction.

(2) If $0 \le H'(\bar{\mu}) \le \frac{1}{\bar{v}_b}$, then, as we assumed that H is differentiable at $\bar{\mu}$, there exist $j_0 \in \mathcal{B}$ and $\epsilon > 0$ such that $H(\mu) = H_{j_0}(\mu)$ for every $\mu \in (\bar{\mu}-\epsilon, \bar{\mu}+\epsilon)$, and thus we must have $0 \le H'_{j_0}(\bar{\mu}) \le \frac{1}{\bar{v}_b}$. We argue that this contradicts $\max_{x\in[0,\bar{v}_b]} H_{j_0}(x) = 1$, which was established in Claim 2 above. To that end, we use the fact that the concavity of $H_{j_0}(\mu)$ implies that $H_{j_0}(\mu) \le H_{j_0}(\bar{\mu}) + H'_{j_0}(\bar{\mu})(\mu-\bar{\mu})$, for any $\mu \in [0,\bar{v}_b]$. Then, for $\mu \in [0,\bar{\mu}]$, we have that $H_{j_0}(\mu) \le H_{j_0}(\bar{\mu}) < G(\bar{\mu}) < 1$. Moreover, for any $\mu \in [\bar{\mu},\bar{v}_b]$, we have that $H'_{j_0}(\bar{\mu})(\mu-\bar{\mu}) \le \frac{1}{\bar{v}_b}(\mu-\bar{\mu})$, which implies that $H_{j_0}(\mu) < G(\bar{\mu}) + \frac{1}{\bar{v}_b}(\tilde{\mu}-\bar{\mu}) \le \frac{\mu}{\bar{v}_b} \le 1$. Therefore, we have that $H_{j_0}(\mu) < 1$ for every $\mu \in [0,\bar{v}_b]$, and thus $\max_{x\in[0,\bar{v}_b]} H_{j_0}(x) < 1$, which is a contradiction.

(3) If $-\frac{1}{\bar{v}_b} \le H'(\bar{\mu}) < 0$, we follow an argument along the lines of the one in case (2). We define $j_0 \in \mathcal{B}$ as in (2), and then prove a contradiction to $\max_{\mu\in[0,\bar{v}_b]} H_{j_0}(\mu) = 1$. For any $\mu \in [0,\bar{\mu}]$, we have that $H'_{j_0}(\bar{\mu})(\mu-\bar{\mu}) \le -\frac{1}{\bar{v}_b}(\mu-\bar{\mu})$. As $H_{j_0}(\bar{\mu}) < G(\bar{\mu}) \le 1 - \frac{\bar{\mu}}{\bar{v}_b}$, we have that $H_{j_0}(\mu) < (1-\frac{\bar{\mu}}{\bar{v}_b}) - \frac{1}{\bar{v}_b}(\mu-\bar{\mu}) \le 1$. For any $\mu \in [\bar{\mu},\bar{v}_b]$, we have that $H'_{j_0}(\bar{\mu})(\mu-\bar{\mu}) \le 0$, which implies that $H_{j_0}(\mu) \le H_{j_0}(\bar{\mu}) < G(\bar{\mu}) < 1$. Therefore, we have that $H_{j_0}(\mu) < 1$ for every $\mu \in [0,\bar{v}_b]$, and thus $\max_{x\in[0,\bar{v}_b]} H_{j_0}(x) < 1$, which is a contradiction.

(4) If $H'(\bar{\mu}) < -\frac{1}{\bar{v}_b}$, the argument is similar to that in case (1). By the concavity of H, we have that $H(\bar{v}_b) \le H(\bar{\mu}) + H'(\bar{\mu})(\bar{v}_b - \bar{\mu})$. Since $H'(\bar{\mu})(\bar{v}_b - \bar{\mu}) < -\frac{1}{\bar{v}_b}(\bar{v}_b - \bar{\mu}) \le -1 + \frac{\bar{\mu}}{\bar{v}_b}$, this leads to a contradiction as $0 = H_{j_0}(\bar{v}_b) < G(\bar{\mu}) - 1 + \frac{\bar{\mu}}{\bar{v}_b} = -1 + 2\frac{\bar{\mu}}{\bar{v}_b} < 0$.

Therefore, we have established that $H(\mu^b) \ge G(\mu^b)$ for all $\mu^b \in [0, \frac{1}{2}\bar{v}_b]$. To conclude the proof, note that

$$\frac{V_h}{V_{opt}} \ge \max_{\mu^b \in [0,\bar{v}_b]} \min_{j \in \mathcal{B}} \tilde{r}_j^\star(\mu^b)\mu^b = \max_{\mu^b \in [0,\bar{v}_b]} H(\mu^b) \ \ge\ H\left(\frac{1}{2}\bar{v}_b\right) \ \ge\ G\left(\frac{1}{2}\bar{v}_b\right) \ =\ \frac{1}{2}, \tag{19}$$

where the first inequality follows from (18). Thus, we obtain that $V_h \ge \frac{1}{2}V_{opt}$, as desired.

Proof of Part (ii). Suppose that the network satisfies the ε-marriage condition for $\varepsilon \in [0,1)$. (The claim holds trivially when $\varepsilon = 1$ because $V_h \ge 0$.) By part (i) of Theorem 3, we have that $V_{max} = (\sum_{j \in \mathcal{B}} b_j) f(\tilde{t})$, where $\tilde{t} := \frac{\sum_{i \in \mathcal{S}} s_i}{\sum_{j \in \mathcal{B}} b_j}$ and f is defined as in (17). Define $\tilde{r} := \arg\max_{r \in [0,\min\{1,\tilde{t}\}]} \left[F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{\tilde{t}})\right] r$, and define $\bar{\mu}^b := F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}})$. To obtain the desired lower bound, we consider the revenue $\bar{V}(\bar{\mu}^b)$ defined in (16) and use lower bounds on $\frac{y_j^\star}{b_j^\star}$ and $r_j^\star(\bar{\mu}^b)$ (defined as in (15)) for all $j \in \mathcal{B}$.

Claim (ii)-1: $\frac{y_j^\star}{b_j^\star} \ge (1-\varepsilon)\tilde{t}$ for all $j \in \mathcal{B}$. Let vector $\boldsymbol{y}^\star > 0$ be the optimal solution to problem (4) and, as before, let $t_j := \frac{y_j^\star}{b_j}$. We further let the distinct values of t_j's to be given by $0 < t_1 < t_2 \cdots < t_l$ and let $\bar{\mathcal{B}}_1 = \{j \in \mathcal{B} : \frac{y_j^\star}{b_j} = t_1\}$. This implies that

$$\frac{y_j^\star}{b_j} \ \ge\ t_1 \ =\ \frac{\sum_{j \in \bar{\mathcal{B}}_1} y_j^\star}{\sum_{j \in \bar{\mathcal{B}}_1} b_j} \ \overset{(e)}{=}\ \frac{\sum_{i \in N_E(\bar{\mathcal{B}}_1)} s_i}{\sum_{j \in \bar{\mathcal{B}}_1} b_j} \ \overset{(f)}{\ge} (1-\varepsilon)\frac{\sum_{i \in \mathcal{S}} s_i}{\sum_{j \in \mathcal{B}} b_j} = (1-\varepsilon)\tilde{t} \qquad \text{for every } j \in \mathcal{B}, \tag{20}$$

where equality (e) follows from the fact that, by Lemma A.2, vector $\boldsymbol{y}^\star$ is a lexicographically optimal base in polymatroid $\mathcal{P} = \{\boldsymbol{y} \ge \boldsymbol{0} : \sum_{j \in B} y_j \le \sum_{j \in B} b_j, \forall B \subset \mathcal{B}\}$ and, by the equivalence of item (i) and item (ii) of Theorem 3.2 in Fujishige (1980), we have that $\sum_{j \in \bar{\mathcal{B}}_1} y_j^\star = \sum_{i \in N_E(\bar{\mathcal{B}}_1)} s_i$. Equality (f) follows directly from the definition of the ε-marriage condition.

Claim (ii)-2: $r_j^\star(\bar{\mu}^b) \ge (1-\varepsilon)\tilde{r}$ for all $j \in \mathcal{B}$. Let $r_j^\star := r_j^\star(\bar{\mu}^b)$ where $r_j^\star(\bar{\mu}^b)$ is defined as in (15). We want to show $r_j^\star \ge (1-\varepsilon)\tilde{r}$ for all $j \in \mathcal{B}$. As the $F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right)$ is decreasing in r, then we have that one of the constraints in (15) must be binding, that is, either $\bar{\mu}^b = F_b^{-1}(1-r_j^\star) - F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j})$ or $r_j^\star = \frac{y_j^\star}{b_j}$. If $r_j^\star = \frac{y_j^\star}{b_j}$, then $r_j^\star = \frac{y_j^\star}{b_j} \ge (1-\varepsilon)\tilde{t}$ follows from (20), and thus $\frac{1}{1-\varepsilon}\frac{r_j^\star}{\tilde{t}} \ge 1 \ge \frac{\tilde{r}}{\tilde{t}}$, where the last inequality follows from the definition of $\tilde{r}$. Hence, if $r_j^\star = \frac{y_j^\star}{b_j}$, we have that $r_j^\star \ge (1-\varepsilon)\tilde{r}$.

Suppose that $\bar{\mu}^b = F_b^{-1}(1-r_j^\star) - F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j})$. Recall that, by the definition of $\bar{\mu}^b$, we also have that $\bar{\mu}^b = F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}})$ and, therefore,

$$F_b^{-1}(1-r_j^\star) - F_s^{-1}\left(\frac{r_j^\star}{y_j^\star/b_j}\right) = F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}}). \tag{21}$$

We show that $r_j^\star \ge (1-\varepsilon)\tilde{r}$ by discussing the two subcases depending on whether $\frac{y_j^\star}{b_j} \ge \tilde{t}$ or $\frac{y_j^\star}{b_j} < \tilde{t}$. If $\frac{y_j^\star}{b_j} \ge \tilde{t}$, then (21) implies that $r_j^\star \ge \tilde{r} \ge (1-\varepsilon)\tilde{r}$. On the other hand, if $\frac{y_j^\star}{b_j} < \tilde{t}$, then (21) implies that $r_j^\star < \tilde{r}$. In this case, to show that $\frac{r_j}{y_j^\star/b_j} \ge \frac{\tilde{r}}{\tilde{t}}$, suppose on the contrary that $\frac{r_j^\star}{y_j^\star/b_j} < \frac{\tilde{r}}{\tilde{t}}$. Then, we have that $F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j}) < F_s^{-1}(\frac{\tilde{r}}{\tilde{t}})$. Given $r_j^\star < \tilde{r}$, we have that $F_b^{-1}(1-r_j^\star) > F_b^{-1}(1-\tilde{r})$. This leads to a contradiction as $\bar{\mu}^b = F_b^{-1}(1-r_j^\star) - F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j}) > F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}}) = \bar{\mu}^b$. Therefore, we have that $\frac{r_j^\star}{y_j^\star/b_j} \ge \frac{\tilde{r}}{\tilde{t}}$. As we established in Claim (ii)-1 that $\frac{y_j^\star}{b_j} \ge (1-\varepsilon)\tilde{t}$, we have that $\frac{1}{1-\varepsilon}\frac{r_j^\star}{\tilde{t}} \ge \frac{r_j^\star}{y_j^\star/b_j} \ge \frac{\tilde{r}}{\tilde{t}}$ or equivalently $r_j^\star \ge (1-\varepsilon)\tilde{r}$.

Finally, note that

$$\begin{aligned} V_h &\overset{(g)}{\ge} \bar{V}_h(\bar{\mu}^b) \ \overset{(h)}{=}\ \sum_{j \in \mathcal{B}} \bar{\mu}^b b_j r_j^\star(\bar{\mu}^b) \\ &\overset{(i)}{\ge} \sum_{j \in \mathcal{B}} \bar{\mu}^b b_j (1-\varepsilon)\tilde{r} \ \overset{(j)}{\ge}\ (1-\varepsilon)\left(\sum_{j \in \mathcal{B}} b_j\right)\left[F_b^{-1}(1-\tilde{r}) - F_s^{-1}\left(\frac{\tilde{r}}{\tilde{t}}\right)\right]\tilde{r} \ \overset{(k)}{=}\ (1-\varepsilon)V_{max}, \end{aligned} \tag{22}$$

where inequality (g) follows from the definition of $\bar{V}_h(\bar{\mu}^b)$, and equality (h) follows from (16). Inequality (i) follows from the fact $r_j \geq (1-\varepsilon)\tilde{r}$ for all $j \in \mathcal{B}$ (Claim (ii)-1). Inequality (j) follows from the fact that $\bar{\mu}^b = F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{t})$. Equality (k) follows from the definition of V_{max}. This completes the proof that $V_h \geq (1-\varepsilon)V_{max}$. Q.E.D.