Online supplement to "Should we wait before outsourcing? Analysis of a revenue-generating blended contact center"

Benjamin Legros[1] • Oualid Jouini[2] • Ger Koole[3]

[1] *EM Normandie, Laboratoire Métis, 64 Rue du Ranelagh, 75016 Paris, France*

[2] *Laboratoire Genie Industriel, CentraleSupélec, Université Paris-Saclay, 9 rue Joliot Curie, 91190, Gif-sur-Yvette, France*

[3] *VU University Amsterdam, Department of Mathematics, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands*

benjamin.legros@centraliens.net • oualid.jouini@centralesupelec.fr • ger.koole@vu.nl

This is an online supplement to the main paper, with the same title.

Proof of Theorem 1

To prove Theorem 1, we use the value iteration technique in order to show the propagation of some properties defining the threshold form of the optimal policy. We only provide the proof for Ω_l. We first reformulate the value function of Section 3.1 of the article as follows:

$$V_{k+1}(x) = \lambda U_k(x) + (s+x)\mu W_k(x-1) + (1-\lambda-(s+x)\mu)W_k(x), \text{ for } -s \le x \le 0, \text{ and,} \tag{1}$$
$$V_{k+1}(x) = -s\mu\frac{\omega r_1}{\gamma}x + \gamma U_k(x) + s\mu F(W_k(x)) + (1-\gamma-s\mu)W_k(x), \text{ for } x>0, \text{ for } k \ge 0, \text{ with}$$

$U_k(x) = \max(F(V_k(x)) - (r_1+L), V_k(x+1))$ if $x \ge 0$, and $U_k(x) = V_k(x+1)$ if $x<0$, $W_k(x) = \max(V_k(x), V_k(x+1)+r_2)$ if $-s \le x < 0$, and $W_k(x) = V_k(x)$ if $x \ge 0$. To show that a threshold policy is optimal we need to show that V_k, U_k and W_k are decreasing and generally concave (*dgcv*) in x for $x \ge -s$ and $k \ge 0$. We define the *dgcv* property for a function $f(x)$, $x \ge -s$, by

$$f(x) \ge f(x+1), \text{ and,}$$
$$f(x+1) + F(f(x+1)) \ge f(x+2) + F(f(x)),$$

where $F(f(x)) = \sum_{h=0}^{x} q_{x,x-h} f(x-h)$, if $x>0$ and $F(f(x)) = f(x)$ if $-s \le x \le 0$. We next prove the *dgcv* property by induction on k. Since $V_0(x) = U_0(x) = W_0(x) = 0$ for $x \ge -s$, V_0, U_0, and W_0 are *dgcv*. We only provide the details for the second order monotonicity property.

Induction from V_k to U_k and W_k for the second order monotonicity property. We assume that V_k is decreasing and generally concave for a given $k \ge 0$, and we want to show that the same property applies for U_k and W_k. The induction from V_k to W_k is identical to the induction for an *a priori* policy. Consider now

U_k. We may write for $x \geq 0$,

$$U_k(x+1) + F(U_k(x+1)) \geq V_k(x+2) + \sum_{h=0}^{x+1} q_{x+1,x+1-h} V_k(x+2-h) = V_k(x+2) + F(V_k(x+2)) + \left(\frac{\gamma}{\gamma+\lambda}\right)^{x+2} (V_k(1) - V_k(0)), \tag{2}$$

$$U_k(x+1) + F(U_k(x+1)) \geq F(V_k(x+1)) - (r_1+L) + \sum_{h=n+1}^{x+1} q_{x+1,x+1-h} V_k(x+2-h) + \sum_{h=0}^{n} q_{x+1,x+1-h} \left(F(V_k(x+1-h)) - (r_1+L)\right), \text{ for } 2 \leq n \leq x. \tag{3}$$

Case 1: $U_k(x+2) + F(U_k(x)) = V_k(x+3) + \sum_{h=0}^{x} q_{x,x-h} V_k(x+1-h) = V_k(x+3) + F(V_k(x+1)) + \left(\frac{\gamma}{\gamma+\lambda}\right)^{x+1} (V_k(1) - V_k(0))$. Since V_k is generally concave, we obtain $V_k(x+2) + F(V_k(x+2)) \geq V_k(x+3) + F(V_k(x+1))$ and since V_k is decreasing in x, $\left(\frac{\gamma}{\gamma+\lambda}\right)^{x+2} (V_k(1) - V_k(0)) \geq \left(\frac{\gamma}{\gamma+\lambda}\right)^{x+1} (V_k(1) - V_k(0))$. Finally, Equation (2) proves that U_k is also generally concave.

Case 2:

$$U_k(x+2) + F(U_k(x)) = F(V_k(x+2)) - (r_1+L) + \sum_{h=n+1}^{x} q_{x,x-h} V_k(x+1-h) + \sum_{h=0}^{n} q_{x,x-h} \left(F(V_k(x-h)) - (r_1+L)\right).$$

One may write

$$\begin{aligned} &F(V_k(x+1)) - (r_1+L) + \sum_{h=n+1}^{x+1} q_{x+1,x+1-h} V_k(x+2-h) + \sum_{h=0}^{n} q_{x+1,x+1-h} \left(F(V_k(x+1-h)) - (r_1+L)\right) \\ &\quad - F(V_k(x+2)) + (r_1+L) - \sum_{h=n+1}^{x} q_{x,x-h} V_k(x+1-h) - \sum_{h=0}^{n} q_{x,x-h} \left(F(V_k(x-h)) - (r_1+L)\right) \\ &= \sum_{h=0}^{n} q_{x+1,x+1-h} \left(F(V_k(x+1-h)) + V_k(x+1-h) - F(V_k(x-h)) - V_k(x+2-h)\right) \\ &\quad - (V_k(1) - V_k(0)) \left(\frac{\gamma}{\gamma+\lambda}\right)^{x+1} \left(\frac{\lambda}{\gamma+\lambda}\right) \geq 0, \end{aligned}$$

since V_k is decreasing and generally concave. Thus, Equation (3) proves that U_k is also generally concave. Moreover, we have proven that if V_k is generally concave we have

$$U_k(x+1) + F(U_k(x+1)) - U_k(x+2) - F(U_k(x)) \geq (V_k(0) - V_k(1)) (1-u)^{x+1} u. \tag{4}$$

Induction from V_k, U_k and W_k to V_{k+1} for the second order monotonicity property. We now assume that U_k, W_k and V_k are *dgcv* and we prove that V_{k+1} is also *dcvg*. The case $x < -1$ has already been proven in the *a priori* case.

For $x = -1$, we may write

$$V_{k+1}(0) + F(V_{k+1}(0)) - F(V_{k+1}(-1)) - V_{k+1}(1) = s\mu\frac{r_1\omega}{\gamma} + \lambda(F(U_k(0)) + U_k(0) - U_k(-1) - U_k(1))$$
$$+ s\mu(2W_k(-1) - W_k(-2) - F(W_k(1))) + (1-\lambda-s\mu)(2W_k(0) - W_k(-1) - W_k(1))$$
$$+ (\gamma-\lambda)(W_k(1) - U_k(1)) + \mu(W_k(-2) - W_k(-1)).$$

We now show that all terms in this expression are positive. Consider the first term proportional with λ. We have $U_k(-1) = V_k(0)$. For $U_k(1)$, two cases should be considered.
Case 1: $U_k(1) = V_k(2)$. Since $U_k(0) \geq V_k(1)$, $F(U_k(0)) + U_k(0) - U_k(-1) - U_k(1) \geq (1+u)V_k(1) - uV_k(0) - V_k(2)$. This last expression is positive due to the general concavity property of V_k.
Case 2: $U_k(1) = V_k(1) - (r_1 + L)$. Since $F(U_k(0)) + U_k(0) \geq F(V_k(1)) + V_k(0) - (r_1 + L)$, $F(U_k(0)) + U_k(0) - U_k(-1) - U_k(1) \geq (1-u)(V_k(0) - V_k(1))$. This last expression is positive because V_k is decreasing. Consider now the term proportional with $s\mu$. We have $F(W_k(1)) = uW_k(1) + (1-u)W_k(0)$. So, $2W_k(-1) - W_k(-2) - F(W_k(1)) = 2W_k(-1) - W_k(-2) - W_k(0) + u(W_k(0) - W_k(1)) \geq 0$, because of the decreasing and concave property of W_k. The term proportional with $1-\lambda-s\mu$ is also positive because of the general concavity property of W_k. The term proportional with $\gamma-\lambda$ is $W_k(1) - U_k(1) = V_k(1) - \max(V_k(2), V_k(1) - (r_1+L)) \geq r_1 + L \geq 0$. Finally, the term proportional with μ is positive because W_k is decreasing. This proves that the generally concave property holds for V_{k+1} for $x = -1$.

For $x = 0$, we may write

$$V_{k+1}(1) + F(V_{k+1}(1)) - F(V_{k+1}(0)) - V_{k+1}(2)$$
$$= \gamma(U_k(1) + F(U_k(1)) - F(U_k(0)) - U_k(2)) + s\mu F(W_k(1) + F(W_k(1)) - F(W_k(0)) - W_k(2))$$
$$+ (1-\gamma-s\mu)(W_k(1) + F(W_k(1)) - F(W_k(0)) - W_k(2)) + (\gamma-\lambda)(F(U_k(0)) - W_k(0)).$$

The terms proportional with $s\mu$ and $1-\gamma-s\mu$ are positive because W_k has the generally concave property. Using Equation (4), we have $\gamma(U_k(1) + F(U_k(1)) - F(U_k(0)) - U_k(2)) \geq \gamma u(1-u)(V_k(0) - V_k(1))$. Moreover, $(\gamma-\lambda)(F(U_k(0)) - W_k(0)) \geq (\gamma-\lambda)u(V_k(1) - V_k(0))$. So, $\gamma(U_k(1) + F(U_k(1)) - F(U_k(0)) - U_k(2)) + (\gamma-\lambda)(F(U_k(0)) - W_k(0)) \geq u(V_k(0) - V_k(1))\frac{\lambda^2}{\lambda+\gamma} \geq 0$. This proves that the generally concave property hold for V_{k+1} for $x = 0$.

For $x > 0$, we have

$$F(V_{k+1}(x)) = -\frac{s\mu r_1\omega}{\gamma}\sum_{h=0}^{x-1} u\,(1-u)^h\,(x-h) + \gamma F(U_k(x)) + (1-\gamma-s\mu)F(V_k(x)) + s\mu F(F(V_k(x-h)))$$
$$+ (\gamma-\lambda)\,(1-u)^x\,(V_k(0) - V_k(1)),\ \text{so,}$$

$$V_{k+1}(x+1)+F(V_{k+1}(x+1))-V_{k+1}(x+2)-F(V_{k+1}(x))=\frac{s\mu r_1\omega}{\gamma}(1-u)^{x+1}$$
$$+\gamma(U_k(x+1)+F(U_k(x+1))-U_k(x+2)-F(U_k(x)))$$
$$+(1-\gamma-s\mu)(V_k(x+1)+F(V_k(x+1))-V_k(x+2)-F(V_k(x)))$$
$$+s\mu\sum_{h=0}^{x-1}u(1-u)^h(V_k(x+1-h)+F(V_k(x+1-h))-V_k(x+2-h)-F(V_k(x-h)))$$
$$+s\mu u(1-u)^x((1+u)V_k(1)-uV_k(0)-V_k(2))+s\mu u(1-u)^{x+1}(V_k(0)-V_k(1))$$
$$-(\lambda-\gamma)u(1-u)^x(V_k(1)-V_k(0)).$$

All terms in this expression are positive, except $-(\lambda-\gamma)u(1-u)^x(V_k(1)-V_k(0))$. This can be compensated by the first term proportional with γ. Using Equation (4), we may write

$$\gamma(U_k(x+1)+F(U_k(x+1))-U_k(x+2)-F(U_k(x)))+s\mu u(1-u)^{x+1}(V_k(0)-V_k(1))$$
$$-(\lambda-\gamma)u(1-u)^x(V_k(1)-V_k(0))$$
$$\geq(V_k(0)-V_k(1))\,u(1-u)^x(\lambda-\gamma+s\mu(1-u)+\gamma(1-u))$$
$$=(V_k(0)-V_k(1))\,u(1-u)^x\frac{\lambda^2+\gamma s\mu}{\lambda+\gamma}\geq 0.$$

This proves that V_k is generally concave for $x>0$ and finishes the proof of Theorem 1. □

Proof of Theorem 2

We prove the monotonicity properties given in Theorem 2. Given the similarities in the performance measures' expressions, we only provide the proof of the results of the reservation threshold for the *a posteriori* policy and of the outsourcing threshold for the *a priori* policy. The remaining cases can be proven in an identical way. Note that we only consider the case $\lambda\neq s\mu$ in the proofs. By letting λ tend to $s\mu$, our results also apply for $\lambda=s\mu$.

Monotonicity properties in the reservation threshold

We first prove that the probability of having $s-c$ customers in the system, $p_{s-c}=[\epsilon+\lambda J]^{-1}$, is strictly decreasing in λ. This result will be useful in the proof of the main results hereafter.

The probability p_{s-c} is strictly decreasing in λ. We have $p_{s-c}^{-1}=\sum_{x=0}^{c}\frac{(s-c)!a^x}{(s-c+x)!}+\frac{(s-c)!a^c}{s!}\frac{a}{s-a}\left(1-e^{-\tau(s\mu-\lambda)}\right)$. It is clear that the quantities $\sum_{x=0}^{c}\frac{(s-c)!a^x}{(s-c+x)!}$ and $\frac{(s-c)!a^c}{s!}$ are strictly increasing in λ. Taking now the derivative of $g(\lambda)=\frac{a}{s-a}\left(1-e^{-\tau(s\mu-\lambda)}\right)$ in λ, we may write

$$\frac{\partial g(\lambda)}{\partial\lambda}=\frac{1}{\mu}\frac{s(1-e^{-\tau\mu(s-a)}-a\tau\mu e^{-\tau\mu(s-a)})+a^2\tau\mu e^{-\tau\mu(s-a)}}{(s-a)^2}.$$

Let us define $h(\lambda)$ by $h(\lambda) = s(1 - e^{-\tau\mu(s-a)} - a\tau\mu e^{-\tau\mu(s-a)}) + a^2\tau\mu e^{-\tau\mu(s-a)}$. We have

$$\frac{\partial h(\lambda)}{\partial \lambda} = \frac{1}{\mu}\left(\tau\mu(a-s)e^{-\tau\mu(s-a)}\left(2+\tau\mu a\right)\right).$$

Thus, for $\lambda < s\mu$, $\frac{\partial h(\lambda)}{\partial \lambda} < 0$; and for $\lambda > s\mu$, $\frac{\partial h(\lambda)}{\partial \lambda} > 0$. Therefore, h is decreasing in λ for $\lambda \in [0, s\mu]$, and h is increasing in λ for $\lambda > s\mu$, thus h has a minimum at $\lambda = s\mu$. Let us define $\zeta = s - a$. We may then write $e^{-\tau\mu(s-a)} = 1 - \tau\mu\zeta + \frac{(\tau\mu\zeta)^2}{2!} + o(\zeta^2)$. Thus,

$$s(1 - e^{-\tau\mu(s-a)} - a\tau\mu e^{-\tau\mu(s-a)}) + a^2\tau\mu e^{-\tau\mu(s-a)} = \zeta^2\left(\tau\mu + s\frac{(\tau\mu)^2}{2}\right) + o(\zeta^2).$$

This leads to $\lim\limits_{\lambda \to s\mu} \frac{\partial g(\lambda)}{\partial \lambda} = \lim\limits_{\zeta \to 0}\left(\frac{1}{\mu}\frac{\zeta^2\left(\tau\mu + s\frac{(\tau\mu)^2}{2}\right)}{\zeta^2} + o(\zeta^2)\right) = \tau\mu + s\frac{(\tau\mu)^2}{2} > 0$. So, g is strictly increasing in λ. We then deduce that p_{s-c}^{-1} is strictly increasing in λ. □

Proof of the monotonicity results in c. We denote the outbound calls expected throughput by $E(T)(c)$ instead of $E(T)$ so as to specify that c is the reservation threshold from which $E(T)$ is derived. Recall that we may write $E(T)(c) = \frac{\mu}{\sum_{x=0}^{c}\frac{(s-c-1)!a^x}{(s-c+x)!} + \frac{(s-c-1)!a^c}{s!}\frac{a}{s}\frac{1}{1-a/s}\left(1-e^{-\tau(s\mu-\lambda)}\right)}$, for $0 \le c < s$ and $E(T)(s) = 0$ for $c = s$. Since $E(T)(s-1) > 0$, we have $E(T)(s-1) > E(T)(s)$. For $0 < c < s$, we have

$$\mu(E(T)(c)^{-1} - E(T)(c-1)^{-1}) = \frac{(s-c-1)!a^c}{s!} + \sum_{x=0}^{c-1}\frac{(s-c-1)!a^x}{(s-c+x)!}\left(1 - \frac{s-c}{s-c+1+x}\right) + \frac{(s-c-1)!a^c}{s!}\frac{a-(s-c)}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right).$$

Since $\sum_{x=0}^{c-1}\frac{(s-c-1)!a^x}{(s-c+x)!}\left(1 - \frac{s-c}{s-c+1+x}\right) > 0$, it remains to show that $\frac{(s-c-1)!a^c}{s!} + \frac{(s-c-1)!a^c}{s!}\frac{a-(s-c)}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) > 0$. This reduces to prove that $1 + \frac{a-(s-c)}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) \ge 0$. If $a > s$, we have $a - (s-c) > 0$ and $1 - e^{-\tau\mu(s-a)} < 0$. Thus, $\frac{a-(s-c)}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) > 0$ and $1 + \frac{a-(s-c)}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) > 0$. If $0 < a < s$, $1 + \frac{a-(s-c)}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) = 1 + \frac{c}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) - \left(1 - e^{-\tau\mu(s-a)}\right)$. Since $0 < 1 - e^{-\tau\mu(s-a)} < 1$ for $0 < a < s$, we get $1 + \frac{c}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) > 1$ and $1 + \frac{c}{s-a}\left(1 - e^{-\tau\mu(s-a)}\right) - \left(1 - e^{-\tau\mu(s-a)}\right) > 0$. In conclusion, $E(T)(c)^{-1} - E(T)(c-1)^{-1} > 0$ for $0 < c < s$ and $E(T)(s-1) > E(T)(s)$. The expected outbound throughput is thus strictly decreasing in c.

We now consider the second order monotonicity properties in c. For this purpose, let us define the sequence f_c as $f_c = \left[\sum_{x=0}^{c}\frac{s!a^{x-c}}{(s-c+x)!} + \frac{a}{s}\frac{1}{1-a/s}\left(1 - e^{-\tau(s\mu-\lambda)}\right)\right]^{-1}$, for $0 \le c \le s$. First we show that f_c is strictly decreasing and strictly convex in c for $0 \le c \le s$. For $0 \le c < s$, we have $f_{c+1}^{-1} - f_c^{-1} = s!\frac{a^{-(c+1)}}{(s-c-1)!} > 0$. Thus f_c is strictly decreasing in c for $0 \le c \le s$. We next focus on the proof of convexity of f_c in c (for $s \ge 2$). We do so by proving that $f_c + f_{c+2} - 2f_{c+1} > 0$, for $0 \le c \le s-2$. Since $f_c + f_{c+2} - 2f_{c+1} = f_c f_{c+1} f_{c+2}\left(f_{c+2}^{-1}f_{c+1}^{-1} + f_{c+1}^{-1}f_c^{-1} - 2f_{c+2}^{-1}f_c^{-1}\right)$, it suffices to prove that $f_{c+2}^{-1}f_{c+1}^{-1} + f_{c+1}^{-1}f_c^{-1} - 2f_{c+2}^{-1}f_c^{-1} > 0$, for $0 \le c \le s-2$. Observing that $f_{c+1}^{-1} = f_{c+2}^{-1} - s!\frac{a^{-(c+2)}}{(s-c-2)!}$ and $f_c^{-1} = f_{c+2}^{-1} - s!\frac{a^{-(c+1)}}{(s-c-1)!} - s!\frac{a^{-(c+2)}}{(s-c-2)!}$, we obtain

$$f_{c+2}^{-1}f_{c+1}^{-1} + f_{c+1}^{-1}f_c^{-1} - 2f_{c+2}^{-1}f_c^{-1} = s!\frac{a^{-(c+2)}}{(s-c-2)!}\left(f_{c+2}^{-1}\left(\frac{a}{s-c-1} - 1\right) + s!\frac{a^{-(c+2)}}{(s-c-2)!}\left(1 + \frac{a}{s-c-1}\right)\right),$$

for $0 \le c \le s-2$. Since $s!\frac{a^{-(c+2)}}{(s-c-2)!} > 0$, we need to prove that $f_{c+2}^{-1}\left(\frac{a}{s-c-1}-1\right) + s!\frac{a^{-(c+2)}}{(s-c-2)!}\left(1+\frac{a}{s-c-1}\right) > 0$, for $0 \le c \le s-2$. We have $f_{c+2}^{-1} = p_{s-(c+2)}^{-1} s!\frac{a^{-(c+2)}}{(s-c-2)!}$, where $p_{s-(c+2)}^{-1}$ is the probability of observing $c+2$ idle agents in a system with $c+2$ agents reserved for inbound calls. Therefore, we only need to show that $p_{s-(c+2)}^{-1}\left(\frac{a}{s-c-1}-1\right)+1+\frac{a}{s-c-1} > 0$, for $0 \le c \le s-2$. We have proven that $p_{s-(c+2)}^{-1}$ is strictly increasing in λ. Because $a = \lambda/\mu$, the quantity $p_{s-(c+2)}^{-1}\left(\frac{a}{s-c-1}-1\right)+1+\frac{a}{s-c-1}$ is also strictly increasing in λ. Moreover, using the expression of $p_{s-(c+2)}^{-1}$, we write

$$\lim_{\substack{\lambda \to 0 \\ \lambda>0}} p_{s-(c+2)}^{-1} = \lim_{\substack{\lambda \to 0 \\ \lambda>0}} \left(\sum_{x=0}^{c+2} \frac{(s-c-2)!a^x}{(s-c-2+x)!} + \frac{(s-c-2)!a^{c+2}}{s!}\frac{a}{s}\frac{1}{1-a/s}\left(1-e^{-\tau(s\mu-\lambda)}\right)\right).$$

Since $\lim_{\substack{\lambda \to 0 \\ \lambda>0}} \left(\frac{(s-c-2)!a^{c+2}}{s!}\frac{a}{s}\frac{1}{1-a/s}\left(1-e^{-\tau(s\mu-\lambda)}\right)\right) = 0$, and $\lim_{\substack{\lambda \to 0 \\ \lambda>0}} \left(\sum_{x=0}^{c+2}\frac{(s-c-2)!a^x}{(s-c-2+x)!}\right) = 1$, we get $\lim_{\substack{\lambda \to 0 \\ \lambda>0}} p_{s-(c+2)}^{-1} = 1$. This implies

$$\lim_{\substack{\lambda \to 0 \\ \lambda>0}} \left(p_{s-(c+2)}^{-1}\left(\frac{a}{s-c-1}-1\right)+1+\frac{a}{s-c-1}\right) = 0,$$

for $0 \le c \le s-2$. Using the fact that $p_{s-(c+2)}^{-1}\left(\frac{a}{s-c-1}-1\right)+1+\frac{a}{s-c-1}$ is strictly increasing in λ, we deduce that $p_{s-(c+2)}^{-1}\left(\frac{a}{s-c-1}-1\right)+1+\frac{a}{s-c-1} > 0$, for $a > 0$ and $0 \le c \le s-2$. This proves that f_c is a strictly decreasing and strictly convex function in c for $0 \le c \le s$. Since $P_{\overline{S}} = f_c e^{-\tau(s\mu-\lambda)}$ and $E(W) = \frac{s\mu}{(s\mu-\lambda)^2}\left(1-(1+\frac{a}{s}\tau(s\mu-\lambda))e^{-\tau(s\mu-\lambda)}\right) f_c$, $P_{\overline{S}}$ and $E(W)$ are strictly decreasing and strictly convex in c, for $0 \le c \le s$.

We may write $E(W_S) = \frac{s\mu}{(s\mu-\lambda)^2}\left(1-(1+\tau(s\mu-\lambda))e^{-\tau(s\mu-\lambda)}\right) f_c P_S^{-1}$, and
$P(W_S > t) = \mathbb{1}_{t<\tau}\frac{1}{1-a/s}\left(e^{-t(s\mu-\lambda)} - e^{-\tau(s\mu-\lambda)}\right) f_c P_S^{-1}$, where $P_S = 1 - P_{\overline{S}}$. Therefore, as a function of c, $E(W_S)$ and $P(W_S > t)$ are positively proportional to $f_c P_S^{-1}$. It then suffices to show that P_S^{-1} is strictly decreasing and strictly convex in c, for $0 \le c \le s$. Since P_S is strictly increasing in c, P_S^{-1} is strictly decreasing in c. We have

$$P_S^{-1}(c+2) - 2P_S^{-1}(c+1) + P_S^{-1}(c) = \frac{P_S(c+2)P_S(c+1) + P_S(c+1)P_S(c) - 2P_S(c+2)P_S(c)}{P_S(c+2)P_S(c+1)P_S(c)},$$

for $0 \le c \le s-2$. Using $P_S(c) = 1 - P_{\overline{S}}(c)$, we next obtain

$$\begin{aligned} &P_S(c+2)P_S(c+1) + P_S(c+1)P_S(c) - 2P_S(c+2)P_S(c) \\ &\quad = P_{\overline{S}}(c+2)(1-P_{\overline{S}}(c)) + P_{\overline{S}}(c)(1-P_{\overline{S}}(c+2)) + P_{\overline{S}}(c+1)(P_{\overline{S}}(c+2) - 2P_{\overline{S}}(c+1) + P_{\overline{S}}(c)). \end{aligned}$$

Since $P_{\overline{S}}(c)$ is strictly convex in c, $P_{\overline{S}}(c+2) - 2P_{\overline{S}}(c+1) + P_{\overline{S}}(c) > 0$, for $0 \le c \le s-2$. Moreover, $0 \le P_{\overline{S}}(c+2) < P_{\overline{S}}(c+1) < P_{\overline{S}}(c) \le 1$. Then, P_S^{-1} is strictly decreasing and strictly convex in c for $0 \le c \le s$. As a consequence, $f_c P_S^{-1}$ is a strictly decreasing and strictly convex function in c. Therefore, $E(W_S)$ and $P(W_S > t)$ are strictly decreasing and strictly convex in c, for $0 \le c \le s$. □

Monotonicity properties in the outsourcing threshold

First, we focus on the probability of outsourcing, $P_{\overline{S}}$.

Case 1, $a/s < 1$: Given that $(a/s)^{n+1}$ decreases in n, then $1-(a/s)^{n+1}$ increases in n. Therefore, $p_{s-c}^{-1} = \sum_{x=0}^{c-1} \frac{(s-c)!a^x}{(s-c+x)!} + \frac{(s-c)!a^c}{s!}\frac{1-(a/s)^{n+1}}{1-a/s}$ is also increasing in n, and p_{s-c} is thus decreasing in n. Observe that as a function of n, $P_{\overline{S}}$ is positively proportional to $(a/s)^n p_{s-c}$. Since $(a/s)^n$ decreases in n, $P_{\overline{S}}$ is decreasing in n. The function $(a/s)^n$ is decreasing and convex. The function $1-(a/s)^{n+1}$ is increasing and concave, thus p_{s-c}^{-1} is also increasing and concave. Using the notations $f' = \frac{\partial f(n)}{\partial n}$ and $f'' = \frac{\partial^2 f(n)}{\partial n^2}$ for a given function f, we have $p''_{s-c} = \frac{2((p_{s-c}^{-1})')^2 - p_{s-c}^{-1}(p_{s-c}^{-1})''}{p_{s-c}^{-3}} > 0$, since p_{s-c}^{-1} is positive, increasing and concave. Thus, $P_{\overline{S}}$ is also convex in n.

Case 2, $a/s > 1$: We may write $P_{\overline{S}} = \frac{(a/s)^n}{A+B(a/s)^n}$, where $A = \sum_{x=0}^{c-1}\frac{s!a^{x-c}}{(s-c+x)!} - \frac{1}{a/s-1}$ and $B = \frac{a/s}{a/s-1}$. We have $A < \frac{1-(a/s)^{-c}}{a/s-1} - \frac{1}{a/s-1} < 0$, given that $\frac{s!}{(s-c+x)!} < s^{c-x}$. After some algebra, we obtain $\frac{\partial P_{\overline{S}}}{\partial n} = \frac{A\ln(a/s)(a/s)^n}{[A+B(a/s)^n]^2} < 0$. Therefore, $P_{\overline{S}}$ is decreasing in n. We also have $\frac{\partial^2 P_{\overline{S}}}{\partial n^2} = \frac{A(\ln(a/s))^2(a/s)^n(A-B(a/s)^n)}{[A+B(a/s)^n]^3}$. Since $A<0$, $B>0$, $A-B(a/s)^n < 0$ and $A+B(a/s)^n > 0$, $\frac{\partial^2 P_{\overline{S}}}{\partial n^2} > 0$. Therefore, $P_{\overline{S}}$ is convex in n.

We now focus on the outbound call throughput. The outbound call throughput is positively proportional to p_{s-c}. For the case $a/s<1$, we already proved that p_{s-c} is decreasing and convex in n, therefore, $E(T)$ is also decreasing and convex in n for $a/s<1$. For the case $a/s>1$, we may write $p_{s-c} = [C + D(a/s)^n]^{-1}$, where $C = \sum_{x=0}^{c-1}\frac{(s-c)!a^x}{(s-c+x)!} - \frac{(s-c)!a^c}{s!(a/s-1)} < 0$ and $D = \frac{(s-c)!a^c}{s!(a/s-1)}\frac{a}{s} > 0$ (the proof of the signs of C and D is identical to that of A and B for the outsourcing probability). We thus have $p'_{s-c} = -\frac{D\ln(a/s)(a/s)^n}{(C+D(a/s)^n)^2} < 0$, so, $E(T)$ is decreasing in n. We also have $p''_{s-c} = \frac{D(\ln(a/s))^2(a/s)^n(-C+D(a/s)^n)}{(C+D(a/s)^n)^3} > 0$, thus, $E(T)$ is convex in n.

We now consider $E(W_S)$. The expected waiting time of served customers is positively proportional to $\phi(n) = \frac{1-(n+1)(a/s)^n + n(a/s)^{n+1}}{E+G\frac{1-(a/s)^n}{1-a/s}}$, where $E = \sum_{x=0}^{c-1}\frac{(s-c)!a^x}{(s-c+x)!} > 0$ and $G = \frac{(s-c)!a^c}{s!} > 0$. We have

$$\begin{aligned}\phi(n+1)-\phi(n) &= \frac{1-(n+2)(a/s)^{n+1}+(n+1)(a/s)^{n+2}}{E+G\frac{1-(a/s)^{n+1}}{1-a/s}} - \frac{1-(n+1)(a/s)^n+n(a/s)^{n+1}}{E+G\frac{1-(a/s)^n}{1-a/s}}\\ &= E\frac{(n+1)(a/s)^n(1-a/s)^2}{\left(E+G\frac{1-(a/s)^{n+1}}{1-a/s}\right)\left(E+G\frac{1-(a/s)^n}{1-a/s}\right)} + G(a/s)^n\frac{(a/s)^{n+1}-(n+1)(a/s)+n}{\left(E+G\frac{1-(a/s)^{n+1}}{1-a/s}\right)\left(E+G\frac{1-(a/s)^n}{1-a/s}\right)}.\end{aligned}$$

The first term in the sum is strictly positive. The sign of the second term depends on the sign of $v(a/s) = (a/s)^{n+1} - (n+1)(a/s) + n$. As a function of a/s, we have $v'(a/s) = (n+1)(a/s)^n - (n+1)$ and $v'(a/s) = 0$ is equivalent to $a/s = 1$. Thus, v has a minimum at $a/s=1$. Since $v(1)=0$, $\phi(n+1)-\phi(n) > 0$ and $E(W_S)$ is strictly increasing in n.

We now focus on $E(W)$. We have $E(W) = P_S E(W_S)$. Since both P_S and $E(W_S)$ are increasing in n, $E(W)$ is also increasing in n. We next focus on $P(W_S>t)$. We denote by $P(W_S>t)_n$ the probability that a customer is served within more than t units of time when the threshold is n. We can write

$$P(W_S>t)_{n+1} - P(W_S>t)_n = \frac{e^{-s\mu t}S\left(\sum_{x=0}^{n}\frac{(s\mu t)^x(a/s)^n}{x!}\left(G(1-(a/s)^x)+C(1-(a/s))\right)\right)}{(1-a/s)\left(E+G\frac{1-(a/s)^n}{1-a/s}\right)\left(E+G\frac{1-(a/s)^{n+1}}{1-a/s}\right)}.$$

We have $\frac{S(1-(a/s)^x)+C(1-(a/s))}{1-a/s} \geq 0$ for $x \geq 0$. Thus, $P(W_S>t)_{n+1} - P(W_S>t)_n \geq 0$ and $P(W_S>t)$ is increasing in n. □

Proof of Theorem 4

First, we assume that the two outsourcing policies have the same reservation thresholds and the same proportion of outsourced calls. Therefore, we can relate τ and n. This leads to $\tau = -n\frac{\ln(a/s)}{s\mu(1-a/s)}$. A difference between the outsourcing parameters is that n is an integer whereas τ is real. Thus, *a posteriori* outsourcing has an advantage in maximizing the call center revenue. We thus assume that n is real in order to obtain a fair comparison. In practice, this is equivalent to allowing randomization between the two adjacent thresholds n and $n+1$. We compare the probability of waiting more than a threshold t for served customers with *a priori* outsourcing and with *a posteriori* outsourcing. We denote by $P(W_S > t)_1$ and $P(W_S > t)_2$ the probability of waiting more than t with *a priori* outsourcing and with *a posteriori* outsourcing, respectively. Using $\tau = -n\frac{\ln(a/s)}{s\mu(1-a/s)}$, we may write

$$P(W_S > t)_1 - P(W_S > t)_2 = \frac{(s-c)!a^c}{s!} \frac{e^{-s\mu t}\sum_{x=0}^{n-1}\frac{(s\mu t)^x((a/s)^x-(a/s)^n)}{x!} - \mathbb{1}_{t<\tau}(e^{-t(s\mu-\lambda)} - \left(\frac{a}{s}\right)^n)}{(1-a/s)\left(\sum_{x=0}^{c-1}\frac{(s-c)!a^x}{(s-c+x)!} + \frac{(s-c)!a^c}{s!}\frac{1-(a/s)^n}{1-a/s}\right)}. \quad (5)$$

If $t \geq \tau$, then $P(W_S > t)_1 - P(W_S > t)_2 \geq 0$. Otherwise if $t < \tau$, we have

$$e^{-s\mu t}\sum_{x=0}^{n-1}\frac{(s\mu t)^x((a/s)^x-(a/s)^n)}{x!} - \mathbb{1}_{t<\tau}\left(e^{-t(s\mu-\lambda)} - \left(\frac{a}{s}\right)^n\right) = e^{-s\mu t}\left(\sum_{x=0}^{n-1}\frac{(\lambda t)^x}{x!} - e^{\lambda t}\right) \quad (6)$$
$$+ (a/s)^n\left(1 - e^{-s\mu t}\sum_{x=0}^{n-1}\frac{(s\mu t)^x}{x!}\right).$$

Since $\sum_{x=0}^{n-1}\frac{(\lambda t)^x}{x!} - e^{\lambda t} = -\sum_{x=n}^{\infty}\frac{(\lambda t)^x}{x!}$ and $1 - e^{-s\mu t}\sum_{x=0}^{n-1}\frac{(s\mu t)^x}{x!} = e^{-s\mu t}\sum_{x=n}^{\infty}\frac{(s\mu t)^x}{x!}$, the expression in the numerator of Equation (6) is equal to $e^{-s\mu t}\left(\sum_{x=n}^{\infty}\frac{(a/s)^n(s\mu t)^x-(\lambda t)^x}{x!}\right) = e^{-s\mu t}\left(\sum_{x=n}^{\infty}\frac{(s\mu t)^x((a/s)^n-(a/s)^x)}{x!}\right)$. This expression is positive for $a < s$ (because $(a/s)^x$ decreases in x) and negative otherwise. The expression of $P(W_S > t)_1 - P(W_S > t)_2$ in Equation (5) also has the term $1 - a/s$ in the denominator. This term is also positive for $a < s$ and is negative otherwise. Therefore, the probability of waiting more than t is higher with *a priori* outsourcing than with *a posteriori* outsourcing when the two policies have the same reservation threshold and the same proportion of outsourced calls. We can proceed similarly to compare $E(W)$ for the two policy classes. The details are omited.

Assume that (n, c) is the optimal threshold couple for *a priori* outsourcing (i.e., the thresholds defining Policy π_e^*). Consider now the threshold couple $(\tau = -n\frac{\ln(a/s)}{s\mu(1-a/s)}, c)$ for a given threshold policy in Ω_l. With this threshold couple, the outbound throughput and the proportion of outsourced calls is identical under both outsourcing policies. However, $E(W_S)$ is lower for the policy considered in Ω_l with the couple. So, we find a threshold couple for which a given threshold policy in Ω_l outperforms the optimal revenue for the optimal policy in Ω_e. This proves that *a posteriori* outsourcing outperforms *a priori* outsourcing in maximizing the expected revenue. □