Online Appendix

The unabridged version of the appendix can be found on SSRN[1]. This online appendix is divided into six parts: In Part A, we present a horizontal differentiation model that considers utility-maximizing consumers, from which we show how the demand function in our model can be derived; Part B presents a preliminary result (Lemma A.1) which is needed for the ensuring proofs in Parts D and E; Part C shows the proof for the benchmark cases (Section 3.2); Parts D and E present the proofs for the exogenous inventory case (Section 4) and the endogenous inventory case (Section 5), respectively; Part F provides additional results that are omitted from the main paper. For easy reference, we summarize the key notations in Table A.1 below.

Table A.1: Key Notations

Notation	Description
I_i	Retailer i's pre-season inventory, $i = 1, 2$
s_i	Retailer i's service-quality level, $i = 1, 2$
k	Cost factor for service quality
c	Marginal cost for pre-season inventory
$\bar{c}$	Marginal cost of quick-response order, $\bar{c} > c$
p	Retail price of the product, $p \geq \bar{c} > c$
β	Demand sensitivity factor to service, $0 < \beta \leq \bar{\beta}$
δ	Unit shipping cost of transferring inventory from one retailer to the other, $0 \leq \delta < \bar{c}$
t	Unit transfer price of shared inventory between the two retailers, $0 \leq t \leq \bar{c} - \delta$
π_i	Retailer i's profit, $i = 1, 2$
ε_i	Random factor that influences retailer i's demand, $i = 1, 2$
μ	Mean of ε_i
g_{12}	Joint pdf of ε_1 and ε_2

A. Horizontal Differentiation Model

Consider a continuum of consumers in the market with a total mass normalized to 1, who are uniformly located on a Hotelling line from 0 to 1. The two retailers are located on the two ends of the line. A consumer on location $x \in [0, 1]$ expects to receive a utility of $u_1(s_1, x) = v + s_1 - p - bx$ if buying from retailer 1, and a utility of $u_2(s_2, x) = v + s_2 - p - b(1 - x)$ if buying from retailer 2, where $v > 0$ is the consumer's valuation for the product component, s_i is the service quality of retailer i, $b > 0$ captures the level of horizontal differentiation between the two retailers, and x follows a uniform distribution on $[0, 1]$ and represents the consumer's horizontal preference. Assume that v is large enough so that the market is fully covered by the two retailers in the equilibrium. A consumer on location x will visit retailer i if and only if $u_i(s_i, x) \geq \max\{u_j(s_j, x), 0\}$, where $i, j = 1, 2$ and $i \neq j$. Each consumer's final purchase decision is subject to the influence of random interruptions. As a result, only a random fraction of consumers will eventually complete the purchase process at the visited stores, and the rest will leave the market without making a purchase. Let $D_i = \varepsilon_i \cdot d_i(s_i, s_j)$ represent the total number of consumers who finally complete purchases from retailer i, where ε_i is the random fraction distributed on $[0, 1]$ and $d_i(s_i, s_j) = \frac{1}{2} + \frac{s_i - s_j}{2b}$ denotes the total number of potential consumers for retailer i, for $i = 1, 2$. Define $\beta = \frac{1}{2b}$. Then the above demand function is identical to that in our model.

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[1] See https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3512712

B. Preliminary Lemma

We summarize and prove a preliminary result in Lemma A.1. In the following Lemma A.1, $Z(\cdot)$ is a general function, and ε_i is a random variable that follows a distribution with a probability density function (pdf) of $g(\cdot)$ and a corresponding cumulative distribution function (cdf) of $G(\cdot)$ on the interval $[0,1]$, for $i=1,2$. The joint distribution of $(\varepsilon_1,\varepsilon_2)$ has a general pdf of $g_{12}(\cdot)$ and cdf of $G_{12}(\cdot)$, which satisfy that $g_{12}(x,y)=g_{12}(y,x)$ and $G_{12}(x,y)=G_{12}(y,x)$.

Lemma A.1. *For any $I\in[0,1]$, the following equations hold:*

(a)$\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2} Z(\varepsilon_1,\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2=\int_{2I}^{1}(\int_0^{4I-\varepsilon_2} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$*;*

(b)$\int_0^{2I}(\int_{4I-\varepsilon_2}^{1} Z(\varepsilon_1,\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2=\int_{2I}^{1}(\int_{4I-\varepsilon_2}^{2I} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$*;*

(c)$\int_0^{1}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1=0$*;*

(d)$\int_0^{2I} g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1=\int_{2I}^{1} g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1$*;*

(e)$\int_0^{2I}(2I-\varepsilon_1)^2g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1=\int_{2I}^{1}(2I-\varepsilon_1)^2g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1$*;*

Proof of Lemma A.1. (a) According to Fubini's Theorem, we have

$$\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2} Z(\varepsilon_1,\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2=\int_0^{2I}(\int_{2I}^{4I-y} Z(x,y)g_{12}(x,y)dx)dy=\int_{2I}^{4I}(\int_0^{4I-x} Z(x,y)g_{12}(x,y)dy)dx.$$

Let $x=\varepsilon_2$ and $y=\varepsilon_1$. Since $g_{12}(x,y)=g_{12}(y,x)$, we have

$$\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2} Z(\varepsilon_1,\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2=\int_{2I}^{4I}(\int_0^{4I-\varepsilon_2} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2.$$

Moreover, $\int_{4I}^{1}(\int_0^{4I-\varepsilon_2} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2=0$ since $g_{12}(\varepsilon_1,\varepsilon_2)=0$ if $(\varepsilon_1,\varepsilon_2)\notin[0,1]\times[0,1]$. Thus, we have

$$\begin{aligned}\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2} Z(\varepsilon_1,\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2&=\int_{2I}^{4I}(\int_0^{4I-\varepsilon_2} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2+\int_{4I}^{1}(\int_0^{4I-\varepsilon_2} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\&=\int_{2I}^{1}(\int_0^{4I-\varepsilon_2} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2.\end{aligned}$$

(b) Similar to the proof of (a), according to Fubini's Theorem, we have

$$\begin{aligned}\int_0^{2I}(\int_{4I-\varepsilon_2}^{1} Z(\varepsilon_1,\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2&=\int_0^{2I}(\int_{4I-y}^{1} Z(x,y)g_{12}(x,y)dx)dy=\int_{2I}^{1}(\int_{4I-x}^{2I} Z(x,y)g_{12}(x,y)dy)dx\\&=\int_{2I}^{1}(\int_{4I-\varepsilon_2}^{2I} Z(\varepsilon_2,\varepsilon_1)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2.\end{aligned}$$

(c) To show Lemma A.1(c), we just need to show:

$$\int_0^{2I}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1+\int_{2I}^{1}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1=0.$$

Since $\int_{4I}^{1}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1=0$, we just need to show:

$$\int_0^{2I}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1+\int_{2I}^{4I}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1=0.$$

Let $x=4I-\varepsilon_1$. Then, we have

$$\begin{aligned}\int_{2I}^{4I}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1&=\int_0^{2I}(x-2I)g_{12}(x,4I-x)dx=\int_0^{2I}(\varepsilon_1-2I)g_{12}(\varepsilon_1,4I-\varepsilon_1)d\varepsilon_1\\&=-\int_0^{2I}(2I-\varepsilon_1)g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1.\end{aligned}$$

(d) Since $\int_{4I}^{1} g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1 = 0$, we just need to show

$$\int_0^{2I} g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1 = \int_{2I}^{4I} g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1.$$

Similar to the proof of (a), let $x = 4I - \varepsilon_1$, and we have

$$\int_0^{2I} g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1 = -\int_{4I}^{2I} g_{12}(x,4I-x)dx = \int_{2I}^{4I} g_{12}(4I-x,x)dx = \int_{2I}^{4I} g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1.$$

(e) Since $\int_{4I}^{1}(2I-\varepsilon_1)^2 g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1 = 0$, we just need to show

$$\int_0^{2I} (2I-\varepsilon_1)^2 g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1 = \int_{2I}^{4I} (2I-\varepsilon_1)^2 g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1.$$

Similar to the proof of (a), let $x = 4I - \varepsilon_1$, and we have

$$\begin{aligned}\int_0^{2I} (2I-\varepsilon_1)^2 g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1 &= \int_{4I}^{2I} (x-2I)^2 g_{12}(x,4I-x)d(4I-x) = \int_{2I}^{4I} (2I-x)^2 g_{12}(x,4I-x)d(x)\\ &= \int_{2I}^{4I} (2I-\varepsilon_1)^2 g_{12}(4I-\varepsilon_1,\varepsilon_1)d\varepsilon_1\end{aligned}$$

This completes the proof of Lemma A.1. □

C. Benchmark Cases

Proof of Lemma 1. We first show the optimal inventory and profit in the centralized case, and then show the equilibrium in the decentralized case without competition. Finally, we prove parts (a)-(c) in Lemma 1.

Centralized System. It is clear that the centralized firm will set zero service quality for each retailer to save cost, i.e., $s_1^c = s_2^c = 0$. Thus, the demand of retailer i is $D_i = \frac{1}{2}\varepsilon_i$. Inventory sharing between the retailers will naturally occur at a unit shipping cost δ. Thus, a centralized firm's problem is given by

$$\begin{aligned}\max_{I_1,I_2} \pi^c(I_1,I_2) =& p\cdot E[\frac{1}{2}\varepsilon_1+\frac{1}{2}\varepsilon_2] - \bar{c}\cdot E[(\frac{1}{2}\varepsilon_1+\frac{1}{2}\varepsilon_2 - I_1 - I_2)^+]\\ &- \delta\cdot(E[(\min\{I_1-\frac{1}{2}\varepsilon_1, \frac{1}{2}\varepsilon_2 - I_2\})^+ + (\min\{I_2-\frac{1}{2}\varepsilon_2,\frac{1}{2}\varepsilon_1 - I_1\})^+]) - c(I_1+I_2).\end{aligned} \quad \text{(A.1)}$$

We first check the second-order condition:

$$\begin{aligned}&\frac{d^2\pi^c(I_1,I_2)}{dI_i^2} = -2\delta g(2I_i) - 2(\bar{c}-\delta)\int_0^1 g_{12}(2I_1+2I_2-\varepsilon,\varepsilon)d\varepsilon \le 0,\\ &\frac{d^2\pi^c(I_1,I_2)}{dI_1dI_2} = -2(\bar{c}-\delta)\int_0^1 g_{12}(2I_1+2I_2-\varepsilon,\varepsilon)d\varepsilon, \text{and}\\ &\frac{d^2\pi^c(I_1,I_2)}{dI_1^2}\frac{d^2\pi^c(I_1,I_2)}{dI_2^2} - (\frac{d^2\pi^c(I_1,I_2)}{dI_1dI_2})^2 = 4\delta^2 g(2I_1)g(2I_2) + 4\delta(\bar{c}-\delta)(g(2I_1)+g(2I_2))\int_0^1 g_{12}(2I_1+2I_2-\varepsilon,\varepsilon)d\varepsilon \ge 0.\end{aligned}$$

Thus, $\pi^c(I_1,I_2)$ is jointly concave in I_1 and I_2. Therefore, the optimal inventory can be obtained by solving the first-order-condition $\frac{d\pi^c(I_1,I_2)}{dI_1} = 0$ and $\frac{d\pi^c(I_1,I_2)}{dI_2} = 0$. So we can get optimal inventory level $I_1^c = I_2^c = I^c$ that satisfies:

$$\begin{aligned}Y^c(I^c) = &\int_0^{2I^c}(\int_{4I^c-\varepsilon_2}^{1} \bar{c} g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{2I^c}^{4I^c-\varepsilon_2} \delta g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 +\\ &\int_{2I^c}^{1}(\int_{2I^c}^{1} \bar{c} g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{4I^c-\varepsilon_2}^{2I^c}(\bar{c}-\delta) g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 - c = 0.\end{aligned} \quad \text{(A.2)}$$

The centralized firm's optimal total profit from the two retailers is given as below:

$$\pi^{c*} = 2\pi_i^{c*} = 2(p\cdot E[\frac{1}{2}\varepsilon_i] - \delta\cdot E[(\min\{\frac{1}{2}\varepsilon_i - I^c, I^c - \frac{1}{2}\varepsilon_j\})^+] - \bar{c}\cdot E[(\frac{1}{2}\varepsilon_i - I^c - (I^c - \frac{1}{2}\varepsilon_j))^+] - cI^c). \quad \text{(A.3)}$$

Decentralized System without Competition. It is obvious that each retailer's service quality will be zero when $\beta = 0$ since it has no impact on demand. Thus, the demand of retailer i is $D_i = \frac{1}{2}\varepsilon_i$. We first solve the case without inventory sharing and then the case with inventory sharing.

Without Inventory Sharing. Retailer i's profit in this case is given by

$$\pi_i^{nc-N}(I_i) = p \cdot E[\frac{1}{2}\varepsilon_i] - \bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I_i)^+] - cI_i, \text{for } i = 1, 2. \tag{A.4}$$

When the inventory is an endogenous decision, retailer i chooses I_i to maximize the above profit. We can verify that $\frac{d^2\pi_i^{nc-N}(I_i)}{dI_i^2} = -2\bar{c}g(2I_i) \leq 0$, i.e., $\pi_i^{nc-N}(I_i)$ is concave in I_i. Thus, retailer i's optimal inventory can be obtained by solving the first-order-condition $\frac{d\pi_i^{nc-N}(I_i)}{dI_i} = \int_0^{2I_i} \bar{c}g(\varepsilon)d\varepsilon - c = 0$. Thus, the optimal inventory level is: $I_1^{nc-N} = I_2^{nc-N} = I^{nc-N} = G^{-1}(1 - \frac{c}{\bar{c}})/2$; and the retailer i's corresponding profit is $\pi_i^{nc-N*} = \pi_i^{nc-N}(I^{nc-N})$.

With Inventory Sharing. Retailer i's profit in this case is given by

$$\begin{aligned} \pi_i^{nc}(t, I_i|I_j) =& p \cdot E[\frac{1}{2}\varepsilon_i] + t \cdot E[(\min\{I_i - \frac{1}{2}\varepsilon_i, \frac{1}{2}\varepsilon_j - I_j\})^+] - (t+\delta) \cdot E[(\min\{\frac{1}{2}\varepsilon_i - I_i, I_j - \frac{1}{2}\varepsilon_j\})^+] \\ & - \bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I_i - (I_j - \frac{1}{2}\varepsilon_j)^+)^+] - cI_i \end{aligned} \tag{A.5}$$

When the the inventory is an endogenous decision, retailer i chooses I_i to maximize the above profit. We can verify that

$$\begin{aligned} \frac{d^2\pi_i^{nc}(t, I_i|I_j)}{dI_i^2} =& -2\int_0^{2I_j} ((t+\delta)g_{12}(2I_i, \varepsilon_j) + (\bar{c} - t - \delta)g_{12}(2I_i + 2I_j - \varepsilon_j, \varepsilon_j))d\varepsilon_j \\ & -2\int_{2I_j}^1 ((\bar{c} - t)g_{12}(2I_i, \varepsilon_j) + tg_{12}(2I_i + 2I_j - \varepsilon_j, \varepsilon_j))d\varepsilon_j < 0. \end{aligned}$$

That is, $\pi_i^{nc}(t, I_i|I_j)$ is concave in I_i. Therefore, the two retailers' optimal inventory levels can be obtained by solving the first-order-condition $\frac{d\pi_1^{nc}(t, I_1|I_2)}{dI_1} = 0$ and $\frac{d\pi_2^{nc}(t, I_2|I_1)}{dI_2} = 0$ together. So the equilibrium inventory level $I_1^{nc} = I_2^{nc} = I^{nc}$ satisfies:

$$\begin{aligned} Y^{nc}(I^{nc}, t) = & \int_0^{2I^{nc}} (\int_{4I^{nc} - \varepsilon_2}^1 \bar{c}g_{12}(\varepsilon_1, \varepsilon_2)d\varepsilon_1 + \int_{2I^{nc}}^{4I^{nc} - \varepsilon_2} (t+\delta)g_{12}(\varepsilon_1, \varepsilon_2)d\varepsilon_1)d\varepsilon_2 + \\ & \int_{2I^{nc}}^1 (\int_{2I^{nc}}^1 \bar{c}g_{12}(\varepsilon_1, \varepsilon_2)d\varepsilon_1 + \int_{4I^{nc} - \varepsilon_2}^{2I^{nc}} tg_{12}(\varepsilon_1, \varepsilon_2)d\varepsilon_1)d\varepsilon_2 - c = 0. \end{aligned} \tag{A.6}$$

Retailer i's equilibrium profit in the sharing case is then $\pi_i^{nc*}(t) = \pi_i^{nc}(t, I^{nc}|I^{nc})$.

Next, we prove parts (a) to (c) in Lemma 1.

(a) We first consider the case when the inventory is exogenous given, and then consider the endogenous-inventory case. In the exogenous-inventory case, we just need to show that, given $I_1 = I_2 = I \in [0, 1]$, $\pi_i^{nc}(t, I|I) > \pi_i^{nc-N}(I)$ holds for any transfer price $t \in [0, \bar{c} - \delta]$, where $\pi_i^{nc-N}(\cdot)$ and $\pi_i^{nc}(\cdot)$ are given by Equations (A.4) and (A.5). We can verify that:

$$\begin{aligned} \pi_i^{nc}(t, I|I) - \pi_i^{nc-N}(I) =& t \cdot E[(\min\{I - \frac{1}{2}\varepsilon_i, \frac{1}{2}\varepsilon_j - I\})^+] - (t+\delta) \cdot E[(\min\{\frac{1}{2}\varepsilon_i - I, I - \frac{1}{2}\varepsilon_j\})^+] \\ & - \bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I - (I - \frac{1}{2}\varepsilon_j)^+)^+] - (-\bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I)^+]) \\ =& (\bar{c} - \delta) \cdot E[(\min\{\frac{1}{2}\varepsilon_i - I, I - \frac{1}{2}\varepsilon_j\})^+] \geq 0. \end{aligned}$$

$$\begin{aligned} (\text{since } E[(\min\{I - \frac{1}{2}\varepsilon_i, \frac{1}{2}\varepsilon_j - I\})^+] =& E[(\min\{\frac{1}{2}\varepsilon_i - I, I - \frac{1}{2}\varepsilon_j\})^+] \text{ and} \\ E[(\frac{1}{2}\varepsilon_i - I)^+] =& E[(\min\{\frac{1}{2}\varepsilon_i - I, I - \frac{1}{2}\varepsilon_j\})^+ + (\frac{1}{2}\varepsilon_i - I - (I - \frac{1}{2}\varepsilon_j)^+)^+] \\ =& E[(\min\{\frac{1}{2}\varepsilon_i - I, I - \frac{1}{2}\varepsilon_j\})^+] + E[(\frac{1}{2}\varepsilon_i - I - (I - \frac{1}{2}\varepsilon_j)^+)^+] \quad) \end{aligned}$$

Thus, the retailers prefer to share inventory for any $t \in [0, \bar{c} - \delta]$ when inventory is exogenously given.

In the endogenous-inventory case, we just need to show that $\pi_i^{nc*}(t) = \pi_i^{nc}(t, I^{nc}|I^{nc}) > \pi_i^{nc-N*} = \pi_i^{nc-N}(I^{nc-N})$ for any $t \in [0, \bar{c} - \delta]$. Note that in the sharing case, given $I_j = I^{nc}$, $I_i = I^{nc}$ is the unique solution that maximizes the retailer i's profit. That is, $\pi_i^{nc*}(t) = \max_{I_i} \pi_i^{nc}(t, I_i|I^{nc}) > \pi_i^{nc}(t, I^{nc-N}|I^{nc})$, given $I^{nc-N} \neq I^{nc}$. Next, we will verify $\pi_i^{nc}(t, I^{nc-N}|I^{nc}) > \pi_i^{nc-N*} = \pi_i^{nc-N}(I^{nc-N})$:

$$\begin{aligned}
\pi_i^{nc*}(t) - \pi_i^{nc-N*} &> \pi_i^{nc}(t, I^{nc-N}|I^{nc}) - \pi_i^{nc-N}(I^{nc-N}) \\
&= t \cdot E[(\min\{I^{nc-N} - \frac{1}{2}\varepsilon_i, \frac{1}{2}\varepsilon_j - I^{nc}\})^+] - (t+\delta) \cdot E[(\min\{\frac{1}{2}\varepsilon_i - I^{nc-N}, I^{nc} - \frac{1}{2}\varepsilon_j\})^+] \\
&\quad - \bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I^{nc-N} - (I^{nc} - \frac{1}{2}\varepsilon_j)^+)^+] - (-\bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I^{nc-N})^+]) \\
&= t \cdot E[(\min\{I^{nc-N} - \frac{1}{2}\varepsilon_i, \frac{1}{2}\varepsilon_j - I^{nc}\})^+] + (\bar{c} - t - \delta) \cdot E[(\min\{\frac{1}{2}\varepsilon_i - I^{nc-N}, I^{nc} - \frac{1}{2}\varepsilon_j\})^+] > 0.
\end{aligned}$$

$$\begin{aligned}
(\text{since } E[(\frac{1}{2}\varepsilon_i - I^{nc-N})^+] &= E[(\min\{\frac{1}{2}\varepsilon_i - I^{nc-N}, I^{nc} - \frac{1}{2}\varepsilon_j\})^+ + (\frac{1}{2}\varepsilon_i - I^{nc-N} - (I^{nc} - \frac{1}{2}\varepsilon_j)^+)^+] \\
&= E[(\min\{\frac{1}{2}\varepsilon_i - I^{nc-N}, I^{nc} - \frac{1}{2}\varepsilon_j\})^+] + E[(\frac{1}{2}\varepsilon_i - I^{nc-N} - (I^{nc} - \frac{1}{2}\varepsilon_j)^+)^+] \quad)
\end{aligned}$$

Therefore, for any $t \in [0, \bar{c} - \delta]$, $\pi_i^{nc*}(t) > \pi_i^{nc-N*}$ holds, and the retailers prefer to share inventory when inventory is an endogenous decision,

(b) Given the exogenous inventory $I_1 = I_2 = I \in [0, 1]$, retailer i's profit in the sharing case can be simplified as below:

$$\begin{aligned}
\pi_i^{nc}(t, I|I) &= p \cdot E[\frac{1}{2}\varepsilon_i] + t \cdot E[(\min\{I - \frac{1}{2}\varepsilon_i, \frac{1}{2}\varepsilon_j - I\})^+] - (t+\delta) \cdot E[(\min\{\frac{1}{2}\varepsilon_i - I, I - \frac{1}{2}\varepsilon_j\})^+] \\
&\quad - \bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I - (I - \frac{1}{2}\varepsilon_j)^+)^+] - cI \\
&= p \cdot E[\frac{1}{2}\varepsilon_i] - \delta \cdot E[(\min\{\frac{1}{2}\varepsilon_i - I, I - \frac{1}{2}\varepsilon_j\})^+] - \bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - I - (I - \frac{1}{2}\varepsilon_j)^+)^+] - cI.
\end{aligned}$$

Thus, $\pi_i^{nc}(t, I|I)$ is independent in the transfer price t.

(c) It is obvious that the optimal inventory in the centralized case, I^c, is independent in t, since it is solved from $Y^c(I) = 0$ (Equation (A.2)). Moreover, we can verify that $Y^c(I)$ continuously decreases in I:

$$\frac{\partial Y^c(I)}{\partial I} = -2\delta g(2I) - 4(\bar{c} - \delta) \int_0^1 g_{12}(4I - \varepsilon, \varepsilon) d\varepsilon < 0.$$

Next, we will show that the inventory in the sharing case of decentralized system, I^{nc}, continuously increases in t. I^{nc} is solved from $Y^{nc}(I, t) = 0$ (Equation (A.6)). Moreover, we can verify that $Y^{nc}(I, t)$ decreases in I and increases in t:

$$\begin{aligned}
\frac{\partial Y^{nc}(I, t)}{\partial I} &= -\int_{2I}^1 (2(\bar{c} - t) g_{12}(2I, \varepsilon_2) + 4t g_{12}(4I - \varepsilon_2, \varepsilon_2)) d\varepsilon_2 \\
&\quad - \int_0^{2I} (2(t+\delta) g_{12}(2I, \varepsilon_2) + 4(\bar{c} - t - \delta) g_{12}(4I - \varepsilon_2, \varepsilon_2)) d\varepsilon_2 < 0, \\
\frac{\partial Y^{nc}(I, t)}{\partial t} &= \int_0^{2I} (\int_{2I}^{4I - \varepsilon_2} g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2 + \int_{2I}^1 (\int_{4I - \varepsilon_2}^{2I} g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2 > 0.
\end{aligned}$$

According to Chain Rule, we have $\frac{\partial I^{nc}}{\partial t} = -\frac{\partial Y^{nc}(I,t)}{\partial t} / \frac{\partial Y^{nc}(I,t)}{\partial I}|_{I = I^{nc}} > 0$. Therefore, I^{nc} continuously increases in t.

Then, we will show that $I^{nc}|_{t=0} \leq I^c$ and $I^{nc}|_{t=\bar{c}-\delta} \geq I^c$: (1) When $t = 0$, one can verify that $Y^{nc}(I, 0) - Y^c(I) = -(\bar{c} - \delta) \int_{2I}^1 (\int_{4I-\varepsilon_2}^{2I} g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 d\varepsilon_2) \leq 0$ for any $I \in [0, 1]$. Moreover, since $\frac{\partial Y^{nc}(I,t)}{\partial I} < 0$ and $\frac{\partial Y^c(I)}{\partial I} < 0$, we have $I^{nc}|_{t=0} \leq I^c$. (2) When $t = \bar{c} - \delta$, one can verify that $Y^{nc}(I, \bar{c} - \delta) - Y^c(I) = (\bar{c} - \delta)(\int_0^{2I} \int_{2I}^{4I-\varepsilon_2} g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 d\varepsilon_2) \geq 0$ for any $I \in [0, 1]$. Since since $\frac{\partial Y^{nc}(I,t)}{\partial I} < 0$ and $\frac{\partial Y^c(I)}{\partial I} < 0$, we have $I^{nc}|_{t=\bar{c}-\delta} \geq I^c$.

Combining the above result with the one that $I^{nc}(t)$ continuously increases in t and I^c is independent in t, we can conclude that there exists a $t^{nc} \in [0, \bar{c} - \delta]$ such that $I^{nc}|_{t=t^{nc}} = I^c$, and thus $\pi_i^{nc*}|_{t=t^{nc}} = \pi_i^{c*}$. $\square$

D. Exogenous Inventory

In the exogenous inventory case with $I_1 = I_2 = I \in [0,1]$, we define $\bar{\beta} = \min\{\hat{\beta}_1(I), \hat{\beta}_2(I)\}$ and assume that $\beta \le \bar{\beta}$ to ensure that each retailer's profit is non-negative in the equilibrium of both inventory sharing and no-sharing cases. Note that:

$$\begin{aligned}
&\hat{\beta}_1(I) = \frac{\sqrt{4k \cdot (p \cdot \mu/2 - \bar{c}\int_{2I}^{1}(\varepsilon_i/2 - I)g(\varepsilon_i)d\varepsilon_i - cI)}}{(p\mu - \bar{c}\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon)} \text{ and} \\
&\hat{\beta}_2(I) = \min\left\{\frac{\sqrt{4k \cdot (p \cdot \mu/2 - F_1 - F_2 - F_3 - cI)}}{(p\mu - A - B)}, \frac{\sqrt{k \cdot [(t+\delta)V_0 + (\bar{c}-t)V_1 + (\bar{c}-\delta)E_0]}}{2\left[I((t+\delta)V_0 + (\bar{c}-t)V_1) + (\bar{c} - 2t - \delta)C_0\right]}\right\},
\end{aligned} \tag{A.7}$$

where $F_1 = \delta\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2 - I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$,
$F_2 = \int_0^{2I}(\int_{4I-\varepsilon_2}^{1}(\delta(I - \varepsilon_2/2) + \bar{c}(\varepsilon_1/2 + \varepsilon_2/2 - 2I))g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$, $F_3 = \bar{c}\int_{2I}^{1}(\int_{2I}^{1}(\varepsilon_1/2 - I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$,
$A = \int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(t+\delta)\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{4I-\varepsilon_2}^{1}(\bar{c}\varepsilon_1 - \bar{c}\varepsilon_2 + (t+\delta)\varepsilon_2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$,
$B = \int_{2I}^{1}(\int_0^{4I-\varepsilon_2} t\varepsilon_2 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{4I-\varepsilon_2}^{2I} t\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{2I}^{1}\bar{c}\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2))d\varepsilon_2$,
$V_0 = \int_0^{2I} g_{12}(2I,\varepsilon)d\varepsilon$, $V_1 = \int_{2I}^{1} g_{12}(2I,\varepsilon)d\varepsilon$, $C_0 = \int_0^{2I}(2I - \varepsilon)g_{12}(4I - \varepsilon,\varepsilon)d\varepsilon$, and $E_0 = \int_0^{2I} g_{12}(4I - \varepsilon,\varepsilon)d\varepsilon$.

Proof of Lemma 2. (a) Retailer i's problem in the case without inventory sharing is given by

$$\max_{s_i} \pi_i^N(s_i, I_i|s_j) = p \cdot E[\varepsilon_i \cdot d_i] - \bar{c} \cdot E[(\varepsilon_i \cdot d_i - I_i)^+] - ks_i^2 - cI_i, \text{ for } i,j = 1,2 \text{ and } i \ne j,$$

where $d_i = \frac{1}{2} + \beta(s_i - s_j)$. We can rewrite $\pi_i^N(s_i, I_i|s_j)$ as: $\pi_i^N(s_i, I_i|s_j) = p \cdot \mu \cdot d_i - \bar{c}\int_{I_i/d_i}^{1}(\varepsilon_i d_i - I_i)g(\varepsilon_i)d\varepsilon_i - ks_i^2 - cI_i$. We can verify: $\frac{d^2\pi_i^N(s_i,I_i|s_j)}{ds_i^2} = -\frac{2b\bar{c}I_i^2 g(\frac{I_i}{d_i})}{(\frac{1}{2\beta} + s_i - s_j)^3} - 2k < 0$, i.e., $\pi_i^N(s_i, I_i|s_j)$ is concave in s_i.

Therefore, retailer i's optimal service quality can be solved from the first-order-condition as below:

$$\frac{d\pi_i^N(s_i, I_i|s_j)}{ds_i} = \beta p\mu - \bar{c}\beta\int_{I/d_i}^{1}\varepsilon_i g(\varepsilon_i)d\varepsilon_i - 2ks_i = 0.$$

Solving $\frac{d\pi_1^N(s_1,I|s_2)}{ds_1} = 0$ and $\frac{d\pi_2^N(s_2,I|s_1)}{ds_2} = 0$ together, we can get service quality in the symmetric equilibrium:

$$\hat{s}_1^N(I) = \hat{s}_2^N(I) = \hat{s}^N(I) = \frac{\beta(p\mu - \bar{c}\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon)}{2k}. \tag{A.8}$$

(b) Taking derivative of $\hat{s}^N(I)$ with respect to I, β, and $\bar{c}$:

$$\frac{\partial\hat{s}^N(I)}{\partial I} = \frac{2\beta\bar{c}Ig(2I)}{k} > 0, \quad \frac{\partial\hat{s}^N(I)}{\partial\beta} = \frac{p\mu - \bar{c}\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon}{2k} > 0, \text{ and } \frac{\partial\hat{s}^N(I)}{\partial\bar{c}} = \frac{-\beta\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon}{2k} < 0.$$

Thus, $\hat{s}^N(I)$ increases in I and β, and decreases in $\bar{c}$.

(c) The corresponding profit of retailer i in the case without inventory sharing is given as below:

$$\hat{\pi}_i^N(I) = \pi_i^N(\hat{s}^N(I), I|\hat{s}^N(I)) = p \cdot \mu/2 - \bar{c}\int_{2I}^{1}(\varepsilon_i/2 - I)g(\varepsilon_i)d\varepsilon_i - k(\hat{s}^N(I))^2 - cI. \tag{A.9}$$

Taking derivative of $\hat{\pi}_i^N(I)$ with respect to I, β, and $\bar{c}$:

$$\begin{aligned}
&\frac{\partial\hat{\pi}_i^N(I)}{\partial I} = \bar{c}\int_{2I}^{1} g(\varepsilon_i)d\varepsilon_i - 2k\hat{s}^N(I)\frac{\partial\hat{s}^N(I)}{\partial I} - c = \bar{c}\int_{2I}^{1} g(\varepsilon_i)d\varepsilon_i - 4\beta\hat{s}^N(I)\bar{c}Ig(2I) - c, \\
&\frac{\partial\hat{\pi}_i^N(I)}{\partial\beta} = -2k\hat{s}^N(I)\frac{\partial\hat{s}^N(I)}{\partial\beta} < 0, \text{ and} \\
&\frac{\partial\hat{\pi}_i^N(I)}{\partial\bar{c}} = -\int_{2I}^{1}(\varepsilon_i/2 - I)g(\varepsilon_i)d\varepsilon_i - 2k\hat{s}^N(I)\frac{\partial\hat{s}^N(I)}{\partial\bar{c}} = -\int_{2I}^{1}(\varepsilon_i/2 - I)g(\varepsilon_i)d\varepsilon_i + \beta\hat{s}^N(I)\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon.
\end{aligned}$$

One can verify that $\frac{\partial\hat{\pi}_i^N(I)}{\partial\beta} < 0$, and $\frac{\partial\hat{\pi}_i^N(I)}{\partial I}$ and $\frac{\partial\hat{\pi}_i^N(I)}{\partial\bar{c}}$ could be positive or negative for different given parameters. Thus, $\hat{\pi}_i^N(I)$ decreases in β, is non-monotone in I and $\bar{c}$. □

Proof of Lemma 3. (a) Retailer i's problem under inventory-sharing case is given by

$$\begin{aligned}\max_{s_i} \pi_i(s_i, I_i|s_j, I_j) =& p \cdot E[\varepsilon_i \cdot d_i] + t \cdot E[(\min\{I_i - \varepsilon_i d_i, \varepsilon_j d_j - I_j\})^+] - (t+\delta) \cdot E[(\min\{\varepsilon_i d_i - I_i, I_j - \varepsilon_j d_j\})^+] \\ & - \bar{c} \cdot E[(\varepsilon_i d_i - I_i - (I_j - \varepsilon_j d_j)^+)^+] - k s_i^2 - c I_i, \text{ for } i, j = 1, 2 \text{ and } i \neq j.\end{aligned}$$

The first-order-condition leads to

$$\begin{aligned}\frac{d\pi_i(s_i, I_i|s_j, I_j)}{ds_i} =& -2ks_i + \beta(p \int_0^1 \varepsilon_i g_i(\varepsilon_i) d\varepsilon_i - t \int_{\frac{I_j}{d_j}}^1 \left[\int_0^{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}} \varepsilon_j g_{12}(\varepsilon_i, \varepsilon_j) d\varepsilon_i + \int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^{\frac{I_i}{d_i}} \varepsilon_i g_{12}(\varepsilon_i, \varepsilon_j) d\varepsilon_i \right] d\varepsilon_j \\ & - (t+\delta) \int_0^{\frac{I_j}{d_j}} \left[\int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^1 \varepsilon_j g_{12}(\varepsilon_i, \varepsilon_j) d\varepsilon_i + \int_{\frac{I_i}{d_i}}^{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}} \varepsilon_i g_{12}(\varepsilon_i, \varepsilon_j) d\varepsilon_i \right] d\varepsilon_j \\ & - \bar{c} \left[\int_0^{\frac{I_j}{d_j}} \int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^1 (\varepsilon_i - \varepsilon_j) g_{12}(\varepsilon_i, \varepsilon_j) d\varepsilon_i d\varepsilon_j + \int_{\frac{I_j}{d_j}}^1 \int_{\frac{I_i}{d_i}}^1 \varepsilon_i g_{12}(\varepsilon_i, \varepsilon_j) d\varepsilon_i d\varepsilon_j \right]) = 0.\end{aligned}$$

Solving $\frac{d\pi_1(s_1, I|s_2, I)}{ds_1} = 0$ and $\frac{d\pi_2(s_2, I|s_1, I)}{ds_2} = 0$ together, we can get the service quality in the unique symmetric equilibrium outcome as below:

$$\hat{s}_1(t, I) = \hat{s}_2(t, I) = \hat{s}(t, I) = \frac{\beta(\int_0^1 p\varepsilon g(\varepsilon) d\varepsilon - A - B)}{2k} = \frac{\beta(p\mu - A - B)}{2k}, \tag{A.10}$$

where $A = \int_0^{2I} (\int_{2I}^{4I-\varepsilon_2} (t+\delta)\varepsilon_1 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 + \int_{4I-\varepsilon_2}^1 (\bar{c}\varepsilon_1 - \bar{c}\varepsilon_2 + (t+\delta)\varepsilon_2) g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2$ and $B = \int_{2I}^1 (\int_0^{4I-\varepsilon_2} t\varepsilon_2 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 + \int_{4I-\varepsilon_2}^{2I} t\varepsilon_1 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 + \int_{2I}^1 \bar{c}\varepsilon_1 g_{12}(\varepsilon_1, \varepsilon_2)) d\varepsilon_2$.

Next, we check the second-order condition to show that $\frac{d^2\pi_i(s_i, I|s_j, I)}{ds_i^2}|_{s_1=s_2=\hat{s}(t,I)} < 0$. Taking the second derivative of $\pi_i(s_i, I_i|s_j, I_j)$ with respective to s_i, we have:

$$\begin{aligned}\frac{d^2\pi_i(s_i, I_i|s_j, I_j)}{ds_i^2} =& -t\beta^2 \left[\int_0^{\frac{I_i}{d_i}} (-H) d\varepsilon_i + \int_{\frac{I_j}{d_j}}^1 (-J) d\varepsilon_j + \int_{\frac{I_j}{d_j}}^1 K d\varepsilon_j \right] - (t+\delta)\beta^2 \left[\int_{\frac{I_i}{d_i}}^1 H d\varepsilon_i + \int_0^{\frac{I_j}{d_j}} J d\varepsilon_j + \int_0^{\frac{I_j}{d_j}} (-K) d\varepsilon_j \right] \\ & - \bar{c}\beta^2 \left[\int_{\frac{I_i}{d_i}}^1 (-H) d\varepsilon_i + \int_{\frac{I_j}{d_j}}^1 J d\varepsilon_j + \int_0^{\frac{I_j}{d_j}} K d\varepsilon_j \right] - 2k,\end{aligned}$$

where $H = \frac{I_j^2}{d_j^3} g_{12}(\varepsilon_i, \frac{I_j}{d_j})$, $J = \frac{I_i^2}{d_i^3} g_{12}(\frac{I_i}{d_i}, \varepsilon_j)$, and $K = \frac{[I_i+I_j-\varepsilon_j(d_i+d_j)]^2}{d_i^3} g_{12}(\frac{I_i+I_j-\varepsilon_j d_j}{d_i}, \varepsilon_j)$. Given that $s_1 = s_2 = \hat{s}(t, I)$ and $I_1 = I_2 = I$, we can obtain the follows: $d_1 = d_2 = \frac{1}{2}$, $H = 8I^2 \cdot g_{12}(\varepsilon_i, 2I)$, $J = 8I^2 \cdot g_{12}(2I, \varepsilon_j) = 8I^2 \cdot g_{12}(\varepsilon_j, 2I)$, and $K = 2(4I - 2\varepsilon_j)^2 g_{12}(4I - \varepsilon_j, \varepsilon_j)$. After simplification, the second-order-condition can be rewritten as below:

$$\begin{aligned}\frac{d^2\pi_i(s_i, I|s_j, I)}{ds_i^2}|_{s_1=s_2=\hat{s}(t,I)} =& (-t\beta^2) \int_{2I}^1 K d\varepsilon_j + (t+\delta-\bar{c})\beta^2 \int_0^{2I} K d\varepsilon_j - \delta\beta^2 \int_0^1 8I^2 \cdot g_{12}(\varepsilon_i, 2I) d\varepsilon_i - 2k. \\ =& -\beta^2(t \int_{2I}^1 K d\varepsilon_j + (\bar{c} - t - \delta) \int_0^{2I} K d\varepsilon_j + \delta \int_0^1 8I^2 \cdot g_{12}(\varepsilon_i, 2I) d\varepsilon_i) - 2k.\end{aligned}$$

Since $0 \leq t \leq t + \delta \leq \bar{c}$, we can show that $\frac{d^2\pi_i(s_i, I|s_j, I)}{ds_i^2}|_{s_1=s_2=\hat{s}(t,I)} < 0$.

(b) Taking derivative of $\hat{s}(t, I)$ with respect to t, δ, $\bar{c}$, I, and β:

$$\begin{aligned}\frac{\partial \hat{s}(t, I)}{\partial t} =& -\beta(\frac{\int_0^{2I} (\int_{2I}^{4I-\varepsilon_2} \varepsilon_1 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 + \int_{4I-\varepsilon_2}^1 \varepsilon_2 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2}{2k} \\ & + \frac{\int_{2I}^1 (\int_0^{4I-\varepsilon_2} \varepsilon_2 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 + \int_{4I-\varepsilon_2}^{2I} \varepsilon_1 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2}{2k}) < 0, \\ \frac{\partial \hat{s}(t, I)}{\partial \delta} =& -\beta \frac{\int_0^{2I} (\int_{2I}^{4I-\varepsilon_2} \varepsilon_1 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1 + \int_{4I-\varepsilon_2}^1 \varepsilon_2 g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2}{2k} < 0, \\ \frac{\partial \hat{s}(t, I)}{\partial \bar{c}} =& -\beta \frac{\int_0^{2I} (\int_{4I-\varepsilon_2}^1 (\varepsilon_1 - \varepsilon_2) g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2 + \int_{2I}^1 (\int_{2I}^1 \varepsilon_1 g_{12}(\varepsilon_1, \varepsilon_2)) d\varepsilon_2}{2k} < 0, \\ \frac{\partial \hat{s}(t, I)}{\partial \beta} =& \frac{(p\mu - A - B)}{2k} > 0, \text{ and } \frac{\partial \hat{s}(t, I)}{\partial I} = \frac{\beta(X_1 + X_2)}{k},\end{aligned}$$

where A and B are given in part (a), $X_1 = \int_0^{2I} 2(2t+\delta)Ig_{12}(\varepsilon_1, 2I) + 4(\bar{c}-t-\delta)(2I-\varepsilon_1)g_{12}(\varepsilon_1, 4I-\varepsilon_1))d\varepsilon_1$, and $X_2 = \int_{2I}^1 2(2\bar{c}-2t-\delta)Ig_{12}(\varepsilon_1, 2I) + 4t(2I-\varepsilon_1)g_{12}(\varepsilon_1, 4I-\varepsilon_1))d\varepsilon_1$.

According to Lemma A.1 (c), we can easily show that $\int_{2I}^1 4t(2I-\varepsilon_1)g_{12}(\varepsilon_1, 4I-\varepsilon_1))d\varepsilon_1 = -\int_0^{2I} 4t(2I-\varepsilon_1)g_{12}(\varepsilon_1, 4I-\varepsilon_1))d\varepsilon_1$. Thus, $\frac{\partial \hat{s}(t,I)}{\partial I}$ can be simplified as below:

$$\frac{\partial \hat{s}(t,I)}{\partial I} = \beta \frac{\int_0^{2I}(2I(2t+\delta)g_{12}(\varepsilon_1, 2I) + 4(\bar{c}-2t-\delta)(2I-\varepsilon_1)g_{12}(\varepsilon_1, 4I-\varepsilon_1))d\varepsilon_1 + \int_{2I}^1 2I(2\bar{c}-2t-\delta)g_{12}(\varepsilon_1, 2I))d\varepsilon_1}{k}.$$

Therefore, $\hat{s}(t,I)$ increases in β, and decreases in t, δ, and $\bar{c}$; and $\hat{s}(t,I)$ increases in I when $t \leq \frac{\bar{c}-\delta}{2}$ and could be non-monotone in I when $t > \frac{\bar{c}-\delta}{2}$.

(c) The corresponding profit of retailer i in the case with inventory sharing is given as below:

$$\begin{aligned}\hat{\pi}_i(t,I) =& \pi_i(\hat{s}(t,I), I|\hat{s}(t,I), I) = p\cdot\mu/2 + t\int_{2I}^1(\int_0^{4I-\varepsilon_2}(\varepsilon_2/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 + t\int_{2I}^1(\int_{4I-\varepsilon_2}^{2I}(I-\varepsilon_1/2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\ &-\int_0^{2I}(\int_{4I-\varepsilon_2}^1((t+\delta)(I-\varepsilon_2/2)+\bar{c}(\varepsilon_1/2+\varepsilon_2/2-2I))g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\ &-(t+\delta)\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 - \bar{c}\int_{2I}^1(\int_{2I}^1(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 - k\hat{s}^2(I) - cI.\end{aligned}$$

According to Lemma A.1(a)-(b), $t\int_{2I}^1(\int_0^{4I-\varepsilon_2}(\varepsilon_2/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 = t\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$ and $t\int_{2I}^1(\int_{4I-\varepsilon_2}^{2I}(I-\varepsilon_1/2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 = t\int_0^{2I}(\int_{4I-\varepsilon_2}^1(I-\varepsilon_2/2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$. After simplification, we can rewrite $\hat{\pi}_i(t,I)$ as below:

$$\begin{aligned}\hat{\pi}_i(t,I) =& p\cdot\mu/2 - \delta\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\ &-\int_0^{2I}(\int_{4I-\varepsilon_2}^1(\delta(I-\varepsilon_2/2)+\bar{c}(\varepsilon_1/2+\varepsilon_2/2-2I))g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\ &-\bar{c}\int_{2I}^1(\int_{2I}^1(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 - k\hat{s}^2(I) - cI.\end{aligned} \tag{A.11}$$

Taking derivative with respect to t, we have: $\frac{\partial\hat{\pi}_i(t,I)}{\partial t} = -2k\hat{s}(t,I)\frac{\partial\hat{s}(t,I)}{\partial t} > 0$. Thus, $\hat{\pi}_i(t,I)$ increases in t.

Taking derivative of $\hat{\pi}_i(t,I)$ with respect to δ:

$$\frac{\partial\hat{\pi}_i(t,I)}{\partial\delta} = \underbrace{-\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{4I-\varepsilon_2}^1(I-\varepsilon_2/2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}_{\text{First Term}} \underbrace{-2k\hat{s}(t,I)\frac{\partial\hat{s}(t,I)}{\partial\delta}}_{\text{Second Term}}.$$

The first term in $\frac{\partial\hat{\pi}_i(t,I)}{\partial\delta}$ is negative and the second term is positive. And one can easily verify that $\frac{\partial\hat{\pi}_i(t,I)}{\partial\delta}$ could be negative or positive. Thus, $\hat{\pi}_i(t,I)$ is non-monotone in δ.

Taking derivative of $\hat{\pi}_i(t,I)$ with respect to $\bar{c}$:

$$\frac{\partial\hat{\pi}_i(t,I)}{\partial\bar{c}} = \underbrace{-\int_0^{2I}(\int_{4I-\varepsilon_2}^1(\varepsilon_1/2+\varepsilon_2/2-2I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}_{\text{First Term}} \underbrace{-\int_{2I}^1(\int_{2I}^1(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}_{\text{Second Term}} \underbrace{-2k\hat{s}(t,I)\frac{\partial\hat{s}(t,I)}{\partial\bar{c}}}_{\text{Third Term}}.$$

The first and second terms in $\frac{\partial\hat{\pi}_i(t,I)}{\partial\bar{c}}$ are negative and the third term is positive. And one can easily verify that $\frac{\partial\hat{\pi}_i(t,I)}{\partial\bar{c}}$ could be negative or positive. Thus, $\hat{\pi}_i(t,I)$ is non-monotone in $\bar{c}$.

Taking derivative of $\hat{\pi}_i(t,I)$ with respect to β, we have $\frac{\partial\hat{\pi}_i(t,I)}{\partial\beta} = -2k\hat{s}(t,I)\frac{\partial\hat{s}(t,I)}{\partial\beta} < 0$. Thus, $\hat{\pi}_i(t,I)$ decreases in β.

Taking derivative of $\hat{\pi}_i(t,I)$ with respect to I:

$$\begin{aligned}\frac{\partial\hat{\pi}_i(t,I)}{\partial I} =& \int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(t+\delta)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{4I-\varepsilon_2}^1(2\bar{c}-t-\delta)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\ &+\int_{2I}^1(-\int_0^{4I-\varepsilon_2}tg_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{4I-\varepsilon_2}^{2I}tg_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{2I}^1\bar{c}g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 - 2k\hat{s}(t,I)\frac{\partial\hat{s}(t,I)}{\partial I} - c.\end{aligned}$$

$\frac{\partial\hat{\pi}_i(t,I)}{\partial I}$ could be positive or negative, and thus $\hat{\pi}_i(t,I)$ is non-monotone in I. □

Proof of Proposition 1. (a) $\hat{s}^N(I)$ and $\hat{s}(t,I)$ are given by Equations (A.8) and (A.10), respectively.

$$\begin{aligned}
\hat{s}(t,I)-\hat{s}^N(I)=&\frac{\beta(p\mu-A-B)}{2k}-\frac{\beta(p\mu-\bar{c}\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon)}{2k}=\frac{\beta(-A-B+\bar{c}\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon)}{2k}\\
=&\beta\frac{-A-B+\bar{c}\int_0^{2I}(\int_{2I}^{1}\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2+\bar{c}\int_{2I}^{1}(\int_{2I}^{1}\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}{2k}\\
=&\beta(\frac{\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\bar{c}-t-\delta)\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1+\int_{4I-\varepsilon_2}^{1}(\bar{c}-t-\delta)\varepsilon_2 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}{2k}\\
&-\frac{\int_{2I}^{1}(\int_0^{4I-\varepsilon_2}t\varepsilon_2 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1+\int_{4I-\varepsilon_2}^{2I}t\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}{2k})\\
=&\beta\frac{\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\bar{c}-2t-\delta)\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1+\int_{4I-\varepsilon_2}^{1}(\bar{c}-2t-\delta)\varepsilon_2 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}{2k}\\
&\text{(according to Lemma A.1 (a) and (b))}\\
=&(\bar{c}-2t-\delta)\beta\frac{\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}\varepsilon_1 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1+\int_{4I-\varepsilon_2}^{1}\varepsilon_2 g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2}{2k},
\end{aligned}$$

where A and B are given in Lemma 3(a).

Thus, $\hat{s}(t,I)>\hat{s}^N(I)$ when $t<\frac{\bar{c}-\delta}{2}$ and $\hat{s}(t,I)<\hat{s}^N(I)$ when $t>\frac{\bar{c}-\delta}{2}$.

(b)&(c) $\hat{\pi}_i^N(I)$ and $\hat{\pi}_i(t,I)$ are given by Equations (A.9) and (A.11), respectively. Thus, we have:

$$\hat{\Delta}(t,I)=\hat{\pi}_i(t,I)-\hat{\pi}_i^N(I)=\underbrace{\hat{R}(t,I)-\hat{R}^N(I)}_{\text{First Term}}\ \underbrace{-k((\hat{s}(t,I))^2-(\hat{s}^N(I))^2)}_{\text{Second Term}}.$$

where $\hat{R}^N(I)=p\cdot\mu/2-\bar{c}\int_{2I}^{1}(\varepsilon_i/2-I)g(\varepsilon_i)d\varepsilon_i$ and $\hat{R}(t,I)=p\cdot\mu/2-\delta\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2-\int_0^{2I}(\int_{4I-\varepsilon_2}^{1}(\delta(I-\varepsilon_2/2)+\bar{c}(\varepsilon_1/2+\varepsilon_2/2-2I))g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2-\bar{c}\int_{2I}^{1}(\int_{2I}^{1}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2$.

After simplification, we have:

$$\begin{aligned}
\hat{R}(t,I)-\hat{R}^N(I)=&-\delta\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\
&-\int_0^{2I}(\int_{4I-\varepsilon_2}^{1}(\delta(I-\varepsilon_2/2)+\bar{c}(\varepsilon_1/2+\varepsilon_2/2-2I))g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\
&-\bar{c}\int_{2I}^{1}(\int_{2I}^{1}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\
&+\bar{c}\int_0^{2I}(\int_{2I}^{1}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2+\bar{c}\int_{2I}^{1}(\int_{2I}^{1}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2\\
=&(\bar{c}-\delta)\int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(\varepsilon_1/2-I)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1+\int_{4I-\varepsilon_2}^{1}(I-\varepsilon_2/2)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2,
\end{aligned}$$

$$(\hat{s}(t,I))^2-(\hat{s}_N(I))^2=\beta^2((\frac{(p\mu-A-B)}{2k})^2-(\frac{(p\mu-\bar{c}\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon)}{2k})^2)=\beta^2((\hat{S}(t,I))^2-(\hat{S}^N(I))^2).$$

where $\hat{S}(t,I)=\frac{(p\mu-A-B)}{2k}$ and $\hat{S}^N(I)=\frac{(p\mu-\bar{c}\int_{2I}^{1}\varepsilon g(\varepsilon)d\varepsilon)}{2k}$.

Thus, the first term in $\hat{\Delta}(t,I)$ is positive and independent in t, i.e., $\hat{R}(t,I)-\hat{R}^N(I)>0$ and is independent in t and β; the second term $-k((\hat{s}(t,I))^2-(\hat{s}^N(I))^2)$ increases in t since $\hat{s}(t,I)$ decreases in t and $\hat{s}^N(I)$ is independent in t. Moreover, according to part (a). the second term is negative (i.e., $\hat{s}(t,I)>\hat{s}^N(I)$) when $t<\frac{\bar{c}-\delta}{2}$ and is positive (i.e., $\hat{s}(t,I)<\hat{s}^N(I)$) when $t>\frac{\bar{c}-\delta}{2}$.

Therefore, $\hat{\Delta}(t,I)$ increases in t, and $\hat{\Delta}(\frac{\bar{c}-\delta}{2},I)>0$. Hence, there exists a unique $\hat{t}(\beta)\in[0,\frac{\bar{c}-\delta}{2})$ such that $\hat{\Delta}(t,I)\geq 0$ when $t\geq\hat{t}(\beta)$, and $\hat{\Delta}(t,I)<0$ otherwise.

In addition, one can also easily verify that the first term in $\hat{\Delta}(t,I)$ is positive and independent in β. According to part (a), $\hat{s}(t,I)=\beta\hat{S}(t,I)>\hat{s}^N(I)=\beta\hat{S}^N(I)>0$ if $t<\frac{\bar{c}-\delta}{2}$ and $0<\hat{s}(t,I)=\beta\hat{S}(t,I)<\hat{s}^N(I)=\beta\hat{S}^N(I)$ if $t>\frac{\bar{c}-\delta}{2}$. Thus, the second term $-k((\hat{s}(t,I))^2-(\hat{s}^N(I))^2)=-k\beta^2((\hat{S}(t,I))^2-(\hat{S}^N(I))^2)$ is negative and decreases in β when $t<\frac{\bar{c}-\delta}{2}$, and it is positive and increases in β when $t>\frac{\bar{c}-\delta}{2}$. Hence, $\hat{\Delta}(t,I)$

decreases in β when $t < \frac{\bar{c}-\delta}{2}$; and $\hat{\Delta}(t,I) > 0$ and increases in β when $t > \frac{\bar{c}-\delta}{2}$. Moreover, when $t < \frac{\bar{c}-\delta}{2}$, there exists $\hat{\beta}(t) = \sqrt{\frac{\hat{R}(t,I)-\hat{R}^N(I)}{k(((\hat{S}(t,I)^2-(\hat{S}^N(I))^2))}}$ such that, $\hat{\Delta}(t,I) < 0$ iff $\beta > \hat{\beta}(t)$ and $\hat{\Delta}(t,I) > 0$ iff $\beta < \hat{\beta}(t)$.

This completes the proof of Proposition 1. ☐

Proof of Corollary 1. The proof follows directly from Proposition 1 (a) and (b). ☐

Proof of Proposition 2. As shown in Lemma 3(c), each retailer's profit in the sharing case $\hat{\pi}_i(t,I)$ increases in t for $t \in [0, \bar{c}-\delta]$. Thus, the optimal transfer price is $\hat{t}^* = \bar{c}-\delta$. As shown in Proposition 1(b), $\hat{\Delta}(t,I) = \hat{\pi}_i(t,I) - \hat{\pi}_i^N(I) > 0$ when $t > \hat{t}(\beta)$. Since $\hat{t}^* = \bar{c}-\delta > \frac{\bar{c}-\delta}{2} > \hat{t}(\beta)$, the retailers prefer to share inventory at the transfer price $\hat{t}^*$. In addition, in the centralized system, $s_1^c = s_2^c = 0$. However, $\hat{s}(t,I) > 0$ for any t given that $\beta > 0$. Thus, one can easily verify that $\hat{\pi}_i(t,I) < \pi_i^c(I,I)$ (where $\pi_i^c(I,I)$ is given by Equation (A.1)) holds for any $t \in [0, \bar{c}-\delta]$, i.e., coordination cannot be achieved. ☐

E. Endogenous Inventory

In the endogenous inventory case, we define $\bar{\beta} = \min\{\hat{\beta}_1(\tilde{I}^N), \hat{\beta}_2(\tilde{I})\}$ and assume that $\beta \le \bar{\beta}$ to ensure that each retailer's profit is non-negative in the equilibrium of both inventory sharing and no-sharing cases. Note that $\hat{\beta}_1(\cdot)$ and $\hat{\beta}_2(\cdot)$ are defined in Equation (A.7), and $\tilde{I}^N$ and $\tilde{I}$ are defined in Equations (A.12) and (A.14).

Proof of Lemma 4. (a) Retailer i's problem in the case without inventory sharing is given by

$$\max_{s_i, I_i} \pi_i^N(s_i, I_i|s_j) = p \cdot E[\varepsilon_i \cdot d_i] - \bar{c} \cdot E[(\varepsilon_i \cdot d_i - I_i)^+] - ks_i^2 - cI_i, \text{ for } i,j = 1,2 \text{ and } i \neq j.$$

Note that $d_i = \frac{1}{2} + \beta(s_i - s_j)$. We can rewrite $\pi_i^N(s_i, I_i|s_j)$ as: $\pi_i^N(s_i, I_i|s_j) = p \cdot \mu \cdot d_i - \bar{c}\int_{I/d_i}^1 (\varepsilon_i d_i - I_i) g(\varepsilon_i) d\varepsilon_i - ks_i^2 - cI_i$. We first check the second-order condition:

$$\frac{d^2\pi_i^N(s_i, I_i|s_j)}{ds_i^2} = -\frac{\beta^2 \bar{c} I_i^2 g(\frac{I_i}{d_i})}{d_i^3} - 2k < 0, \quad \frac{d^2\pi_i^N(s_i, I_i|s_j)}{dI_i^2} = -\frac{\bar{c}}{d_i} g(\frac{I_i}{d_i}) < 0,$$

$$\frac{d^2\pi_i^N(s_i, I_i|s_j)}{dI_i ds_i} = \frac{\beta \bar{c} I_i g(\frac{I_i}{d_i})}{d_i^2}, \quad \text{and } \frac{d^2\pi_i^N(s_i, I_i|s_j)}{ds_i^2}\frac{d^2\pi_i(s_i, I_i|s_j)}{dI_i^2} - (\frac{d^2\pi_i^N(s_i, I_i|s_j)}{dI_i ds_i})^2 = \frac{2k\bar{c}g(\frac{I_i}{d_i})}{d_i} \ge 0.$$

Thus, $\pi_i^N(s_i, I_i|s_j)$ is jointly concave in s_i and I_i. Therefore, each retailer's optimal inventory and service quality can be solved by considering the first-order-condition as below:

$$\frac{d\pi_i^N(s_i, I_i|s_j)}{dI_i} = \bar{c}\int_{I/d_i}^1 g(\varepsilon)d\varepsilon - c = \bar{c}(1 - G(I/d_i)) - c = 0 \text{ and } \frac{d\pi_i^N(s_i, I_i|s_j)}{ds_i} = \beta p\mu - \bar{c}\beta\int_{I/d_i}^1 \varepsilon_i g(\varepsilon_i)d\varepsilon_i - 2ks_i = 0.$$

Solving $\frac{d\pi_1^N(s_1,I_1|s_2)}{dI_1} = 0$, $\frac{d\pi_1^N(s_1,I_1|s_2)}{ds_1} = 0$, $\frac{d\pi_2^N(s_2,I_2|s_1)}{dI_2} = 0$, and $\frac{d\pi_2^N(s_2,I_2|s_1)}{ds_2} = 0$ together (simultaneously), we can get the unique symmetric equilibrium outcome as below:

$$I_1 = I_2 = \tilde{I}^N = \frac{G^{-1}(1-\frac{c}{\bar{c}})}{2} \text{ and } s_1 = s_2 = \tilde{s}^N = \hat{s}^N(\tilde{I}^N) = \frac{\beta(p\mu - \bar{c}\int_{2\tilde{I}^N}^1 \varepsilon g(\varepsilon)d\varepsilon)}{2k}. \tag{A.12}$$

(b) According to the proof of part (a), $\tilde{I}^N$ is solved from $\bar{c}(1-G(2I))-c = 0$. Define $Y^N(I,\bar{c}) = \bar{c}(1-G(2I))-c$. Obviously, $\tilde{I}^N$ is independent in β. According to Chain Rule,

$$\frac{\partial \tilde{I}^N}{\partial \bar{c}} = -\frac{\partial Y^N(I,\bar{c})}{\partial \bar{c}} / \frac{\partial Y^N(I,\bar{c})}{\partial I}|_{I=\tilde{I}^N} = \frac{1-G(2I)}{2\bar{c}g(2I)}|_{I=\tilde{I}^N} \ge 0.$$

Therefore, $\tilde{I}^N$ increases in $\bar{c}$.

(c) Taking derivative of $\tilde{s}^N$ with respective to $\bar{c}$ and β: $\frac{\partial \tilde{s}^N}{\partial \beta} = \frac{\partial \hat{s}^N(\tilde{I}^N)}{\partial \beta} > 0$ (According to Lemma 2), and

$$\begin{aligned}\frac{\partial \tilde{s}^N}{\partial \bar{c}} &= \frac{\partial \hat{s}^N(\tilde{I}^N)}{\partial \bar{c}} = (\frac{\partial \hat{s}^N(I)}{\partial \bar{c}} + \frac{\partial \hat{s}^N(I)}{\partial I}\frac{\partial \tilde{I}^N}{\partial \bar{c}})|_{I=\tilde{I}^N} \\ &= (\frac{-\beta\int_{2I}^1 \varepsilon g(\varepsilon)d\varepsilon}{2k} + \frac{\bar{c}\beta I g(2I)}{k}\frac{1-G(2I)}{\bar{c}g(2I)})|_{I=\tilde{I}^N} = -(\frac{\beta\int_{2I}^1(\frac{\varepsilon}{2} - I)g(\varepsilon)d\varepsilon}{k})|_{I=\tilde{I}^N} \le 0.\end{aligned}$$

Therefore, $\tilde{s}^N$ decrease in $\bar{c}$ and increases in β.

(d) The corresponding profit of retailer i in the case without inventory sharing is given as below:

$$\tilde{\pi}_i^N = \pi_i^N(\tilde{s}^N, \tilde{I}^N|\tilde{s}^N) = p\cdot\mu/2 - \bar{c}\int_{2\tilde{I}^N}^1 (\varepsilon/2 - \tilde{I}^N)g(\varepsilon)d\varepsilon - k(\tilde{s}^N)^2 - c\tilde{I}^N = \hat{\pi}^N(\tilde{I}^N). \tag{A.13}$$

Taking derivative of $\tilde{\pi}_i^N$ with respect to β and $\bar{c}$: $\frac{\partial\tilde{\pi}_i^N}{\partial\beta} = -2k\tilde{s}^N\frac{\partial\tilde{s}^N}{\partial\beta} < 0$ and

$$\begin{aligned}\frac{\partial\tilde{\pi}_i^N}{\partial\bar{c}} &= \frac{\partial\hat{\pi}_i^N(\tilde{I}^N)}{\partial\bar{c}} = \Big(\frac{\partial\hat{\pi}_i^N(I)}{\partial\bar{c}} + \frac{\partial\hat{\pi}_i^N(I)}{\partial I}\frac{\partial\tilde{I}^N}{\partial\bar{c}}\Big)\Big|_{I=\tilde{I}^N}\\ &= \Big(-\int_{2I}^1(\varepsilon_i/2 - I)g(\varepsilon_i)d\varepsilon_i + \beta\hat{s}^N(I)\int_{2I}^1\varepsilon g(\varepsilon)d\varepsilon + \big(\bar{c}\int_{2I}^1 g(\varepsilon_i)d\varepsilon_i - 4\beta\hat{s}^N(I)\bar{c}Ig(2I) - c\big)\frac{1-G(2I)}{2\bar{c}g(2I)}\Big)\Big|_{I=\tilde{I}^N}.\end{aligned}$$

Since $\frac{\partial\tilde{\pi}_i^N}{\partial\bar{c}}$ can be positive or negative and $\frac{\partial\tilde{\pi}_i^N}{\partial\beta} < 0$, $\tilde{\pi}_i^N$ is non-monotone in $\bar{c}$ and decreases in β.

This completes the proof of Lemma 4. □

Proof of Lemma 5. (a) Retailer i's problem in the case with inventory sharing is given by

$$\begin{aligned}\max_{I_i,s_i}\pi_i(s_i,I_i|s_j,I_j) =& p\cdot E[\varepsilon_i\cdot d_i] + t\cdot E[(\min\{I_i-\varepsilon_i d_i, \varepsilon_j d_j - I_j\})^+] - (t+\delta)\cdot E[(\min\{\varepsilon_i d_i - I_i, I_j - \varepsilon_j d_j\})^+]\\ &- \bar{c}\cdot E[(\varepsilon_i d_i - I_i - (I_j - \varepsilon_j d_j)^+)^+] - ks_i^2 - cI_i,\ \text{for } i,j = 1,2 \text{ and } i\neq j.\end{aligned}$$

The first-order-condition leads to

$$\begin{aligned}\frac{d\pi_i(s_i,I_i|s_j,I_j)}{ds_i} =& -2ks_i + \beta p\int_0^1\varepsilon_i g_i(\varepsilon_i)d\varepsilon_i - \beta t\int_{\frac{I_j}{d_j}}^1\left[\int_0^{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}\varepsilon_j g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i + \int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^{\frac{I_i}{d_i}}\varepsilon_i g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i\right]d\varepsilon_j\\ &-\beta(t+\delta)\int_0^{\frac{I_j}{d_j}}\left[\int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^1\varepsilon_j g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i + \int_{\frac{I_i}{d_i}}^{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}\varepsilon_i g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i\right]d\varepsilon_j\\ &-\beta\bar{c}\left[\int_0^{\frac{I_j}{d_j}}\int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^1(\varepsilon_i-\varepsilon_j)g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i d\varepsilon_j + \int_{\frac{I_j}{d_j}}^1\int_{\frac{I_i}{d_i}}^1\varepsilon_i g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i d\varepsilon_j\right] = 0 \quad \text{and}\\ \frac{d\pi_i(s_i,I_i|s_j,I_j)}{dI_i} =& t\int_{\frac{I_j}{d_j}}^1\int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^{\frac{I_i}{d_i}}g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i d\varepsilon_j + (t+\delta)\int_0^{\frac{I_j}{d_j}}\int_{\frac{I_i}{d_i}}^{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i d\varepsilon_j\\ &+\bar{c}\left[\int_0^{\frac{I_j}{d_j}}\int_{\frac{I_i+I_j-\varepsilon_j d_j}{d_i}}^1 g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i d\varepsilon_j + \int_{\frac{I_j}{d_j}}^1\int_{\frac{I_i}{d_i}}^1 g_{12}(\varepsilon_i,\varepsilon_j)d\varepsilon_i d\varepsilon_j\right] - c = 0.\end{aligned}$$

Solving $\frac{d\pi_1(s_1,I_1|s_2,I_2)}{ds_1} = 0$, $\frac{d\pi_1(s_1,I_1|s_2,I_2)}{dI_1} = 0$, $\frac{d\pi_2(s_2,I_2|s_1,I_1)}{ds_2} = 0$, and $\frac{d\pi_2(s_2,I_2|s_1,I_1)}{dI_2} = 0$ together (simultaneously), we can get the retailers' inventory and service quality in the symmetric equilibrium: $I_1 = I_2 = \tilde{I}(t)$ and $s_1 = s_2 = \tilde{s}(t) = \hat{s}(t, \tilde{I}(t))$, where $\hat{s}(\cdot)$ is given by Equation (A.10) and $\tilde{I}(t)$ is solved from:

$$\begin{aligned}Y(\tilde{I},\bar{c},t,\delta) =& \int_0^{2\tilde{I}}\Big(\int_{4\tilde{I}-\varepsilon_2}^1\bar{c}g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{2\tilde{I}}^{4\tilde{I}-\varepsilon_2}(t+\delta)g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1\Big)d\varepsilon_2\\ &+\int_{2\tilde{I}}^1\Big(\int_{2\tilde{I}}^1\bar{c}g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1 + \int_{4\tilde{I}-\varepsilon_2}^{2\tilde{I}}tg_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1\Big)d\varepsilon_2 - c = 0.\end{aligned} \tag{A.14}$$

Next, we check the second-order condition to verify that $\pi_i(s_i, I_i|s_j, I_j)$ is jointly concave in s_i and I_i when $s_1 = s_2 = \tilde{s}(t)$ and $I_1 = I_2 = \tilde{I}(t)$. For expositional brevity, in the following proof, we use $\pi_i(s_i, I_i)$ to represent $\pi_i(s_i, I_i|s_j, I_j)$, and we use $\tilde{I}$ and $\tilde{s}$ to represent $\tilde{I}(t)$ and $\tilde{s}(t)$, respectively. The Hessian matrix of $\pi_i(s_i, I_i)$ is given by

$$H = \begin{bmatrix}\frac{\partial^2\pi_i(s_i,I_i)}{\partial I_i^2} & \frac{\partial^2\pi_i(s_i,I_i)}{\partial I_i\partial s_i}\\ \frac{\partial^2\pi_i(s_i,I_i)}{\partial I_i\partial s_i} & \frac{\partial^2\pi_i(s_i,I_i)}{\partial s_i^2},\end{bmatrix},$$

in which:

$$
\begin{aligned}
\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i^2} =& -\frac{1}{d_i}\left[\int_0^{\frac{I_j}{d_j}} \left((t+\delta)g_{12}(\frac{I_i}{d_i}, \varepsilon_j) + (\bar{c}-t-\delta)g_{12}(\frac{I_1+I_2-\varepsilon_j d_j}{d_i}, \varepsilon_j)\right)\varepsilon_j\right] \\
& -\frac{1}{d_i}\left[\int_{\frac{I_j}{d_j}}^1 \left((\bar{c}-t)g_{12}(\frac{I_i}{d_i}, \varepsilon_j) + tg_{12}(\frac{I_1+I_2-\varepsilon_j d_j}{d_i}, \varepsilon_j)\right)\varepsilon_j\right], \\
\frac{\partial^2 \pi_i(s_i, I_i)}{\partial s_i^2} =& -2k + \beta^2 \frac{I_j^2}{d_j^3}\left[\int_0^{\frac{I_i}{d_i}} tg_{12}(\varepsilon_i, \frac{I_j}{d_j})d\varepsilon_i + \int_{\frac{I_i}{d_i}}^1 (\bar{c}-t-\delta)g_{12}(\varepsilon_i, \frac{I_j}{d_j})d\varepsilon_i\right] \\
& -\beta^2 \frac{I_i^2}{d_i^3}\left[\int_0^{\frac{I_j}{d_j}} (t+\delta)g_{12}(\frac{I_i}{d_i}, \varepsilon_j)d\varepsilon_j + \int_{\frac{I_j}{d_j}}^1 (\bar{c}-t)g_{12}(\frac{I_i}{d_i}, \varepsilon_j)d\varepsilon_j\right] \\
& -\frac{\beta^2}{d_i^3}\left[\int_0^{\frac{I_j}{d_j}} (\bar{c}-t-\delta)(I_1+I_2-\varepsilon_j)^2 g_{12}(\frac{I_1+I_2-\varepsilon_j d_j}{d_i}, \varepsilon_j)d\varepsilon_j + \int_{\frac{I_j}{d_j}}^1 t(I_1+I_2-\varepsilon_j)^2 g_{12}(\frac{I_1+I_2-\varepsilon_j d_j}{d_i}, \varepsilon_j)d\varepsilon_j\right], \\
\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i \partial s_i} =& \frac{\beta}{d_i^2}\left[\int_0^{\frac{I_j}{d_j}} \left(I_i(t+\delta)g_{12}(\frac{I_i}{d_i}, \varepsilon_j) + (\bar{c}-t-\delta)(I_1+I_2-\varepsilon_j)g_{12}(\frac{I_1+I_2-\varepsilon_j d_j}{d_i}, \varepsilon_j)\right)d\varepsilon_j\right] \\
& +\frac{\beta}{d_i^2}\left[\int_{\frac{I_j}{d_j}}^1 \left(I_i(\bar{c}-t)g_{12}(\frac{I_i}{d_i}, \varepsilon_j) + t(I_1+I_2-\varepsilon_j)g_{12}(\frac{I_1+I_2-\varepsilon_j d_j}{d_i}, \varepsilon_j)\right)d\varepsilon_j\right].
\end{aligned}
$$

Note that $d_i = 1/2 + \beta(s_i - s_j)$ for $i, j = 1, 2$ and $i \neq j$. We want to verify that the Hessian matrix H is negative definite at the point $I_1 = I_2 = \tilde{I}$ and $s_1 = s_2 = \tilde{s}$. Given that $t \leq \bar{c} - \delta$, it is obvious $\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i^2} < 0$ at any points. Thus, we just need to verify that $det(H) > 0$ when $I_1 = I_2 = \tilde{I}$ and $s_1 = s_2 = \tilde{s}$, i.e.,

$$
\left[\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i^2}\frac{\partial^2 \pi_i(s_i, I_i)}{\partial s_i^2} - (\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i \partial s_i})^2\right]|_{I_1=I_2=\tilde{I}, s_1=s_2=\tilde{s}} > 0.
$$

Let $V_0 = \int_0^{2I} g_{12}(2I, \varepsilon)d\varepsilon$, $V_1 = \int_{2I}^1 g_{12}(2I, \varepsilon)d\varepsilon$, $C_0 = \int_0^{2I}(2I-\varepsilon)g_{12}(4I-\varepsilon, \varepsilon)d\varepsilon$, $C_1 = \int_{2I}^1(2I-\varepsilon)g_{12}(4I-\varepsilon, \varepsilon)d\varepsilon$, $D_0 = \int_0^{2I}(2I-\varepsilon)^2 g_{12}(4I-\varepsilon, \varepsilon)d\varepsilon$, $D_1 = \int_{2I}^1(2I-\varepsilon)^2 g_{12}(4I-\varepsilon, \varepsilon)d\varepsilon$, $E_0 = \int_0^{2I} g_{12}(4I-\varepsilon, \varepsilon)d\varepsilon$, $E_1 = \int_{2I}^1 g_{12}(4I-\varepsilon, \varepsilon)d\varepsilon$. We can write $det(H)$ as follows:

$$
\begin{aligned}
&\left[\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i^2}\frac{\partial^2 \pi_i(s_i, I_i)}{\partial s_i^2} - (\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i \partial s_i})^2\right]|_{I_1=I_2=\tilde{I}, s_1=s_2=\tilde{s}} \\
&= \Bigg\{ [-2((t+\delta)V_0 + (\bar{c}-t-\delta)E_0 + (\bar{c}-t)V_1 + tE_1)] \times \left[-2k - 8\beta^2\tilde{I}^2\delta(V_0+V_1) - 8\beta^2((\bar{c}-t-\delta)D_0 + tD_1)\right] \\
&- \left[4\beta\tilde{I}((t+\delta)V_0 + (\bar{c}-t)V_1) + 4\beta((\bar{c}-t-\delta)C_0 + tC_1)\right]^2 \Bigg\}|_{I=\tilde{I}}.
\end{aligned}
$$

Given Lemma A.1, we have $C_1 = -C_0$, $D_0 = D_1$, and $E_0 = E_1$. Thus, we can simplify $det(H)$ as below:

$$
\begin{aligned}
&\left[\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i^2}\frac{\partial^2 \pi_i(s_i, I_i)}{\partial s_i^2} - (\frac{\partial^2 \pi_i(s_i, I_i)}{\partial I_i \partial s_i})^2\right]|_{I_1=I_2=\tilde{I}, s_1=s_2=\tilde{s}} \\
&= \Bigg\{ [2((t+\delta)V_0 + (\bar{c}-t)V_1 + (\bar{c}-\delta)E_0)] \times \left[2k + 8\beta^2\tilde{I}^2\delta(V_0+V_1) + 8\beta^2(\bar{c}-\delta)D_0\right] \qquad \text{(A.15)} \\
&- 16\beta^2\left[\tilde{I}((t+\delta)V_0 + (\bar{c}-t)V_1) + (\bar{c}-2t-\delta)C_0\right]^2 \Bigg\}|_{I=\tilde{I}}.
\end{aligned}
$$

After simple transformation of Equation (A.15), it is clear that $det(H)|_{I_1=I_2=\tilde{I}, s_1=s_2=\tilde{s}} > 0$ is equivalent to:

$$
\frac{k}{4\beta^2} > \left\{\frac{\left[\tilde{I}((t+\delta)V_0 + (\bar{c}-t)V_1) + (\bar{c}-2t-\delta)C_0\right]^2}{(t+\delta)V_0 + (\bar{c}-t)V_1 + (\bar{c}-\delta)E_0} - (\tilde{I}^2\delta(V_0+V_1) + (\bar{c}-\delta)D_0)\right\}|_{I=\tilde{I}}.
$$

In the endogenous-inventory case, we assume that $\beta < \bar{\beta} = \min\{\hat{\beta}_1(\tilde{I}^N), \hat{\beta}_2(\tilde{I})\}$ (see the definition in the beginning of Part E), which can guarantee that

$$\frac{k}{4\beta^2} > \left\{ \frac{\left[\tilde{I}((t+\delta)V_0 + (\bar{c}-t)V_1) + (\bar{c}-2t-\delta)C_0\right]^2}{(t+\delta)V_0 + (\bar{c}-t)V_1 + (\bar{c}-\delta)E_0} \right\}|_{I=\tilde{I}}.$$

Thus, given $0 \leq \delta \leq \bar{c}$, it is clear that $det(H)|_{I_1=I_2=\tilde{I}, s_1=s_2=\tilde{s}} > 0$ holds. Therefore, we have shown that $\pi_i(s_i, I_i)$ is jointly concave in s_i and I_i when $s_1 = s_2 = \tilde{s}$ and $I_1 = I_2 = \tilde{I}$.

(b) According to the proof of part (a), $\tilde{I}(t)$ is solved from $Y(I, \bar{c}, t, \delta) = 0$, where $Y(\cdot)$ is given by Equation (A.14). Taking derivative of $Y(I, \bar{c}, t, \delta)$ with respect to I, $\bar{c}$, t, and δ, we have:

$$\begin{aligned}
\frac{\partial Y(I,\bar{c},t,\delta)}{\partial I} &= -\int_{2I}^{1} (2(\bar{c}-t)g_{12}(2I,\varepsilon_2) + 4tg_{12}(4I-\varepsilon_2,\varepsilon_2))d\varepsilon_2 \\
&\quad - \int_{0}^{2I} (2(t+\delta)g_{12}(2I,\varepsilon_2) + 4(\bar{c}-t-\delta)g_{12}(4I-\varepsilon_2,\varepsilon_2))d\varepsilon_2 \leq 0, \\
\frac{\partial Y(I,\bar{c},t,\delta)}{\partial \bar{c}} &= \int_{0}^{2I} (\int_{4I-\varepsilon_1}^{1} g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 + \int_{2I}^{1}(\int_{2I}^{1} g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 \geq 0, \\
\frac{\partial Y(I,\bar{c},t,\delta)}{\partial t} &= \int_{0}^{2I} (\int_{2I}^{4I-\varepsilon_2} g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 + \int_{2I}^{1}(\int_{4I-\varepsilon_2}^{2I} g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 \geq 0, \text{ and} \\
\frac{\partial Y(I,\bar{c},t,\delta)}{\partial \delta} &= \int_{0}^{2I} (\int_{2I}^{4I-\varepsilon_2} g_{12}(\varepsilon_1,\varepsilon_2)d\varepsilon_1)d\varepsilon_2 \geq 0,
\end{aligned}$$

Obviously, $\tilde{I}(t)$ is independent in β. According to Chain Rule, we have:

$$\frac{\partial \tilde{I}(t)}{\partial \bar{c}} = -\frac{\partial Y(I,\bar{c},t,\delta)}{\partial \bar{c}} / \frac{\partial Y(I,\bar{c},t,\delta)}{\partial I}|_{I=\tilde{I}(t)} \geq 0, \quad \frac{\partial \tilde{I}(t)}{\partial t} = -\frac{\partial Y(I,\bar{c},t,\delta)}{\partial t} / \frac{\partial Y(I,\bar{c},t,\delta)}{\partial I}|_{I=\tilde{I}(t)} \geq 0, \text{ and}$$
$$\frac{\partial \tilde{I}(t)}{\partial \delta} = -\frac{\partial Y(I,\bar{c},t,\delta)}{\partial \delta} / \frac{\partial Y(I,\bar{c},t,\delta)}{\partial I}|_{I=\tilde{I}(t)} \geq 0.$$

Therefore, $\tilde{I}(t)$ is increasing in $\bar{c}$, t, and δ.

(c) Taking derivative of $\tilde{s}(t)$ with respective to $\bar{c}$, t, δ, and β:

$$\frac{\partial \tilde{s}(t)}{\partial \bar{c}} = \frac{\partial \hat{s}(t,\tilde{I}(t))}{\partial \bar{c}} = (\frac{\partial \hat{s}(t,I)}{\partial \bar{c}} + \frac{\partial \hat{s}(t,I)}{\partial I}\frac{\partial \tilde{I}(t)}{\partial \bar{c}})|_{I=\tilde{I}(t)}, \quad \frac{\partial \tilde{s}(t)}{\partial t} = \frac{\partial \hat{s}(t,\tilde{I}(t))}{\partial t} = (\frac{\partial \hat{s}(t,I)}{\partial t} + \frac{\partial \hat{s}(t,I)}{\partial I}\frac{\partial \tilde{I}(t)}{\partial t})|_{I=\tilde{I}(t)},$$
$$\frac{\partial \tilde{s}(t)}{\partial \delta} = \frac{\partial \hat{s}(t,\tilde{I}(t))}{\partial \delta} = (\frac{\partial \hat{s}(t,I)}{\partial \delta} + \frac{\partial \hat{s}(t,I)}{\partial I}\frac{\partial \tilde{I}(t)}{\partial \delta})|_{I=\tilde{I}(t)}, \quad \text{and} \quad \frac{\partial \tilde{s}(t)}{\partial \beta} = \frac{\partial \hat{s}(t,\tilde{I}(t))}{\partial \beta} > 0.$$

Since $\frac{\partial \hat{s}(t,I)}{\partial \bar{c}} < 0$, $\frac{\partial \hat{s}(t,I)}{\partial t} < 0$, $\frac{\partial \hat{s}(t,I)}{\partial \delta} < 0$, $\frac{\partial \tilde{I}(t)}{\partial \bar{c}} \geq 0$, $\frac{\partial \tilde{I}(t)}{\partial t} \geq 0$, $\frac{\partial \tilde{I}(t)}{\partial \delta} \geq 0$, and $\frac{\partial \hat{s}(t,I)}{\partial I}$ could be positive and negative, one can verify that $\frac{\partial \tilde{s}(t)}{\partial \bar{c}}$, $\frac{\partial \tilde{s}(t)}{\partial t}$, and $\frac{\partial \tilde{s}(t)}{\partial \delta}$ could be positive or negative. Therefore, $\tilde{s}(t)$ is non-monotone in $\bar{c}$, t, and δ, and $\tilde{s}(t)$ increases in β.

(d) Retailer i's equilibrium profit is $\tilde{\pi}_i(t) = \hat{\pi}_i(t, \tilde{I}(t))$, where $\hat{\pi}_i(\cdot)$ is given in Equation (A.11). Taking derivative of $\tilde{\pi}_i(t)$ with respective to $\bar{c}$, t, δ, and β: $\frac{\partial \tilde{\pi}_i(t)}{\partial \bar{c}} = \frac{\partial \hat{\pi}_i(t,\tilde{I}(t))}{\partial \bar{c}} = (\frac{\partial \hat{\pi}_i(t,I)}{\partial \bar{c}} + \frac{\partial \hat{\pi}_i(t,I)}{\partial I}\frac{\partial \tilde{I}(t)}{\partial \bar{c}})|_{I=\tilde{I}(t)}$, $\frac{\partial \tilde{\pi}_i(t)}{\partial t} = \frac{\partial \hat{\pi}_i(t,\tilde{I}(t))}{\partial t} = (\frac{\partial \hat{\pi}_i(t,I)}{\partial t} + \frac{\partial \hat{\pi}_i(t,I)}{\partial I}\frac{\partial \tilde{I}(t)}{\partial t})|_{I=\tilde{I}(t)}$, $\frac{\partial \tilde{\pi}_i(t)}{\partial \delta} = \frac{\partial \hat{\pi}_i(t,\tilde{I}(t))}{\partial \delta} = (\frac{\partial \hat{\pi}_i(t,I)}{\partial \delta} + \frac{\partial \hat{\pi}_i(t,I)}{\partial I}\frac{\partial \tilde{I}(t)}{\partial \delta})|_{I=\tilde{I}(t)}$, and $\frac{\partial \tilde{\pi}_i(t)}{\partial \beta} = -2k\tilde{s}(t)\frac{\partial \tilde{s}(t)}{\partial \beta} < 0$.

Note that $\hat{\pi}_i(t, I)$ increases in t, is non-monotone in I, $\bar{c}$, and δ; and $\tilde{I}(t)$ is increasing in $\bar{c}$, t, and δ. Thus, one can easily verify that $\frac{\partial \tilde{\pi}_i(t)}{\partial \bar{c}}$, $\frac{\partial \tilde{\pi}_i(t)}{\partial t}$, and $\frac{\partial \tilde{\pi}_i(t)}{\partial \delta}$ could be positive or negative. Therefore, $\tilde{\pi}_i(t)$ is non-monotone in $\bar{c}$, t, and δ, and it decreases in β. ☐

Proof of Proposition 3. According to Lemmas 4 and 5, $\tilde{I}^N$ is solved from $Y^N(I, \bar{c}) = \bar{c}(1 - G(2I)) - c$, and $\tilde{I}(t)$ is solved from $Y(I, \bar{c}, t, \delta) = 0$ (see Equation (A.14)).

When $t = 0$, one can verify that $Y(I, \bar{c}, 0, \delta) - Y^N(I, \bar{c}) = \int_0^{2I}(\int_{2I}^{4I-\varepsilon_2}(t + \delta - \bar{c})g_{12}(\varepsilon_1, \varepsilon_2)d\varepsilon_1)d\varepsilon_2 \leq 0$. Moreover, since $\frac{\partial Y(I,\bar{c},t,\delta)}{\partial I} \leq 0$ and $\frac{\partial Y^N(I,\bar{c})}{\partial I} \leq 0$, we have $\tilde{I}(0) \leq \tilde{I}^N$.

When $t = \bar{c} - \delta$, one can verify that $Y(I, \bar{c}, \bar{c} - \delta, \delta) - Y^N(I, \bar{c}) = \int_{2I}^{1}(\int_{4I-\varepsilon_2}^{2I}(r - \delta)g_{12}(\varepsilon_1, \varepsilon_2)d\varepsilon_1)d\varepsilon_2 \geq 0$. Moreover, since $\frac{\partial Y(I,\bar{c},t,\delta)}{\partial I} \leq 0$ and $\frac{\partial Y^N(I,\bar{c})}{\partial I} \leq 0$, we have $\tilde{I}(\bar{c} - \delta) \geq \tilde{I}^N$.

In addition, according to Lemma 5, $\tilde{I}(t)$ is increasing t; and obviously, $\tilde{I}^N$ is independent in t. Therefore, there exists $\tilde{t}_I \in [0, \bar{c} - \delta]$ such that $\tilde{I}(t) \geq \tilde{I}^N$ when $t \geq \tilde{t}_I$ and $\tilde{I}(t) < \tilde{I}^N$ otherwise. $\square$

Proof of Proposition 4. Given that $g(\varepsilon) = 1$ for $\varepsilon \in [0, 1]$ and $g_{12}(\varepsilon_1, \varepsilon_2) = 1$ for $\varepsilon_1, \varepsilon_2 \in [0, 1]$, we can simplify $\tilde{I}^N$, $\tilde{s}^N$, $\tilde{\pi}_i^N$, $\tilde{I}(t)$, $\tilde{s}(t)$, and $\tilde{\pi}_i(t)$ (see the proof of Lemmas 4-5) as below:

$$\tilde{I}^N = \frac{1}{2}(1 - \frac{c}{\bar{c}}) \tag{A.16a}$$

$$\tilde{s}^N = \hat{s}^N(\tilde{I}^N) = \beta \frac{p - \bar{c}(1 - 4(\tilde{I}^N)^2)}{4k} = \beta \cdot \frac{p - 2c + c^2/\bar{c}}{4k} \tag{A.16b}$$

$$\tilde{\pi}_i^N = \hat{\pi}_i^N(\tilde{I}^N) = p \cdot \int_0^1 \frac{1}{2}\varepsilon_i g(\varepsilon_i) d\varepsilon_i - \bar{c} \int_{2\tilde{I}^N}^1 (\frac{1}{2}\varepsilon_i - \tilde{I}^N) g(\varepsilon_i) d\varepsilon_i - c\tilde{I}^N - k(\tilde{s}^N)^2 \tag{A.16c}$$

$$= p \cdot \frac{1}{4} - \bar{c} \cdot \frac{(1 - 2\tilde{I}^N)^2}{4} - c\tilde{I}^N - k(\tilde{s}^N)^2 = \frac{((p-2c)\bar{c} + c^2)(4k\bar{c} - ((p-2c)\bar{c} + c^2)\beta^2)}{16k\bar{c}^2} \tag{A.16d}$$

$$\tilde{I}(t) = \begin{cases} \frac{t - \bar{c} + \sqrt{3\bar{c}^2 + 2\bar{c}t + t^2 - 2c(\bar{c} + 2t - \delta) - 2\bar{c}\delta}}{2(\bar{c} + 2t - \delta)}, & \text{if } t < \frac{8c - 3\bar{c} - \delta}{2} \\ \frac{3\bar{c} - t - 2\delta - \sqrt{c(6\bar{c} - 4t - 6\delta) + (t+\delta)^2}}{6\bar{c} - 4t - 6\delta}, & \text{if } t \geq \frac{8c - 3\bar{c} - \delta}{2} \end{cases} \tag{A.16e}$$

$$\tilde{s}(t) = \hat{s}(t, \tilde{I}(t)) = \begin{cases} \beta \cdot \frac{4\tilde{I}^2(2\bar{c} - \delta - 2t) + p - \bar{c}}{4k}, & \text{if } t < \frac{8c - 3\bar{c} - \delta}{2} \\ \beta \cdot \frac{p + (2\tilde{I} - 1)(2\tilde{I}(3\delta + 6t - 2\bar{c}) + (2\bar{c} - \delta - 2t))}{4k}, & \text{if } t \geq \frac{8c - 3\bar{c} - \delta}{2} \end{cases} \tag{A.16f}$$

$$\tilde{\pi}_i(t) = \hat{\pi}_i(t, \tilde{I}(t)) = p \cdot E[\frac{1}{2}\varepsilon_i] + t \cdot E[(\min\{\tilde{I} - \frac{1}{2}\varepsilon_i, \frac{1}{2}\varepsilon_j - \tilde{I}\})^+] - (t + \delta) \cdot E[(\min\{\frac{1}{2}\varepsilon_i - \tilde{I}, \tilde{I} - \frac{1}{2}\varepsilon_j\})^+] \tag{A.16g}$$

$$- \bar{c} \cdot E[(\frac{1}{2}\varepsilon_i - \tilde{I} - (\tilde{I} - \frac{1}{2}\varepsilon_j))^+] - c\tilde{I} - k(\tilde{s}(t))^2 \tag{A.16h}$$

$$= \begin{cases} p \cdot \frac{1}{4} - \bar{c}(\frac{1}{4} - \tilde{I} + \frac{8}{3}\tilde{I}^3) - \delta(\tilde{I}^2 - \frac{8}{3}\tilde{I}^3) - c\tilde{I} - k(\tilde{s}(t))^2, & \text{if } t < \frac{8c - 3\bar{c} - \delta}{2} \\ p \cdot \frac{1}{4} - \frac{(1 - 2\tilde{I})^2}{12}(\bar{c}(4 - 8\tilde{I}) + \delta(8\tilde{I} - 1)) - c\tilde{I} - k(\tilde{s}(t))^2, & \text{if } t \geq \frac{8c - 3\bar{c} - \delta}{2} \end{cases} \tag{A.16i}$$

$$= \begin{cases} p \cdot \frac{1}{4} - \bar{c}(\frac{1}{4} - \tilde{I} + \frac{8}{3}\tilde{I}^3) - \delta(\tilde{I}^2 - \frac{8}{3}\tilde{I}^3) - c\tilde{I} - \beta^2 \cdot \frac{(4\tilde{I}^2(2\bar{c} - \delta - 2t) + p - \bar{c})^2}{16k}, & \text{if } t < \frac{8c - 3\bar{c} - \delta}{2} \\ p \cdot \frac{1}{4} - \frac{(1 - 2\tilde{I})^2}{12}(\bar{c}(4 - 8\tilde{I}) + \delta(8\tilde{I} - 1)) - c\tilde{I} - \beta^2 \cdot \frac{(p + (2\tilde{I} - 1)(2\tilde{I}(3\delta + 6t - 2\bar{c}) + (2\bar{c} - \delta - 2t)))^2}{16k}, & \text{if } t \geq \frac{8c - 3\bar{c} - \delta}{2} \end{cases} \tag{A.16j}$$

(a) It is easy to verify that $\tilde{s}(t)$ is continuous in t. In addition, though tedious but straightforward algebraic analysis, we can show: If $c \leq \frac{11\bar{c}^3 - 11\bar{c}^2\delta + \bar{c}\delta^2 + \delta^3}{2(3\bar{c} - 2\delta)^2}$, $\tilde{s}(t)$ decreases in t, i.e., $\frac{\partial \tilde{s}(t)}{\partial t} \leq 0$ for any $t \in [0, \bar{c} - \delta]$; otherwise, $\tilde{s}(t)$ first increases then decreases in t, i.e., $\frac{\partial \tilde{s}(t)}{\partial t} > 0$ when $t < t_1$ and $\frac{\partial \tilde{s}(t)}{\partial t} < 0$ when $t > t_1$, where t_1 is the second root of the polynomial equation $2t^3 + (3\bar{c} - \delta)t^2 + (14\bar{c}\delta - 4\delta^2 - 12\bar{c}^2)t + 18\bar{c}^2c - 11\bar{c}^3 - 24\bar{c}c\delta + 11\bar{c}^2\delta + 8c\delta^2 - \bar{c}\delta^2 - \delta^3 = 0$. Thus, $\tilde{s}(t)$ either decreases in t or first increases then decreases in t.

Moreover, it is easy to verify that $\tilde{s}(0) > \tilde{s}^N$ always hold, and that $\tilde{s}(\bar{c} - \delta) > \tilde{s}^N$ when $c > \tilde{c}$ and $\tilde{s}(\bar{c} - \delta) \leq \tilde{s}^N$ when $c \leq \tilde{c}$, where $\tilde{c} = \frac{3\bar{c}^2 - 5\bar{c}\delta + 2\delta\sqrt{\bar{c}\delta}}{3(\bar{c} - \delta)}$. Combining the result that $\tilde{s}^N$ is independent in t and $\tilde{s}(t)$ either decreases in t or first increases then decreases in t, we have: When $c > \tilde{c}$, $\tilde{s}(t) > \tilde{s}^N$ for any $t \in [0, \bar{c} - \delta]$; when $c \leq \tilde{c}$, there exists a unique $\tilde{t}_s \in [0, \bar{c} - \delta]$ such that $\tilde{s}(t) > \tilde{s}^N$ when $t < \tilde{t}_s$ and $\tilde{s}(t) < \tilde{s}^N$ when $t > \tilde{t}_s$, where $\tilde{t}_s$ is the unique solution of $\tilde{s}(t) = \tilde{s}^N$.

We further define that $\tilde{t}_s > \bar{c} - \delta$ when $c > \tilde{c}$ and $\tilde{t}_s$ as the unique solution of $\tilde{s}(t) = \tilde{s}^N$ when $c \leq \tilde{c}$. Thus, we can conclude that $\tilde{s}(t) > \tilde{s}^N$ when $t < \tilde{t}_s$ and $\tilde{s}(t) < \tilde{s}^N$ when $t > \tilde{t}_s$.

(b)&(c) Similar to the proof of Proposition 1(b)&(c), we can show that:

$$\tilde{\Delta}(t) = \tilde{\pi}_i(t) - \tilde{\pi}_i^N = \hat{\pi}_i(t, \tilde{I}(t)) - \hat{\pi}_i^N(\tilde{I}^N) = \underbrace{\hat{R}(t, \tilde{I}(t)) - \hat{R}^N(\tilde{I}^N)}_{\text{First Term}} \quad \underbrace{-k((\hat{s}(t, \tilde{I}(t)))^2 - (\hat{s}^N(\tilde{I}^N))^2)}_{\text{Second Term}}.$$

where $\hat{R}^N(I) = p \cdot \mu/2 - \bar{c} \int_{2I}^1 (\varepsilon_i/2 - I) g(\varepsilon_i) d\varepsilon_i$ and $\hat{R}(t, I) = p \cdot \mu/2 - \delta \int_0^{2I} (\int_{2I}^{4I - \varepsilon_2} (\varepsilon_1/2 - I) g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2 - \int_0^{2I} (\int_{4I - \varepsilon_2}^1 (\delta(I - \varepsilon_2/2) + \bar{c}(\varepsilon_1/2 + \varepsilon_2/2 - 2I)) g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2 - \bar{c} \int_{2I}^1 (\int_{2I}^1 (\varepsilon_1/2 - I) g_{12}(\varepsilon_1, \varepsilon_2) d\varepsilon_1) d\varepsilon_2$. Moreover, $(\hat{s}(t, \tilde{I}(t)))^2 - (\hat{s}_N(\tilde{I}^N))^2 = \beta^2((\hat{S}(t, \tilde{I}(t)))^2 - (\hat{S}^N(\tilde{I}^N))^2)$, where $\hat{S}(t, I) = \frac{(p\mu - A - B)}{2k}$ and $\hat{S}^N(I) = \frac{(p\mu - \bar{c} \int_{2I}^1 \varepsilon g(\varepsilon) d\varepsilon)}{2k}$.

Comparing with the proof of Lemma 1, we can find that $\tilde{I}(t) = I^{nc}$, $\tilde{I}^N = I^{nc-N}$, $\hat{R}(t, \tilde{I}(t)) = \pi_i^{nc*}(t)$, and $\hat{R}^N(\tilde{I}^N) = \pi_i^{nc-N*}$. As shown in Lemma 1, we can know that the firm term in $\tilde{\Delta}(t)$ is positive, i.e., $\hat{R}(t, \tilde{I}(t)) - \hat{R}^N(\tilde{I}^N) = \pi_i^{nc*}(t) - \pi_i^{nc-N*} > 0$. And it is easy to verify that the first term is independent in β.

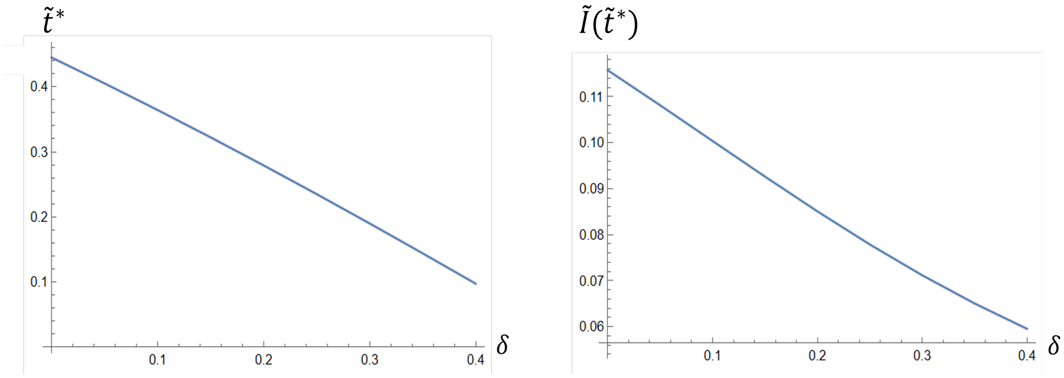
Similar to the logic in the proof of Proposition 1(b)&(c), we can show that the second term in $\tilde{\Delta}(t)$ is negative and decreases in β when $t < \tilde{t}_s$, and it is positive and increases in β when $t > \tilde{t}_s$. Hence, $\tilde{\Delta}(t)$ decreases in β when $t < \tilde{t}_s$; and $\tilde{\Delta}(t) > 0$ and increases in β when $t > \tilde{t}_s$. Moreover, when $t < \tilde{t}_s$, there exists $\tilde{\beta}(t) = \sqrt{\frac{\hat{R}(t,\tilde{I}(t)) - \hat{R}^N(\tilde{I}^N)}{k((\hat{S}(t,\tilde{I}(t)))^2 - (\hat{S}^N(\tilde{I}^N))^2)}}$ such that, $\tilde{\Delta}(t) < 0$ iff $\beta > \tilde{\beta}(t)$, and $\tilde{\Delta}(t) > 0$ iff $\beta < \tilde{\beta}(t)$. □

Proof of Proposition 5. (a) From Equation (A.16), it is easy to verify that $\tilde{\pi}_i(t)$ is continuous in t for $t \in [0, \bar{c} - \delta]$. Moreover, $\frac{d\tilde{\pi}_i(t)}{dt} = \frac{d\hat{R}(t,\tilde{I}(t))}{dt} - 2k\tilde{s}(t)\frac{d\tilde{s}(t)}{dt} = \frac{d\hat{R}(t,\tilde{I}(t))}{dI}|_{I=\tilde{I}(t)}\frac{d\tilde{I}(t)}{dt} - 2k\tilde{s}(t)\frac{d\tilde{s}(t)}{dt}$. Note that $\hat{R}(t, \tilde{I}(t))$ is concave in I, $\frac{d\tilde{I}(t)}{dt} > 0$, and $\tilde{s}(t)$ either decrease in t or first increases then decreases in t. Thus $\tilde{\pi}_i(t)$ might be concave or convex in t. Thus, there always exists (maybe not unique) an optimal transfer price $\tilde{t}^* \in [0, \bar{c} - \delta]$ that maximizes each retailer's profit, and $\tilde{t}^*$ could be strictly smaller than $\bar{c} - \delta$ (see the illustrated example in Figure 9).
(b) From the proof of Proposition 4, we have shown that $\tilde{\Delta}(t) < 0$ iff $\beta > \tilde{\beta}(t)$ and $t < \tilde{t}_s$, where $\tilde{\beta}(t) = \sqrt{\frac{\hat{R}(t,\tilde{I}(t)) - \hat{R}^N(\tilde{I}^N)}{k((\hat{S}(t,\tilde{I}(t)))^2 - (\hat{S}^N(\tilde{I}^N))^2)}}$.

Note that when $c > \tilde{c}$, $\tilde{t}_s > \bar{c} - \delta$ holds. That is, given $c > \tilde{c}$, $\tilde{\Delta}(t) < 0$ iff $\beta > \tilde{\beta}(t)$ for any $t \in [0, \bar{c} - \delta]$. We define $\bar{\tilde{\beta}} = \max_{t \in [0, \bar{c}-\delta]} \tilde{\beta}(t)$. Then, given $c > \tilde{c}$ and $\beta > \bar{\tilde{\beta}}$, $\tilde{\Delta}(t) < 0$ holds for any $t \in [0, \bar{c} - \delta]$. One example of parameters that satisfies this condition is provided in Figure 9.
(c) In the centralized system, $s_1^c = s_2^c = 0$. However, $\tilde{s}(t) > 0$ for any t given that $\beta > 0$. Thus, one can easily verify that $\tilde{\pi}_i(t) < \pi_i^{c*}$ holds for any $t \in [0, \bar{c} - \delta]$, i.e., coordination cannot be achieved. □

F. Additional Results

In this part, we provide some additional results that are omitted from the main paper. First, we conduct numerical studies to examine the impact of shipping cost (δ) on the equilibrium transfer price and inventory in the endogenous-inventory case, which complements the numerical analysis of Section 5.4 in the main paper. We have tested many different parameter regions; here, we present the results in Figure A.1, using a representative example with $k = 1, p = 1, c = 0.45, \bar{c} = 0.5$, and $\beta = 0.4$. Through numerical results, we can observe that the optimal transfer price ($\tilde{t}^*$) and the full equilibrium inventory ($\tilde{I}(\tilde{t}^*)$) both decrease in δ.



(a) Equilibrium Transfer Price (b) Equilibrium Inventory

Figure A.1: Impact of δ under Endogenous Inventory

Note: This figure is illustrated with $g(x) = 1$ for $x \in [0, 1]$ and $g_{12}(x, y) = g(x)g(y)$. Moreover, $k = 1, p = 1, c = 0.45, \bar{c} = 0.5$, and $\beta = 0.4$.

Second, we show the effect of demand uncertainty on the retailers' inventory-sharing decisions in the exogenous-inventory case. To simplify the analysis, we consider a special case of the random demand factors $(\varepsilon_1, \varepsilon_2)$. Suppose that ε_1 and ε_2 are independent and follow the same uniform distribution on the interval $[\mu-\sigma, \mu+\sigma]$, where $\mu \in [0,1]$ is the mean and $\sigma \in [0, min\{\mu, 1-\mu\}]$ measures the level of demand uncertainty. Thus, the pdf of ε_i can be written as $g(x) = 1/(2\sigma)$ for $x \in [\mu-\sigma, \mu+\sigma]$, and the pdf for the joint distribution of $(\varepsilon_1, \varepsilon_2)$ is given by $g_{12}(x,y) = g(x)g(y)$. We define $\hat{\Delta}(t,I) = \hat{\pi}_i(t,I) - \hat{\pi}_i^N(I)$ as the profit difference between the sharing and the no-sharing cases. When $\hat{\Delta}(t,I) > 0$, the two retailers would strictly prefer to agree on inventory sharing; when $\hat{\Delta}(t,I) < 0$, they would prefer not entering into an inventory-sharing agreement. As the level of demand uncertainty σ changes, the sign of $\hat{\Delta}(t,I)$ can change, indicating that the retailers' preference for inventory-sharing will change. The following Proposition A.1 summarizes the effect of the level of demand uncertainty (i.e., σ) on the retailers' preferences for sharing.

Proposition A.1. *Suppose that ε_1 and ε_2 are independent and follow the same uniform distribution on $[\mu-\sigma, \mu+\sigma]$. Given the retailers' exogenous inventory levels $I_1 = I_2 = I \in [0,1]$, when $t \geq \frac{\bar{c}-\delta}{2}$, then $\hat{\Delta}(t,I) \geq 0$; when $t < \frac{\bar{c}-\delta}{2}$, then there exists $\hat{\sigma}(t,I)$ such that: $\hat{\Delta}(t,I) = 0$ if $\sigma \leq |2I-\mu|$, $\hat{\Delta}(t,I) < 0$ if $|2I-\mu| < \sigma < \hat{\sigma}(t,I)$, and $\hat{\Delta}(t,I) > 0$ if $\sigma > \hat{\sigma}(t,I)$.*

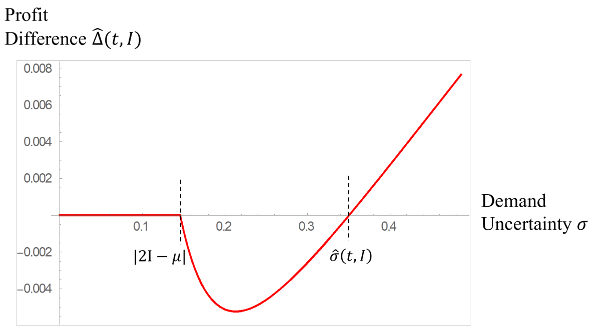


Figure A.2: Effect of Demand Uncertainty under Exogenous Inventory for $t < \frac{\bar{c}-\delta}{2}$.

Note: This figure is illustrated with $g(x) = \frac{1}{2\delta}$ for $x \in [\mu-\delta, \mu+\delta]$ and $g(x) = 0$ otherwise; $g_{12}(x,y) = g(x)g(y)$. $p = 1$, $\beta = 0.5$, $k = 5$, $c = 0.1$, $\bar{c} = 0.5$, $t = 0.05$, $\delta = 0.1$, $I_1 = I_2 = 0.32$, and $\mu = 0.5$.

Figure A.2 provides an illustrative example for Proposition A.1. From Proposition 1, we know that if the transfer price is high (i.e., $t \geq \frac{\bar{c}-\delta}{2}$), the retailers will always prefer having an inventory-sharing agreement, i.e., $\hat{\Delta}(t,I) \geq 0$ holds for all levels of demand uncertainty. So, our ensuing analysis will focus on the case of $t < \frac{\bar{c}-\delta}{2}$, where the sign of $\hat{\Delta}(t,I)$ depends on the level of demand uncertainty σ. Specifically, when σ is low (i.e., $\sigma \leq |2I-\mu|$), no-sharing and sharing cases result in the same profits ($\hat{\Delta}(t,I) = 0$) because this situation is equivalent to that of $I \geq (\mu+\sigma)/2$ or $I \leq (\mu-\sigma)/2$, where the two retailers will either both overstock (i.e., $I \geq (\mu+\sigma)/2$) or both understock (i.e., $I \leq (\mu-\sigma)/2$) for all demand realizations such that inventory sharing never occurs even if a sharing agreement was accepted in Stage 1. When the level of demand uncertainty is moderate (i.e., $|2I-\mu| < \sigma < \hat{\sigma}(t,I)$), sharing inventory is less profitable than no-sharing because the benefit of risk pooling in this situation is dominated by the intensified service competition, which will make the retailers worse off. By contrast, under a high level of demand uncertainty (i.e., $\sigma > \hat{\sigma}(t,I)$), sharing inventory becomes more profitable because the retailers are more likely to face very high or very low demand, which makes the risk-pooling benefit of inventory sharing dominate the negative effect of intensified service competition (caused by inventory sharing at a low transfer price). This result shows that inventory sharing at low transfer prices can improve the retailers' profits only when they face a high level of demand uncertainty.

Proof of Proposition A.1. See the detailed proof on SSRN: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3512712`. ☐