A. Online Supplement: Proofs of Results

Proof of Lemma 1. In Case P, demands are as in (3.2). In Case F, demands are as in (3.3). In either case, the supply of platform i is as in (4.1) and $\rho_i = \sqrt{\frac{d_i}{\alpha p_i - c}}$ for $i \in \{0,1\}$. Replacing the expression for ρ_i in each demand function in Case P and solving for d_i leads to the balanced demand quantities in (4.3). Plugging these into (4.1) leads to the balanced supply quantities in (4.6). Replacing the expressions for ρ_0 and ρ_1 in both demand functions in Case F results in a system of two equations with two unknowns, namely, d_0 and d_1. Solving this system of equations leads to the balanced demand quantities in (4.4). From these, we obtain the balanced supply quantities in (4.7). Finally, (3.2) implies that the condition for Case P (i.e., $\underline{v}_d \le \bar{v}_d$) holds if $d_0^{\mathrm{P}}(p_0) + d_1^{\mathrm{P}}(p_1) \le \lambda$, and the condition for Case F (i.e., $\underline{v}_d > \bar{v}_d$) holds if $d_0^{\mathrm{P}}(p_0) + d_1^{\mathrm{P}}(p_1) > \lambda$. Q.E.D.

Proof of Theorem 1. For $i \in \{0,1\}$, define $x_i = \alpha p_i - c$, which represents the margin for platform i drivers (net of commission). We start by showing that $D_i(p_i, p_j)$ is concave in p_i, which is equivalent to showing that $D_i(x_i, x_j)$ is concave in x_i. It is easy to verify that $d_i^{\mathrm{P}}(x_i) = \lambda \frac{(\alpha - c)x_i - x_i^2}{\alpha x_i + \lambda\gamma\alpha}$ and $d_i^{\mathrm{F}}(x_i, x_j) = \lambda \frac{\lambda\gamma\alpha x_i + (\alpha - x_i + x_j)x_i x_j}{2\alpha x_i x_j + \lambda\gamma\alpha(x_i + x_j)}$ are concave in x_i. For a given value of x_j, say $\hat{x}_j$, let us define x_i^l (resp., x_i^u) as the lowest (resp., highest) value of x_i such that $d_i^{\mathrm{P}}(x_i) + d_j^{\mathrm{P}}(\hat{x}_j) = \lambda$. If such values do not exist, then it must be that $d_i^{\mathrm{P}}(x_i) + d_j^{\mathrm{P}}(\hat{x}_j) < \lambda$ for all x_i, and therefore, $D_i(x_i, x_j) = d_i^{\mathrm{P}}(x_i)$, which is concave. Otherwise, to establish concavity of $D_i(x_i, \hat{x}_j)$ in x_i, it remains to show that

$$\left.\frac{\partial d_i^{\mathrm{P}}(x_i)}{\partial x_i^-}\right|_{x_i = x_i^l} \ge \left.\frac{\partial d_i^{\mathrm{F}}(x_i, \hat{x}_j)}{\partial x_i^+}\right|_{x_i = x_i^l} \ge 0 \quad \text{and} \quad \left.\frac{\partial d_i^{\mathrm{P}}(x_i)}{\partial x_i^+}\right|_{x_i = x_i^u} \le \left.\frac{\partial d_i^{\mathrm{F}}(x_i, \hat{x}_j)}{\partial x_i^-}\right|_{x_i = x_i^u} \le 0.$$

We first show that $\lambda - d_j^{\mathrm{P}}(\hat{x}_j) \le d_i^{\mathrm{F}}(x_i, \hat{x}_j) \le d_i^{\mathrm{P}}(x_i)$ for $x_i \in [x_i^l, x_i^u]$. It is easy to verify through algebraic calculations that $\lambda - d_j^{\mathrm{P}}(\hat{x}_j) = d_i^{\mathrm{F}}(x_i^l, \hat{x}_j) = d_i^{\mathrm{P}}(x_i^l)$ and $\lambda - d_j^{\mathrm{P}}(\hat{x}_j) = d_i^{\mathrm{F}}(x_i^u, \hat{x}_j) = d_i^{\mathrm{P}}(x_i^u)$. By concavity of $d_i^{\mathrm{F}}(x_i, \hat{x}_j)$ and $d_i^{\mathrm{P}}(x_i)$ in x_i, these equations imply that $\lambda - d_j^{\mathrm{P}}(\hat{x}_j) < d_i^{\mathrm{F}}(x_i, \hat{x}_j)$ and $\lambda - d_j^{\mathrm{P}}(\hat{x}_j) < d_i^{\mathrm{P}}(x_i)$ for $x_i \in (x_i^l, x_i^u)$, and equality in these expressions holds for $x_i = x_i^l$ and $x_i = x_i^u$. Now fix a value $\hat{x}_i \in (x_i^l, x_i^u)$. Because $\lambda - d_j^{\mathrm{P}}(\hat{x}_j) < d_i^{\mathrm{P}}(\hat{x}_i)$, there exists an interval around $\hat{x}_j$ such that $\lambda - d_j^{\mathrm{P}}(x_j) < d_i^{\mathrm{P}}(\hat{x}_i)$, or equivalently $\lambda - d_i^{\mathrm{P}}(\hat{x}_i) < d_j^{\mathrm{P}}(x_j)$, for all x_j in that interval. That means that there exist $x_j^l < \hat{x}_j < x_j^u$ such that $\lambda - d_i^{\mathrm{P}}(\hat{x}_i) = d_j^{\mathrm{F}}(\hat{x}_i, x_j^l) = d_j^{\mathrm{P}}(x_i^l)$ and $\lambda - d_i^{\mathrm{P}}(\hat{x}_i) = d_j^{\mathrm{F}}(x_j^u, \hat{x}_i) = d_j^{\mathrm{P}}(x_i^u)$. By concavity of $d_j^{\mathrm{F}}(x_j, \hat{x}_i)$ in x_j, this implies that $d_i^{\mathrm{F}}(\hat{x}_i, \hat{x}_j) < d_i^{\mathrm{P}}(\hat{x}_i)$, where the inequality follows because in a fully covered market, both platforms cover the entire market for any pair of prices. This argument is valid for any fixed value $\hat{x}_i \in (x_i^l, x_i^u)$. We therefore have that $d_i^{\mathrm{F}}(x_i, \hat{x}_j) < d_i^{\mathrm{P}}(x_i)$ for all $x_i \in (x_i^l, x_i^u)$. Now note the following:

$$\left.\frac{\partial d_i^{\mathrm{F}}(x_i, \hat{x}_j)}{\partial x_i^+}\right|_{x_i = x_i^l} = \lim_{x_i \to x_i^+} \frac{d_i^{\mathrm{F}}(x_i, \hat{x}_j) - d_i^{\mathrm{F}}(x_i^l, \hat{x}_j)}{x_i - x_i^l} = \lim_{x_i \to x_i^+} \frac{d_i^{\mathrm{F}}(x_i, \hat{x}_j) - d_i^{\mathrm{P}}(x_i^l)}{x_i - x_i^l} \le \lim_{x_i \to x_i^+} \frac{d_i^{\mathrm{P}}(x_i) - d_i^{\mathrm{P}}(x_i^l)}{x_i - x_i^l} = \left.\frac{\partial d_i^{\mathrm{P}}(x_i)}{\partial x_i^-}\right|_{x_i = x_i^l},$$

$$\left.\frac{\partial d_i^{\mathrm{F}}(x_i, \hat{x}_j)}{\partial x_i^+}\right|_{x_i = x_i^l} = \lim_{x_i \to x_i^+} \frac{d_i^{\mathrm{F}}(x_i, \hat{x}_j) - d_i^{\mathrm{F}}(x_i^l, \hat{x}_j)}{x_i - x_i^l} = \lim_{x_i \to x_i^+} \frac{d_i^{\mathrm{F}}(x_i, \hat{x}_j) - [\lambda - d_j^{\mathrm{P}}(\hat{x}_j)]}{x_i - x_i^l} \ge \lim_{x_i \to x_i^+} \frac{\lambda - d_j^{\mathrm{P}}(\hat{x}_j) - [\lambda - d_j^{\mathrm{P}}(\hat{x}_j)]}{x_i - x_i^l} = 0.$$

Similarly, we show that $\left.\frac{\partial d_i^{\mathrm{P}}(x_i)}{\partial x_i^+}\right|_{x_i = x_i^u} \le \left.\frac{\partial d_i^{\mathrm{F}}(x_i, \hat{x}_j)}{\partial x_i^-}\right|_{x_i = x_i^u} \le 0$. This establishes the concavity of $D_i(p_i, p_j)$ in p_i for $i \in \{0,1\}$, and it implies that there exists a Nash equilibrium of the game where platform i's objective function is $D_i(p_i, p_j)$. Moreover, it can be shown that there exists a unique symmetric Nash equilibrium with objective function $D_i(p_i, p_j)$ for $i \in \{0,1\}$. Denote by p^{*D} the unique symmetric Nash equilibrium price of that game. We now examine the game with platform i's objective function given by its profit $\Pi_i(p_i, p_j) = [(1-\alpha)p_i + c]D_i(p_i, p_j)$. Note that $\frac{\partial}{\partial p_i}\Pi_i(p_i, p_j) = (1-\alpha)D_i(p_i, p_j) +$

$[(1-\alpha)p_i + c]\frac{\partial}{\partial p_i}D_i(p_i,p_j)$, and moreover, $\frac{\partial^2}{\partial p_i^2}\Pi_i(p_i,p_j) = 2\big[(1-\alpha)\frac{\partial}{\partial p_i}D_i(p_i,p_j)\big] + [(1-\alpha)p_i + c]\frac{\partial^2}{\partial p_i^2}D_i(p_i,p_j)$. We have that $\frac{\partial}{\partial p_i}D_i(p_i,p_j) < 0$ for $p_i > p^{*D}$ and $\frac{\partial}{\partial p_i}D_i(p_i,p_j) > 0$ for $p_i < p^{*D}$. Moreover, $\frac{\partial}{\partial p_i}\Pi_i(p_i,p_j)\big|_{(p_i,p_j)=(p^{*D},p^{*D})} > 0$. Therefore, there exists $p_i^l < p^{*D}$ such that $\Pi_i(p_i,p_j)$ is concave in p_i for $p_i > p_i^l$ with its maximizer greater than p^{*D}. Thus, by restricting attention to prices in $[p_0^l,\overline{p}]\times[p_1^l,\overline{p}]$, it follows that there exists a Nash equilibrium of the game with platform i's objective function given by $\Pi_i(p_i,p_j)$. (This restriction is without loss of optimality in the sense that it eliminates only suboptimal price choices, as $\Pi_i(p_i,p_j)$ is strictly increasing in p_i for $p_i \in [0,p_i^l]$.) Uniqueness of a symmetric Nash equilibrium is verified by showing that there is a unique solution to $\frac{\partial}{\partial p_i}\big\{[(1-\alpha)p_i + c]d_i^{\mathrm{P}}(p_i)\big\} = 0$ and a unique solution to $\frac{\partial}{\partial p_i}\big\{[(1-\alpha)p_i + c]d_i^{\mathrm{F}}(p_i,p_j)\big\}\big|_{p_i=p_j=p} = 0$.   Q.E.D.

Proof of Proposition 1. To calculate the Nash equilibrium of the game between the platforms, we first consider the maximization of $d_i^{\mathrm{P}}(x_i)$ over x_i subject to the restriction that $d_i^{\mathrm{P}}(x_i) \le s_i^{\mathrm{P}}(x_i)$. These maximization problems depend on the platforms' own variables only. The maximum of $d_i^{\mathrm{P}}(x_i)$ over x_i is achieved at $x_i = x^{*a} = \sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} - \lambda\gamma$. It follows from simple algebra that $d_0^{\mathrm{P}}(x^{*a}) + d_1^{\mathrm{P}}(x^{*a}) \le \lambda$ if and only if $\gamma \ge \frac{(\frac{1}{2}-c)^2}{2\lambda}$. Therefore, if the latter inequality holds, we have that $D_i(x_i,x_j) = d_i^{\mathrm{P}}(x_i)$. Moreover, the condition $d_i^{\mathrm{P}}(x_i) \le s_i^{\mathrm{P}}(x_i)$ is equivalent to $x_i \ge \frac{\lambda(1-\gamma-c)}{1+\lambda}$. It thus follows that $x^{*a} \ge \frac{\lambda(1-\gamma-c)}{1+\lambda}$ if and only if $\gamma \ge \frac{\lambda(1-c)}{1+2\lambda}$. This establishes solution (a) in the statement of the result. If, on the other hand, $\gamma < \frac{\lambda(1-c)}{1+2\lambda}$ implying that $x^{*a} > \frac{\lambda(1-\gamma-c)}{1+\lambda}$, then the supply constraint is not satisfied at $x_i = x^{*a}$, and the maximum for platform i is achieved at $x_i = x^{*b} = \frac{\lambda(1-\gamma-c)}{1+\lambda}$, which solves $d_i^{\mathrm{P}}(x_i) = s_i^{\mathrm{P}}(x_i)$. This solution verifies the condition $d_0^{\mathrm{P}}(x^{*b}) + d_1^{\mathrm{P}}(x^{*b}) \le \lambda$ if and only if $\gamma \ge \frac{1}{2} - \frac{\lambda}{2} - c$, in which case we again have that $D_i(x_i,x_j) = d_i^{\mathrm{P}}(x_i)$ for $i,j \in \{0,1\}$ with $i \ne j$. Collecting these conditions, we establish solution (b) in the statement of the theorem.

If $\gamma < \frac{(\frac{1}{2}-c)^2}{2\lambda}$ for $\gamma \in [0,\frac{1-2c}{4}]$ and $\gamma < \frac{1}{2} - \frac{\lambda}{2} - c$ for $\gamma \ge \frac{1-2c}{4}$, then we have that $d_0^{\mathrm{P}}(x^{*a}) + d_1^{\mathrm{P}}(x^{*a}) > \lambda$ and $d_0^{\mathrm{P}}(x^{*b}) + d_1^{\mathrm{P}}(x^{*b}) > \lambda$, respectively. Thus, in these regions (Regions (c) and (d) in Figure 4), $D_i(x_i,x_j) = d_i^{\mathrm{F}}(x_i,x_j)$ for $i,j \in \{0,1\}$ with $i \ne j$; as a result, each platform's payoff function depends on the other platform's pricing decision as well. We therefore study the game between the platforms with payoff functions $cd_i^{\mathrm{F}}(x_i,x_j)$ for (λ,γ) in Regions (c) and (d). Solving the system of equations given by the first-order conditions, one can verify that there exists a unique symmetric Nash equilibrium for (λ,γ) in Regions (c) and (d). Ignoring the supply constraint $d_i^{\mathrm{F}}(x_i,x_j) \le s_i^{\mathrm{F}}(x_i,x_j)$, the unique symmetric equilibrium is given by $x_0 = x_1 = x^{*d} = \sqrt{\frac{\lambda\gamma}{2}}$. The equilibrium satisfies the supply constraint condition $d_i^{\mathrm{F}}(x^{*d},x^{*d}) \le s_i^{\mathrm{F}}(x^{*d},x^{*d})$ if and only if $d_i^{\mathrm{F}}(x^{*d},x^{*d}) \le x^{*d}$ for $i \in \{0,1\}$. In turn, this condition is equivalent to $\gamma \ge \frac{\lambda}{2}$. Under this condition, $x_0 = x_1 = x^{*d}$ is the unique symmetric Nash equilibrium establishing part (d) in the theorem. On the other hand, if $\gamma < \frac{\lambda}{2}$, then the supply constraint is binding and the Nash equilibrium solves the equations $d_i^{\mathrm{F}}(x_i,x_j) = s_i^{\mathrm{F}}(x_i,x_j)$ for $i \in \{0,1\}$. The unique symmetric solution to these equations is given by $x_0 = x_1 = x^{*c} = \frac{\lambda}{2}$. This establishes solution (c) and completes the proof.   Q.E.D.

Proof of Proposition 2. Note that if (λ,γ) is in regions of partial market coverage, then the equilibrium outcomes for competition and monopoly are identical, implying that the desired result weakly holds. Thus, in the remainder of the proof, suppose that (λ,γ) is in regions of full market coverage, either constrained or unconstrained supply (i.e., Regions (c) or (d) in Figure 4).

To establish the desired result, we compare equilibrium outcomes in Regions (c) and (d) to outcomes that would result if one of the competing platforms were removed from the market (resulting in a true monopoly for the remaining platform). Removing a platform corresponds to removing the boundary separating Region (b) from Region (c) and the boundary separating Region (d) from Region (a) in Figure 4. The parameter space would then be split into two regions by the boundary separating Regions (a) and (b), which would now extend to the vertical axis. Parameters above (below) this boundary would lead to outcomes matching those for Region (a) (Region (b)) in Theorem 1. Since this boundary passes slightly above the current boundary between Regions (c) and (d), there are three comparisons to consider: (i) equilibrium outcomes for parameters in Region (c) vs. Region (b) (monopoly) outcomes for the same parameters; (ii) equilibrium outcomes for parameters in Region (d) that fall above the (a)-(b) boundary curve vs. Region (a) (monopoly) outcomes for the same parameters; and (iii) equilibrium outcomes for parameters in Region (d) that fall below the (a)-(b) boundary curve vs. Region (b) (monopoly) outcomes for the same parameters.

To facilitate these comparisons, let p^{*c} and p^{*d} denote the equilibrium prices (with competition) that arise at parameter vectors in Regions (c) and (d), respectively. Let p^{*a} and p^{*b} denote the prices that would arise at the same parameter vectors but *without competition*, i.e., in the "extended" Regions (a) and (b), respectively (depending on which side of the (a)-(b) boundary the parameters fall). Thus, for example, we will compare p^{*c} to p^{*b} for parameters in Region (c). We will also use the same superscripts to denote other quantities at those prices—e.g., we will compare demands d^{*c} and d^{*b}, customer utilities u_d^{*b} and u_d^{*c}, etc. In light of platform symmetry, we will generally suppress platform indices to simplify the notation.

(i) In Region (c), $\gamma < \frac{1}{2} - \frac{\lambda}{2} - c$, which can be rewritten as $\frac{1}{2}(1+\lambda) < 1-\gamma-c$. Therefore, $\frac{\lambda(1-\gamma-c)}{1+\lambda} > \frac{\lambda}{2}$. Thus, $p^{*c} < p^{*b}$. Since the supply constraint is binding in both Regions (b) and (c), we have the following for utilization: $\rho^{*b} = \rho^{*c} = 1$. Combining this fact with (4.1), we deduce that that $d^{*b} = p^{*b} - c$ and $d^{*c} = p^{*c} - c$. Since $p^{*c} < p^{*b}$, it immediately follows that $d^{*c} < d^{*b}$, and thus $s^{*c} < s^{*b}$. Finally, since utilization is the same in both regions, (3.1), (3.4) and the ordering of prices implies that $u_d^{*c} > u_d^{*b}$ and $u_s^{*c} < u_s^{*b}$. This establishes the result for comparison (i).

(ii) In this region, we have $\gamma < \frac{(\frac{1}{2}-c)^2}{2\lambda}$, which is equivalent to $1-c > \frac{1}{2} + \sqrt{2\lambda\gamma}$. Therefore, $p^{*a} - c + \lambda\gamma = \sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} > \sqrt{\lambda^2\gamma^2 + \lambda\gamma(\frac{1}{2} + \sqrt{2\lambda\gamma})} = \sqrt{\frac{\lambda\gamma}{2}} + \lambda\gamma = p^{*d} - c + \lambda\gamma$, and hence $p^{*a} > p^{*d}$. To compare demands, note that d^{*d} is the demand each platform experiences under competition and equilibrium prices p^{*d}. Removing one platform while keeping the price at p^{*d} causes the remaining platform's demand to increase, since the platform now attracts all positive-utility customers, not just those whose utility is higher than it was for the competing platform. Now adjusting to the optimal (monopoly) price p^{*a} causes demand to increase further, and hence $d^{*a} > d^{*d}$. From (4.1) one can show that supply is increasing in demand. Combining this with the fact $p^{*a} > p^{*d}$ yields $s^{*a} = \sqrt{(p^{*a}-c)d^{*a}} > \sqrt{(p^{*d}-c)d^{*d}} = s^{*d}$. We next show that $\rho^{*d} < \rho^{*a}$. To that end, note that $\rho^{*d} = \sqrt[4]{\frac{\lambda}{2\gamma}}$ and $\rho^{*a} = \sqrt{-\lambda + \frac{1}{\gamma}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}}$. Therefore, $\rho^{*d} < \rho^{*a}$ if and only if $\lambda\gamma + \sqrt{\frac{\lambda\gamma}{2}} < \sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}$. By elementary algebra, this is equivalent to the following:

$$\frac{1}{2}\left(\frac{1}{2} - c\right) > \sqrt{\frac{\lambda\gamma}{2}}. \tag{A.1}$$

Because $\gamma < \frac{\left(\frac{1}{2}-c\right)^2}{2\lambda}$ in Region (d), it follows that, for parameters in Region (d) that fall above the (a)-(b) boundary curve, (A.1) holds, and hence, $\rho^{*d} < \rho^{*a}$. Finally, the fact that prices are lower and utilization is lower under competition implies that $u_d^{*d} > u_d^{*a}$ and $u_s^{*d} < u_s^{*a}$. This establishes the result for comparison (ii).

(iii) In this region, we have $\gamma < \frac{\lambda(1-c)}{1+2\lambda}$, which holds if and only if $\gamma < \frac{\lambda(1-\gamma-c)}{1+\lambda}$. Since $\gamma \geq \frac{\lambda}{2}$ in Region (d), the preceding inequality implies that $\sqrt{\frac{\lambda\gamma}{2}} < \frac{\lambda(1-\gamma-c)}{1+\lambda}$, which is equivalent to $p^{*d} < p^{*b}$. To compare demands, we note that $\gamma < \frac{1}{2} - \frac{\lambda}{2} - c$ for parameters in Region (d) that fall below the (a)-(b) boundary curve. By elementary algebra, this holds if and only if $\frac{\lambda}{2} < \frac{\lambda(1-c-\gamma)}{1+\lambda}$, implying that $d^{*d} < d^{*b}$. Moreover, $\rho^{*b} = 1 > \rho^{*d} = \sqrt[4]{\frac{\lambda}{2\gamma}}$. Recall that $\gamma \geq \frac{\lambda}{2}$ in Region (d), from which we have that

$$\gamma \geq \sqrt{\frac{\lambda\gamma}{2}} = \frac{\lambda}{2}\sqrt{\frac{2\gamma}{\lambda}}. \tag{A.2}$$

For purchasing customers' utilities, note that $\gamma \geq \frac{\lambda}{2}$ and $\gamma < \frac{\lambda(1-c)}{1+2\lambda}$ for parameters in Region (d) that fall below the (a)-(b) boundary curve. These inequalities imply that $\lambda(1-c)+\gamma > 2(1+\lambda)\gamma > 2(1+\lambda)\sqrt{\frac{\lambda\gamma}{2}}$, which is equivalent to $p^{*b}+\gamma(\rho^{*b})^2 > p^{*d}+\gamma(\rho^{*d})^2$. By (3.1), this implies that $u_d^{*b} < u_d^{*d}$. Regarding supply and active drivers' utilities, (A.2) is equivalent to the following:

$$\frac{(1+2\lambda)\gamma-\lambda\gamma}{1+\lambda} \geq \frac{\lambda}{2}\sqrt{\frac{2\gamma}{\lambda}}. \tag{A.3}$$

For parameters in Region (d) that fall below the (a)-(b) boundary curve, we have $\gamma < \frac{\lambda(1-c)}{1+2\lambda}$; i.e., $\lambda(1-c) > (1+2\lambda)\gamma$. Combining this fact with (A.3), we have that

$$\frac{\lambda(1-\gamma-c)}{1+\lambda} > \frac{\lambda}{2}\sqrt{\frac{2\gamma}{\lambda}} \geq \frac{\lambda}{2}\sqrt[4]{\frac{2\gamma}{\lambda}}, \tag{A.4}$$

where the last inequality follows from $\gamma \geq \frac{\lambda}{2}$. This implies that $s^{*d} < s^{*b}$ and $u_s^{*d} < u_s^{*b}$. Q.E.D.

Proof of Proposition 3. (i) It follows from simple algebra that $\frac{\partial x^{*a}}{\partial\lambda} > 0$, $\frac{\partial x^{*d}}{\partial\lambda} > 0$, where $x^{*a} = \sqrt{\lambda^2\gamma^2+\lambda\gamma(1-c)} - \lambda\gamma$ and $x^{*d} = \sqrt{\frac{\lambda\gamma}{2}}$ are the equilibrium margins (equilibrium prices minus commission c) in Regions (a) and (d), respectively, depicted in Figure 4. Below we prove parts (i) and (ii) for the cases in which (λ,γ) and $(\hat{\lambda},\gamma)$ are either both in Region (a) or both in Region (d). The result extends to the case where λ and $\hat{\lambda}$ are such that the pairs fall in Regions (d) and (a), respectively, by defining $(\tilde{\lambda},\gamma)$ as the point that falls on the curve $\gamma = \frac{\left(\frac{1}{2}-c\right)^2}{2\lambda}$ and using the proof below to compare performance between (λ,γ) and $(\tilde{\lambda},\gamma)$ and then between $(\tilde{\lambda},\gamma)$ and $(\hat{\lambda},\gamma)$ to derive the final comparison.

We first focus on Region (a). In this region, $\rho_i = \frac{d_i^{\mathrm{P}}(x^{*a})}{s_i^{\mathrm{P}}(x^{*a})} = \sqrt{\frac{\lambda(1-c-x^{*a})}{x^{*a}+\lambda\gamma}}$; thus, $\frac{\partial}{\partial\lambda}\rho_i > 0$ and $\frac{\partial}{\partial\lambda}\{p^{*a} + \gamma\rho_i^2\} > 0$. We also have that $\frac{\partial}{\partial\lambda}\{x^{*a}\rho_i\} > 0$. These inequalities imply that, evaluated at the equilibrium price, customer utility decreases and driver utility increases with an increase in λ. Because in Region (d), $\frac{\partial}{\partial\lambda}x^{*d} > 0$ and $\rho_i = \sqrt[4]{\frac{\lambda}{2\gamma}}$, which is increasing in λ, the same conclusions apply to customer utility and driver utility for parameters in this region. This proves part (i) of the proposition.

(ii) In this scenario, we compare customer and driver utilities at an increased market size $\hat{\lambda}$ when evaluated at the resulting equilibrium prices $p^*(\hat{\lambda})$ versus at the original equilibrium prices $p^*(\lambda)$.

We first focus on Region (a). In this region, the market is partially covered and a platform's demand depends only on its own price. To explicitly capture dependence on λ, we apply the same slight abuse of notation that was introduced prior to the statement of Proposition 3 to demand and profit, i.e., $\Pi_i(p,\lambda)$

and $D_i(p,\lambda)$ are the profit and demand, respectively, experienced by platform i given a price p and market size λ. Thus this part of the proof involves comparing $D_i(p^*(\lambda),\hat{\lambda}) = d_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda})$ with $D_i(p^*(\hat{\lambda}),\hat{\lambda}) = d_i^{\mathrm{P}}(p^*(\hat{\lambda}),\hat{\lambda})$ and $\Pi_i(p^*(\lambda),\hat{\lambda}) = cd_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda})$ with $\Pi_i(p^*(\hat{\lambda}),\hat{\lambda}) = cd_i^{\mathrm{P}}(p^*(\hat{\lambda}),\hat{\lambda})$ (under the assumption that $\alpha = 1$). First note that, in order for $p^*(\lambda)$ to be feasible under market size $\hat{\lambda}$, we must have $d_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda}) \leq s_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda})$. This holds as long as $p^*(\lambda) \geq \frac{\hat{\lambda}(1-\gamma)+c}{1+\hat{\lambda}}$. If $\gamma > 1$, then this inequality holds for any $\hat{\lambda} > \lambda$ and we therefore define $\Lambda(\lambda,\gamma) = \infty$. Otherwise, we define $\Lambda(\lambda,\gamma) = \frac{c-p^*(\lambda)}{\gamma-1+p^*(\lambda)}$ and it follows that $p^*(\lambda)$ is feasible if $\hat{\lambda} \leq \Lambda(\lambda,\gamma)$. Because $p^*(\hat{\lambda})$ is platform i's optimal price at $\hat{\lambda}$, it follows that $\Pi_i(p^*(\lambda),\hat{\lambda}) < \Pi_i(p^*(\hat{\lambda}),\hat{\lambda})$, which implies that $d_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda}) < d_i^{\mathrm{P}}(p^*(\hat{\lambda}),\hat{\lambda})$. From (3.1) and (3.2), the latter inequality directly implies that $u_{di}(p^*(\lambda),\hat{\lambda}) < u_{di}(p^*(\hat{\lambda}),\hat{\lambda})$. We also know that the equilibrium price increases with λ. Combining that with $d_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda}) < d_i^{\mathrm{P}}(p^*(\hat{\lambda}),\hat{\lambda})$ and the expressions in (3.2a) and (3.2b), we must have that $\rho_i(p^*(\lambda),\hat{\lambda}) > \rho_i(p^*(\hat{\lambda}),\hat{\lambda})$. The latter inequality together with $d_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda}) < d_i^{\mathrm{P}}(p^*(\hat{\lambda}),\hat{\lambda})$ imply that for supply, $s_i^{\mathrm{P}}(p^*(\lambda),\hat{\lambda}) < s_i^{\mathrm{P}}(p^*(\hat{\lambda}),\hat{\lambda})$. Therefore, from (3.4) and (3.5), we have that $u_{si}(p^*(\lambda),\hat{\lambda}) < u_{si}(p^*(\hat{\lambda}),\hat{\lambda})$.

We now consider Region (d). First, note that $D_i(p^*(\lambda),\hat{\lambda}) = d_i^{\mathrm{F}}(p^*(\lambda),\hat{\lambda}) = \frac{\hat{\lambda}}{2} = d_i^{\mathrm{F}}(p^*(\hat{\lambda}),\hat{\lambda}) = D_i(p^*(\hat{\lambda}),\hat{\lambda})$. From (4.7), we have that $s_i^{\mathrm{F}}(p,\hat{\lambda}) = \sqrt{(p-c)d_i^{\mathrm{F}}(p,\hat{\lambda})}$ and $\rho_i = \sqrt{\frac{d_i^{\mathrm{F}}(p,\hat{\lambda})}{p-c}}$, for any price p. Therefore, $p^*(\lambda)$ is feasible under market size $\hat{\lambda}$ if and only if $\frac{\hat{\lambda}}{2} \leq p^*(\lambda) - c$. We therefore define $\Lambda(\lambda,\gamma) = 2\,(p^*(\lambda)-c)$. In terms of customer utility, we aim to show that $u_{di}(p^*(\lambda),\hat{\lambda}) < u_{di}(p^*(\hat{\lambda}),\hat{\lambda})$, which is equivalent to $c + \sqrt{\frac{\lambda\gamma}{2}} + \hat{\lambda}\sqrt{\frac{\gamma}{2\lambda}} > c + \sqrt{\frac{\hat{\lambda}\gamma}{2}} + \gamma\sqrt{\frac{\hat{\lambda}}{2\gamma}}$. It follows from simple algebra that this inequality holds. In terms of driver utility, we aim to show that $u_{si}(p^*(\lambda),\hat{\lambda}) < u_{si}(p^*(\hat{\lambda}),\hat{\lambda})$. This is equivalent to $\sqrt[4]{\frac{\lambda\gamma}{2}}\sqrt{\frac{\hat{\lambda}}{2}} < \sqrt{\frac{\hat{\lambda}\gamma}{2}}\sqrt[4]{\frac{\hat{\lambda}}{2\gamma}}$, which holds as long as $\hat{\lambda} > \lambda$. Q.E.D.

Proof of Lemma 2. As explained earlier, in Case P, demands are as in (3.2), whereas in Case F, demands are as in (3.3). In both cases, the supply of the platforms is as in (5.4), and $\rho_0 = \rho_1 = \frac{d_0+d_1}{\sqrt{2[(\alpha p_0-c)d_0+(\alpha p_1-c)d_1]}}$. Plugging in the expressions for ρ_0 and ρ_1 in the demand functions in Case P leads to a nonlinear system of two equations with two unknowns, namely, d_0 and d_1. According to this system of equations, the quantity $\frac{d_0-d_1}{\lambda}$ is equal to $p_1 - p_0$, whereas the quantity $\xi = \frac{d_0+d_1}{\lambda}$ satisfies the following parabolic equation: $\eta(\xi;p_0,p_1) = A(p_0,p_1)\,\xi^2 + B(p_0,p_1)\,\xi + C(p_0,p_1) = 0$, where $A(p_0,p_1) = 2[\lambda\gamma + \frac{\alpha(p_0+p_1)}{2} - c]$, $B(p_0,p_1) = 4[c - \frac{(\alpha+c)(p_0+p_1)}{2} + \alpha p_0 p_1]$, and $C(p_0,p_1) = \alpha(2-p_0-p_1)(p_0-p_1)^2$. If the parabolic equation has at least one non-negative root, we use its largest non-negative root to derive an expression of the sum of platform demands, d_0+d_1. Combining this expression with the fact that $d_0 - d_1 = \lambda(p_1-p_0)$, we obtain the balanced demand quantities in Case P, expressed in (5.6). Using these demand expressions in (5.4), we obtain the balanced supply quantities in (5.9). By elementary algebra, we also deduce that, if $p_0 = p_1 \in [\underline{p},\overline{p}]$, then the parabolic equation $\eta(\xi;p_0,p_1) = 0$ has at least one non-negative root, namely $\xi_+(p_0,p_1)$, and $\frac{\partial^2}{\partial p_i^2}\xi_+(p_0,p_1) < 0$ for $i \in \{0,1\}$. Thus, the set $\mathcal{P}$ described in the statement of the lemma exists. Plugging in the expressions for ρ_0 and ρ_1 in the demand functions in Case F leads to the balanced demand quantities in (5.7). Replacing these demand expressions in (5.4), we derive the balanced supply quantities in (5.10). As before, by (3.2), the condition for Case P (i.e., $\underline{v}_d \leq \bar{v}_d$) holds if $d_0^{\mathrm{P}}(p_0,p_1) + d_1^{\mathrm{P}}(p_1,p_0) \leq \lambda$, and the condition for Case F (i.e., $\underline{v}_d > \bar{v}_d$) holds if $d_0^{\mathrm{P}}(p_0,p_1) + d_1^{\mathrm{P}}(p_1,p_0) > \lambda$. Q.E.D.

Proof of Theorem 2. Let $x_i = \alpha p_i - c$ for $i \in \{0,1\}$ and $\mathcal{X} = \{(\alpha p_0 - c, \alpha p_1 - c) : (p_0,p_1) \in \mathcal{P}\}$. As in the proof of Theorem 1, we first prove the concavity of $D_i(p_i,p_j)$ in p_i, or equivalently, the concavity of $D_i(x_i,x_j)$ in x_i. For all $(x_0,x_1) \in \mathcal{X}$, we know from Lemma 2 that $\xi_+(x_0,x_1)$ is concave in x_0 and x_1. Thus, $d_i^{\mathrm{P}}(x_i,x_j) = \frac{\lambda}{2}\left[\xi_+(x_0,x_1) - x_i + x_j\right]$ and $d_i^{\mathrm{F}}(x_i,x_j) = \frac{\lambda}{2}(1 - x_i + x_j)$ are concave in x_i. Given a value of x_j, say $\hat{x}_j$, let x_i^l (resp., x_i^u) be the lowest (resp., highest) value of x_i satisfying $d_i^{\mathrm{P}}(x_i,\hat{x}_j) + d_j^{\mathrm{P}}(\hat{x}_j,x_i) = \lambda$. If x_i^l and x_i^u do not exist, then $d_i^{\mathrm{P}}(x_i,\hat{x}_j) + d_j^{\mathrm{P}}(\hat{x}_j,x_i) < \lambda$ for all x_i, in which case $D_i(x_i,x_j)$ is equal to $d_i^{\mathrm{P}}(x_i,x_j)$ and hence concave. Now, suppose that x_i^l and x_i^u exist. Note that if $x_i \in (x_i^l, x_i^u)$ then $\xi_+(x_0,x_1) > 1$. This implies that $\frac{\partial}{\partial x_i}\xi_+(x_0,x_1)\big|_{x_i=x_i^l} \geq 0$ and $\frac{\partial}{\partial x_i}\xi_+(x_0,x_1)\big|_{x_i=x_i^u} \leq 0$ because $\xi_+(x_0,x_1)$ is concave in x_0 and x_1. Consequently,

$$\frac{\partial d_i^{\mathrm{P}}(x_i,\hat{x}_j)}{\partial x_i^-}\bigg|_{x_i=x_i^l} = \frac{\lambda}{2}\left[\frac{\partial}{\partial x_i}\xi_+(x_0,x_1) - 1\right]\bigg|_{x_i=x_i^l} \geq -\frac{\lambda}{2} = \frac{\partial d_i^{\mathrm{F}}(x_i,\hat{x}_j)}{\partial x_i^+}\bigg|_{x_i=x_i^l},$$
$$\frac{\partial d_i^{\mathrm{P}}(x_i,\hat{x}_j)}{\partial x_i^+}\bigg|_{x_i=x_i^u} = \frac{\lambda}{2}\left[\frac{\partial}{\partial x_i}\xi_+(x_0,x_1) - 1\right]\bigg|_{x_i=x_i^u} \leq -\frac{\lambda}{2} = \frac{\partial d_i^{\mathrm{F}}(x_i,\hat{x}_j)}{\partial x_i^-}\bigg|_{x_i=x_i^u}.$$

Therefore, based on the arguments used in the proof of Theorem 1, we deduce that $D_i(p_i,p_j)$ in concave p_i for $i \in \{0,1\}$ and there exists a unique symmetric Nash equilibrium p^{*D} for the game with platform i's objective function given by $D_i(p_i,p_j)$. Moreover, as argued in the proof of Theorem 1, we restrict attention to prices in $[p_0^l, \overline{p}] \times [p_1^l, \overline{p}]$, where $p_i^l < p^{*D}$ for $i \in \{0,1\}$, and deduce that the concavity of $D_i(p_i,p_j)$ in p_i implies the concavity of $\Pi_i(p_i,p_j) = [(1-\alpha)p_i + c]D_i(p_i,p_j)$ in p_i. (As before, this restriction is without loss of optimality in the sense that it eliminates only suboptimal price choices.) Thus, there exists a Nash equilibrium of the game with platform i's objective function given by $\Pi_i(p_i,p_j)$. Finally, uniqueness of a symmetric Nash equilibrium is verified by showing that there is a unique solution to $\frac{\partial}{\partial p_i}\big\{[(1-\alpha)p_i + c]d_i^{\mathrm{P}}(p_i,p_j)\big\}\big|_{p_i=p_j=p} = 0$ and a unique solution to $\frac{\partial}{\partial p_i}\big\{[(1-\alpha)p_i + c]d_i^{\mathrm{F}}(p_i,p_j)\big\}\big|_{p_i=p_j=p} = 0$. This equilibrium is well-defined only when the equilibrium demand d^* is non-negative and does not exceed the total market size, i.e., when $(\lambda,\gamma,\alpha,c) \in \Theta = \{(\lambda,\gamma,\alpha,c) : d^* \geq 0\} \cap \{(\lambda,\gamma,\alpha,c) : 2d^* \leq \lambda\}$. Q.E.D.

Proof of Proposition 4. Denote by P^{unc} and P^{con} the subcases of Case P where the supply constraint is non-binding and binding, respectively. Note that $d_i^{\mathrm{P}}(x_i,x_j) = \frac{\lambda}{2}[\xi_+(x_0,x_1) - x_i + x_j]$ for $i,j \in \{0,1\}$ with $i \neq j$, where $\xi_+(x_0,x_1) = \frac{-B(x_0,x_1) + \sqrt{B(x_0,x_1)^2 - 4A(x_0,x_1)C(x_0,x_1)}}{2A(x_0,x_1)}$, $A(x_0,x_1) = 2\lambda\gamma + x_0 + x_1$, $B(p_0,p_1) = 2[2x_0x_1 - (1-c)(x_0+x_1)]$, and $C(p_0,p_1) = [2(1-c) - x_0 - x_1](x_0 - x_1)^2$. Based on the solution to the system of equations given by the first-order conditions and the concavity of $d_i^{\mathrm{P}}(p_i,p_j)$ in p_i for $i \in \{0,1\}$, there exists a unique candidate for symmetric Nash equilibria in Case P^{unc}, given by $x_0 = x_1 = x^{*\tilde{a}} = \sqrt{\frac{\lambda^2\gamma^2 + \lambda\gamma(1-c)}{2}} - \lambda\gamma$. We note that $d_i^{\mathrm{P}}(x^{*\tilde{a}}, x^{*\tilde{a}})$ has a non-negative real value if and only if $2(1-c)^2 \geq \lambda\gamma(1-c) + \lambda^2\gamma^2$, which holds because $(\lambda,\gamma,c) \in \Theta_1$. By elementary algebra, we also have $d_0^{\mathrm{P}}(x^{*\tilde{a}}, x^{*\tilde{a}}) + d_1^{\mathrm{P}}(x^{*\tilde{a}}, x^{*\tilde{a}}) \leq \lambda$ if and only if $\gamma \geq \frac{3\sqrt{3-2c+c^2}+c-5}{2\lambda}$, and $d_0^{\mathrm{P}}(x^{*\tilde{a}}, x^{*\tilde{a}}) + d_1^{\mathrm{P}}(x^{*\tilde{a}}, x^{*\tilde{a}}) \leq s_0^{\mathrm{P}}(x^{*\tilde{a}}, x^{*\tilde{a}}) + s_1^{\mathrm{P}}(x^{*\tilde{a}}, x^{*\tilde{a}})$ if and only if $\gamma \geq \frac{2\lambda(1-c)}{1+2\lambda-\lambda^2}$. Thus, if the problem parameters are in Region $(\tilde{a})$, $\big($i.e., if $\gamma \geq \frac{3\sqrt{3-2c+c^2}+c-5}{2\lambda}$ and $\gamma \geq \frac{2\lambda(1-c)}{1+2\lambda-\lambda^2}\big)$ then the unique symmetric Nash equilibrium is given by $x_0 = x_1 = x^{*\tilde{a}}$. In Case P^{con}, the supply constraint (5.11) is binding; thus, the utilization is 1. Consequently, the demand for platform $i \in \{0,1\}$ is $\lambda(1-\gamma-c-x_i)$. Using this fact and the supply equation (5.4), we deduce that there is a unique candidate for symmetric Nash equilibria in Case P^{con}, given by $x_0 = x_1 = x^{*\tilde{b}} = \frac{\lambda(1-\gamma-c)}{1+\lambda}$. We deduce from elementary algebra that $d_0^{\mathrm{P}}(x^{*\tilde{b}}, x^{*\tilde{b}}) + d_1^{\mathrm{P}}(x^{*\tilde{b}}, x^{*\tilde{b}}) \leq \lambda$ if and only if $\gamma \geq \frac{1}{2} - \frac{\lambda}{2} - c$, and $d_0^{\mathrm{P}}(x^{*\tilde{b}}, x^{*\tilde{b}}) + d_1^{\mathrm{P}}(x^{*\tilde{b}}, x^{*\tilde{b}}) < s_0^{\mathrm{P}}(x^{*\tilde{b}}, x^{*\tilde{b}}) + s_1^{\mathrm{P}}(x^{*\tilde{b}}, x^{*\tilde{b}})$ if and only if $\gamma > \frac{2\lambda(1-c)}{1+2\lambda-\lambda^2}$. As a result, if the problem parameters are in Region $(\tilde{b})$, $\big($i.e., if $\gamma \geq \frac{1}{2} - \frac{\lambda}{2} - c$ and $\gamma < \frac{2\lambda(1-c)}{1+2\lambda-\lambda^2}\big)$ then the unique symmetric Nash equilibrium is given by $x_0 = x_1 = x^{*\tilde{b}}$.

In Case F, the demand for platform $i \in \{0,1\}$ is $\frac{\lambda}{2}(1 - x_i + x_j)$, where $j \in \{0,1\} \setminus \{i\}$. Suppose towards a contradiction that there is a symmetric Nash equilibrium candidate for Case F such that the supply constraint (5.11) is not binding. Then, platform $i \in \{0,1\}$ would maximize its balanced demand by setting $x_i = 0$, and thus, the resulting equilibrium candidate would be $x_0 = x_1 = 0$. But, by (5.11), this would make the total platform supply equal 0, which contradicts with the assumption that the supply constraint is not binding. As a result, there are no symmetric Nash equilibrium candidates for Case F such that the supply constraint is not binding, and it is sufficient to restrict attention to the subcase of Case F in which the supply constraint is binding. In this subcase, we deduce by (5.11) and elementary algebra that there exists a unique candidate for symmetric Nash equilibria, given by $x_0 = x_1 = x^{*\tilde{c}} = \frac{\lambda}{2}$. We have by elementary algebra that $d_0^{\mathrm{P}}(x^{*\tilde{c}}, x^{*\tilde{c}}) + d_1^{\mathrm{P}}(x^{*\tilde{c}}, x^{*\tilde{c}}) > \lambda$ if and only if $\gamma < \frac{1}{2} - \frac{\lambda}{2} - c$. Consequently, from (5.5), if the problem parameters are in Region $(\tilde{c})$, the unique symmetric Nash equilibrium is given by $x_0 = x_1 = x^{*\tilde{c}}$. Q.E.D.

Proof of Proposition 5. We will prove the proposition's statement in each of the regions in Figure 7.

Region $(a\text{-}\tilde{a})$: In this region, the equilibrium outcomes for the single-homing setting are as follows: $p^*_{\mathrm{SH}} = c + \sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} - \lambda\gamma$, $D^*_{\mathrm{SH}} = \lambda\big(1 - c + 2\lambda\gamma - 2\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big)$, and $\rho^*_{\mathrm{SH}} = \sqrt{-\lambda + \frac{1}{\gamma}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}}$. The equilibrium outcomes for the multi-homing setting are the following: $p^*_{\mathrm{MH}} = c + \frac{1}{\sqrt{2}}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} - \lambda\gamma$, $D^*_{\mathrm{MH}} = \lambda\big(1 - c + 2\lambda\gamma - \frac{3}{\sqrt{2}}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big)$, and $\rho^*_{\mathrm{MH}} = \sqrt{-\lambda + \frac{1}{\gamma}\sqrt{2\lambda^2\gamma^2 + 2\lambda\gamma(1-c)}}$. Observe that $p^*_{\mathrm{SH}} > p^*_{\mathrm{MH}}$ and $D^*_{\mathrm{SH}} > D^*_{\mathrm{MH}}$ in this region. For purchasing customers, note that $u^*_{d,\mathrm{SH}} > u^*_{d,\mathrm{MH}}$ if and only if $p^*_{\mathrm{MH}} + \gamma(\rho^*_{\mathrm{MH}})^2 > p^*_{\mathrm{SH}} + \gamma(\rho^*_{\mathrm{SH}})^2$. By elementary algebra, this is equivalent to $\frac{1}{2}(3\sqrt{2} - 4)\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} > 0$, which holds since $3\sqrt{2} > 4$. For active drivers, note that $u^*_{s,\mathrm{SH}} > u^*_{s,\mathrm{MH}}$ if and only if $(p^*_{\mathrm{SH}} - c)\rho^*_{\mathrm{SH}} > (p^*_{\mathrm{MH}} - c)\rho^*_{\mathrm{MH}}$. It follows from elementary algebra that the preceding inequality is equivalent to $\gamma\big[-\lambda + \frac{1}{\gamma}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big]^{3/2} + \big[\lambda\gamma - \frac{1}{\sqrt{2}}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big]\sqrt{-\lambda + \frac{1}{\gamma}\sqrt{2\lambda^2\gamma^2 + 2\lambda\gamma(1-c)}} > 0$. Using elementary algebra and the fact that $\frac{1}{\sqrt{2}} + \sqrt{2} > 2$, we deduce that this inequality holds.

Region $(a\text{-}\tilde{b})$: For the single-homing setting, we have the following in this region: $p^*_{\mathrm{SH}} = c + \sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} - \lambda\gamma$, $D^*_{\mathrm{SH}} = \lambda\big(1 - c + 2\lambda\gamma - 2\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big)$, and $\rho^*_{\mathrm{SH}} = \sqrt{-\lambda + \frac{1}{\gamma}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}}$. whereas, for the multi-homing setting, we have $p^*_{\mathrm{MH}} = \frac{\lambda(1-\gamma)+c}{1+\lambda}$, $D^*_{\mathrm{MH}} = \frac{\lambda(1-\gamma-c)}{1+\lambda}$, and $\rho^*_{\mathrm{MH}} = 1$. Note that $p^*_{\mathrm{SH}} > p^*_{\mathrm{MH}}$ if and only if $2\lambda\gamma + \gamma > \lambda(1-c)$, which holds inside Region $(a\text{-}\tilde{b})$ since $\gamma > \frac{\lambda(1-c)}{1+2\lambda}$ in this region. We also deduce from elementary algebra that $D^*_{\mathrm{SH}} > D^*_{\mathrm{MH}}$ if and only if $(1+\lambda)^2\gamma^2 - 2(1+\lambda)\gamma\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} + \lambda^2\gamma^2 + \lambda\gamma(1-c) > 0$, which can be re-expressed as $[\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} - (1+\lambda)\gamma]^2 > 0$. Consequently, $D^*_{\mathrm{SH}} > D^*_{\mathrm{MH}}$ inside Region $(a\text{-}\tilde{b})$. Regarding purchasing customers, we deduce from elementary algebra that $u^*_{d,\mathrm{SH}} > u^*_{d,\mathrm{MH}}$ if and only if $[\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} - (1+\lambda)\gamma]^2 > 0$. Hence, inside Region $(a\text{-}\tilde{b})$, we have $u^*_{d,\mathrm{SH}} > u^*_{d,\mathrm{MH}}$ for all purchasing customers. Regarding active drivers, we obtain from elementary algebra that $u^*_{s,\mathrm{SH}} > u^*_{s,\mathrm{MH}}$ if and only if $\gamma\big[-\lambda + \frac{1}{\gamma}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big]^{3/2} - \frac{\lambda(1-\gamma-c)}{1+\lambda} > 0$, which holds because $\gamma > \frac{\lambda(1-c)}{1+2\lambda}$ in Region $(a\text{-}\tilde{b})$.

Region (b-$\tilde{b}$): In this region, $p^*_{\text{SH}} = p^*_{\text{MH}} = \frac{\lambda(1-\gamma)+c}{1+\lambda}$, $D^*_{\text{SH}} = D^*_{\text{MH}} = \frac{\lambda(1-\gamma-c)}{1+\lambda}$, and $\rho^*_{\text{SH}} = \rho^*_{\text{MH}} = 1$; hence, the equilibrium outcomes are identical for the single-homing and multi-homing settings.

Region (c-$\tilde{c}$): We have $p^*_{\text{SH}} = p^*_{\text{MH}} = c + \frac{\lambda}{2}$, $D^*_{\text{SH}} = D^*_{\text{MH}} = \frac{\lambda}{2}$, and $\rho^*_{\text{SH}} = \rho^*_{\text{MH}} = 1$ for this region. Thus, the equilibrium outcomes are identical for the single-homing and multi-homing settings.

Region (d-$\tilde{a}$): In this region, we have the following regarding the single-homing setting: $p^*_{\text{SH}} = c + \sqrt{\frac{\lambda\gamma}{2}}$, $D^*_{\text{SH}} = \frac{\lambda}{2}$, and $\rho^*_{\text{SH}} = \sqrt[4]{\frac{\lambda}{2\gamma}}$. Moreover, we have the following regarding the multi-homing setting: $p^*_{\text{MH}} = c + \frac{1}{\sqrt{2}}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} - \lambda\gamma$, $D^*_{\text{MH}} = \lambda\big(1 - c + 2\lambda\gamma - \frac{3}{\sqrt{2}}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big)$, and $\rho^*_{\text{MH}} = \sqrt{-\lambda + \frac{1}{\gamma}\sqrt{2\lambda^2\gamma^2 + 2\lambda\gamma(1-c)}}$. By elementary algebra, $p^*_{\text{SH}} > p^*_{\text{MH}}$ if and only if $\lambda\gamma + 2\sqrt{2\lambda\gamma} + c > 0$, which always holds. Note that $D^*_{\text{SH}} > D^*_{\text{MH}}$ if and only if $2D^*_{\text{MH}} < \lambda$. Inside Region ($d$-$\tilde{a}$), we have partial market coverage in equilibrium in the multi-homing setting, implying that $2D^*_{\text{MH}} < \lambda$; thus, $D^*_{\text{SH}} > D^*_{\text{MH}}$ in this region. For purchasing customers, we have $u^*_{d,\text{SH}} > u^*_{d,\text{MH}}$ if and only if $\frac{3}{\sqrt{2}}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)} > 2\lambda\gamma + \sqrt{2\lambda\gamma}$. Because $D^*_{\text{MH}} = \lambda\big(1 - c + 2\lambda\gamma - \frac{3}{\sqrt{2}}\sqrt{\lambda^2\gamma^2 + \lambda\gamma(1-c)}\big) < \frac{\lambda}{2}$ in this region, we deduce that $u^*_{d,\text{SH}} > u^*_{d,\text{MH}}$ if $\gamma < \frac{(\frac{1}{2}-c)^2}{2\lambda}$, which holds inside Region (d-$\tilde{a}$). For active drivers, we deduce from elementary algebra that $u^*_{s,\text{SH}} > u^*_{s,\text{MH}}$ if $\lambda\gamma + 2\sqrt{2\lambda\gamma} + c > 0$, which is valid in general. As a result, in Region (d-$\tilde{a}$), $u^*_{s,\text{SH}} > u^*_{s,\text{MH}}$ for all active drivers.

Region (d-$\tilde{b}$): In this region, the equilibrium outcomes for the single-homing setting are the following: $p^*_{\text{SH}} = c + \sqrt{\frac{\lambda\gamma}{2}}$, $D^*_{\text{SH}} = \frac{\lambda}{2}$, and $\rho^*_{\text{SH}} = \sqrt[4]{\frac{\lambda}{2\gamma}}$, whereas the equilibrium outcomes for the multi-homing setting are as follows: $p^*_{\text{MH}} = \frac{\lambda(1-\gamma)+c}{1+\lambda}$, $D^*_{\text{MH}} = \frac{\lambda(1-\gamma-c)}{1+\lambda}$, and $\rho^*_{\text{MH}} = 1$. Because $\gamma > \frac{\lambda}{2}$ in Region (d-$\tilde{b}$), $p^*_{\text{SH}} > p^*_{\text{MH}}$ if $\frac{\lambda}{2} > \frac{\lambda(1-c-\gamma)}{1+\lambda}$, which is equivalent to $\gamma > \frac{1}{2} - \frac{\lambda}{2} - c$, which holds in Region (d-$\tilde{b}$). Consequently, $p^*_{\text{SH}} > p^*_{\text{MH}}$ in Region (d-$\tilde{b}$). We deduce from elementary algebra that $D^*_{\text{SH}} > D^*_{\text{MH}}$ if and only if $\gamma > \frac{1}{2} - \frac{\lambda}{2} - c$, which holds in Region (d-$\tilde{b}$); therefore, $D^*_{\text{SH}} > D^*_{\text{MH}}$ in this region. For purchasing customers, we note that $u^*_{d,\text{SH}} > u^*_{d,\text{MH}}$ if and only if $\frac{\lambda(1-c)+\gamma}{1+\lambda} > \sqrt{2\lambda\gamma}$, which holds in Region (d-$\tilde{b}$) because $\gamma > \frac{1}{2} - \frac{\lambda}{2} - c$ and $\gamma < \frac{(\frac{1}{2}-c)^2}{2\lambda}$ in this region. For active drivers, we deduce from elementary algebra that $u^*_{s,\text{SH}} > u^*_{s,\text{MH}}$ if $\frac{\lambda}{2} > \frac{(1-\gamma-c)\lambda}{1+\lambda}$, which holds in Region (d-$\tilde{b}$) because $\gamma > \frac{1}{2} - \frac{\lambda}{2} - c$ in this region.

Region (d-$\tilde{c}$): In this region, we have the following for the single-homing setting: $p^*_{\text{SH}} = c + \sqrt{\frac{\lambda\gamma}{2}}$, $D^*_{\text{SH}} = \frac{\lambda}{2}$, and $\rho^*_{\text{SH}} = \sqrt[4]{\frac{\lambda}{2\gamma}}$, and for the multi-homing setting, we have $p^*_{\text{MH}} = c + \frac{\lambda}{2}$, $D^*_{\text{MH}} = \frac{\lambda}{2}$, and $\rho^*_{\text{MH}} = 1$. Since $\gamma > \frac{\lambda}{2}$ inside Region (d-$\tilde{c}$), we deduce that $p^*_{\text{SH}} > p^*_{\text{MH}}$ in this region. In addition, $D^*_{\text{SH}} = D^*_{\text{MH}}$ in Region (d-$\tilde{c}$). For purchasing customers, note that $u^*_{d,\text{SH}} > u^*_{d,\text{MH}}$ if and only if $\big(\sqrt{\gamma} - \sqrt{\frac{\lambda}{2}}\big)^2 > 0$, which holds inside Region (d-$\tilde{c}$). For active drivers, note that $u^*_{s,\text{SH}} > u^*_{s,\text{MH}}$ if and only if $\gamma > \frac{\lambda}{2}$, which holds inside Region (d-$\tilde{c}$). Q.E.D.

Proof of Proposition 6. From (3.4), a single-homing driver experiences utility $u_{si,\text{SH}} = -k_{si} + (p_i - c)\rho_i$, while from (5.2) at the single-homing symmetric price equilibrium a multi-homing driver experiences utility $u_{si,\text{MH}} = -k_{si} + [p_i - (c+\delta)]\mathfrak{P}$, where $\mathfrak{P}$ is the probability that a single driver defecting to multi-homing would successfully obtain a customer. For single-homing to be preferred, we must have $u_{si,\text{SH}} > u_{si,\text{MH}}$. Since $u_{si,\text{MH}}$ is increasing in $\mathfrak{P}$ and $\mathfrak{P} \le 1$, it is sufficient to have $-k_{si} + (p_i - c)\rho_i > -k_{si} + (p_i - c) - \delta$, or, equivalently, $\delta > (1-\rho_i)(p_i - c)$. Q.E.D.