

Electronic Companion

Designing Sustainable Products under Co-Production Technology

Appendix D: Proofs in Section 4

LEMMA D.1 (GF Strategy). Under GF strategy of Lemma 2, the firm's problem (P) reduces to

$$\pi_{\text{GF}}(c, v, n, q_t) := \max_{F^{-1}\left(\frac{(F(q_t)-n)^+}{1-n}\right) \leq q_o \leq q_t} \Pi_{\text{GF}}(q_o) := \frac{1}{\bar{F}(q_o)} \{q_t \bar{F}(q_t) - c + (\bar{F}(q_o) - \bar{F}(q_t)) [q_o + v(q_t - q_o)]\}. \quad (\text{GF})$$

The co-product's quality solving (GF) is given by

$$q_o^{\text{GF}} := \begin{cases} q_{\text{GF}}^*, & \text{if } c < c_{\text{GF}}, \\ F^{-1}\left(\frac{(F(q_t)-n)^+}{1-n}\right), & \text{if } c \geq c_{\text{GF}}, \end{cases} \quad (\text{D.1})$$

where $q_{\text{GF}}^* \in \left[F^{-1}\left(\frac{(F(q_t)-n)^+}{1-n}\right), q_t\right)$ uniquely satisfies

$$\bar{F}(q_{\text{GF}}^*) - \bar{F}(q_t) + \left[\bar{F}(q_t)(q_t - q_{\text{GF}}^*) - \frac{c}{1-v}\right] f(q_{\text{GF}}^*) / \bar{F}(q_{\text{GF}}^*) = 0, \quad (\text{D.2})$$

and

$$c_{\text{GF}} := (1-v) \left\{ \bar{F}(q_t) \left[q_t - F^{-1}\left(\frac{(F(q_t)-n)^+}{1-n}\right) \right] + \frac{\left[F(q_t) - \frac{(F(q_t)-n)^+}{1-n} \right] \left[1 - \frac{(F(q_t)-n)^+}{1-n} \right]}{f\left(\frac{(F(q_t)-n)^+}{1-n}\right)} \right\}. \quad (\text{D.3})$$

Accordingly, the co-product's price and the material quantity are given by $p_o^{\text{GF}} = q_o^{\text{GF}} + v(q_t - q_o^{\text{GF}})$ and $Q^{\text{GF}} = 1/(1 - F(q_o^{\text{GF}}))$, respectively.

Proof of Lemma D.1. By substituting $p_o = q_o + v(q_t - q_o)$ and $Q = 1/(1 - F(q_o))$ into (P), we immediately obtain (GF). Direct calculation reveals that

$$\Pi'_{\text{GF}}(q_o) = \frac{1}{\bar{F}(q_o)} \{ (1-v) [\bar{F}(q_o) - \bar{F}(q_t)] + [(1-v)\bar{F}(q_t)(q_t - q_o) - c] f(q_o) / \bar{F}(q_o) \}.$$

Therefore, we have

- For $c \leq (1-v)\bar{F}(q_t)(q_t - q_o)$ or equivalently $q_o \leq \left[q_t - \frac{c}{(1-v)\bar{F}(q_t)} \right]^+$, we have $\Pi'_{\text{GF}}(q_o) \geq 0$.
- For $c > (1-v)\bar{F}(q_t)(q_t - q_o)$ or equivalently $q_o > \left[q_t - \frac{c}{(1-v)\bar{F}(q_t)} \right]^+$, The log-concavity of $F(\cdot)$ implies that $\bar{F}(q_o)\Pi'_{\text{GF}}(q_o) = (1-v) [\bar{F}(q_o) - \bar{F}(q_t)] + [(1-v)\bar{F}(q_t)(q_t - q_o) - c] f(q_o) / \bar{F}(q_o)$ is decreasing in q_o and takes negative value at $q_o = q_t$.

Thus, $\Pi_{\text{GF}}(q_o)$ is quasi-concave in $q_o \in [0, q_t]$ and its maximum is determined by the following two cases:

1. If $\Pi'_{\text{GF}}(q_o) \leq 0$ at $q_o = F^{-1}\left(\frac{(F(q_t)-n)^+}{1-n}\right)$ or equivalently $c \geq c_{\text{GF}}$, then $\Pi_{\text{GF}}(q_o)$ is maximized at $q_o^{\text{GF}} = F^{-1}\left(\frac{(F(q_t)-n)^+}{1-n}\right)$, yielding (D.1).

2. Otherwise, $\Pi'_{\text{GF}}(q_o) > 0$ at $q_o = F^{-1}\left(\frac{(F(q_t)-n)^+}{1-n}\right)$. Then, $\Pi_{\text{GF}}(q_o)$ is maximized where $\Pi'_{\text{GF}}(q_o) = 0$, i.e., at $q_o^{\text{GF}} = q_{\text{GF}}^*$ given by (D.2), yielding (D.1). \square

LEMMA D.2 (GP Strategy). GP strategy of Lemma 2 is only feasible for $n \leq F(q_t)$, under which the firm's problem (P) reduces to

$$\pi_{\text{GP}}(c, v, n, q_t) := \max_{0 \leq q_o \leq F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)} \Pi_{\text{GP}}(q_o) := n \left\{ \frac{q_t \bar{F}(q_t) - c}{\bar{F}(q_o) - \bar{F}(q_t)} + q_o + v(q_t - q_o) \right\}. \quad (\text{GP})$$

The co-product's quality solving (GP) is given by

$$q_o^{\text{GP}} := \begin{cases} F^{-1}\left(\frac{F(q_t)-n}{1-n}\right), & \text{if } c \leq c_{\text{GP}}, \\ q_{\text{GP}}^*, & \text{if } c > c_{\text{GP}}, \end{cases} \quad (\text{D.4})$$

where $q_{\text{GP}}^* \in \left[0, F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)\right)$ is given by

$$q_{\text{GP}}^* = \inf \left\{ q \geq 0 : (1-v) (\bar{F}(q) - \bar{F}(q_t))^2 - (c - q_t \bar{F}(q_t)) f(q) \leq 0 \right\}, \quad (\text{D.5})$$

and

$$c_{\text{GP}} := q_t \bar{F}(q_t) + (1-v) \frac{n^2 \bar{F}^2(q_t)}{(1-n)^2 f\left(F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)\right)}. \quad (\text{D.6})$$

Accordingly, the co-product's price and the material quantity are given by $p_o^{\text{GP}} = q_o^{\text{GP}} + v(q_t - q_o^{\text{GP}})$ and $Q^{\text{GP}} = n/(F(q_t) - F(q_o^{\text{GP}}))$, respectively.

Proof of Lemma D.2. The feasibility of GP strategy necessitates $F(q_o) \leq (F(q_t) - n)/(1-n)$, which requires $n \leq F(q_t)$. By straightforward verification, substituting $p_o = q_o + v(q_t - q_o)$ and $Q = n/(F(q_t) - F(q_o))$ reduces (P) to (GP). Direct calculation reveals that

$$\Pi'_{\text{GP}}(q_o) = \frac{n}{\bar{F}(q_o) - \bar{F}(q_t)} \left\{ \frac{(q_t \bar{F}(q_t) - c) f(q_o)}{\bar{F}(q_o) - \bar{F}(q_t)} + (1-v) (\bar{F}(q_o) - \bar{F}(q_t)) \right\}$$

Therefore, we have

- For $c \leq q_t \bar{F}(q_t)$, we have $\Pi'_{\text{GP}}(q_o) > 0$ and hence the maximum of $\Pi_{\text{GP}}(q_o)$ is achieved at $q_o = F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)$.
- For $c > q_t \bar{F}(q_t)$, the log-concavity of $F(\cdot)$ implies that $\frac{(q_t \bar{F}(q_t) - c) f(q_o)}{\bar{F}(q_o) - \bar{F}(q_t)} + (1-v) (\bar{F}(q_o) - \bar{F}(q_t))$ is decreasing in q_o and takes
 1. nonnegative value for all $q_o \in \left[0, F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)\right]$ if $q_t \bar{F}(q_t) < c \leq c_{\text{GP}}$, implying that $\Pi_{\text{GP}}(q_o)$ is maximized at $q_o^{\text{GP}} = F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)$ and yielding (D.4);
 2. positive value at $q_o = 0$ but negative value at $q_o = F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)$ if $c_{\text{GP}} < c < q_t \bar{F}(q_t) + (1-v)F^2(q_t)/f(0)$, implying that $\Pi_{\text{GP}}(q_o)$ is maximized where $\Pi'_{\text{GP}}(q_o) = 0$, i.e., $q_o^{\text{GP}} = q_{\text{GP}}^*$ given by (D.5) and yielding (D.4);
 3. non-positive value for all $q_o \in \left[0, F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)\right]$ if $c \geq q_t \bar{F}(q_t) + (1-v)F^2(q_t)/f(0)$, implying that $\Pi_{\text{GP}}(q_o)$ is maximized at $q_o^{\text{GP}} = 0 = q_{\text{GP}}^*$, again given by (D.5), yielding (D.4). \square

LEMMA D.3 (**LF Strategy**). *LF strategy of Lemma 2 is only feasible for $n \leq F(q_t)$, under which the firm's problem (P) reduces to*

$$\pi_{\text{LF}}(c, n, q_t) := \max_{0 \leq q_o \leq F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)} \Pi_{\text{LF}}(q_o) := \{q_t \bar{F}(q_t) - c + (\bar{F}(q_o) - \bar{F}(q_t)) q_o\} / \bar{F}(q_o), \quad (\text{LF})$$

independent of v . The co-product's quality solving (LF) is given by

$$q_o^{\text{LF}} := \begin{cases} F^{-1}\left(\frac{F(q_t)-n}{1-n}\right), & \text{if } c \leq c_{\text{LF}}, \\ q_{\text{LF}}^*, & \text{if } c > c_{\text{LF}}, \end{cases} \quad (\text{D.7})$$

where $q_{\text{LF}}^* \in \left(0, F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)\right)$ uniquely satisfies

$$\bar{F}(q_{\text{LF}}^*) - \bar{F}(q_t) + [\bar{F}(q_t)(q_t - q_{\text{LF}}^*) - c] f(q_{\text{LF}}^*) / \bar{F}(q_{\text{LF}}^*) = 0, \quad (\text{D.8})$$

and

$$c_{\text{LF}} := \bar{F}(q_t) \left[q_t - F^{-1}\left(\frac{F(q_t)-n}{1-n}\right) \right] + \frac{n \bar{F}^2(q_t)}{(1-n)^2 f\left(F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)\right)}. \quad (\text{D.9})$$

Accordingly, the co-product's price and the material quantity are given by $p_o^{\text{LF}} = q_o^{\text{LF}}$ and $Q^{\text{LF}} = 1/(1 - F(q_o^{\text{LF}}))$, respectively.

Proof of Lemma D.3. The feasibility of GP strategy necessitates $F(q_o) \leq (F(q_t) - n)/(1-n)$, which requires $n \leq F(q_t)$. By straightforward verification, substituting $p_o = q_o$ and $Q = 1/(1 - F(q_o))$ reduces (P) to (LF). Direct calculation reveals that

$$\Pi'_{\text{LF}}(q_o) = \frac{1}{\bar{F}(q_o)} \{ \bar{F}(q_o) - \bar{F}(q_t) + [\bar{F}(q_t)(q_t - q_o) - c] f(q_o) / \bar{F}(q_o) \}.$$

Therefore, we have

- For $\bar{F}(q_t)(q_t - q_o) \geq c$ or equivalently $q_o \leq \left[q_t - \frac{c}{\bar{F}(q_t)} \right]^+$, we have $\Pi'_{\text{LF}}(q_o) \geq 0$.
- For $\bar{F}(q_t)(q_t - q_o) < c$ or equivalently $q_o > \left[q_t - \frac{c}{\bar{F}(q_t)} \right]^+$, the log-concavity of $F(\cdot)$ implies that $\bar{F}(q_o) \Pi'_{\text{LF}}(q_o) = \bar{F}(q_o) - \bar{F}(q_t) + [\bar{F}(q_t)(q_t - q_o) - c] f(q_o) / \bar{F}(q_o)$ is decreasing in q_o and takes nonnegative value at $q_o = F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)$ if and only if $c < \bar{F}(q_t) \left[q_t - F^{-1}\left(\frac{F(q_t)-n}{1-n}\right) \right] + \frac{n \bar{F}^2(q_t)}{(1-n)^2 f\left(F^{-1}\left(\frac{F(q_t)-n}{1-n}\right)\right)}$.

Thus, $\Pi_{LF}(q_o)$ is quasi-concave in $q_o \in [0, q_t]$ and its maximum can be of the following cases:

1. For $c \leq \bar{F}(q_t) \left[q_t - F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right) \right] \leq c_{LF}$ or equivalently $F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right) \leq q_t - \frac{c}{\bar{F}(q_t)}$, the maximum of $\Pi_{LF}(q_o)$ is achieved at $q_o^{LF} = F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right)$, yielding (D.7).
2. For $\bar{F}(q_t) \left[q_t - F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right) \right] < c \leq c_{LF} \bar{F}(q_t) \left[q_t - F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right) \right]$, we have $F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right) > \left[q_t - \frac{c}{\bar{F}(q_t)} \right]^+$ and $\Pi'_{LF}(q_o) \geq 0$ at $q_o = F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right)$, where $\Pi_{LF}(q_o)$ is maximized. Thus, we obtain (D.7).
3. For $c > c_{LF}$, we have $F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right) > \left[q_t - \frac{c}{\bar{F}(q_t)} \right]^+$ and $\Pi'_{LF}(q_o) < 0$ at $q_o = F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right)$. Hence, $\Pi_{LF}(q_o)$ is maximized at $p_o^{LF} = p_{LF}^*$ given by (D.8), again yielding (D.7). \square

Proof of Theorem 1. We compare the optimal profits of the three subproblems (GF), (GP) and (LF), the largest of which determines the global optimal solution to the firm's problem (P). To that end, we first make the following claims, which follow from straightforward verification.

Claim 1. $\pi_{GF}(c, v, n, q_t) \geq 0$ or $\pi_{GP}(c, v, n, q_t) \geq 0$ if and only if condition (3) holds (i.e., $c \leq \bar{c}$), while $\pi_{LF}(c, n, q_t) \geq 0$ if and only if $c \leq \underline{c} < \bar{c}$, where \underline{c} is defined by (4) and clearly smaller than \bar{c} in contrast with the definition of \bar{c} in (3). This claim follows from the fact that $\Pi_{GF}(q_o)$ and $\Pi_{GP}(q_o)$ are of the same sign as $q_t \bar{F}(q_t) + (\bar{F}(q_o) - \bar{F}(q_t)) [q_o + v(q_t - q_o)] - c$, which is nonnegative if and only if condition (3) holds. Instead, $\Pi_{LF}(q_o)$ is of the same sign as $q_t \bar{F}(q_t) + (\bar{F}(q_o) - \bar{F}(q_t)) q_o - c$, which is nonnegative if and only if condition (4) holds.

Claim 2. When $n > F(q_t)$, only subproblem (GF) is feasible and hence the solution to (P) is given by (D.1). In this case, we let $c_1^* = c_{GF} \wedge \bar{c}$ and $c_2^* = \bar{c}$. For the rest of the proof, we will focus on the case $n \leq F(q_t)$.

Claim 3. When $n \leq F(q_t)$, we have $c_{GF} = (1 - v)c_{LF} \leq c_{LF}$, which allows us to divide the proof into three different parametric scenarios. In particular, if we will denote $q_o^\sharp := F^{-1} \left(\frac{F(q_t) - n}{1 - n} \right)$, then

$$\Pi_{GF}(q_o^\sharp) = \Pi_{GP}(q_o^\sharp), \quad \text{and} \quad \Pi_{GF}(q_o) \geq \Pi_{LF}(q_o) \text{ for all } q_o. \quad (\text{D.10})$$

Scenario 1: $c_{GF} \leq c_{LF} \leq c_{GP}$.

- For $c \leq c_{GF}$, Lemmas D.1, D.2 and D.3 imply that

$$\pi_{GF}(c, v, n, q_t) = \Pi_{GF}(q_{GF}^*) \geq \Pi_{GF}(q_o^\sharp) \begin{cases} = \Pi_{GP}(q_o^\sharp) = \pi_{GP}(c, v, n, q_t), \\ \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t), \end{cases}$$

where we use the optimality of q_{GF}^* to obtain the first inequality and (D.10) to obtain the other (in)equalities. Thus, GF strategy is optimal and the optimal co-product's quality is q_{GF}^* .

- For $c_{GF} < c \leq c_{LF}$, similarly, Lemmas D.1, D.2 and D.3 imply that

$$\pi_{GF}(c, v, n, q_t) = \Pi_{GF}(q_o^\sharp) \begin{cases} = \Pi_{GP}(q_o^\sharp) = \pi_{GP}(c, v, n, q_t) \\ \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t). \end{cases}$$

Therefore, GF strategy is optimal and the optimal co-product's quality is q_o^\sharp .

- For $c_{LF} < c \leq c_{GP}$, Lemmas D.1, D.2 and D.3 imply that the larger profit between $\pi_{GF}(c, v, n, q_t) = \Pi_{GF}(q_o^\sharp) = \Pi_{GP}(q_o^\sharp) = \pi_{GP}(c, v, n, q_t)$ and $\pi_{LF}(c, n, q_t) = \Pi_{LF}(q_{LF}^*)$ determines the solution. By the Envelope Theorem,

$$\frac{\partial \pi_{GF}(c, v, n, q_t)}{\partial c} = \frac{\partial \pi_{GP}(c, v, n, q_t)}{\partial c} = -1/\bar{F}(q_o^\sharp) \leq -1/\bar{F}(q_{LF}^*) = \frac{\partial \pi_{LF}(c, n, q_t)}{\partial c}.$$

That is, $\pi_{GP}(c, v, n, q_t) - \pi_{LF}(c, n, q_t)$ is decreasing in $c \in [c_{LF}, c_{GP}]$. Let

$$\hat{c} := \max \{c \in [c_{LF}, c_{GP}] : \pi_{GP}(c, v, n, q_t) \geq \pi_{LF}(c, n, q_t)\}.$$

Then, for $c_{LF} < c \leq \hat{c}$, the optimal co-product's quality is q_o^\sharp and GF strategy is optimal. For $\hat{c} < c \leq c_{GP}$, subproblem (LF) dominates the other two subproblems; thus, LF strategy is optimal and the optimal co-product's quality is q_{LF}^* .

- For $c > c_{GP}$, Lemmas D.1, D.2 and D.3 imply that $\pi_{LF}(c, n, q_t) = \Pi(q_{LF}^*)$ and $\pi_{GP}(c, v, n, q_t) = \Pi_{GP}(q_{GP}^*) \geq \Pi_{GF}(q_o^\sharp) = \Pi_{GP}(q_o^\sharp) = \pi_{GF}(c, v, n, q_t)$. Thus, if $\pi_{GP}(c, v, n, q_t) \geq \pi_{LF}(c, n, q_t)$, then GP strategy is optimal with q_{GP}^* being the optimal co-product's quality; otherwise, LF strategy is optimal with q_{LF}^* being the optimal co-product's quality. In particular, according to Claim 1, the former case must be true for $c \geq \underline{c}$.

In sum, we obtain the theorem by letting $c_1^* = \hat{c} \wedge \bar{c}$ and $c_2^* = (\hat{c} \vee \underline{c}) \wedge \bar{c}$. In particular, for sufficiently large $v \leq 1$, we have $\hat{c} < \bar{c}$ and hence $c_2^* < \bar{c}$, because $\hat{c} \leq c_{GP} \rightarrow q_t \bar{F}(q_t) < q_t$ while $\bar{c} \rightarrow q_t$ as $v \rightarrow 1$.

Scenario 2: $c_{GF} \leq c_{GP} \leq c_{LF}$.

- For $c \leq c_{GF}$, following arguments similar to the first scenario, Lemmas D.1, D.2 and D.3 imply that

$$\pi_{GF}(c, v, n, q_t) = \Pi_{GF}(q_{GF}^*) \geq \Pi_{GF}(q_o^\sharp) \begin{cases} = \Pi_{GP}(q_o^\sharp) = \pi_{GP}(c, v, n, q_t), \\ \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t). \end{cases}$$

Thus, GF strategy is optimal and the optimal co-product's quality is q_{GF}^* .

- For $c_{GF} < c \leq c_{GP}$, Lemmas D.1, D.2 and D.3 again imply that

$$\pi_{GF}(c, v, n, q_t) = \Pi_{GF}(q_o^\sharp) \begin{cases} = \Pi_{GP}(q_o^\sharp) = \pi_{GP}(c, v, n, q_t) \\ \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t). \end{cases}$$

Therefore, GF strategy is optimal and the optimal co-product's quality is q_o^\sharp .

- For $c_{GP} < c \leq c_{LF}$, Lemmas D.1, D.2 and D.3 imply that

$$\pi_{GP}(c, v, n, q_t) = \Pi_{GP}(q_{GP}^*) \geq \Pi_{GP}(q_o^\sharp) = \Pi_{GF}(q_o^\sharp) = \pi_{GF}(c, v, n, q_t) \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t).$$

Therefore, GP strategy is optimal and the optimal co-product's quality is q_{GP}^* .

- For $c > c_{LF}$, Lemmas D.1, D.2 and D.3 imply that $\pi_{LF}(c, n, q_t) = \Pi_{LF}(q_{LF}^*)$ and $\pi_{GP}(c, v, n, q_t) = \Pi_{GP}(q_{GP}^*) \geq \Pi_{GP}(q_o^\sharp) = \Pi_{GF}(q_o^\sharp) = \pi_{GF}(c, v, n, q_t)$. Thus, if $\pi_{GP}(c, v, n, q_t) \geq \pi_{LF}(c, n, q_t)$, then GP strategy is optimal with q_{GP}^* being the optimal co-product's quality; otherwise, FL strategy is optimal with q_{LF}^* being the optimal co-product's quality. In particular, according to Claim 1, the former case must be true for $c \geq \underline{c}$.

In sum, we obtain the theorem by letting $c_1^* = c_{GP} \wedge \bar{c}$ and $c_2^* = (c_{GP} \vee \underline{c}) \wedge \bar{c}$. (In fact, if $c_{LF} \geq \underline{c}$, we can set $c_2^* = c_{GP} \wedge \bar{c}$.) In particular, for large enough $v \leq 1$, we have $c_{GP} < \bar{c}$ and hence $c_2^* < \bar{c}$, because $c_{GP} \rightarrow q_t \bar{F}(q_t) < q_t$ while $\bar{c} \rightarrow q_t$ as $v \rightarrow 1$.

Scenario 3: $c_{GP} \leq c_{GF} \leq c_{LF}$.

- For $c \leq c_{GP}$, following arguments similar to the first scenario, Lemmas D.1, D.2 and D.3 imply that

$$\pi_{GF}(c, v, n, q_t) = \Pi_{GF}(q_{GF}^*) \geq \Pi_{GF}(q_o^\sharp) \begin{cases} = \Pi_{GP}(q_o^\sharp) = \pi_{GP}(c, v, n, q_t), \\ \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t). \end{cases}$$

Thus, GF strategy is optimal with q_{GF}^* being the optimal co-product's quality.

- For $c_{GP} < c \leq c_{GF}$, Lemmas D.1, D.2 and D.3 imply that that $\pi_{GP}(c, v, n, q_t) = \Pi_{GP}(q_{GP}^*)$ and $\pi_{GF}(c, v, n, q_t) = \Pi_{GF}(q_{GF}^*) \geq \Pi_{GF}(q_o^\sharp) \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t)$. Therefore, we need to compare $\pi_{GF}(c, v, n, q_t)$ and $\pi_{GP}(c, v, n, q_t)$. By the Envelope Theorem,

$$\frac{\partial \pi_{GF}(c, v, n, q_t)}{\partial c} = -1/\bar{F}(q_{GF}^*) < -1/\bar{F}(q_{GP}^*) \leq -n/(\bar{F}(q_{GP}^*) - \bar{F}(q_t)) = \frac{\partial \pi_{GP}(c, v, n, q_t)}{\partial c}$$

where the first inequality follows from $q_{GF}^* > q_{GP}^*$ and the second inequality from the fact that $q_{GP}^* \leq q_o^\sharp = F^{-1}\left(\frac{F(q_t) - n}{1 - n}\right)$ or equivalently $F(q_{GP}^*) \leq \frac{F(q_t) - n}{1 - n}$. Namely, $\pi_{GF}(c, v, n, q_t) - \pi_{GP}(c, v, n, q_t)$ is decreasing in $c \in [c_{GP}, c_{GF}]$. On the other hand, as shown above, $\pi_{GF}(c_{GP}, v, n, q_t) - \pi_{GP}(c_{GP}, v, n, q_t) \geq 0$, whereas as shown below $\pi_{GF}(c_{GF}, v, n, q_t) - \pi_{GP}(c_{GF}, v, n, q_t) \leq 0$. Therefore, there exists a unique $\hat{c} \in [c_{GP}, c_{GF}]$ such that GF strategy is optimal for $c_{GP} \leq c \leq \hat{c}$ with q_{GF}^* being optimal co-product's quality, and GP strategy is optimal for $\hat{c} \leq c \leq c_{GF}$ with q_{GP}^* being optimal co-product's quality.

- For $c_{GF} < c \leq c_{LF}$, Lemmas D.1, D.2 and D.3 imply that

$$\pi_{GP}(c, v, n, q_t) = \Pi_{GP}(q_{GP}^*) \geq \Pi_{GP}(q_o^\sharp) = \Pi_{GF}(q_o^\sharp) = \pi_{GF}(c, v, n, q_t) \geq \Pi_{LF}(q_o^\sharp) = \pi_{LF}(c, n, q_t).$$

Therefore, GP strategy is optimal with q_{GP}^* being optimal co-product's quality.

- For $c > c_{LF}$, Lemmas D.1, D.2 and D.3 imply that $\pi_{GP}(c, v, n, q_t) = \Pi_{GP}(q_{GP}^*) \geq \Pi_{GP}(q_o^\sharp) = \Pi_{GF}(q_o^\sharp) = \pi_{GF}(c, v, n, q_t)$ and $\pi_{LF}(c, n, q_t) = \Pi_{LF}(q_{LF}^*)$. Therefore, if $\pi_{GP}(c, v, n, q_t) \geq \pi_{LF}(c, n, q_t)$, then GP strategy is optimal with q_{GP}^* being the optimal co-product's quality; otherwise, LF strategy is optimal with q_{LF}^* being the optimal co-product's quality. In particular, according to Claim 1, the former case must be true for $c \geq \underline{c}$.

In sum, we obtain the theorem by letting $c_1^* = \hat{c} \wedge \bar{c}$ and $c_2^* = (\hat{c} \vee \underline{c}) \wedge \bar{c}$. (In fact, if $c_{LF} \geq \underline{c}$, we can set $c_1^* = c_2^* = \hat{c} \wedge \bar{c}$.) In particular, for sufficiently large $v \leq 1$, we have $\hat{c} < \bar{c}$ and hence $c_2^* < \bar{c}$, because $\hat{c} \leq c_{GF} \rightarrow 0$ while $\bar{c} \rightarrow q_t$ as $v \rightarrow 1$.

In all the three scenarios above, the existence of \bar{v} follows from the fact that c_{LF} and $\pi_{LF}(c, n, q_t)$ are independent of v while c_{GP} is decreasing in v and $\pi_{GP}(c, v, n, q_t)$ is increasing in v . \square

Proof of Corollary 1. If $q_o = q_t$, then the marginal revenue from one unit of material is $q_t \bar{F}(q_t) \leq \underline{c}$ and hence the firm enters the market if and only if $c \leq q_t \bar{F}(q_t)$. If the co-product is introduced as a low-end product (i.e., $p_o = q_o$), then the marginal revenue is given by \underline{c} in (4) and hence the firm enters the market if and only if $c \leq \underline{c}$. Therefore, Theorem 1 implies that GP strategy is optimal for $c \in [\underline{c}, \bar{c}]$. \square

Proof of Theorem 2. We specialize Lemmas D.1, D.2, and D.3 with $F(q) = q$ to obtain

$$\pi_{GF}(c, v, n, q_t) = \max_{q_o \in \left[\frac{q_t - n}{1 - n}, q_t\right]} \Pi_{GF}(q_o) \equiv \frac{1 - q_t}{1 - q_o} q_t + \frac{q_t - q_o}{1 - q_o} [q_o + v(q_t - q_o)] - \frac{c}{1 - q_o}, \quad (D.11)$$

$$\pi_{GP}(c, v, n, q_t) = \max_{q_o \in \left[0, \frac{q_t - n}{1 - n}\right]} \Pi_{GP}(q_o) \equiv (1 - q_t) \left(\frac{n}{q_t - q_o}\right) q_t + n[q_o + v(q_t - q_o)] - \frac{nc}{q_t - q_o}, \quad (D.12)$$

$$\pi_{LF}(c, n, q_t) = \max_{q_o \in \left[0, \frac{q_t - n}{1 - n}\right]} \Pi_{LF}(q_o) \equiv \frac{1 - q_t}{1 - q_o} q_t + \frac{q_t - q_o}{1 - q_o} q_o - \frac{c}{1 - q_o}. \quad (D.13)$$

Solution to (D.11). Straightforward calculation yields $\frac{\partial^2 \Pi_{GF}}{\partial q_o^2} = -\frac{2(c+(1-q_t)^2(1-v))}{(1-q_o)^3} < 0$, so the unconstrained optimal solution can be obtained below by the first-order-condition, which leads to $q_o = 1 - \sqrt{(c+(1-q_t)^2(1-v))/(1-v)}$. With this solution, the constraint $q_o \geq \frac{q_t-n}{1-n}$ is equivalent to $c \leq c_{GF} \equiv \frac{n(2-n)(1-q_t)^2(1-v)}{(1-n)^2}$. Therefore, we have $q_o \geq \frac{q_t-n}{1-n}$.

$$q_o^{GF} = \begin{cases} 1 - \sqrt{(c+(1-q_t)^2(1-v))/(1-v)}, & \text{if } c \leq c_{GF}, \\ (q_t-n)/(1-n), & \text{if } c > c_{GF}. \end{cases} \quad (\text{D.14})$$

Solution to (D.12). Straightforward calculation yields $\frac{\partial^2 \Pi_{GP}}{\partial q_o^2} = -\frac{2n(c-(1-q_t)q_t)}{(q_t-q_o)^3}$. Thus, when $c-(1-q_t)q_t > 0$, thereby $\frac{\partial^2 \Pi_{GP}}{\partial q_o^2} < 0$, the unconstrained optimal solution can be obtained by the first-order-condition, which leads to $q_o = q_t - \sqrt{(c-(1-q_t)q_t)/(1-v)}$. With this solution, the constraint $0 \leq q_o \leq \frac{q_t-n}{1-n}$ is equivalent to $c_{GP} \leq c \leq \bar{c}_{GP}$, where $c_{GP} \equiv \frac{(1-q_t)(q_t(1-2n)+n^2(1-(1-q_t)v))}{(1-n)^2}$ and $\bar{c}_{GP} \equiv q_t(1-vq_t)$. It is also straightforward to show that the requirement $c-(1-q_t)q_t > 0$ holds when $c_{GF} \leq c \leq c_{GP}$. Hence the concavity of Π_{GP} also implies that the optimal solution is given by

$$q_o^{GP} = \begin{cases} (q_t-n)/(1-n), & \text{if } c < c_{GP}, \\ q_t - \sqrt{(c-(1-q_t)q_t)/(1-v)}, & \text{if } c_{GP} \leq c \leq \bar{c}_{GP}, \\ 0, & \text{if } c > \bar{c}_{GP}. \end{cases} \quad (\text{D.15})$$

Solution to (D.13). Straightforward calculation yields $\frac{\partial^2 \Pi_{LF}}{\partial q_o^2} = -\frac{2(c+(1-q_t)^2)}{(1-q_o)^3} < 0$. So the unconstrained optimal solution can be obtained by the first-order-condition, which leads to $q_o^* = 1 - \sqrt{c+(1-q_t)^2}$. With this solution, the constraint $0 \leq q_o^* \leq \frac{q_t-n}{1-n}$ is equivalent to $c_{LF} \leq c \leq \bar{c}_{LF}$, where $c_{LF} \equiv \frac{(2-n)n(1-q_t)^2}{(1-n)^2}$ and $\bar{c}_{LF} \equiv q_t(2-q_t)$. Thus, by the concavity of Π_{LF} , we obtain the following optimal solution for the LF subproblem:

$$q_o^{LF} = \begin{cases} (q_t-n)/(1-n), & \text{if } c \leq c_{LF}, \\ 1 - \sqrt{c+(1-q_t)^2}, & \text{if } c_{LF} < c \leq \bar{c}_{LF}, \\ 0, & \text{if } c > \bar{c}_{LF}. \end{cases} \quad (\text{D.16})$$

Combining (D.14) and (D.15), we obtain the following joint optimal solution for GF and GP strategies:

$$q_o^{GF-GP} = \begin{cases} 1 - \sqrt{(c+(1-q_t)^2(1-v))/(1-v)}, & \text{if } c \leq c_{GF}, \\ (q_t-n)/(1-n), & \text{if } c_{GF} < c \leq c_{GP}, \\ q_t - \sqrt{(c-(1-q_t)q_t)/(1-v)}, & \text{if } c_{GP} < c \leq \bar{c}_{GP}, \\ 0, & \text{if } c > \bar{c}_{GP}. \end{cases} \quad (\text{D.17})$$

Using (D.16) and (D.17), we compute and compare $\max\{\pi_{GF}(c, v, n, q_t), \pi_{GP}(c, v, n, q_t)\}$ with $\pi_{LF}(c, n, q_t)$. Specifically, solving $\Pi_{LF}(1 - \sqrt{c+(1-q_t)^2}) > \Pi_{GF}((q_t-n)/(1-n))$ yields $c > c_{GF-LF} \equiv \frac{n(1-q_t)^2(2-n(1-\sqrt{v}))(1+\sqrt{v})}{(1-n)^2}$; solving $\Pi_{LF}(1 - \sqrt{c+(1-q_t)^2}) > \Pi_{GP}(q_t - \sqrt{(c-(1-q_t)q_t)/(1-v)})$ yields $v < \bar{v} \equiv \frac{(4c(1-n^2)-8(1-A)-4(n^2-3-n(1-A)+A)q_t-(5+2n-3n^2)q_t^2)}{4n^2(c-(1-q_t)q_t)}$ where $A \equiv \sqrt{c+(1-q_t)^2}$. It can be shown that \bar{v} intersects with \bar{c} at $(v, c) = (0, q_t - \frac{3q_t^2}{4})$. Therefore LF region exists only if $\frac{\partial \bar{v}}{\partial c} < 0$ at $c = q_t - \frac{3q_t^2}{4}$, which is equivalent to $n < \frac{q_t}{2-q_t}$. Similarly, the boundary $n = (2 - 2\sqrt{1-q_t})/q_t - 1$ is obtained by solving for n such that c_{GP} and c_{GF-LF} intersect at $v = 1$. These lead to the following threshold values in the corollary:

- (i) For $0 \leq n \leq (2 - 2\sqrt{1-q_t})/q_t - 1$, $\hat{c} = c_{GP}$ and \hat{v} is the relevant solution to $\Pi_{GP}((q_t - \sqrt{(c-(1-q_t)q_t)/(1-v)})^+) = \Pi_{LF}(1 - \sqrt{c+(1-q_t)^2})$, which is implicitly defined by $v = \bar{v}$.
- (ii) For $(2 - 2\sqrt{1-q_t})/q_t - 1 \leq n < q_t/(2-q_t)$, $\hat{c} = \min\{c_{GP}, c_{GF-LF}\}$ and \bar{v} is given above.
- (iii) For $q_t/(2-q_t) \leq n < q_t$, $\hat{c} = c_{GF}$.

Subsequently, using the optimal strategy identified above, we can compute the marginal revenue, which leads to (5). Finally, the expressions for q_o^* in Table 2 follow from (D.14)-(D.16); the inequalities on Q_t^* and Q_o^* in Table 2 follow from Lemma 2. \square

Appendix E: Proofs in Section 5.1

Proof of Proposition 1. Recall that \bar{v} , given in proof of Theorem 2, is the boundary between LF and GP regions and \hat{c} is the boundary between LF and GF regions. Then the co-product positioning can only moves from low-end to green because it can be shown that (1) \bar{v} is concave in c and decreasing in n , and (2) \hat{c} increases in v and n .

For part (a), Table 2 implies that in LF region, $q_o^* = 1 - \sqrt{c+(1-q_t)^2}$, $Q^* = \frac{1}{1-q_o^*}$ and $S^* = 1$, which in turn imply that $\frac{\partial Q^*}{\partial n} = \frac{\partial Q^* q_o^*}{\partial n} = \frac{\partial Q^*/S^*}{\partial n} = \frac{\partial Q^* q_o^*/S^*}{\partial n} = 0$ and $\frac{\partial Q^*}{\partial v} = \frac{\partial Q^* q_o^*}{\partial v} = \frac{\partial Q^*/S^*}{\partial v} = \frac{\partial Q^* q_o^*/S^*}{\partial v} = 0$.

For part (b), Table 2 implies that in GP region, $q_o^* = \left(q_t - \sqrt{\frac{c-(1-q_t)q_t}{1-v}}\right)^+$, $Q^* = \frac{n}{q_t-q_o^*}$ and $S^* = n + Q^*(1-q_t)$, which in turn imply that $\frac{\partial Q^*}{\partial n} = \max\left\{\frac{1}{q_t}, \sqrt{\frac{1-v}{c-(1-q_t)q_t}}\right\} > 0$ and $\frac{\partial q_o^*}{\partial n} = 0$, suggesting $\frac{\partial Q^* q_o^*}{\partial n} > 0$. Also, it can be shown

that $\frac{\partial Q^*}{\partial v} = \frac{n}{(q_t - q_o^*)^2} \frac{\partial q_o^*}{\partial v}$, and $\frac{\partial q_o^*}{\partial v} \leq 0$ implies that $\frac{\partial Q^*}{\partial v} \leq 0$, suggesting $\frac{\partial Q^* q_o^*}{\partial v} \leq 0$. Also, we have $Q^*/S^* = \frac{1}{1 - q_o^*}$, which implies that $\frac{\partial Q^*/S^*}{\partial n} = 0$, suggesting $\frac{\partial Q^* q_o^*/S^*}{\partial n} = 0$. Furthermore it can be shown that $\frac{\partial Q^*/S^*}{\partial v} = \frac{1}{(1 - q_o^*)^2} \frac{\partial q_o^*}{\partial v} \leq 0$ and $\frac{\partial Q^* q_o^*/S^*}{\partial v} = \frac{\partial q_o^*}{\partial v} \frac{1}{1 - q_o^*} + q_o^* \frac{\partial Q^*/S^*}{\partial v} \leq 0$. Following similar approach, we can demonstrate the monotonicity of all environmental metrics in GF region.

For part (c), we hereby only identify the condition for the upward discontinuity of the total consumption Q^* in v ; all other properties can be established by following a similar argument. We note that the total consumption is $Q_{LF} = 1/\sqrt{c + (1 - q_t)^2}$ under LF strategy, $Q_{GF} = \frac{1}{1 - q_o^*}$ under GF strategy, and $Q_{GP} = \frac{n}{q_t - q_o^*}$ under GP strategy, where q_o^* is provided correspondingly in Table 2. First, we consider the comparison between Q_{LF} and Q_{GF} along the boundary \hat{c} between LF and GF regions. It can be shown that (1) the boundary \hat{c} between LF and GF increases in v , (2) when $c = \hat{c}|_{v=0} = \frac{(2-n)n(1-q_t)^2}{(1-n)^2}$ we have $Q_{GF} > Q_{LF}$, and (3) $\frac{\partial(Q_{GF} - Q_{LF})}{\partial c} > 0$. Thus, we must have $Q_{GF} > Q_{LF}$ along the entire LF-GF boundary. Second, we consider the comparison between Q_{LF} and Q_{GP} . By Theorem 2, for $n \leq (2 - 2\sqrt{1 - q_t})/q_t - 1$, LF and GP regions share boundary \check{c} , along which it can be shown that $Q_{LF} > Q_{GP}$. For $(2 - 2\sqrt{1 - q_t})/q_t - 1 \leq n < q_t/(2 - q_t)$, LF and GP regions share boundary \bar{v} , which, as a function of c , reaches its maximum at $c = \theta_v(q_t, n) \equiv \frac{(1+n)(1-q_t)^2 q_t (4 - q_t - n q_t)}{(2 - q_t - n q_t)^2}$. It can be verified that $Q_{LF} < Q_{GF}$ along the boundary \bar{v} if $c < \theta_v(q_t, n)$. \square

Proof of Proposition 2. Consumer surplus is zero in GF and GP regions because $p_t^* = q_t$ which is equal to the utility a traditional consumer would earn and $p_o^* = q_o + v(q_t - q_o) =$ which is equal to the utility a green consumer would earn in those cases. In LF region, we have $p_t^* = q_t$ so the traditional consumers earn zero utility, but $p_o^* = q_o^*$ which implies that a green consumer earns a utility $q_o^* + v(q_t - q_o^*) - p_o^* = v(q_t - q_o^*) > 0$, hence the total consumer surplus is $nv(q_t - q_o^*) > 0$. Notice that $q_o^* = 1 - \sqrt{c + (1 - q_t)^2}$ which is independent in v and n . Also since $q_t - q_o^* > 0$ by definition, it proceeds that $nv(q_t - q_o^*)$ increases in both n and v . With the optimal decision given in Theorem 2, it is straightforward to show that $\frac{\partial \Pi^*}{\partial n} \geq 0$ and $\frac{\partial \Pi^*}{\partial v} \geq 0$. \square

Appendix F: Proofs in Section 5.2

In this subsection, we first characterize the firm's optimal product line strategy under binding quota constraint, i.e., material quantity is given by \bar{Q} . The pricing decision characterized by Lemma C.1 still holds. Thus, given the the co-product's quality \bar{q}_o and price \bar{p}_o , the firm's profit function is given by

$$\pi(\bar{q}_o, \bar{p}_o) = \bar{Q}[(1 - q_t)q_t + (q_t - \bar{q}_o)\bar{p}_o - c]. \quad (\text{F.1})$$

Following similar argument as in Lemma 2, the lemma below derives three subproblems for the firm.

LEMMA F.1. *Under binding quota \bar{Q} , the firm enters the market if and only if $c \leq (1 - q_t)q_t + (q_t - \bar{q}_o)\bar{p}_o$, whereby the optimal (\bar{q}_o, \bar{p}_o) must solve one of the following three subproblems that yields the highest profit:*

$$\text{If } \bar{Q} \geq \frac{1-n}{1-q_t}, \text{ then } \max \pi(\bar{q}_o, \bar{p}_o), \text{ subject to } (1 - 1/\bar{Q})^+ \leq \bar{q}_o \leq q_t \text{ and } \bar{p}_o = \bar{q}_o + v(q_t - \bar{q}_o); \quad (\bar{S}_1)$$

$$\text{If } \bar{Q} \leq \frac{1-n}{1-q_t}, \text{ then } \max \pi(\bar{q}_o, \bar{p}_o), \text{ subject to } (q_t - n/\bar{Q})^+ \leq \bar{q}_o \leq q_t \text{ and } \bar{p}_o = \bar{q}_o + v(q_t - \bar{q}_o); \quad (\bar{S}_2)$$

$$\text{If } \bar{Q} \leq \frac{1-n}{1-q_t}, \text{ then } \max \pi(\bar{q}_o, \bar{p}_o), \text{ subject to } (1 - 1/\bar{Q})^+ \leq \bar{q}_o \leq (q_t - n/\bar{Q})^+ \text{ and } \bar{p}_o = \bar{q}_o. \quad (\bar{S}_3)$$

Clearly, each subproblem reduces to a constrained optimization problem with a single decision variable \bar{q}_o and hence can be solved following standard procedure. The lemma below documents their solutions.

LEMMA F.2. *The optimal co-product's quality $\bar{q}_o^{(i)}$ and price $\bar{p}_o^{(i)}$ of subproblem (\bar{S}_i) is given by*

$$\bar{q}_o^{(1)} = \begin{cases} 1 - 1/\bar{Q}, & \text{if } \bar{Q} \geq \max \left\{ \frac{2(1-v)}{2(1-v) - q_t(1-2v)}, \frac{1-n}{1-q_t} \right\}, \\ \frac{q_t(1-2v)}{2(1-v)}, & \text{if } \frac{1-n}{1-q_t} \leq \bar{Q} \leq \frac{2(1-v)}{2(1-v) - q_t(1-2v)}, \end{cases} \quad \text{and } \bar{p}_o^{(1)} = \bar{q}_o^{(1)} + v(q_t - \bar{q}_o^{(1)}); \quad (\text{F.2})$$

$$\bar{q}_o^{(2)} = \begin{cases} q_t - n/\bar{Q}, & \text{if } \frac{2n(1-v)}{q_t} \leq \bar{Q} \leq \frac{1-n}{1-q_t}, \\ \frac{q_t(1-2v)}{2(1-v)}, & \text{if } 0 \leq \bar{Q} \leq \frac{2n(1-v)}{q_t}, \end{cases} \quad \text{and } \bar{p}_o^{(2)} = \bar{q}_o^{(2)} + v(q_t - \bar{q}_o^{(2)}) \quad (\text{F.3})$$

$$\bar{q}_o^{(3)} = \begin{cases} 1 - 1/\bar{Q}, & \text{if } \frac{2}{2 - q_t} \leq \bar{Q} \leq \frac{1-n}{1-q_t}, \\ q_t/2, & \text{if } \frac{2n}{q_t} \leq \bar{Q} \leq \frac{2}{2 - q_t}, \\ q_t - n/\bar{Q}, & \text{if } \frac{n}{q_t} \leq \bar{Q} \leq \min \left\{ \frac{2n}{q_t}, \frac{1-n}{1-q_t} \right\}, \\ 0, & \text{if } 0 \leq \bar{Q} \leq \frac{n}{q_t}, \end{cases} \quad \text{and } \bar{p}_o^{(3)} = \bar{q}_o^{(3)}. \quad (\text{F.4})$$

By comparing the optimal profits of the three subproblems, we obtain the global optimal co-product's quality and price in the next proposition.

PROPOSITION F.1. If $\bar{Q} \geq \frac{1-n}{1-qt}$, the optimal co-product is given by (F.2), which corresponds to full coverage of the market if $\bar{Q} \geq \max\left\{\frac{2(1-v)}{2(1-v)-qt(1-2v)}, \frac{1-n}{1-qt}\right\}$ and to partial coverage if $\frac{1-n}{1-qt} \leq \bar{Q} \leq \frac{2(1-v)}{2(1-v)-qt(1-2v)}$. If $\bar{Q} < \frac{1-n}{1-qt}$ and $\frac{1-n}{1-qt} \leq \frac{2n}{qt}$, the optimal co-product is given by (F.3), corresponding to partial coverage. If $\bar{Q} < \frac{1-n}{1-qt}$ and $\frac{1-n}{1-qt} > \frac{2n}{qt}$, the optimal co-product is given by (F.4) for $\min\left\{\frac{2n(1+\sqrt{v})}{qt}, \frac{B-\sqrt{B^2-4AC}}{2A}\right\} \leq \bar{Q} \leq \frac{B+\sqrt{B^2-4AC}}{2A}$ and otherwise by (F.3), where $A \equiv 1-qt$, $B \equiv 2-(1+n)qt$ and $C \equiv 1-n^2(1-v)$. For $\bar{Q} \leq 2/(2-qt)$, (F.4) corresponds to partial coverage; otherwise, (F.4) corresponds to full coverage.

Proof of Proposition F.1. When $\bar{Q} \geq \frac{1-n}{1-qt}$, only (\bar{S}_1) is feasible and hence the optimal co-product is given by (F.2). For $\bar{Q} \leq \frac{1-n}{1-qt}$, both (\bar{S}_2) and (\bar{S}_3) are feasible and the optimal co-product is determined by the one that yields higher profit. Given \bar{Q} , to compare the profits between (\bar{S}_2) and (\bar{S}_3) , it suffices to only compare the revenues generated by the co-product, which we respectively denote as $\Pi_o^2(\bar{q}_o) = \bar{Q}(qt - \bar{q}_o)[\bar{q}_o + v(qt - \bar{q}_o)]$ and $\Pi_o^3(\bar{q}_o) = \bar{Q}(qt - \bar{q}_o)\bar{q}_o$. In particular, we note that

$$\Pi_o^2(\bar{q}_o) \geq \Pi_o^3(\bar{q}_o) \text{ for all } \bar{q}_o. \quad (\text{F.5})$$

- When $\frac{1-n}{1-qt} \leq \frac{2n}{qt}$, we have, by (F.4), $\bar{q}_o^{(3)} = (qt - n/\bar{Q})^+$. Hence, $\Pi_o^2(\bar{q}_o^{(2)}) \geq \Pi_o^2((qt - n/\bar{Q})^+) \geq \Pi_o^3((qt - n/\bar{Q})^+)$, where the first inequality follows from the optimality of (F.4) and the second one from (F.5). Therefore, the optimal co-product is given by (F.3).

- When $\frac{1-n}{1-qt} > \frac{2n}{qt}$, similar to the last case, the optimal co-product is given by (F.3) for $\bar{Q} \in [0, \frac{2n}{qt}]$. For $\bar{Q} \in [\frac{2n}{qt}, \frac{1-n}{1-qt}]$, (F.3) implies that $\Pi_o^2(\bar{q}_o^{(2)}) = n[qt - (1-v)n/\bar{Q}]$, while (F.4) implies that

$$\Pi_o^3(\bar{q}_o^{(3)}) = \begin{cases} \bar{Q}q_t^2/4, & \text{if } \bar{Q} \in \left[\frac{2n}{qt}, \frac{2}{2-qt}\right], \\ (\bar{Q}-1)(qt - 1 + 1/\bar{Q}) & \text{if } \bar{Q} \in \left[\frac{2}{2-qt}, \frac{1-n}{1-qt}\right]. \end{cases}$$

A direct comparison reveals that $\Pi_o^2(\bar{q}_o^{(2)}) \leq \Pi_o^3(\bar{q}_o^{(3)})$ if and only if $\min\left\{\frac{2n(1+\sqrt{v})}{qt}, \frac{B-\sqrt{B^2-4AC}}{2A}\right\} \leq \bar{Q} \leq \frac{B+\sqrt{B^2-4AC}}{2A}$, hence concluding the proof.

The demand fulfillment patterns under (F.2)-(F.4) follow from straightforward verification. \square

Proof of Proposition 3. (a) We first demonstrate the monotonicity of each environmental metric $P \in \{Q^*, Q^*q_o^*, Q^*/S^*, Q^*q_o^*/S^*\}$ in c , which follows by showing $\frac{\partial P}{\partial c} \leq 0$ within each strategy region. For instance, in LF region we have $\frac{\partial Q^*}{\partial c} = -\frac{1}{2(\sqrt{c+(1-qt)^2})^3} < 0$. The discontinuity of $P \in \{Q^*, Q^*q_o^*, Q^*/S^*, Q^*q_o^*/S^*\}$ between LF and GP regions follows from Proposition 1. To demonstrate the monotonicity of each P in \bar{Q} , we first note that total material consumption is simply \bar{Q} , which is increasing in \bar{Q} . By (F.2)-(F.4), the total waste $\bar{Q}q_o^{(i)}$ is also increasing in \bar{Q} within each strategic region. Since the total units sold is given by $S^* = \bar{Q}(1 - \bar{q}_o^{(i)})$, the per-unit consumption and waste are given by $\bar{Q}/S^* = 1/(1 - \bar{q}_o^{(i)})$ and $\bar{Q}q_o^{(i)}/S^* = \bar{q}_o^{(i)}/(1 - \bar{q}_o^{(i)})$, respectively. By (F.2)-(F.4), it is then straightforward to verify that they are increasing in \bar{Q} within each strategic region. As \bar{Q} decreases, the optimal strategy changes from (F.2) to (F.3), then to (F.4), and finally back to (F.3). The corresponding optimal co-product's quality decreases in all switching places except for the last one, i.e., from (F.4) to (F.3), which occurs when $\bar{Q} \in [2n/qt, 2/(2-qt)]$. At this switch, the co-product's quality changes from $\bar{q}_o^{(3)} = qt/2$ to $\bar{q}_o^{(2)} = qt - n/\bar{Q}$, and it is straightforward to show that $qt - n/\bar{Q} \geq qt/2$ (i.e., an upward jump) for $\bar{Q} \geq 2n/qt$.

(b) In the absence of quota, Proposition 2 implies positive consumer surplus in LF region, which is given by $nv(qt - q_o^*) = nv(\sqrt{c+(1-qt)^2} - (1-qt))$ and is increasing in c . Under quota, only when the firm prices the co-product as a low-end product (i.e., $\bar{p}_o^{(3)} = \bar{q}_o^{(3)}$), the consumer surplus is positive and is given by $nv(qt - \bar{q}_o^{(3)})$. By Proposition F.1, when (F.4) corresponds to full coverage, the optimal co-product's quality is $1 - 1/\bar{Q}$ and hence the consumer surplus is $nv(qt - 1 + 1/\bar{Q})$, increasing in \bar{Q} ; when (F.4) corresponds to partial coverage, the optimal co-product's quality is $qt/2$ and hence the consumer surplus is $nvqt/2$, independent of \bar{Q} .

(c) The monotonicity of firm's profit in c and \bar{Q} is straightforward. \square

Proof of Proposition 4. By Proposition F.1, the optimal co-product's quality in GP region is given by $\bar{q}_o^{(2)}$ in (F.3). In GP region, the total material consumption is \bar{Q} , the traditional product's revenue is $\bar{Q}(1-qt)qt$, and the total material cost is $c\bar{Q}$, which are all independent of n or v . Thus, to show the monotonicity of the firm's profit, we just need to examine the co-product's revenue, which is given by

$$\Pi_o^2(\bar{q}_o^{(2)}) = \bar{Q}(qt - \bar{q}_o^{(2)})[\bar{q}_o^{(2)} + v(qt - \bar{q}_o^{(2)})] = \begin{cases} n[qt - (1-v)n/\bar{Q}], & \text{if } \frac{2n(1-v)}{qt} \leq \bar{Q} \leq \frac{1-n}{1-qt}, \\ \frac{q_t^2 \bar{Q}}{4(1-v)}, & \text{if } 0 \leq \bar{Q} \leq \frac{2n(1-v)}{qt}, \end{cases}$$

which can be verified to be weakly increasing in n or v . Next, we compute the per-unit consumption

$$1/(1 - \bar{q}_o^{(2)}) = \begin{cases} \frac{\bar{Q}}{(1-qt)\bar{Q}+n}, & \text{if } \frac{2n(1-v)}{qt} \leq \bar{Q} \leq \frac{1-n}{1-qt}, \\ \frac{2(1-v)}{2(1-v)-(1-2v)qt}, & \text{if } 0 \leq \bar{Q} \leq \frac{2n(1-v)}{qt}, \end{cases}$$

which can be verified to be weakly decreasing in n or v . Similarly, we compute the total waste

$$\bar{Q}\bar{q}_o^{(2)} = \begin{cases} \bar{Q}q_t - n, & \text{if } \frac{2n(1-v)}{q_t} \leq \bar{Q} \leq \frac{1-n}{1-q_t}, \\ \frac{\bar{Q}q_t(1-2v)}{2(1-v)}, & \text{if } 0 \leq \bar{Q} \leq \frac{2n(1-v)}{q_t}, \end{cases}$$

which again can be verified to be weakly decreasing in n or v . Finally, we compute the per-unit waste

$$\bar{Q}\bar{q}_o^{(2)}/(1-\bar{q}_o^{(2)}) = \begin{cases} \frac{\bar{Q}q_t - n}{(1-q_t)\bar{Q} + n}, & \text{if } \frac{2n(1-v)}{q_t} \leq \bar{Q} \leq \frac{1-n}{1-q_t}, \\ \frac{q_t(1-2v)}{2(1-v)-(1-2v)q_t}, & \text{if } 0 \leq \bar{Q} \leq \frac{2n(1-v)}{q_t}, \end{cases}$$

which can also be verified to be weakly decreasing in n or v . \square

Appendix G: Proofs in Section B

Proof of Proposition B.1. We first note that the constraint $Q_o = 1 - G^\alpha(v_o) = 1 - v_o^\alpha$ is equivalent to

$$q_o = q_t - (1 - v_o^\alpha)/Q \geq 0, \quad (\text{G.1})$$

which implies that $Q \geq (1 - v_o^\alpha)/q_t$ and

$$p_o = q_o + v_o(q_t - q_o) = q_t - (1 - v_o)(1 - v_o^\alpha)/Q. \quad (\text{G.2})$$

Substituting (G.1) and (G.2) into the objective function of (6) reduces it to

$$[q_t - (1 - v_o)(1 - v_o^\alpha)/Q](1 - v_o^\alpha) + q_t \min\{(1 - q_t)Q, v_o^\alpha\} - cQ,$$

which further reduces to $q_t - (1 - v_o)(1 - v_o^\alpha)/Q - cQ$ and is obviously increasing in v_o for $v_o^\alpha \leq (1 - q_t)Q$. Hence, the optimal solution to (6) must lie within $v_o^\alpha \geq (1 - q_t)Q$, which reduces the seller's problem (6) to

$$\max_{v_o \in [(1-q_t)^{1/\alpha}, 1], (1-v_o^\alpha)/q_t \leq Q \leq v_o^\alpha/(1-q_t)} \Psi(v_o, Q) = q_t(1 - v_o^\alpha) - (1 - v_o)(1 - v_o^\alpha)^2/Q + [q_t(1 - q_t) - c]Q. \quad (\text{G.3})$$

Here, in order to (G.3) to be feasible, we require $v_o \geq (1 - q_t)^{1/\alpha}$ so that $(1 - v_o^\alpha)/q_t \leq v_o^\alpha/(1 - q_t)$.

To solve (G.3), we examine the following cases:

- Suppose $c \leq q_t - q_t^2$. Then, $\Psi(v_o, Q)$ is increasing in Q , which implies that $\Psi(v_o, Q)$ is maximized at $Q = v_o^\alpha/(1 - q_t)$ to $\Psi(v_o, v_o^\alpha/(1 - q_t)) = \psi_1(v_o)$, whose maximum must be nonnegative because $\psi_1(1) = q_t - c/(1 - q_t) \geq 0$. Therefore, we obtain the first result in the proposition.

- Suppose $c > q_t - q_t^2$. For any given v_o , it is straightforward to verify that $\Psi(v_o, Q)$ is increasing in $Q \leq (1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}}$ and is decreasing in $Q \geq (1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}}$. Thus, we further divide this case into the two parameter ranges.

1. For $q_t - q_t^2 < c \leq q_t - q_t^2(1 - q_t)^{1/\alpha}$, we note that function $(1 - v_o^\alpha)/v_o^\alpha \sqrt{1 - v_o^\alpha}$ is monotonically decreasing in v_o , taking value $q_t/(1 - q_t) \sqrt{1 - (1 - q_t)^{1/\alpha}} \geq \sqrt{c - q_t + q_t^2}/(1 - q_t)$ at $v_o = (1 - q_t)^{1/\alpha}$ and taking value smaller than $\sqrt{c - q_t + q_t^2}/(1 - q_t)$ at $v_o = (q_t - c)/q_t^2$. Thus, there exists a unique solution $\bar{v}_o \in [(1 - q_t)^{1/\alpha}, (q_t - c)/q_t^2]$ to (B.4) such that

$$v_o \leq \bar{v}_o \Leftrightarrow v_o^\alpha/(1 - q_t) \leq (1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}}.$$

Hence,

- for $v_o \in [(1 - q_t)^{1/\alpha}, \bar{v}_o]$, we have $\Psi(v_o, Q)$ increasing in $Q \in [(1 - v_o^\alpha)/q_t, v_o^\alpha/(1 - q_t)]$, which implies that $\Psi(v_o, Q)$ is maximized at $Q = v_o^\alpha/(1 - q_t)$ to $\Psi(v_o, v_o^\alpha/(1 - q_t)) = \psi_1(v_o)$;

- for $v_o \in [\bar{v}_o, (q_t - c)/q_t^2]$, we have $(1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}} \in [(1 - v_o^\alpha)/q_t, v_o^\alpha/(1 - q_t)]$, and hence $\Psi(v_o, Q)$ is maximized at $Q = (1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}}$ to $\Psi(v_o, (1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}}) = \psi_2(v_o)$;

- and for $v_o \in [(q_t - c)/q_t^2, 1]$, we must have $(1 - v_o^\alpha)/q_t \geq (1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}}$, which implies that $\Psi(v_o, Q)$ is maximized at $Q = (1 - v_o^\alpha)/q_t$ to $\Psi(v_o, (1 - v_o^\alpha)/q_t) = \psi_3(v_o)$, whose maximum must be nonnegative because $\psi_3(1) = 0$.

Therefore, we obtain the second result in the proposition. In particular, we note that

2. Suppose $c > q_t - q_t^2(1 - q_t)^{1/\alpha}$. It is straightforward to verify that $(1 - v_o^\alpha)/q_t > (1 - v_o^\alpha) \sqrt{\frac{1 - v_o^\alpha}{c - q_t + q_t^2}}$ for all $v_o \in [(1 - q_t)^{1/\alpha}, 1]$. Therefore, $\Psi(v_o, Q)$ is maximized at $Q = (1 - v_o^\alpha)/q_t$ to $\Psi(v_o, (1 - v_o^\alpha)/q_t) = \psi_3(v_o)$, whose maximum again must be nonnegative because $\psi_3(1) = 0$. In particular, if $c \geq q_t$, we must have $\psi_3(v_o) < 0$ for all $v_o < 1$, because $1 - v_o^\alpha > 0$ and $v_o - (c - q_t + q_t^2)/q_t^2 < 0$. Therefore, for $c \geq q_t$, the optimal $v_o^* = 1$ and $Q^* = 0$, i.e., the firm stops entering production. \square