Vehicle Maintenance Contracting in Developing Economies: The Role of Social Enterprise

Online Appendix: Proofs and Auxiliary Results

Lemma A1 *The function*

$$S(t) \equiv h(t)\,M(t) - (1/p)\,F(t), \tag{A1}$$

has the following properties: $S'(t) = h'(t)\,M(t) > 0$ *with* $S(0) = 0$ *and* $S(\infty) = \infty$. *Consequently,* $S(t) \geq 0$. *Moreover,* $\frac{\partial S(t)}{\partial p} < 0$ *for any* $t > 0$.

Proof Recall $H(t) = \int_0^t h(z)\,dz$ and $\overline{F}(t) = \exp(-pH(t))$ and the assumptions $h(0) = 0$, $h(\infty) = \infty$, and $h'(t) > 0$. Clearly, $S(0) = 0$ and $S(\infty) = \infty$. Using the identity $\overline{F}'(t) = -ph(t)\,\overline{F}(t)$ we obtain $S'(t) = h'(t)\,M(t) > 0$ for $\tau > 0$. The results $S(0) = 0$ and $S'(t) > 0$ together imply $S(t) > 0$ for all $t > 0$. Differentiating $S(t)$ with respect to p yields $\frac{\partial S(t)}{\partial p} = \frac{\omega(t)}{p^2}$ where $\omega(t) \equiv 1 - \overline{F}(t) - pH(t)\,\overline{F}(t) - p^2 h(t) \int_0^t H(z)\,\overline{F}(z)\,dz$. Since $h(0) = H(0) = 0$ and $\overline{F}(0) = 1$, we have $\omega(0) = 0$. Moreover, using the identities $H'(t) = h(t)$ and $\overline{F}'(t) = -ph(t)\,\overline{F}(t)$ we obtain $\omega'(t) = -p^2 h'(t) \int_0^t H(z)\,\overline{F}(z)\,dz < 0$. The results $\omega(0) = 0$ and $\omega'(t) < 0$ together imply $\omega(t) < 0$ for all $t > 0$. Therefore, $\frac{\partial S(t)}{\partial p} = \frac{\omega(t)}{p^2} < 0$ for $t > 0$. □

Lemma A2 *Let* $\beta_1 \equiv \frac{kq}{Lp+lq}$ *and*

$$Q(\tau) \equiv (\theta(Lp + lq) + kq)\,S(\tau) - K(Lp + lq)\,h(\tau) - K. \tag{A2}$$

Recall $A(\tau)$, $C(\tau)$, $R(\tau)$ *defined in expression (2). The function* $U(\tau) \equiv \theta A(\tau) - C(\tau) - R(\tau)$ *is quasiconcave, starting from* $U(0) = -\infty$ *and converging to* $U(\infty) = \frac{(\theta\mu - K)p - kq}{(\mu + L)p + lq}$. *Moreover:*

- *(a) If* $\theta + \beta_1 \leq \frac{K}{\mu}$, *then* $U'(\tau) > 0$ *for all* $\tau \geq 0$.
- *(b) If* $\theta + \beta_1 > \frac{K}{\mu}$, *then* $U(\tau)$ *peaks at* $\tau = \widehat{\tau}$ *where* $\widehat{\tau}$ *is a unique root of the function* $Q(\tau)$ *that satisfies* $Q'(\widehat{\tau}) > 0$.

Proof We first prove the following properties of $Q(\tau)$ defined in expression (A2): *(i) If* $\theta + \beta_1 \leq \frac{K}{\mu}$, *then* $Q(\tau) < 0$ *and* $Q'(\tau) < 0$ *for all* $\tau \geq 0$; *(ii) If* $\theta + \beta_1 > \frac{K}{\mu}$, *then* $Q(\tau)$ *crosses zero exactly once at* $\tau = \widehat{\tau} > 0$ *that satisfies* $Q(\widehat{\tau}) = 0$ *and* $Q'(\widehat{\tau}) > 0$. Using the definition of $S(\tau)$ in (A1), we can rewrite $Q(\tau)$ as $Q(\tau) = \chi(\tau)\,h(\tau) - (\theta L + (\theta l + k)\,q/p)\,F(\tau) - K$ where $\chi(\tau) \equiv (\theta Lp + (\theta l + k)\,q)\,M(\tau) - K(Lp + lq)$. Using the relation $\overline{F}'(t) = -ph(t)\,\overline{F}(t)$, we obtain $Q'(\tau) = \chi(\tau)\,h'(\tau)$. Since $h'(\tau) > 0$, we see that the sign of $Q'(\tau)$ is equal to the sign of $\chi(\tau)$. Observe $\chi'(\tau) = (\theta Lp + (\theta l + k)\,q)\,\overline{F}(\tau) > 0$ with $\chi(0) = -K(Lp + lq) < 0$ and $\chi(\infty) = (Lp + lq)\,\mu\left(\theta + \beta_1 - \frac{K}{\mu}\right)$, where $\mu = M(\infty)$. From the latter expression, we see that $\chi(\infty) > 0$ iff $\theta + \beta_1 > \frac{K}{\mu}$. Suppose $\theta + \beta_1 \leq \frac{K}{\mu}$. Then $\chi(\tau) < 0$ for all $\tau \geq$

0 and therefore $Q(\tau)<0$ and $Q'(\tau)<0$. Suppose $\theta+\beta_1>\frac{K}{\mu}$. Then $\chi(\tau)$ crosses zero exactly once from below at $\tau=\tau^o\equiv M^{-1}\left(\frac{K(Lp+lq)}{\theta Lp+(\theta l+k)q}\right)>0$, which implies $Q'(\tau)<0$ for $\tau<\tau^o$ and $Q'(\tau)>0$ for $\tau>\tau^o$, i.e., $Q(\tau)$ is a quasiconvex function with a unique interior minimum occurring at $\tau=\tau^o$. Combining this with $Q(0)=-K<0$ and $Q(\infty)=\chi(\infty)h(\infty)-(\theta L+(\theta l+k)q/p)-K=\infty>0$, we conclude that $Q(\tau)$ crosses zero exactly once at $\tau=\widehat{\tau}>\tau^o$ such that $Q(\widehat{\tau})=0$ and $Q'(\widehat{\tau})>0$.

Rewriting $U(\tau)\equiv\theta A(\tau)-C(\tau)-R(\tau)$ using expression (2) yields $U(\tau)=\frac{\theta M(\tau)-(kq/p)F(\tau)-K}{M(\tau)+(L+lq/p)F(\tau)}$, where $M(\tau)=\int_0^\tau\overline{F}(t)\,dt$. Taking the limit $\tau\to 0$ that results in $H(0)=F(0)=0$ and substituting them in $U(\tau)$, we find $U(0)=-\infty$. Taking the limit $\tau\to\infty$ that results in $H(\infty)=\infty$, $F(\infty)=1$, and $M(\infty)=\mu$, we find $U(\infty)=\frac{(\theta\mu-K)p-kq}{(\mu+L)p+lq}$. Differentiating $U(\tau)$ using the identity $\overline{F}'(\tau)=-ph(\tau)\overline{F}(\tau)$ yields $U'(\tau)=-\frac{\overline{F}(\tau)Q(\tau)}{T(\tau)^2}$. The statements of the lemma follow from the properties of $Q(\tau)$ proved above. □

Proof (**Lemma 1**) The expected length of a vehicle replacement cycle consists of three components: (i) expected vehicle age at the time of replacement; (ii) expected cumulative repair downtimes until replacement; (iii) expected downtime after an unscheduled replacement. With finite retirement age $\tau<\infty$, a vehicle is replaced at age $\min\{Y,\tau\}$, where Y denotes vehicle age at the time of first major failure conditional on no vehicle retirement ($\tau=\infty$). Hence, the expected replacement age is equal to $E[\min\{Y,\tau\}]=\int_0^\tau\overline{F}(t)\,dt=M(\tau)$, which also represents the expected vehicle uptime in a single cycle. This is the first component of the expected cycle length. The second component, expected cumulative repair downtimes until replacement, is equal to $l\times N(\tau)$ where l is the expected repair lead time and $N(\tau)$ is the expected number of minor failures until replacement at vehicle age $\min\{Y,\tau\}$. The latter is evaluated as $N(\tau)=\int_0^\tau qh(t)\Pr(Y>t)\,dt=(q/p)F(\tau)$ (see Beichelt 2006, pp. 138-140). Finally the last component, expected downtime after an unscheduled replacement, is equal to $L\times\Pr(Y<\tau)=LF(\tau)$ where L is the expected replacement lead time. Adding up the three components yields the expected cycle length $T(\tau)=M(\tau)+\left(L+\frac{lq}{p}\right)F(\tau)$. It then follows that $A(\tau)=\frac{M(\tau)}{T(\tau)}$, the expected vehicle uptime in a cycle, $C(\tau)=\frac{kN(\tau)}{T(\tau)}=\frac{(kq/p)F(\tau)}{T(\tau)}$, fixed repair cost k times the expected number of repairs in a cycle, and $R(\tau)=\frac{K}{T(\tau)}$, fixed replacement cost K times the number of replacements in a cycle, which is exactly one because each renewal cycle ends with a replacement.

Differentiating $A(\tau)$, $C(\tau)$, and $R(\tau)$ in expression (2) using the identity $\overline{F}'(\tau)=-ph(\tau)\overline{F}(\tau)$ yields $A'(\tau)=-\frac{(Lp+lq)\overline{F}(\tau)S(\tau)}{T(\tau)^2}$, $C'(\tau)=\frac{kq\overline{F}(\tau)S(\tau)}{T(\tau)^2}$, and $R'(\tau)=-\frac{K\overline{F}(\tau)[1+(Lp+lq)h(\tau)]}{T(\tau)^2}$, where $S(\tau)$ and $T(\tau)$ are defined in expressions (A1) and (1). Since $S(\tau)>0$ (see Lemma A1), it follows that $A'(\tau)<0$, $C'(\tau)>0$, and $R'(\tau)<0$. It is straightforward to evaluate the limiting values at $\tau=0$ and $\tau\to\infty$. □

Proof (**Lemma 2**) It is straightforward to verify that this is a special case of Lemma A2(b) with $\theta = 0$. Therefore, the cost-minimizing $\tau^0 = \widehat{\tau}$, as given in Lemma A2(b). $\square$

Proof (**Proposition 1**) It is straightforward to verify that this is a special case of Lemma A2(b) with $\theta = \frac{\gamma v}{1-\gamma}$. It is also straightforward to verify that the unconstrained optimal solution $\tau^*(\gamma) = \widehat{\tau}$ is decreasing in γ. Therefore, there must exist a threshold $0 < \overline{\gamma} \leq 1$, such that $\tau^*(\gamma) > \underline{\tau}$ if and only if $0 \leq \gamma < \overline{\gamma}$. Hence, the results of (a) and (b) hold. Moreover, in case (a), we have $\underline{\tau} < \tau^*(\gamma) < \overline{\tau}$, implying that SE earns positive profit and achieves less than maximum availability. Also note that when $\gamma = 0$, this case reduces to that of Lemma 2, which means the SE earns maximum profit. In case (b), we have $\tau^*(\gamma) = \underline{\tau} < \overline{\tau}$, implying that SE earns zero profit and achieves maximum availability. $\square$

Proof (**Proposition 2**) To achieve the first-best performance under the decentralized decision setting, it is necessary to induce the government to choose $\tau_G^* = \tau^*(\gamma)$. This can be achieved by the following nonlinear payment scheme:

$$P^*(\tau_G) = \begin{cases} r^* A(\tau_G) \text{ if } \tau_G \leq \tau^*(\gamma), \\ \quad \infty \quad\;\; \text{otherwise,} \end{cases}$$

where $r^* = \frac{b_0 - R(\tau^*(\gamma))}{A(\tau^*(\gamma))}$. Under the above payment scheme, the government's cost function is given by

$$\left(\frac{b_0 - R(\tau^*(\gamma))}{A(\tau^*(\gamma))} \right) A(\tau_G) + R(\tau_G).$$

From Lemma 1, we know $A(\tau)$ and $R(\tau)$ are both decreasing in τ. Therefore, the government is induced to set τ_G at the maximum possible value $\tau^*(\gamma)$ to minimize total cost. Moreover, it can be verified that the total cost is b_0 at $\tau_G = \tau^*(\gamma)$, which meets the budget constraint. Thus, the cost-minimizing $\tau_G = \tau^*(\gamma)$ is feasible. This completes the proof. $\square$

Proof (**Proposition 3**) Under the linear ROC contract, the government's cost function is $rA(\tau) + R(\tau)$. From Lemma 1, we know $A(\tau)$ and $R(\tau)$ are both decreasing in τ. Therefore, the government sets τ_G at the maximum possible value $\overline{\tau}$ to minimize cost. It follows that $\tau_G^\dagger = \overline{\tau} > \tau^*(\gamma)$. Given the government's optimal decision, SE will set r at the maximum possible value of $\frac{b_0 - R(\overline{\tau})}{A(\overline{\tau})}$, where the government budget constraint is binding. Because $\overline{\tau} > \tau^*(\gamma)$ and $A(\tau)$ and $R(\tau)$ are both decreasing in τ, it follows that $r^\dagger = \frac{b_0 - R(\overline{\tau})}{A(\overline{\tau})} > \frac{b_0 - R(\tau^*(\gamma))}{A(\tau^*(\gamma))} = r^*$. Finally, SE's equilibrium profit is given by $b_0 - C(\overline{\tau}) - R(\overline{\tau}) > 0$ for any $0 \leq \gamma \leq 1$. $\square$

Proof (**Proposition 4**) Under the linear RRC contract, it is straightforward to verify the problem is equivalent to the first-best case, except for the reduced budget constraint $b_0 - \psi$. Thus, by Proposition 1, it follows that $\tau^{\ddagger}(\gamma, \psi)$ is decreasing in γ and increasing in ψ. Therefore, there must exist a threshold $0 < \gamma_1(\psi) \leq 1$, such that $\tau^{\ddagger}(\gamma, \psi)$ is an interior solution if and only if $0 \leq \gamma < \gamma_1(\psi)$. Hence, in case (a), we have $\tau^{\ddagger}(\gamma, \psi) = \tau^*(\gamma) < \tau_G^{\dagger}$, and in case (b), we have $\tau^*(\gamma) < \tau^{\ddagger}(\gamma, \psi) < \tau_G^{\dagger}$. From Lemma 1, we know $A(\tau)$ is decreasing in τ, it follows that $r^{\ddagger} = \frac{b_0}{A(\tau^{\ddagger}(\gamma,\psi))} > r^{\dagger} > r^*$. Finally, observe that under linear ROC, SE's equilibrium utility is given by $\gamma v A(\overline{\tau}) + (1-\gamma)(b_0 - C(\overline{\tau}) - R(\overline{\tau}))$, and under linear RRC, SE's equilibrium utility is given by $\gamma v A(\tau^{\ddagger}) + (1-\gamma)(b_0 - \psi - C(\tau^{\ddagger}) - R(\tau^{\ddagger}))$. By comparing the above utility functions and also using the fact that $\tau^{\ddagger}$ is decreasing in γ, it follows that there exists an increasing threshold $\gamma_2(\psi)$ in ψ, such that, SE prefers the linear RRC over linear ROC if and only if $\gamma \geq \gamma_2(\psi)$. ☐

Proof (**Lemma 3**) For notational convenience suppress the arguments (x, y). The results in part (b) are straightforward, obtained by inspecting the expression (2) with the substitution $l = l(y)$. Consider part (a). Let $\ell \equiv L + l\frac{1-\phi}{\phi} > 0$ and $m(\tau) \equiv \frac{M(\tau)}{F(\tau)}$. With $\overline{F}(t) = \exp(-pH(t)) = \exp(-\phi\lambda(x)H(t))$ and $M(\tau) = \int_0^\tau \overline{F}(t)\,dt$, we have $\frac{\partial F(\tau)}{\partial x} = \phi\lambda'(x)H(\tau)\overline{F}(\tau) < 0$ and $\frac{\partial M(\tau)}{\partial x} = -\phi\lambda'(x)\int_0^\tau H(t)\overline{F}(t)\,dt > 0$. Then $\frac{\partial m(\tau)}{\partial x} = \frac{\partial}{\partial x}\frac{M(\tau)}{F(\tau)} = -\phi\lambda'(x)\frac{\int_0^\tau H(t)\overline{F}(t)dt + H(\tau)\overline{F}(\tau)m(\tau)}{F(\tau)} > 0$. Substituting $p = \phi\lambda(x)$ and $q = (1-\phi)\lambda(x)$ into expressions (2) and (1) yields $T(\tau) = M(\tau) + \ell F(\tau)$, $A(\tau) = \left(1 + \frac{\ell}{m(\tau)}\right)^{-1}$, and $C(\tau) = \frac{1-\phi}{\phi}\frac{k}{\ell + m(\tau)}$. From these expressions we find $\frac{\partial A(\tau)}{\partial x} = \frac{\partial A(\tau)}{\partial m(\tau)}\frac{\partial m(\tau)}{\partial x} > 0$ and $\frac{\partial C(\tau)}{\partial x} = \frac{\partial C(\tau)}{\partial m(\tau)}\frac{\partial m(\tau)}{\partial x} < 0$. Finally, $\frac{\partial T(\tau)}{\partial x} = \phi\lambda'(x)\left[-\int_0^\tau H(t)\overline{F}(t)\,dt + \ell H(\tau)\overline{F}(\tau)\right]$ together with $R(\tau) = \frac{K}{T(\tau)}$ imply $\frac{\partial R(\tau)}{\partial x} < 0$ if and only if $\ell < \frac{\int_0^\tau H(t)\overline{F}(t)dt}{H(\tau)\overline{F}(\tau)}$. To prove the last result in part (a), note $C(\tau) + R(\tau) \to \frac{kqH(\tau)+K}{\tau + lqH(\tau)}$ in the limit $\phi \to 0$ which leads to $p = 0$ and $q = \lambda(x)$. Since $\frac{\partial}{\partial q}\left[C(\tau) + R(\tau)\right] = \frac{(k\tau - Kl)H(\tau)}{[\tau + lqH(\tau)]^2}$, we have $\frac{\partial}{\partial x}\left[C(\tau) + R(\tau)\right] = \lambda'(x)\frac{\partial}{\partial q}\left[C(\tau) + R(\tau)\right] < 0$ if and only if $\frac{\tau}{l} > \frac{K}{k}$. ☐

Proof (**Proposition 5**) For notational convenience suppress the arguments (x, y). Consider linear ROC. According to Proposition 3, the government sets $\tau^{\dagger} = \overline{\tau}$ in equilibrium where $\overline{\tau}$ is determined at the binding availability constraint $A(\tau) = a_0$. Since $A'(\tau) < 0$ and $\frac{\partial}{\partial y}A(\tau) > 0$ everywhere (Lemma 3), it follows that $\frac{\partial \overline{\tau}}{\partial y} > 0$. Now consider linear RRC. It follows from Proposition 4 that if $\gamma = 1$, SE sets $\tau^{\ddagger} = \underline{\tau}$ where $\underline{\tau} = \min\{\tau : C(\tau) + R(\tau) = b_0 - \psi - g(x, y)\}$. Since $C(\tau) + R(\tau)$ is quasiconvex (Proposition 1), $\underline{\tau}$ is determined at a point where $C'(\tau) + R'(\tau) < 0$. This, combined with the result $\frac{\partial}{\partial y}\left[C(x, y, \tau) + R(x, y, \tau)\right] > 0$ from Lemma 3 and the assumption $\frac{\partial}{\partial y}g(x, y) > 0$ implies $\frac{\partial \underline{\tau}}{\partial y} > 0$ such that the constraint $C(\underline{\tau}) + R(\underline{\tau}) = b_0 - \psi - g(x, y)$ remains binding. If $\gamma = 0$, on the other hand, $\tau^{\ddagger}$ is determined at the interior point $\hat{\tau}$ that minimizes $C(\tau) + R(\tau)$. The result

$\frac{\partial\hat{\tau}}{\partial y} < 0$ is obtained by implicitly differentiating the first-order condition of $C(\tau) + R(\tau)$. $\square$

Proof (**Proposition 6**) Recall from Proposition 3 that the equilibrium τ is set at $\overline{\tau}$ where the availability constraint binds, i.e., $A(\overline{\tau}) = a_0$ and that the payment rate r is set at the value where the budget constraint $rA(\overline{\tau}) + R(\overline{\tau}) \leq b_0$ is binding at the equilibrium. These remain true even with the addition of efforts x and y and effort cost $g(x, y)$. As a result, SE's objective function in the problem $(\mathcal{P}_1')$ in §4.2, with the addition of effort cost $g(x, y)$ and $\overline{\tau}$ expressed as a function of x and y, is equivalent to $\gamma v a_0 + (1-\gamma)\left[b_0 - C(x, y, \overline{\tau}(x, y)) - R(x, y, \overline{\tau}(x, y)) - g(x, y)\right]$ It is clear from this expression that the optimal values of x^* and y^* are independent of γ. $\square$