

On Queue-Length Information when Customers Travel to a Queue: Online Appendices

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Appendix A: Proofs

THEOREM 1. (*Existence of equilibrium*) For any set of parameters λ, μ, η, V , and C , a symmetric equilibrium strategy exists.

Proof of Theorem 1. Existence of an equilibrium in this model follows from using a fixed-point theorem. This is a game of countably many players. A strategy in this game consists of a vector $p = [p_0, p_1, \dots, p_{n_e}]$, where p_s is the probability of traveling to the queue after observing s customers in it. Let X be the space of all mixed strategy vectors:

$$X = \left\{ [p_0, p_1, \dots, p_{n_e}] : \forall s = 0, 1, \dots, n_e, \quad p_s \in [0, 1] \right\}. \quad (1)$$

Therefore, X is the n_e -dimensional cube $X = [0, 1]^{n_e}$. Thus, X is a compact space.

Let $F : X \rightarrow X$ be the function that generates the best-response strategy:

$$F(x) = \left\{ y \in X : y = p^*(x) \right\}, \quad (2)$$

where

$$p^*(x) = [p^*(0), p^*(1), \dots, p^*(n_e)] : p^*(s) \in \{0, 1\} \quad (3)$$

is the best response vector strategy, as was defined in (7).

Let $y_1, y_2 \in F(x)$. Let $y_3 = \alpha y_1 + (1 - \alpha)y_2$, where $\alpha \in (0, 1)$, be a point on the straight line segment that joins y_1 and y_2 . If $y_1 = y_2$ then it is clear that $y_3 \in F(x)$. If $y_1 \neq y_2$, then for every component s for which $y_1(s) \neq y_2(s)$, the customer is indifferent between traveling and not traveling, and therefore for every α , $y_3 \in F(x)$. Thus, F is convex.

Given a symmetric strategy, the steady-state probabilities are derived from the linear balance equations (2), which are continuous for any symmetric strategy vector. The utility function (Equation (6)) is also continuous. The function that assigns the best response to each steady-state probabilities (Equation (7)) is continuous, and F is continuous as the composition of the two. Therefore the graph of F , $\left\{ \{x, y\} \in X \times X : y \in F(x) \right\}$, is a closed set.

By Kakutani's fixed point theorem, the best response correspondence F has a fixed point p^e . This strategy is a best response of a player, when all other players use p^e , which defines a symmetric equilibrium. ■

THEOREM 2. *For any set of parameters λ, μ, η, V, C , a unique symmetric equilibrium strategy exists.*

Proof of Theorem 2. The proof is based on the continuity of the best-response function, which follows from the continuity of the utility function. $U_N(q)$ is monotonically decreasing in q (see Lemma 1 below): as more customers travel, the queue tends to be more congested, and the expected utility decreases, which implies that the best-response strategy is to Avoid-The-Crowd (ATC). As a result, only one fixed point of the best-response strategy exists, which defines a unique equilibrium (see Hassin and Haviv (1997)). By assumption 1, $U(0) = V - \frac{C}{\mu} - \frac{2C}{\eta} > 0$. If $U(1) \geq 0$ then $q^e = 1$ is the unique equilibrium. Otherwise, if $U(1) < 0$, there exists a unique equilibrium $0 < q^e < 1$ which is the single solution to the equation $U_N(q) = 0$. ■

LEMMA 1. *$U_N(q)$ is monotonically decreasing in q .*

Proof of Lemma 1. $U_N(q)$ as defined in Equation (12) can be written as a function of the product ρq . Denote $\xi = \rho q$. Then (12) can be rewritten as:

$$U(\xi) = V \left(\frac{1 - \xi^{n_e}}{1 - \xi^{n_e+1}} \right) - \frac{C}{\mu} \left(\frac{1 - (n_e + 1)(\xi)^{n_e} + n_e \xi^{n_e+1}}{(1 - \xi)(1 - \xi^{n_e+1})} \right) - \frac{2C}{\eta}. \quad (4)$$

Since ξ is increasing in q , it is enough to show that U is decreasing in ξ .

To show that U is decreasing in ξ , we show that its first order derivative with respect to ξ is negative:

$$\begin{aligned} \frac{dU}{d\xi} &= V \frac{d}{d\xi} \left[\frac{1 - \xi^{n_e}}{1 - \xi^{n_e+1}} \right] - \frac{C}{\mu} \frac{d}{d\xi} \left[\frac{1 - (n_e + 1)(\xi)^{n_e} + n_e \xi^{n_e+1}}{(1 - \xi)(1 - \xi^{n_e+1})} \right] \\ &= \frac{C}{\mu} \left[\frac{V\mu}{C} \frac{d}{d\xi} \left[\frac{1 - \xi^{n_e}}{1 - \xi^{n_e+1}} \right] - \frac{d}{d\xi} \left[\frac{1 - (n_e + 1)(\xi)^{n_e} + n_e \xi^{n_e+1}}{(1 - \xi)(1 - \xi^{n_e+1})} \right] \right] \\ &< \frac{C}{\mu} \left[(n_e + 1) \frac{d}{d\xi} \left[\frac{1 - \xi^{n_e}}{1 - \xi^{n_e+1}} \right] - \frac{d}{d\xi} \left[\frac{1 - (n_e + 1)(\xi)^{n_e} + n_e \xi^{n_e+1}}{(1 - \xi)(1 - \xi^{n_e+1})} \right] \right] \\ &= \frac{C}{\mu} \frac{-1 + \xi^{n_e} (1 + 2n_e(1 - \xi)^2 + n_e^2(1 - \xi)^2 + \xi^2 - \xi^{n_e+2})}{(1 - \xi)^2(1 - \xi^{n_e+1})^2}. \end{aligned} \quad (5)$$

Since $\frac{C}{\mu} > 0$ and the denominator of the second term in (5) is always positive, it is enough to show that the numerator of the second term of (5) is negative. For $\xi \neq 1$:

$$\begin{aligned}
 & -1 + \xi^{n_e} (1 + 2n_e(1 - \xi)^2 + n_e^2(1 - \xi)^2 + \xi^2 - \xi^{n_e+2}) < 0 \\
 \Leftrightarrow & (1 - \xi)^2(2n_e + n_e^2)\xi^{n_e} + \xi^{n_e+2}(1 - \xi^{n_e}) < (1 - \xi^{n_e}) \\
 \Leftrightarrow & (1 - \xi)^2(2n_e + n_e^2)\xi^{n_e} < (1 - \xi^{n_e})(1 - \xi^{n_e+2}) \\
 \Leftrightarrow & (2n_e + n_e^2)\xi^{n_e} < \left(\frac{1 - \xi^{n_e}}{1 - \xi}\right) \left(\frac{1 - \xi^{n_e+2}}{1 - \xi}\right) \\
 \Leftrightarrow & (n_e + 2)n_e\xi^{n_e} < \left(\sum_{s=0}^{n_e-1} \xi^s\right) \left(\sum_{s=0}^{n_e+1} \xi^s\right) \\
 \Leftrightarrow & \xi^{n_e} < \left(\frac{\sum_{s=0}^{n_e-1} \xi^s}{n_e}\right) \left(\frac{\sum_{s=0}^{n_e+1} \xi^s}{(n_e + 2)}\right). \tag{6}
 \end{aligned}$$

Both terms in the right-hand side of (6) are arithmetic means of two series: $\{\xi^s : s = 0, 1, \dots, n - 1\}$ and $\{\xi^s : s = 0, 1, \dots, n + 1\}$. The left-hand side can be read as a product of the geometric means of these series:

$$\xi^{n_e} = \xi^{\frac{n_e-1}{2}} \xi^{\frac{n_e+1}{2}} = \sqrt[n_e]{\xi^{\frac{n_e(n_e-1)}{2}}} \sqrt[n_e+2]{\xi^{\frac{(n_e+1)(n_e+2)}{2}}}.$$

Because an arithmetic mean of non-equal terms is greater than their geometric mean, the inequality (6) holds.

It follows that $\frac{dU}{d\xi} < 0$, and therefore $\frac{dU}{dq} < 0$. For $\xi \rightarrow 1$, the derivative exists and equals $-\frac{C}{\mu} \frac{n_e(n_e+2)}{12} < 0$. ■

THEOREM 3. *For fixed μ, η, V , and C , $\bar{\lambda}$ exists such that in equilibrium, for $\lambda \leq \bar{\lambda}$, customers travel to the queue with probability 1, and for $\lambda > \bar{\lambda}$, customers travel with probability $q^e = \frac{\bar{\lambda}}{\lambda}$. Moreover, the expected utility in equilibrium is monotonically decreasing for $\lambda < \bar{\lambda}$, and equals zero for $\lambda \geq \bar{\lambda}$.*

Proof of Theorem 3. We use Lemma 2 to prove Theorem 3. From Lemma 2(1.) it follows that when $\lambda \rightarrow 0$, the equilibrium is $q = 1$. From Lemma 2(2.), as long as $q = 1$, U decreases in λ , and by Lemma 2(3.) when $\lambda \rightarrow \infty, U(1) < 0$. Because $U|_{q=1}$ is a continuous function that is defined for any $\lambda > 0$, then by the mean-value theorem, there exists $\bar{\lambda}$ such that $U|_{q=1}(\bar{\lambda}) = 0$. For every $\lambda > \bar{\lambda}$, $U|_{q=1}(\bar{\lambda}) < 0$, and therefore by (13), $q = 1$ is no longer an equilibrium. For every $\lambda > \bar{\lambda}$, $q = \frac{\bar{\lambda}}{\lambda}$ is an equilibrium strategy, because $U_N(q) = 0$ and $0 < q < 1$ satisfy (13). ■

LEMMA 2. *When $q = 1$,*

1. $\lim_{\lambda \rightarrow 0} U(1) > 0$.
2. $\frac{dU(1)}{d\lambda} < 0$.
3. $\lim_{\lambda \rightarrow \infty} U(1) < 0$.

Proof of Lemma 2 .

1. $\lim_{\lambda \rightarrow 0} U|_{q=1} = V - C\left(\frac{1}{\mu} + \frac{2}{\eta}\right) > 0$, following Assumption 1.

2.

$$\frac{dU(1)}{d\lambda} = \frac{C}{\mu} \left[\nu \left(\frac{1 - (\rho)^{n_e}}{1 - (\rho)^{n_e+1}} \right)'_{\lambda} - \left(\frac{1 - (n_e + 1)(\rho)^{n_e} + n_e(\rho)^{n_e+1}}{(1 - \rho)(1 - (\rho)^{n_e+1})} \right)'_{\lambda} \right] \quad (7)$$

$$< \frac{C}{\mu} \left[(n_e + 1) \left(\frac{1 - (\rho)^{n_e}}{1 - (\rho)^{n_e+1}} \right)'_{\lambda} - \left(\frac{1 - (n_e + 1)(\rho)^{n_e} + n_e(\rho)^{n_e+1}}{(1 - \rho)(1 - (\rho)^{n_e+1})} \right)'_{\lambda} \right] \quad (8)$$

$$= \frac{C}{\mu^2} \left[\frac{-1 + \rho^{n_e} ((1 - \rho)^2 (n_e + 1)^2 + 2\rho - \rho^{n_e+2})}{(1 - \rho)^2 (1 - \rho^{n_e+1})^2} \right]. \quad (9)$$

The second term in the product in (9) is similar to the second term in the product in (6), hence the rest of the proof is similar to the proof of Lemma 1.

$$3. \lim_{\lambda \rightarrow \infty} U|_{q=1} = -\frac{2C}{\eta} < 0. \quad \blacksquare$$

THEOREM 4. For fixed λ, μ, V , and C , $\bar{\eta} > \frac{2C\mu}{V\mu - C}$ exists such that for $\eta \geq \bar{\eta}$, customers travel to the queue with probability 1, and for $\frac{2C\mu}{V\mu - C} < \eta < \bar{\eta}$, customers mix between traveling and not traveling. The expected utility is monotonically decreasing in η when $\eta \geq \bar{\eta}$, and equals zero when $\frac{2C\mu}{V\mu - C} < \eta < \bar{\eta}$.

Proof of Theorem 4. To prove Theorem 4, we write the utility function in (11) as $U_N(q, \eta) = F(q) - G(\eta)$, where

$$F(q) = \sum_{s=0}^{n_e-1} \pi_s(q) \left(V - C \frac{s+1}{\mu} \right),$$

$$G(\eta) = \frac{2C}{\eta}.$$

Note that $F(q) > 0$ for every $q \in [0, 1]$ and it is independent of η , while $G(\eta)$ decreases with η and it is independent of q . When $\eta \rightarrow \infty$, $G \rightarrow 0$ and therefore customers travel in equilibrium with probability 1. We denote by $\bar{\eta}$ the threshold for which $F(1) = G(\bar{\eta})$. At this point $U(1, \bar{\eta}) = 0$. For $\eta < \bar{\eta}$, $U(1, \eta) > 0$. By Lemma 1, $U_N(q, \eta)$, and thus also $F(q)$, decrease in q , and therefore to satisfy $U_N(q, \eta) = 0$ for $\eta < \bar{\eta}$, it must be that $q < 1$. The same argument implies that as η further decreases, the equilibrium travel probability q also decreases and the associated expected utility increases. \blacksquare

THEOREM 5. (1) For $\lambda < \bar{\lambda}$, throughput is increasing with λ . For $\lambda \geq \bar{\lambda}$, throughput is fixed and equals $\bar{\lambda} \frac{1 - \bar{\rho}^{n_e}}{1 - \bar{\rho}^{n_e+1}}$, where $\bar{\rho} = \frac{\bar{\lambda}}{\mu}$. (2) For $\frac{2C\mu}{V\mu - C} < \eta < \bar{\eta}$, throughput is increasing with η . For $\eta \geq \bar{\eta}$, throughput is fixed and equals $\lambda \frac{1 - \rho^{n_e}}{1 - \rho^{n_e+1}}$.

Proof of Theorem 5. From Theorem 3 it follows that for $\lambda \geq \bar{\lambda}$, $\lambda q = \bar{\lambda}$. Because we define $\bar{\rho} = \frac{\bar{\lambda}}{\mu}$, it follows that $T_N = \bar{\lambda} \frac{1 - (\bar{\rho})^{n_e}}{1 - (\bar{\rho})^{n_e+1}}$ is fixed for $\lambda \geq \bar{\lambda}$.

For $\lambda < \bar{\lambda}$, we show that the derivative of (14) is positive. Recall from Theorem 3 that $q = 1$ in this case. We use the following:

$$T_N(1) = \lambda \sum_{s=0}^{n_e-1} \pi_s(1) = \begin{cases} \lambda \frac{1 - \rho^{n_e}}{1 - \rho^{n_e+1}} & \rho \neq 1 \\ \lambda \frac{n_e}{n_e+1} & \rho = 1. \end{cases} \quad (10)$$

For $\rho = 1$, $T_N(1)$ is linearly increasing in λ . For $\rho \neq 1$, we take the derivative of $T_N(1)$ with respect to λ .

$$\begin{aligned} \frac{dT_N(1)}{d\lambda} &= \frac{1 - \rho^{n_e}}{1 - \rho^{n_e+1}} + \lambda \frac{d}{d\rho} \left(\frac{1 - \rho^{n_e}}{1 - \rho^{n_e+1}} \right) \frac{d\rho}{d\lambda} \\ &= \frac{1 - \rho^{n_e}}{1 - \rho^{n_e+1}} + \rho \frac{\rho^{n_e-1}(\rho - \rho^{n_e+1} - n_e(1 - \rho))}{(1 - \rho^{n_e+1})^2} \\ &= \frac{1 - \rho^{n_e} - \rho^{n_e} n_e (1 - \rho)}{(1 - \rho^{n_e+1})^2}. \end{aligned} \quad (11)$$

For $\rho \neq 1$, the denominator of (11) is positive. To show that the numerator is also positive, we look at two cases:

- When $\rho < 1$, proving $1 - \rho^{n_e} - \rho^{n_e} n_e (1 - \rho) > 0$ is equivalent to proving $\frac{1 - \rho^{n_e}}{1 - \rho} > \rho^{n_e} n_e$. The left-hand side is the sum $\sum_{s=0}^{n_e-1} \rho^s$. There are n_e terms in the sum, each term is greater than ρ^{n_e} . Therefore the inequality holds.

- When $\rho > 1$, proving $1 - \rho^{n_e} - \rho^{n_e} n_e (1 - \rho) > 0$ is equivalent to proving $\sum_{s=0}^{n_e-1} \rho^s < \rho^{n_e} n_e$. There are n_e terms in the sum, each term is smaller than ρ^{n_e} . Therefore the inequality holds.

The second part of Theorem 5, which refers to the change in throughput as a function of η , is similar in proof, because following Theorem 4, $q = 1$ when $\eta \geq \bar{\eta}$ and $q < 1$ otherwise, and q decreases as η decreases in the latter case. Therefore when $\eta \geq \bar{\eta}$ the throughput is fixed and equals $T_N(1)$, and when $\frac{2C\mu}{V\mu - C} < \eta < \bar{\eta}$ the throughput decreases as η decreases because q decreases. ■

COROLLARY 1. *The maximizer of SW_N with respect to λ is in $(0, \bar{\lambda})$, and with respect to η is $\eta \rightarrow \infty$.*

Proof of Corollary 1. (1) From Theorem 3, SW_N is strictly positive for $\lambda \in (0, \bar{\lambda})$, and 0 otherwise. Hence $\arg \max_{\lambda} SW_N \in (0, \bar{\lambda})$. (2) From Theorem 4, SW_N is positive and strictly increasing when $\eta \geq \bar{\eta}$, and 0 otherwise. Hence, $\arg \max_{\eta} SW_N = \infty$. ■

THEOREM 6. *For fixed V, μ, η , and C , $\lambda_P \geq \bar{\lambda}$ exists such that for $\lambda \leq \lambda_P$, $T_P \leq T_N$, and for $\lambda > \lambda_P$, $T_P > T_N$. Moreover, $SW_P > SW_N$.*

Proof of Theorem 6. We start with proving the first part of the Theorem that compares throughput. In the parking model, the throughput equals to

$$T_P = \begin{cases} \lambda \frac{1 - \rho^{n_p}}{1 - \rho^{n_p+1}}, & \text{if } \rho \neq 1, \\ \frac{n_p}{n_p+1}, & \text{if } \rho = 1. \end{cases} \quad (12)$$

Consider first $\lambda \leq \bar{\lambda}$. By Theorem 3, customers travel with probability $q^e = 1$ under the no-information model. Because $n_e \geq n_p$, it follows immediately that the throughput under the no-information model is higher.

When $\lambda > \bar{\lambda}$, by Theorem 5 the throughput under the no-information model is fixed and equals to $\bar{\lambda} \frac{1 - \bar{\rho}^n}{1 - \bar{\rho}^{n_e+1}}$, where $\bar{\rho} = \frac{\bar{\lambda}}{\mu}$. Next, we show that T_P strictly increases with λ , and that $\lim_{\lambda \rightarrow \infty} T_P > \bar{\lambda} \frac{1 - \bar{\rho}^n}{1 - \bar{\rho}^{n_e+1}}$. If $n_e = n_p$, then for $\lambda > \bar{\lambda}$ we get $T_P > T_N$, and the Theorem statement follows from setting $\lambda_P = \bar{\lambda}$. If $n_e > n_p$, then there exists $\lambda_P > \bar{\lambda}$ such that $T_P > T_N$ for $\lambda > \lambda_P$. To prove that, we take the first derivative of T_P with respect to λ . For $\rho \neq 1$,

$$\frac{d}{d\lambda} \frac{\lambda(1 - \rho^n)}{1 - \rho^{n+1}} = \frac{1 - \rho^n - n(1 - \rho)\rho^n}{(1 - \rho^{n+1})^2} = \frac{1 - \rho}{(1 - \rho^{n+1})^2} (1 + \rho + \dots + \rho^{n-1} - n\rho^n). \quad (13)$$

If $\rho < 1$, both terms in the product are positive, for $\rho > 1$ both are negative, and when $\rho \rightarrow 1$ 13 goes to $\frac{n}{2n+2}$. Hence the derivative is positive, and we conclude that T_P strictly increases with λ , and therefore for $\lambda > \bar{\lambda}$, $T_P > \bar{\lambda} \frac{1-\rho^n}{1-\rho^{n_e+1}} = T_N$.

Last, we calculate the upper limit of T_P as $\lambda \rightarrow \inf$.

$$\lim_{\lambda \rightarrow \infty} T_P = \lim_{\lambda \rightarrow \infty} \frac{\lambda(1-\rho^{n_p})}{1-\rho^{n_p+1}} = \lim_{\rho \rightarrow \infty} \frac{\rho\mu(1-\rho^{n_p})}{1-\rho^{n_p+1}} = \mu. \quad (14)$$

Next, we prove the second part of Theorem 6, which compares social welfare. The social welfare under the parking model equals

$$SW_P = \lambda \sum_{s=0}^{n_p-1} \pi_s^P \left(V - C \frac{s+1}{\mu} - \frac{2C}{\eta} \right). \quad (15)$$

We start by considering $\lambda \leq \bar{\lambda}$, in which by Theorem 3 customers travel with probability 1 under the no-information model. Then social welfare under the no-information model equals

$$\begin{aligned} SW_N(q=1) &= \lambda U_N(q=1) = \lambda \left(\sum_{s=0}^{n_e-1} \pi_s^N(1) \left(V - C \frac{s+1}{\mu} \right) - \frac{2C}{\eta} \right) \\ &= \lambda \left(\sum_{s=0}^{n_p-1} \pi_s^N(1) \left(V - C \frac{s+1}{\mu} - \frac{2C}{\eta} \right) + \sum_{s=n_p}^{n_e-1} \pi_s^N(1) \left(V - C \frac{s+1}{\mu} - \frac{2C}{\eta} \right) - \pi_{n_e}^N(1) \frac{2C}{\eta} \right). \end{aligned} \quad (16)$$

The second term in (16) is strictly negative because the utility from joining the queue is less than the cost of traveling, which is the reason why customers in the parking model do not join the queue when observing n_p customers or more. The third term is also negative. We compare the first term in (16) with the parking model social welfare in (15). The only difference between the two terms is in the steady-state probabilities π_s^P and $\pi_s^N(1)$. Comparing the two, we have for $\rho \neq 1$ and for all $s = 0, 1, \dots, n_p - 1$,

$$\pi_s^N(1) = \rho^s \frac{1-\rho}{1-\rho^{n_e+1}} < \rho^s \frac{1-\rho}{1-\rho^{n_p+1}} = \pi_s^P. \quad (17)$$

For $\rho = 1$ we have $\pi_s^N(1) = \frac{1}{n_e+1} < \frac{1}{n_p+1} = \pi_s^P$.

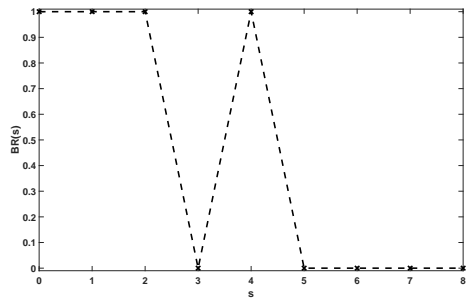
From (17) it follows that $SW_P > SW_N$.

Next, assume $\lambda > \bar{\lambda}$. In this case, under the no-information model customers travel to the queue with probability $q^e < 1$, and their expected utility is zero, and so is SW_N . Because SW_P is strictly positive, $SW_P > SW_N$. ■

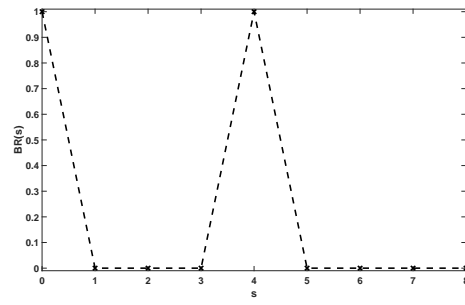
Appendix B: Generalized mixed-threshold strategies

More examples for non-threshold best-response strategies to threshold strategies are given in Figure 1, for $V = 8, \mu = 1, C = 1$.

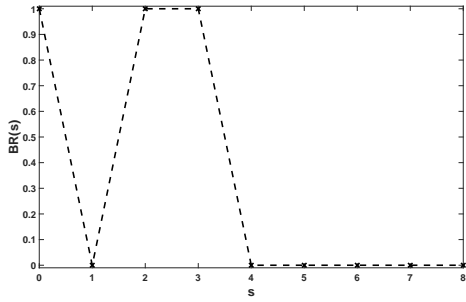
Figure 1(a) shows that best-response strategy to the threshold strategy $[1, 1, 1, 0.8, 0, 0, 0, 0]$ when $\lambda = 6, \eta = 0.7$ is $[1, 1, 1, 0, 1, 0, 0, 0]$. Figure 1(b) shows that the best-response strategy to the threshold strategy $[1, 1, 1, 0.6, 0, 0, 0, 0]$ when $\lambda = 15, \eta = 0.8$ is $[1, 0, 0, 0, 1, 0, 0, 0]$. When $\lambda = 5.5, \eta = 0.5$, Figure 1(c) shows that the best-response strategy to the threshold strategy $[1, 1, 0.4, 0, 0, 0, 0, 0]$ is $[1, 0, 1, 1, 0, 0, 0, 0]$, and Figure 1(d) shows that the best-response strategy to the threshold strategy $[1, 1, 0.44, 0, 0, 0, 0, 0]$ is $[1, 0, 1, 0, 0, 0, 0, 0]$.



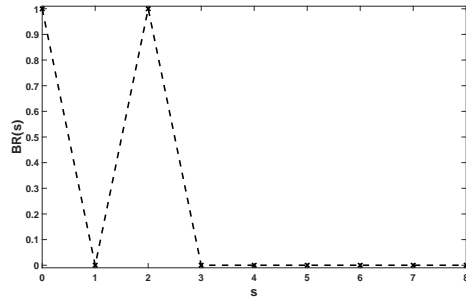
(a) $\lambda = 6, \eta = 0.7$



(b) $\lambda = 15, \eta = 0.8$



(c) $\lambda = 5.5, \eta = 0.5$



(d) $\lambda = 5.5, \eta = 0.5$

Figure 1 Examples of non-threshold best response strategies to threshold strategies, for $V = 8, \mu = 1, C = 1$.