Supplementary Document A

This document contains some proofs for the main model. It is intended to be the paper's online supplement. All the other proofs are contained in another supplementary document, which is available upon request.

Proof of Lemma 2. First we note that, if $\omega > q$, then the retailer cannot make any positive profit regardless of combating. In this case, trivially, she will not combat or sell in the market. Next, we focus on the case $\omega \le q$.

Considering the counterfeiter's profit function, we obtain

$$p_f^*(\gamma, p_a) = \begin{cases} \frac{\beta}{2}p_a, & if\ p_a < \frac{2q(1-\beta)}{2-\beta}, \\ p_a - (1-\beta)q, & if\ \frac{2q(1-\beta)}{2-\beta} \le p_a \le \frac{2-\beta}{2}q, \\ \frac{\beta q}{2}, & otherwise. \end{cases}$$

Then, we have

$$D_a(\gamma, p_a) = \begin{cases} \gamma\left(1 - \frac{p_a}{q}\right) + (1-\gamma)\left(1 - \frac{2-\beta}{2q(1-\beta)}p_a\right), & if\ p_a < \frac{2q(1-\beta)}{2-\beta}, \\ \gamma\left(1 - \frac{p_a}{q}\right), & if\ p_a \ge \frac{2q(1-\beta)}{2-\beta}. \end{cases}$$

Then,

$$\begin{aligned} &\pi_R(\gamma, p_a; \omega) \\ := &\begin{cases} \pi_R^I(\gamma, p_a; \omega) = (p_a - \omega)\left(\gamma\left(1 - \frac{p_a}{q}\right) + (1-\gamma)\left(1 - \frac{2-\beta}{2q(1-\beta)}p_a\right)\right) - \gamma c(\beta), & if\ p_a < \frac{2q(1-\beta)}{2-\beta}, \\ \pi_R^{II}(\gamma, p_a; \omega) = (p_a - \omega)\gamma\left(1 - \frac{p_a}{q}\right) - \gamma c(\beta), & if\ p_a \ge \frac{2q(1-\beta)}{2-\beta}. \end{cases} \end{aligned} \tag{S.1}$$

As such, given ω, the profit function is linear in γ for any p_a. Then, the optimal combat level can be determined by comparing the two extreme points $\gamma = 0$ and $\gamma = 1$. Let $p_a^*(\gamma; \omega) := \arg\max_{p_a} \pi_R(\gamma, p_a; \omega)$ and $\Pi_R(\gamma; \omega) := \pi_R(\gamma, p_a^*(\gamma; \omega); \omega)$. It suffices to compare $\Pi_R(0; \omega)$ and $\Pi_R(1; \omega)$. Specifically, by the proof of Proposition 1, we have

$$p_a^*(0; \omega) = \begin{cases} \frac{1-\beta}{2-\beta}q + \frac{1}{2}\omega, & if\ q > \frac{2-\beta}{2(1-\beta)}\omega, \\ \frac{q+\omega}{2}, & if\ q \le \frac{2-\beta}{2(1-\beta)}\omega, \end{cases} \quad \text{and} \quad p_a^*(1; \omega) = \frac{q+\omega}{2}. \tag{S.2}$$

Then,

$$\Pi_R(0; \omega) = \pi_R^I(0, p_a^*(0; \omega); \omega) = \begin{cases} \frac{2-\beta}{2(1-\beta)q}\left(\frac{1-\beta}{2-\beta}q - \frac{1}{2}\omega\right)^2, & if\ q > \frac{2-\beta}{2(1-\beta)}\omega, \\ 0, & if\ q \le \frac{2-\beta}{2(1-\beta)}\omega, \end{cases}$$

and

$$\Pi_R(1; \omega) = \pi_R^{II}(1, p_a^*(1; \omega); \omega) = \frac{(q-\omega)^2}{4q} - c(\beta).$$

To compare $\Pi_R(0; \omega)$ and $\Pi_R(1; \omega)$, we first analyze the case of $q > \frac{2-\beta}{2(1-\beta)}\omega$. In this case, $\Pi_R(1; \omega) \ge \Pi_R(0; \omega)$ if and only if $q \ge \frac{4c(\beta)(2-\beta)(1-\beta) + \sqrt{16c^2(\beta)(2-\beta)^2(1-\beta)^2 + 2(2-\beta)(1-\beta)\beta^2\omega^2}}{2(1-\beta)\beta}$. Moreover, $\frac{4c(\beta)(2-\beta)(1-\beta) + \sqrt{16c^2(\beta)(2-\beta)^2(1-\beta)^2 + 2(2-\beta)(1-\beta)\beta^2\omega^2}}{2(1-\beta)\beta} > \frac{2-\beta}{2(1-\beta)}\omega$ holds if and only if $\omega < \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}$. Specifically, if $\omega \ge \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}$, $\Pi_R(1; \omega) \ge \Pi_R(0; \omega)$ holds for all $q > \frac{2-\beta}{2(1-\beta)}\omega$.

Next, we examine the case of $\omega \le q \le \frac{2-\beta}{2(1-\beta)}\omega$. In this case, we have $\Pi_R(1; \omega) \ge \Pi_R(0; \omega)$ if and only if $\frac{(q-\omega)^2}{4q} - c(\beta) \ge 0$. This in turn implies $q \ge \omega + 2c(\beta) + 2\sqrt{\omega c(\beta) + c^2(\beta)}$. In addition, $\omega + 2c(\beta) + 2\sqrt{\omega c(\beta) + c^2(\beta)} \le \frac{2-\beta}{2(1-\beta)}\omega$ holds if and only if $\omega \ge \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}$. Specifically, if $\omega < \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}$, we have $\Pi_R(1; \omega) < \Pi_R(0; \omega)$ for any $\omega \le q \le \frac{2-\beta}{2(1-\beta)}\omega$.

Thus, given ω, there exists a critical function $\overline{q}(\omega)$ such that if $q \geq \overline{q}(\omega)$, then $\gamma^*(\omega)=1$; otherwise $\gamma^*(\omega)=0$. Specifically,

$$\overline{q}(\omega) = \begin{cases} \frac{4c(\beta)(2-\beta)(1-\beta)+\sqrt{16c^2(\beta)(2-\beta)^2(1-\beta)^2+2(2-\beta)(1-\beta)\beta^2\omega^2}}{2(1-\beta)\beta}, & if\ \omega < \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}, \\ \omega+2c(\beta)+2\sqrt{\omega c(\beta)+c^2(\beta)}, & if\ \omega \geq \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}. \end{cases}$$

Moreover,

$$\frac{d\overline{q}(\omega)}{d\omega} = \begin{cases} \frac{(2-\beta)\beta\omega}{\sqrt{16c^2(\beta)(2-\beta)^2(1-\beta)^2+2(2-\beta)(1-\beta)\beta^2\omega^2}} > 0, & if\ \omega < \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}, \\ 1+\sqrt{\frac{c(\beta)}{\omega+c(\beta)}} > 0, & if\ \omega \geq \frac{8c(\beta)(2-\beta)(1-\beta)}{\beta^2}. \end{cases}$$

Thus, $\overline{q}(\omega)$ increases in ω and the lemma follows. □

Proof of Lemma 3. It is straightforward to obtain $\frac{\widehat{q}^R}{\widehat{q}^M} = 2\left(\frac{-3+4\beta+\sqrt{\beta(2-\beta)}}{17\beta-9}\right)$. To show $\widehat{q}^R < \widehat{q}^M$, we have three cases:

(i) For $\beta > \frac{9}{17}$, it is equivalent to show $2(-3+4\beta+\sqrt{\beta(2-\beta)} < 17\beta-9$, or $y(\beta)=85\beta^2-62\beta+9>0$. Note that $\frac{dy(\beta)}{d\beta}=170\beta-62>170\cdot\frac{9}{17}-62=28>0$. Then, $y(\beta)$ increases in β and $y(\beta)>85\cdot(\frac{9}{17})^2-62\cdot\frac{9}{17}+9=\frac{405}{17}-\frac{558}{17}+9=0$.

(ii) For $\beta=\frac{9}{17}$, $\frac{\widehat{q}^R}{\widehat{q}^M}=\lim_{\beta\to\frac{9}{17}} 2\left(\frac{-3+4\beta+\sqrt{\beta(2-\beta)}}{17\beta-9}\right)=\lim_{\beta\to\frac{9}{17}} 2\left(\frac{4+\frac{1-\beta}{\sqrt{\beta(2-\beta)}}}{17}\right)=\frac{136}{255}<1$.

(iii) For $\beta<\frac{9}{17}$, it is equivalent to show $2(-3+4\beta+\sqrt{\beta(2-\beta)}>17\beta-9$, or $y(\beta)=2\sqrt{\beta(2-\beta)}-3(3\beta-1)>0$. Note that $\frac{dy(\beta)}{d\beta}=\frac{2(1-\beta)}{\sqrt{\beta(2-\beta)}}-9=2\sqrt{-1+\frac{1}{\beta(2-\beta)}}-9<2\sqrt{-1+\frac{1}{\frac{9}{17}(2-\frac{9}{17})}}-9=-\frac{119}{15}<0$. Then, $y(\beta)$ decreases in β and $y(\beta)>2\sqrt{\frac{9}{17}(2-\frac{9}{17})}-3(3\cdot\frac{9}{17}-1)=\frac{30}{17}-\frac{30}{17}=0$.

Hence, $\widehat{q}^R<\widehat{q}^M$ is proven, and the retailer in the DR model is more keen to combat the counterfeit than the manufacturer in the DM model. □

Proof of Proposition 3. We compare profits under DR and DM from the perspectives of the manufacturer and the supply chain:

(i) We first examine the manufacturer's optimal choice between DR and DM. To compare DR and DM, we define the difference of the manufacturer's profits under DR and under DM as Δ_M. Note that the manufacturer's profit is a continuous function of q under both DR and DM. Then, Δ_M is also continuous. Start with $\beta>\frac{2}{3}$, where $\widehat{q}^R<\frac{4c(\beta)(2-\beta)^2}{\beta^2}<\widehat{q}^M<16c(\beta)$. Then, by the manufacturer's profit functions in Proposition 1 (Table 2) and Proposition 2 (Table A1), we have

$$\Delta_M = \begin{cases} 0, & if\ q<\widehat{q}^R, \\ \frac{\omega_1^R(q-\omega_1^R)}{2q}-\frac{1-\beta}{4(2-\beta)}q, & if\ \widehat{q}^R\leq q\leq\frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{\omega_2^R(q-\omega_2^R)}{2q}-\frac{1-\beta}{4(2-\beta)}q, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2}<q<\widehat{q}^M, \\ \frac{\omega_2^R(q-\omega_2^R)}{2q}-\frac{q}{8}+c(\beta), & if\ \widehat{q}^M\leq q\leq 16c(\beta), \\ c(\beta), & otherwise. \end{cases}$$

Thus, when $q<\widehat{q}^R$, we have $\Delta_M=0$; when $\widehat{q}^M\leq q\leq 16c(\beta)$, we have

$$\Delta_M = \frac{\omega_2^R(q-\omega_2^R)}{2q}-\frac{q}{8}+c(\beta)=\sqrt{qc(\beta)}-\frac{q}{8}-c(\beta), \text{ and}$$

$$\frac{d\Delta_M}{dq} = \frac{1}{2}\sqrt{\frac{c(\beta)}{q}}-\frac{1}{8}\geq\frac{1}{2}\sqrt{\frac{c(\beta)}{16c(\beta)}}-\frac{1}{8}=\frac{1}{8}-\frac{1}{8}=0,$$

which means that Δ_M increases in q, and hence $\Delta_M>0$; when $q>16c(\beta)$, $\Delta_M>0$. Now we consider $\widehat{q}^R\leq q\leq\widehat{q}^M$. By Proposition 1 and Proposition 2, $\gamma^*=0$ under DM but $\gamma^*=1$ under DR. We have the following three cases: when $\widehat{q}^R \le q < \frac{8c(\beta)(2-\beta)^2}{\beta(3-\beta)}$, by (i) in the proof of Proposition 2, we have $\Delta_M = \frac{\omega_1^R(q-\omega_1^R)}{2q} - \frac{1-\beta}{4(2-\beta)}q > 0$; when $\frac{8c(\beta)(2-\beta)^2}{\beta(3-\beta)} \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}$, by (ii) in the proof of Proposition 2, we have $\Delta_M = \frac{\omega_1^R(q-\omega_1^R)}{2q} - \frac{1-\beta}{4(2-\beta)}q > \frac{\frac{1-\beta}{2-\beta}q(q-\frac{1-\beta}{2-\beta}q)}{2q} - \frac{1-\beta}{4(2-\beta)}q = \frac{\beta(1-\beta)}{4(2-\beta)^2}q > 0$; when $\frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < \widehat{q}^M$, by (ii) in the proof of Proposition 2, we have $\Delta_M = \frac{\omega_2^R(q-\omega_2^R)}{2q} - \frac{1-\beta}{4(2-\beta)}q > \frac{\frac{1-\beta}{2-\beta}q(q-\frac{1-\beta}{2-\beta}q)}{2q} - \frac{1-\beta}{4(2-\beta)}q = \frac{\beta(1-\beta)}{4(2-\beta)^2}q > 0$. Then, we have $\Delta_M \ge 0$ for any q and the manufacturer prefers DR over DM. The case of $\beta \le \frac{2}{3}$ can be shown similarly.

(ii) We then examine the supply chain's preference between DR and DM. Define the difference of the supply chain profit (the manufacturer's profit plus the retailer's profit) under DR and under DM as Δ_{sc}. Note that the manufacturer's profit is a continuous function of q under both DR and DM, but the retailer's profit is discontinuous at $q = \widehat{q}^R$ under DR and at $q = \widehat{q}^M$ under DM. Then, Δ_{sc} is also discontinuous at $q = \widehat{q}^R$ and $q = \widehat{q}^M$. Start with $\beta > \frac{2}{3}$, where $\widehat{q}^R < \frac{4c(\beta)(2-\beta)^2}{\beta^2} < \widehat{q}^M < 16c(\beta)$. Then, by the manufacturer's and the retailer's profit functions in Proposition 1 (Table 2) and Proposition 2 (Table A1), we have

$$\Delta_{sc} = \begin{cases} 0, & if\ q < \widehat{q}^R, \\ \frac{\omega_1^R(q-\omega_1^R)}{2q} + \frac{(q-\omega_1^R)^2}{4q} - c(\beta) - \frac{3(1-\beta)}{8(2-\beta)}q, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{\omega_2^R(q-\omega_2^R)}{2q} - \frac{3(1-\beta)}{8(2-\beta)}q, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < \widehat{q}^M, \\ \frac{\omega_2^R(q-\omega_2^R)}{2q} - \frac{3}{16}q + c(\beta), & if\ \widehat{q}^M \le q \le 16c(\beta), \\ 0, & otherwise, \end{cases}$$
$$= \begin{cases} 0, & if\ q < \widehat{q}^R, \\ \frac{5\beta-3}{8(2-\beta)}q + \frac{2-3\beta}{\beta}c(\beta), & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \sqrt{qc(\beta)} - 2c(\beta) - \frac{3(1-\beta)}{8(2-\beta)}q, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < \widehat{q}^M, \\ \sqrt{qc(\beta)} - c(\beta) - \frac{3}{16}q, & if\ \widehat{q}^M \le q \le 16c(\beta), \\ 0, & otherwise, \end{cases} \quad \text{and}$$

$$\frac{d\Delta_{sc}}{dq} = \begin{cases} 0, & if\ q < \widehat{q}^R, \\ \frac{5\beta-3}{8(2-\beta)} > 0, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{1}{2}\sqrt{\frac{c(\beta)}{q}} - \frac{3(1-\beta)}{8(2-\beta)} > \frac{1}{2}\sqrt{\frac{c(\beta)}{16c(\beta)}} - \frac{3(1-\beta)}{8(2-\beta)} = \frac{-1+2\beta}{8(2-\beta)} > 0, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < \widehat{q}^M, \\ \frac{1}{2}\sqrt{\frac{c(\beta)}{q}} - \frac{3}{16} \le \frac{1}{2}\sqrt{\frac{\beta}{8(2-\beta)}} - \frac{3}{16} < \frac{\sqrt{2}}{8} - \frac{3}{16} < 0, & if\ \widehat{q}^M \le q \le 16c(\beta), \\ 0, & otherwise. \end{cases}$$

Then, $\Delta_{sc} = 0$ when $q < \widehat{q}^R$ and $q > 16c(\beta)$. Furthermore,

$$\Delta_{sc} = \begin{cases} \left[\frac{-3+4\beta+\sqrt{\beta(2-\beta)}}{17\beta-9}\frac{2(5\beta-3)}{\beta} + \frac{2-3\beta}{\beta}\right]c(\beta) > 0, & when\ q = \widehat{q}^R, \\ \left[\sqrt{\frac{8(2-\beta)}{\beta}} + \frac{1}{2} - \frac{3}{\beta}\right]c(\beta) > 0, & when\ q = \widehat{q}^M, \\ 0, & when\ q = 16c(\beta). \end{cases}$$

Since Δ_{sc} increases in q when $\widehat{q}^R \le q < \widehat{q}^M$, we have $\Delta_{sc} \ge 0$ for any $\widehat{q}^R \le q < \widehat{q}^M$. Since Δ_{sc} decreases in q when $\widehat{q}^M \le q \le 16c(\beta)$, we have $\Delta_{sc} \ge 0$ for any $\widehat{q}^M \le q \le 16c(\beta)$. Therefore, $\Delta_{sc} \ge 0$ for any q and DR generates a more profitable supply chain than DM. The case of $\beta \le \frac{2}{3}$ can be shown similarly. □

Proof of Theorem 1. (i) Without the counterfeit, it can be shown that the wholesale price is $\frac{q}{2}$, the retail price is $\frac{3}{4}q$ and the demand is $\frac{1}{4}$. Thus, the retailer's profit is $\frac{q}{16}$, the manufacturer's profit is $\frac{q}{8}$ and the total profit for the supply chain is $\frac{3}{16}q$.

Define the difference between results with and without the counterfeit as $\Delta\Pi_{SC}$. Note that the manufacturer's profit is a continuous function of q, but the retailer's profit is discontinuous at $q = \widehat{q}^R$. Then, the difference of supply chain profit (the manufacturer's profit plus the retailer's profit) with and without the counterfeit, $\Delta\Pi_{SC}$, is also discontinuous at $q=\widehat{q}^R$.

For $\beta>\frac{2}{3}$, from the manufacturer's and the retailer's profits in Proposition 2 (Table A1), we have

$$\Delta\Pi_{SC}=\begin{cases}\frac{3(1-\beta)}{8(2-\beta)}q-\frac{3}{16}q, & if\ q<\widehat{q}^R,\\ \frac{\omega_1^R(q-\omega_1^R)}{2q}+\frac{(q-\omega_1^R)^2}{4q}-c(\beta)-\frac{3}{16}q, & if\ \widehat{q}^R\le q\le\frac{4c(\beta)(2-\beta)^2}{\beta^2},\\ \frac{\omega_2^R(q-\omega_2^R)}{2q}-\frac{3}{16}q, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2}<q<16c(\beta),\\ \frac{3}{16}q-c(\beta)-\frac{3}{16}q, & otherwise,\end{cases}$$
$$=\begin{cases}-\frac{3\beta}{16(2-\beta)}q, & if\ q<\widehat{q}^R,\\ \frac{7\beta-6}{16(2-\beta)}q+\frac{2-3\beta}{\beta}c(\beta), & if\ \widehat{q}^R\le q\le\frac{4c(\beta)(2-\beta)^2}{\beta^2},\\ \sqrt{qc(\beta)}-2c(\beta)-\frac{3}{16}q, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2}<q<16c(\beta),\\ -c(\beta), & otherwise.\end{cases}$$

Thus, $\Delta\Pi_{SC}<0$ for $q<\widehat{q}^R$ and $q\ge16c(\beta)$. Next, we will examine the sign of $\Delta\Pi_{SC}$ at $q=\widehat{q}^R$ and $q=\frac{4c(\beta)(2-\beta)^2}{\beta^2}$:

At $q=\widehat{q}^R$, from $\omega_1^R\frac{\widehat{q}^R-\omega_1^R}{2\widehat{q}^R}=\frac{1-\beta}{4(2-\beta)}\widehat{q}^R$, we have $\omega_1^R=\frac{1-\sqrt{\beta/(2-\beta)}}{2}\widehat{q}^R$. Then,

$$\Delta\Pi_{SC}(\widehat{q}^R)=\frac{\omega_1^R(\widehat{q}^R-\omega_1^R)}{2\widehat{q}^R}+\frac{(\widehat{q}^R-\omega_1^R)^2}{4\widehat{q}^R}-c(\beta)-\frac{3}{16}\widehat{q}^R=\frac{1}{16}\left(2\sqrt{\frac{\beta}{2-\beta}}-\frac{\beta}{2-\beta}\right)\widehat{q}^R-c(\beta)$$
$$=\left[\left(2\sqrt{\frac{2-\beta}{\beta}}-1\right)\frac{1-\beta}{3-4\beta+\sqrt{\beta(2-\beta)}}-1\right]c(\beta),$$

which is positive if $\beta<\widehat{\beta}\simeq0.32$, equal to 0 if $\beta=\widehat{\beta}$ and negative otherwise. Thus, $\Delta\Pi_{SC}(\widehat{q}^R)<0$ for $\beta>\frac{2}{3}$.

Now, at $q=\frac{4c(\beta)(2-\beta)^2}{\beta^2}$, we have $\Delta\Pi_{SC}=\frac{7\beta-6}{16(2-\beta)}\frac{4c(\beta)(2-\beta)^2}{\beta^2}+\frac{2-3\beta}{\beta}c(\beta)=\frac{-19\beta^2+28\beta-12}{4\beta^2}c(\beta)<0$.

When $\widehat{q}^R\le q\le\frac{4c(\beta)(2-\beta)^2}{\beta^2}$, since $\Delta\Pi_{SC}$ linearly changes with q, we have $\Delta\Pi_{SC}<0$ for any q within this interval.

When $\frac{4c(\beta)(2-\beta)^2}{\beta^2}<q<16c(\beta)$, we have $\frac{d\Delta\Pi_{SC}}{dq}=\frac{1}{2}\sqrt{\frac{c(\beta)}{q}}-\frac{3}{16}$ decreases in q and hence $\Delta\Pi_{SC}$ is concave with respect to q. We note that at $q=16c(\beta)$, $\frac{d\Delta\Pi_{SC}}{dq}=\frac{1}{2}\sqrt{\frac{c(\beta)}{16c(\beta)}}-\frac{3}{16}=-\frac{1}{16}<0$; at $q=\frac{4c(\beta)(2-\beta)^2}{\beta^2}$, $\frac{d\Delta\Pi_{SC}}{dq}=\frac{1}{2}\sqrt{\frac{c(\beta)\beta^2}{4c(\beta)(2-\beta)^2}}-\frac{3}{16}=\frac{7\beta-6}{16(2-\beta)}<0$ if and only if $\beta<\frac{6}{7}$. Then, if $\frac{2}{3}<\beta<\frac{6}{7}$, we have $\frac{d\Delta\Pi_{SC}}{dq}<0$ and $\Delta\Pi_{SC}$ decreases in q. Hence, we have $\Delta\Pi_{SC}<0$. If $\beta\ge\frac{6}{7}$, we have $\frac{d\Delta\Pi_{SC}}{dq}=\frac{1}{2}\sqrt{\frac{c(\beta)}{q}}-\frac{3}{16}$ is positive when $\frac{4c(\beta)(2-\beta)^2}{\beta^2}<q<\frac{64}{9}c(\beta)$, equal to 0 if $q=\frac{64}{9}c(\beta)$ and negative otherwise. Then, we have $\Delta\Pi_{SC}\le\sqrt{\frac{64}{9}}c(\beta)-2c(\beta)-\frac{3}{16}\frac{64}{9}c(\beta)=-\frac{2}{3}c(\beta)<0$ if $\beta\ge\frac{6}{7}$.

For $\beta\le\frac{2}{3}$, from the manufacturer's and the retailer's profits in Proposition 2 (Table A1), we have

$$\Delta\Pi_{SC}=\begin{cases}\frac{3(1-\beta)}{8(2-\beta)}q-\frac{3}{16}q, & if\ q<\widehat{q}^R,\\ \frac{\omega_1^R(q-\omega_1^R)}{2q}+\frac{(q-\omega_1^R)^2}{4q}-c(\beta)-\frac{3}{16}q, & if\ \widehat{q}^R\le q\le\frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)},\\ \frac{3}{16}q-c(\beta)-\frac{3}{16}q, & otherwise,\end{cases}$$
$$=\begin{cases}-\frac{3\beta}{16(2-\beta)}q<0, & if\ q<\widehat{q}^R,\\ \frac{7\beta-6}{16(2-\beta)}q+\frac{2-3\beta}{\beta}c(\beta), & if\ \widehat{q}^R\le q\le\frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)},\\ -c(\beta), & otherwise,\end{cases}\quad\text{and}$$

$$\frac{d\Delta\Pi_{SC}}{dq}=\begin{cases}-\frac{3\beta}{16(2-\beta)}<0, & if\ q<\widehat{q}^R,\\ \frac{7\beta-6}{16(2-\beta)}<0, & if\ \widehat{q}^R\le q\le\frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)},\\ 0, & otherwise.\end{cases}$$

Thus, $\Delta\Pi_{SC}<0$ for $q<\widehat{q}^R$ and $\Delta\Pi_{SC}$ decreases in q for $q\ge\widehat{q}^R$. Furthermore, $\Delta\Pi_{SC}(\widehat{q}^R)>0$ if and only if $\beta<\widehat{\beta}$. Thus, if $\beta\ge\widehat{\beta}$, $\Delta\Pi_{SC}<0$ for all q; if $\beta<\widehat{\beta}$, $\Delta\Pi_{SC}>0$ when $\widehat{q}^R\le q<\frac{16c(\beta)(2-\beta)(2-3\beta)}{\beta(6-7\beta)}$.

(ii) Now we examine how the counterfeit affects the manufacturer's profit. For $\beta > \frac{2}{3}$, by the manufacturer's profit function in Proposition 2 (Table A1), we obtain the difference of the manufacturer's profits with and without the counterfeit,

$$\Delta\pi_M^* = \begin{cases} \frac{1-\beta}{4(2-\beta)}q - \frac{q}{8} = -\frac{\beta}{8(2-\beta)}q < 0, & if\ q < \widehat{q}^R, \\ \frac{\omega_1^R(q-\omega_1^R)}{2q} - \frac{q}{8} \le \frac{\frac{q}{2}(q-\frac{q}{2})}{2q} - \frac{q}{8} = 0, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{\omega_2^R(q-\omega_2^R)}{2q} - \frac{q}{8} \le \frac{\frac{q}{2}(q-\frac{q}{2})}{2q} - \frac{q}{8} = 0, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ 0, & otherwise. \end{cases}$$

For $\beta \le \frac{2}{3}$, by the manufacturer's profit function in Proposition 2 (Table A1), we have

$$\Delta\pi_M^* = \begin{cases} \frac{1-\beta}{4(2-\beta)}q - \frac{q}{8} = -\frac{\beta}{8(2-\beta)}q < 0, & if\ q < \widehat{q}^R, \\ \frac{\omega_1^R(q-\omega_1^R)}{2q} - \frac{q}{8} \le \frac{\frac{q}{2}(q-\frac{q}{2})}{2q} - \frac{q}{8} = 0, & if\ \widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}, \\ 0, & otherwise. \end{cases}$$

Thus, $\Delta\pi_M^* \le 0$ for any q.

Next, we examine how the counterfeit affects the retailer's profit. For $\beta > \frac{2}{3}$, by the retailer's profit function in Proposition 2 (Table A1), we have the difference of the retailer's profits with and without the counterfeit,

$$\Delta\pi_R^* = \begin{cases} \frac{1-\beta}{8(2-\beta)}q - \frac{q}{16} = -\frac{\beta}{16(2-\beta)}q < 0, & if\ q < \widehat{q}^R, \\ \frac{(q-\omega_1^R)^2}{4q} - c(\beta) - \frac{q}{16}, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ -\frac{q}{16} < 0, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ -c(\beta) < 0, & otherwise. \end{cases}$$

Next, we will examine the sign of $\Delta\pi_R^*$ at $q = \widehat{q}^R$. At $q = \widehat{q}^R$, from $\omega_1^R \frac{\widehat{q}^R - \omega_1^R}{2\widehat{q}^R} = \frac{1-\beta}{4(2-\beta)}\widehat{q}^R$, we have $\omega_1^R = \frac{1-\sqrt{\beta/(2-\beta)}}{2}\widehat{q}^R$. Then,

$$\begin{aligned} \Delta\pi_R^*(\widehat{q}^R) &= \frac{(\widehat{q}^R - \omega_1^R)^2}{4\widehat{q}^R} - c(\beta) - \frac{\widehat{q}^R}{16} = \frac{1}{16}\left(2\sqrt{\frac{\beta}{2-\beta}} + \frac{\beta}{2-\beta}\right)\widehat{q}^R - c(\beta) \\ &= \left[\left(2\sqrt{\frac{2-\beta}{\beta}} + 1\right)\frac{1-\beta}{3-4\beta+\sqrt{\beta(2-\beta)}} - 1\right]c(\beta), \end{aligned}$$

which is positive if $\beta < \frac{2}{3}$, equal to 0 if $\beta = \frac{2}{3}$ and negative otherwise. Thus, for $\beta > \frac{2}{3}$, $\Delta\pi_R^*(\widehat{q}^R) < 0$. Furthermore, when $\widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}$, by noting that

$$\begin{aligned} \frac{d\omega_1^R}{dq} &= \frac{1}{2}\left[\sqrt{\frac{2(1-\beta)q(\beta q - 4c(\beta)(2-\beta))}{\beta(2-\beta)}}\right]^{-1} \frac{4(1-\beta)(\beta q - 2c(\beta)(2-\beta))}{\beta(2-\beta)} \\ &= 2\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1 + \frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}}, \end{aligned}$$

we have

$$\begin{aligned} \frac{d\Delta\pi_R^*}{dq} &= \frac{14-11\beta}{16(2-\beta)} - \sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1 + \frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}} \\ &\le \frac{14-11\beta}{16(2-\beta)} - \sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1 + \frac{(2c(\beta)(2-\beta)/\beta)^2}{4c(\beta)(2-\beta)^2/\beta^2(4c(\beta)(2-\beta)^2/\beta^2 - 4c(\beta)(2-\beta)/\beta)}} \\ &= \frac{14-11\beta}{16(2-\beta)} - \frac{4-3\beta}{4(2-\beta)} = -\frac{1}{16} < 0. \end{aligned}$$

The first inequality holds because $q(q - \frac{4c(\beta)(2-\beta)}{\beta})$ increases in q when $q > \frac{4c(\beta)(2-\beta)}{\beta}$. Thus, $\Delta\pi_R^*$ strictly decreases in q. Hence, we have $\Delta\pi_R^* < 0$ for any q for $\beta > \frac{2}{3}$.

For $\beta \le \frac{2}{3}$, by the retailer's profit function in Proposition 2 (Table A1), we have

$$\Delta\pi_R^* = \begin{cases} \frac{1-\beta}{8(2-\beta)}q - \frac{1}{16}q = -\frac{\beta}{16(2-\beta)}q < 0, & if\ q < \widehat{q}^R, \\ \frac{(q-\omega_1^R)^2}{4q} - c(\beta) - \frac{1}{16}q, & if\ \widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}, \\ -c(\beta) < 0, & otherwise. \end{cases}$$

When $\widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}$, we have

$$\frac{d\Delta\pi_R^*}{dq} = \frac{14-11\beta}{16(2-\beta)} - \sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1+\frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}}$$

$$\le \frac{14-11\beta}{16(2-\beta)} - \sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1+\frac{(2c(\beta)(2-\beta)/\beta)^2}{(32c(\beta)(1-\beta)(2-\beta)/\beta/(6-7\beta))((32c(\beta)(1-\beta)(2-\beta)/\beta/(6-7\beta))-4c(\beta)(2-\beta)/\beta)}}$$

$$= \frac{14-11\beta}{16(2-\beta)} - \frac{10-9\beta}{8(2-\beta)} = \frac{-6+7\beta}{16(2-\beta)} < 0.$$

The first inequality holds because $q(q-\frac{4c(\beta)(2-\beta)}{\beta})$ increases in q when $q > \frac{4c(\beta)(2-\beta)}{\beta}$. Thus, $\Delta\pi_R^*$ strictly decreases in q. Specifically, $\Delta\pi_R^*(\widehat{q}^R) > 0$ for any $\beta \le \frac{2}{3}$. Hence, for $\beta \le \frac{2}{3}$, we have $\Delta\pi_R^* > 0$ when $\widehat{q}^R \le q < \widetilde{q}$, where $\Delta\pi_R^*(\widetilde{q}) = 0$, i.e., $\widetilde{q} = \frac{-2+5\beta+4\sqrt{2(2-\beta)(1-\beta)}}{-7\beta^2+76\beta-60}\frac{16c(\beta)(2-\beta)^2}{\beta}$.

Therefore, the manufacturer is always worse off by the counterfeit while the retailer can be better off if and only if $\beta \le \frac{2}{3}$ and $\widehat{q}^R \le q < \widetilde{q}$. □

Proof of Proposition 4. Without the counterfeit, it can be shown that the wholesale price is $\frac{q}{2}$, the retail price is $\frac{3}{4}q$ and the demand is $\frac{1}{4}$. Thus, the retailer's profit is $\frac{q}{16}$, the manufacturer's profit is $\frac{q}{8}$, the total profit for the supply chain is $\frac{3}{16}q$ and the consumer surplus (CS) is $\frac{q}{32}$. Then, the social welfare (SW), consumer surplus plus profits for the supply chain and the counterfeiter, is $\frac{7}{32}q$.

With the counterfeit, we begin our analysis with consumer surplus. If $q \ge \widehat{q}^R$, there exists $\hat{\theta}^F = \frac{q+\omega^*}{2q}$ such that consumers with $\theta \ge \hat{\theta}^F$ buy the authentic products. Correspondingly, $CS = \int_{\hat{\theta}^F}^1(\theta q - \frac{q+\omega^*}{2})d\theta$. If $q < \widehat{q}^R$, there exist $\hat{\theta}_a^N = \frac{3}{4}$ and $\hat{\theta}_f^N = \frac{3(1-\beta)}{4(2-\beta)}$ such that consumers with $\theta \ge \hat{\theta}_a^N$ buy the authentic products and those with $\hat{\theta}_f^N \le \theta < \hat{\theta}_a^N$ buy the counterfeit. As a result, by Table A1, for $\beta > \frac{2}{3}$,

$$CS = \begin{cases} \int_{\hat{\theta}_a^N}^1(\theta q - \frac{3(1-\beta)}{2(2-\beta)}q)d\theta + \int_{\hat{\theta}_f^N}^{\hat{\theta}_a^N}(\theta\beta q - \frac{3(1-\beta)}{4(2-\beta)}\beta q)d\theta, & if\ q < \widehat{q}^R, \\ \int_{\hat{\theta}^F}^1(\theta q - \frac{q+\omega^*}{2})d\theta, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \int_{\hat{\theta}^F}^1(\theta q - \frac{q+\omega^*}{2})d\theta, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ \int_{\hat{\theta}^F}^1(\theta q - \frac{q+\omega^*}{2})d\theta, & otherwise, \end{cases}$$

$$= \begin{cases} \frac{4+17\beta-5\beta^2}{32(2-\beta)^2}q, & if\ q < \widehat{q}^R, \\ \frac{(q-\omega_1^R)^2}{8q}, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{c(\beta)}{2}, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ \frac{q}{32}, & otherwise. \end{cases}$$

Then,

$$\Delta CS = \begin{cases} \frac{4+17\beta-5\beta^2}{32(2-\beta)^2}q - \frac{q}{32} = \frac{3\beta(7-2\beta)}{32(2-\beta)^2}q > 0, & if\ q < \widehat{q}^R, \\ \frac{(q-\omega_1^R)^2}{8q} - \frac{q}{32} > \frac{(q-\frac{q}{2})^2}{8q} - \frac{q}{32} = 0, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{c(\beta)}{2} - \frac{q}{32} > \frac{c(\beta)}{2} - \frac{16c(\beta)}{32} = 0, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ \frac{q}{32} - \frac{q}{32} = 0, & otherwise. \end{cases}$$

By the counterfeiter's, the manufacturer's and the retailer's profits in Proposition 2 (Table A1), we have

$$\Delta SW = \begin{cases} \frac{1-\beta}{8(2-\beta)}q + \frac{1-\beta}{4(2-\beta)}q + \frac{9\beta(1-\beta)}{16(2-\beta)^2}q + \frac{4+17\beta-5\beta^2}{32(2-\beta)^2}q - \frac{7}{32}q, & if\ q < \widehat{q}^R, \\ \frac{\omega_1^R(q-\omega_1^R)}{2q} + \frac{3(q-\omega_1^R)^2}{8q} - c(\beta) - \frac{7}{32}q, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{\omega_2^R(q-\omega_2^R)}{2q} + \frac{c(\beta)}{2} - \frac{7}{32}q, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ -c(\beta), & otherwise, \end{cases}$$

$$= \begin{cases} \frac{9\beta(3-2\beta)}{32(2-\beta)^2}q > 0, & if\ q < \widehat{q}^R, \\ \frac{\omega_1^R(q-\omega_1^R)}{2q} + \frac{3(q-\omega_1^R)^2}{8q} - c(\beta) - \frac{7}{32}q, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \sqrt{qc(\beta)} - \frac{3}{2}c(\beta) - \frac{7}{32}q, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ -c(\beta) < 0, & otherwise. \end{cases}$$

Then, by noting that

$$\frac{d\omega_1^R}{dq} = \frac{1}{2}\left[\sqrt{\frac{2(1-\beta)q(\beta q - 4c(\beta)(2-\beta))}{\beta(2-\beta)}}\right]^{-1} \frac{4(1-\beta)(\beta q - 2c(\beta)(2-\beta))}{\beta(2-\beta)}$$
$$= 2\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1+\frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}},$$

we have

$$\frac{d\Delta SW}{dq} = \begin{cases} \frac{9\beta(3-2\beta)}{32(2-\beta)^2}, & if\ q < \widehat{q}^R, \\ \frac{2-5\beta}{32(2-\beta)} - \frac{1}{2}\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1+\frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}}, & if\ \widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}, \\ \frac{1}{2}\sqrt{\frac{c(\beta)}{q}} - \frac{7}{32}, & if\ \frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta), \\ 0, & otherwise. \end{cases}$$

Next, we will examine the sign of ΔSW at $q = \widehat{q}^R$. At $q = \widehat{q}^R$, from $\omega_1^R \frac{\widehat{q}^R - \omega_1^R}{2\widehat{q}^R} = \frac{1-\beta}{4(2-\beta)}\widehat{q}^R$, we have $\omega_1^R = \frac{1-\sqrt{\beta/(2-\beta)}}{2}\widehat{q}^R$. Then,

$$\Delta SW(\widehat{q}^R) = \frac{\omega_1^R(\widehat{q}^R - \omega_1^R)}{2\widehat{q}^R} + \frac{3(\widehat{q}^R-\omega_1^R)^2}{8\widehat{q}^R} - c(\beta) - \frac{7}{32}\widehat{q}^R = \frac{1}{16}\left(3\sqrt{\frac{\beta}{2-\beta}} - \frac{\beta}{2(2-\beta)}\right)\widehat{q}^R - c(\beta)$$
$$= \left[\left(3\sqrt{\frac{2-\beta}{\beta}} - 1\right)\frac{1-\beta}{3-4\beta+\sqrt{\beta(2-\beta)}} - 1\right]c(\beta),$$

which is positive if $\beta < \widehat{\beta}' \simeq 0.56$, equal to 0 if $\beta = \widehat{\beta}'$ and negative otherwise. Then, $\Delta SW(\widehat{q}^R) < 0$ for $\beta > \frac{2}{3}$.

When $\widehat{q}^R \le q \le \frac{4c(\beta)(2-\beta)^2}{\beta^2}$, we have

$$\frac{d\Delta SW}{dq} = \frac{2-5\beta}{32(2-\beta)} - \frac{1}{2}\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1+\frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}}$$
$$\le \frac{2-5\beta}{32(2-\beta)} - \frac{1}{2}\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1+\frac{(2c(\beta)(2-\beta)/\beta)^2}{4c(\beta)(2-\beta)^2/\beta^2(4c(\beta)(2-\beta)^2/\beta^2 - 4c(\beta)(2-\beta)/\beta)}}$$
$$= \frac{2-5\beta}{32(2-\beta)} - \frac{4-3\beta}{8(2-\beta)} = -\frac{7}{32} < 0.$$

The first inequality holds because $q(q - \frac{4c(\beta)(2-\beta)}{\beta})$ increases in q when $q > \frac{4c(\beta)(2-\beta)}{\beta}$. Thus, ΔSW strictly decreases in q and $\Delta SW < 0$ within this interval.

When $\frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < 16c(\beta)$, we have $\frac{dSW}{dq} = \frac{1}{2}\sqrt{\frac{c(\beta)}{q}} - \frac{7}{32}$ decreases in q and hence ΔSW is concave with respect to q. We note that at $q = 16c(\beta)$, $\frac{d\Delta SW}{dq} = \frac{1}{2}\sqrt{\frac{c(\beta)}{16c(\beta)}} - \frac{7}{32} = -\frac{3}{32} < 0$; at $q = \frac{4c(\beta)(2-\beta)^2}{\beta^2}$, $\frac{d\Delta SW}{dq} = \frac{1}{2}\sqrt{\frac{c(\beta)\beta^2}{4c(\beta)(2-\beta)^2}} - \frac{7}{32} = \frac{15\beta-14}{32(2-\beta)} < 0$ if and only if $\beta < \frac{14}{15}$. Then, if $\frac{2}{3} < \beta < \frac{14}{15}$, we have $\frac{d\Delta SW}{dq} < 0$ and ΔSW decreases in q. Hence, we have $\Delta SW < 0$. If $\beta \ge \frac{14}{15}$, we have $\frac{d\Delta SW}{dq} = \frac{1}{2}\sqrt{\frac{c(\beta)}{q}} - \frac{3}{16}$ is positive when $\frac{4c(\beta)(2-\beta)^2}{\beta^2} < q < \frac{256}{49}c(\beta)$, equal to 0 if $q = \frac{256}{49}c(\beta)$ and negative otherwise. Then, we have $\Delta SW \le \sqrt{\frac{256}{49}}c(\beta) - \frac{3}{2}c(\beta) - \frac{7}{32}\frac{256}{49}c(\beta) = -\frac{5}{14}c(\beta) < 0$ if $\beta \ge \frac{14}{15}$. Hence, for $\beta > \frac{2}{3}$, $\Delta SW > 0$ when $q < \widehat{q}^R$ and $\Delta SW < 0$ when $q \ge \widehat{q}^R$.

For $\beta \le \frac{2}{3}$, similar to the analysis above,

$$CS = \begin{cases} \frac{4+17\beta-5\beta^2}{32(2-\beta)^2}q, & if\ q < \widehat{q}^R, \\ \frac{(q-\omega_1^R)^2}{8q}, & if\ \widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}, \\ \frac{q}{32}, & otherwise. \end{cases}$$

Then,

$$\Delta CS = \begin{cases} \frac{4+17\beta-5\beta^2}{32(2-\beta)^2}q - \frac{q}{32} = \frac{3\beta(7-2\beta)}{32(2-\beta)^2}q > 0, & if\ q < \widehat{q}^R, \\ \frac{(q-\omega_1^R)^2}{8q} - \frac{q}{32} > \frac{(q-\frac{q}{2})^2}{8q} - \frac{q}{32} = 0, & if\ \widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}, \\ \frac{q}{32} - \frac{q}{32} = 0, & otherwise. \end{cases}$$

By the counterfeiter's, the manufacturer's and the retailer's profit functions in Proposition 2 (Table A1), we have

$$\Delta SW = \begin{cases} \frac{9\beta(3-2\beta)}{32(2-\beta)^2}q > 0, & if\ q < \widehat{q}^R, \\ \frac{\omega_1^R(q-\omega_1^R)}{2q} + \frac{3(q-\omega_1^R)^2}{8q} - c(\beta) - \frac{7}{32}q, & if\ \widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}, \\ -c(\beta) < 0, & otherwise, \end{cases}$$

and

$$\frac{d\Delta SW}{dq} = \begin{cases} \frac{9\beta(3-2\beta)}{32(2-\beta)^2}, & if\ q < \widehat{q}^R, \\ \frac{2-5\beta}{32(2-\beta)} - \frac{1}{2}\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1 + \frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}}, & if\ \widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}, \\ 0, & otherwise. \end{cases}$$

When $\widehat{q}^R \le q \le \frac{32c(\beta)(1-\beta)(2-\beta)}{\beta(6-7\beta)}$, we have

$$\begin{aligned} \frac{d\Delta SW}{dq} &= \frac{2-5\beta}{32(2-\beta)} - \frac{1}{2}\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1 + \frac{(2c(\beta)(2-\beta)/\beta)^2}{q(q-4c(\beta)(2-\beta)/\beta)}} \\ &\le \frac{2-5\beta}{32(2-\beta)} - \frac{1}{2}\sqrt{\frac{1-\beta}{2(2-\beta)}}\sqrt{1 + \frac{(2c(\beta)(2-\beta)/\beta)^2}{(32c(\beta)(1-\beta)(2-\beta)/\beta/(6-7\beta))((32c(\beta)(1-\beta)(2-\beta)/\beta/(6-7\beta))-4c(\beta)(2-\beta)/\beta)}} \\ &= \frac{2-5\beta}{32(2-\beta)} - \frac{10-9\beta}{16(2-\beta)} = \frac{-18+13\beta}{32(2-\beta)} < 0. \end{aligned}$$

The first inequality holds because $q(q - \frac{4c(\beta)(2-\beta)}{\beta})$ increases in q when $q > \frac{4c(\beta)(2-\beta)}{\beta}$. Thus, ΔSW strictly decreases in q within this interval. Moreover, $\Delta SW(\widehat{q}^R) > 0$ if $\beta < \widehat{\beta}'$ and $\Delta SW(\widehat{q}^R) \le 0$ otherwise. Hence, if $\widehat{\beta}' \le \beta \le \frac{2}{3}$, $\Delta SW > 0$ when $q < \widehat{q}^R$ and $\Delta SW \le 0$ otherwise; if $\beta < \widehat{\beta}'$, there exists a unique $\widetilde{q}' > \widehat{q}^R$ such that $\Delta SW > 0$ when $q < \widetilde{q}'$ and $\Delta SW \le 0$ otherwise. For convenience, for $\beta \ge \widehat{\beta}'$, we define $\widetilde{q}' = \widehat{q}^R$. Then we have $\Delta SW > 0$ when $q < \widetilde{q}'$ and $\Delta SW \le 0$ when $q \ge \widetilde{q}'$.

To summarize, for any β, the counterfeit improves customer surplus; there exists a unique $\widetilde{q}'$ such that the counterfeit improves social welfare if $q < \widetilde{q}'$ and the counterfeit reduces social welfare otherwise. □