

## Electronic Companion to “Robust Optimization Approach to Process Flexibility Designs with Contribution Margin Differentials”

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### Appendix A: Proofs

*Proof of Lemma 1.* Since the dual problem (3) is a linear program, basic solutions are achieved at the corners of the polyhedron defined by  $\{y_i + z_j = p_i \mid (a_j, b_i) \in \mathcal{A}\}$ ,  $\{y_i = 0 \mid 1 \leq i \leq n\}$  and  $\{z_j = 0 \mid 1 \leq j \leq m\}$ . From the above  $|\mathcal{A}| + m + n$  equations, we arbitrarily choose  $m + n$  linearly independent binding constraints that identify a unique basic solution. In the chosen  $m + n$  equations, denote  $C_1$  as the set of equations that belong to  $\{y_i = 0 \mid 1 \leq i \leq n\}$  or  $\{z_j = 0 \mid 1 \leq j \leq m\}$ , and denote  $C_2$  as the set of equations that belong to  $\{y_i + z_j = p_i \mid (a_j, b_i) \in \mathcal{A}\}$ . Note that  $\{y_i = 0 \mid 1 \leq i \leq n\} \cup \{z_j = 0 \mid 1 \leq j \leq m\}$  is not a feasible solution, and therefore we assume that  $|C_2| > 0$ . Let  $G = (A, B, \mathcal{G})$  be the graph corresponding to the chosen  $m + n$  equations, where the set of edges  $\mathcal{G}$  is defined by  $C_2$ . In other words,  $(a_j, b_i) \in \mathcal{G}$  if equation  $y_i + z_j = p_i$  belongs to  $C_2$ . Suppose  $G$  consists of  $g$  components, each of which is a connected subgraph. For each subgraph  $G_r = (A_r, B_r, \mathcal{G}_r)$  where  $1 \leq r \leq g$ , there are three cases to consider: (1) there exists some  $j$  such that  $z_j = 0$ ; (2) there exists some  $i$  such that  $y_i = 0$ ; and (3)  $z_j \neq 0$  and  $y_i \neq 0$  for all  $a_j \in A_r, b_i \in B_r$ .

**Case 1:** Suppose  $z_{j^*} = 0$  for some  $a_{j^*} \in A_r$ . For any  $a_{\hat{j}} \in A_r$ , since  $G_r$  is connected, there exists a path  $(a_{j^*}, b_{i_1}, a_{j_1}, b_{i_2}, \dots, b_{i_k}, a_{\hat{j}})$  connecting  $a_{j^*}$  and  $a_{\hat{j}}$ . Since  $y_i + z_j = p_i$  for any pair  $(i, j)$  on this path,  $y_{i_1} = p_{i_1}, z_{j_1} = 0, \dots, y_{i_{k-1}} = p_{i_{k-1}}, z_{j_{k-1}} = 0, y_{i_k} = p_{i_k}$  and  $z_{\hat{j}} = 0$ . Therefore,  $z_j = 0$  for all  $a_j \in A_r$ . For any  $b_{\hat{i}} \in B_r$ , since  $G_r$  is connected, there exists a path  $(a_{j^*}, b_{i_1}, a_{j_1}, \dots, a_{j_k}, b_{\hat{i}})$  connecting  $a_{j^*}$  and  $b_{\hat{i}}$ . Since  $y_i + z_j = p_i$  for any pair  $(i, j)$  on this path,  $y_{i_1} = p_{i_1}, z_{j_1} = 0, \dots, z_{j_k} = 0$ , and  $y_{\hat{i}} = p_{\hat{i}}$ . Therefore,  $y_i = p_i$  for any  $b_i \in B_r$ .

**Case 2:** Suppose  $y_{i^*} = 0$  for some  $b_{i^*} \in B_r$ . For any  $a_{\hat{j}} \in A_r$ , since  $G_r$  is connected, there exists a path  $(b_{i^*}, a_{j_1}, b_{i_1}, \dots, b_{i_k}, a_{\hat{j}})$  connecting  $a_{\hat{j}}$  and  $b_{i^*}$ . Since  $y_i + z_j = p_i$  for any pair  $(i, j)$  on this path,  $z_{j_1} = p_{i^*}, y_{i_1} = p_{i_1} - p_{i^*}, \dots, z_{j_k} = p_{i^*}, y_{i_k} = p_{i_k} - p_{i^*}$ , and  $z_{\hat{j}} = p_{i^*}$ . Therefore,  $z_j = p_{i^*}$  for each  $a_j \in A_r$ . Similarly, there exists a path  $(b_{i^*}, a_{j_1}, b_{i_1}, \dots, a_{j_k}, b_{\hat{i}})$  connecting  $b_{\hat{i}}$  and  $b_{i^*}$  for any  $b_{\hat{i}} \in B_r$ . Since  $y_i + z_j = p_i$  for each pair  $(i, j)$  on this path,  $z_{j_1} = p_{i^*}, y_{i_1} = p_{i_1} - p_{i^*}, \dots, z_{j_k} = p_{i^*}$  and  $y_{\hat{i}} = p_{\hat{i}} - p_{i^*}$ . Therefore,  $y_i = p_i - p_{i^*}$  for each  $a_j \in A_r$ . Notice  $y_i + z_j = p_i$  for all  $(a_j, b_i) \in G_r$ , and hence  $b_{i^*}$  has the lowest margin and  $p_i \geq p_{i^*}$  for all  $b_i \in B_r$ .

**Case 3:** Suppose  $z_j \neq 0$  and  $y_i \neq 0$  for all  $a_j \in A_r, b_i \in B_r$ . In this case, the chosen equations in  $G_r$  all come from subset  $C_2$ . Since there are  $|A_r| + |B_r|$  variables, the number of arcs in  $G_r$  is also  $|A_r| + |B_r|$ . Notice that a connected graph with  $n$  vertices is a tree if it has  $n - 1$  edges. It immediately follows that there must

exist an even cycle in  $G_r$  since the number of the nodes is the same as that of the edges. But the arcs in an even cycle are linearly dependent, contradicting the fact that we have chosen  $m + n$  linearly independent equations to identify a basic solution.  $\square$

*Proof of Theorem 1.* To prove Theorem 1, we first show a result that focuses on the worst-case profit, and then we generalize to the broader class of robust measures in  $\Gamma$ .

PROPOSITION 3. For any uncertainty set  $U$  that is part-wise independently symmetric around  $\boldsymbol{\mu}$ , the worst-case profit  $R^g(\mathcal{A}, U)$  is given by

$$R^g(\mathcal{A}, U) = \min_{K \in \mathcal{Q}, \boldsymbol{\epsilon} \in E} \left\{ \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \epsilon_{s_t, i} \right) \right) \right\}, \quad (\text{EC.1})$$

where  $\boldsymbol{\epsilon}_{s_t} := \mathbf{d}_{s_t} - \boldsymbol{\mu}_{s_t}$  denotes the vector of demand residuals for each product category  $S_t$ .

*Proof of Proposition 3.* We first show that for any fixed  $\mathbf{d} \in U$  and any given parameter set  $K = \{k_{tr} \in \mathbb{Z}^+ : t = 1, \dots, T, r = 0, \dots, t-1\}$ , the worst-case profit  $R^g(\mathcal{A}, U)$  is bounded from above by the following:

$$R^g(\mathcal{A}, U) \leq \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \min^i \boldsymbol{\epsilon}_{s_t} \right), \quad (\text{EC.2})$$

where  $\boldsymbol{\epsilon}_{s_t} := \mathbf{d}_{s_t} - \boldsymbol{\mu}_{s_t}$  denotes the vector of demand residuals for each product category  $S_t$ . Then we shall show that there always exist some demand vector  $\mathbf{d}^*$  and parameter set  $K^*$  such that the inequality in (EC.2) is tight, which in turn implies the representation (EC.1) of the worst-case profit.

To see (EC.2), notice that by the definition of  $\delta_{\boldsymbol{\mu}}^K(\mathcal{A})$ , there exist  $\mathbf{y}, \mathbf{z}$  that are feasible to problem (3) such that  $\sum_{i=1}^n \mu_i y_i + \sum_{j=1}^m C_j z_j = \delta_{\boldsymbol{\mu}}^K(\mathcal{A})$ , and  $\sum_{b_i \in S_t} \mathbb{1}\{y_i = \beta_{tr}\} = k_{tr}$  for all  $1 \leq t \leq T, 0 \leq r \leq t-1$ . By the assumption on  $U$ , the uncertainty set for  $\boldsymbol{\epsilon}_{s_t}$ , denoted by  $E_t := \{\boldsymbol{\epsilon}_{s_t} | \boldsymbol{\epsilon} \in E\}$ , is symmetric. Let  $\sigma^t$  be a permutation of the index set  $s_t$  such that  $y_i = \beta_{tr}$  if and only if  $\epsilon_{\sigma^t(i)} \in \{\min^i \boldsymbol{\epsilon}_{s_t} | H_{tr} - k_{tr} < i \leq H_{tr}\}$ . All such permutations  $\sigma^t$  for each  $s_t$  combined together form a permutation  $\sigma$  of  $[n]$ , and let  $\mathbf{d}_{\sigma} = \boldsymbol{\mu} + \boldsymbol{\epsilon}_{\sigma}$ . It then follows that

$$\sum_{i=1}^n d_{\sigma(i)} y_i + \sum_{j=1}^m C_j z_j = \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{i=1}^n \epsilon_{\sigma(i)} y_i = \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \min^i \boldsymbol{\epsilon}_{s_t} \right). \quad (\text{EC.3})$$

By definition, we have  $g(\mathcal{A}, \mathbf{d}_{\sigma}) \leq \sum_{i=1}^n d_{\sigma(i)} y_i + \sum_{j=1}^m C_j z_j$ . Since  $U$  is a part-wise independently symmetric perturbation uncertainty set, we have  $\boldsymbol{\epsilon}_{\sigma} \in E$ , which then implies that  $R^g(\mathcal{A}, U) \leq g(\mathcal{A}, \mathbf{d}_{\sigma}) \leq \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \min^i \boldsymbol{\epsilon}_{s_t} \right)$ .

Next, we prove there always exist some demand vector  $\mathbf{d}^*$  and parameter set  $K^*$  such that the inequality in (EC.2) is tight. Let  $\mathbf{d}^* = \arg \min_{\mathbf{d} \in U} g(\mathcal{A}, \mathbf{d})$  and define  $\boldsymbol{\tau} = \mathbf{d}^* - \boldsymbol{\mu}$ . By definition, we have

$$g(\mathcal{A}, \boldsymbol{\mu} + \boldsymbol{\tau}) = \min_{\mathbf{y}, \mathbf{z}} \sum_{i=1}^n (\mu_i + \tau_i) y_i + \sum_{j=1}^m C_j z_j \quad (\text{EC.4})$$

$$\text{s.t. } y_i + z_j \geq p_i, \quad \forall (a_j, b_i) \in \mathcal{A},$$

$$y_i \geq 0, \forall 1 \leq i \leq n, \text{ and } z_j \geq 0, \forall 1 \leq j \leq m.$$

By Lemma 1, there exists an optimal solution  $(\mathbf{y}^*, \mathbf{z}^*)$  to problem (EC.4) that satisfy  $y_i^* \in \{\beta_{tr} \mid 0 \leq r \leq t\}$  for all  $b_i \in S_t$  and  $1 \leq t \leq T$ . Let  $k_{tr}^* = \sum_{b_i \in S_t} \mathbb{1}\{y_i^* = \beta_{tr}\}$  for all  $0 \leq r \leq t-1$  and  $1 \leq t \leq T$ , and define  $K^* = \{k_{tr}^* : t = 1, \dots, T, r = 0, \dots, t-1\}$ . Let  $H_{tr}^* := \sum_{j=0}^r k_{tj}^*$  for all  $0 \leq r \leq t-1$  and  $1 \leq t \leq T$ . Then we must have  $\sum_{i=1}^n y_i^* \tau_i \geq \sum_{t=1}^T \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}^* - k_{tr}^* + 1}^{H_{tr}^*} \min^i \boldsymbol{\tau}_{s_t} \right)$  and  $\sum_{j=1}^m C_j z_j^* + \sum_{i=1}^n \mu_i y_i^* \geq \delta_{\boldsymbol{\mu}}^{K^*}(\mathcal{A})$ . It then follows that  $R^g(\mathcal{A}, U) = g(\mathcal{A}, \boldsymbol{\mu} + \boldsymbol{\tau}) \geq \delta_{\boldsymbol{\mu}}^{K^*}(\mathcal{A}) + \sum_{t=1}^T \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}^* - k_{tr}^* + 1}^{H_{tr}^*} \min^i \boldsymbol{\tau}_{s_t} \right)$ . On the other hand, it follows from (EC.2) that  $R^g(\mathcal{A}, U) = g(\mathcal{A}, \boldsymbol{\mu} + \boldsymbol{\tau}) \leq \delta_{\boldsymbol{\mu}}^{K^*}(\mathcal{A}) + \sum_{t=1}^T \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}^* - k_{tr}^* + 1}^{H_{tr}^*} \min^i \boldsymbol{\tau}_{s_t} \right)$ . Therefore, we have

$$R^g(\mathcal{A}, U) = \delta_{\boldsymbol{\mu}}^{K^*}(\mathcal{A}) + \sum_{t=1}^T \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}^* - k_{tr}^* + 1}^{H_{tr}^*} \min^i \boldsymbol{\tau}_{s_t} \right) \quad (\text{EC.5})$$

Now we are ready to complete the proof of (EC.1). By (EC.2) and (EC.5), we get

$$\begin{aligned} R^g(\mathcal{A}, U) &= \min_{K \in \mathcal{Q}} \left\{ \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \min_{\boldsymbol{\epsilon}_{s_t} \in E_t} \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr} - k_{tr} + 1}^{H_{tr}} \min^i \boldsymbol{\epsilon}_{s_t} \right) \right) \right\} \\ &= \min_{K \in \mathcal{Q}} \left\{ \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \min_{\boldsymbol{\epsilon}_{s_t} \in E_t} \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr} - k_{tr} + 1}^{H_{tr}} \boldsymbol{\epsilon}_{s_t, i} \right) \right) \right\} \\ &= \min_{K \in \mathcal{Q}, \boldsymbol{\epsilon}_{s_t} \in E_t, \forall 1 \leq t \leq T} \left\{ \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr} - k_{tr} + 1}^{H_{tr}} \boldsymbol{\epsilon}_{s_t, i} \right) \right) \right\} \\ &= \min_{K \in \mathcal{Q}, \boldsymbol{\epsilon} \in E} \left\{ \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr} - k_{tr} + 1}^{H_{tr}} \boldsymbol{\epsilon}_{s_t, i} \right) \right) \right\}, \end{aligned}$$

where the second equality holds because  $E$  is part-wise independently symmetric.  $\square$

We now complete the proof of Theorem 1. For each  $1 \leq t \leq T$ , let  $E_t^{w_t} := \{\boldsymbol{\epsilon}_{s_t} \in E_t \mid \sum \boldsymbol{\epsilon}_{s_t} = w_t\}$  for any given  $w_t \in \mathbb{R}_+$ . Note that both  $E_t^{w_t}$  and  $(E_1^{w_1} \times E_2^{w_2} \times \dots \times E_T^{w_T}) \cap E$  are part-wise independently symmetric.

For notation brevity, denote  $\prod_{t=1}^T E_t^{w_t} = E_1^{w_1} \times E_2^{w_2} \times \dots \times E_T^{w_T}$ . Now we have

$$R^f(\mathcal{A}, U) = \min_{\mathbf{d} \in U} \left\{ h \left( g(\mathcal{A}, \mathbf{d}), \sum_{i \in s_1} d_i, \dots, \sum_{i \in s_T} d_i \right) \right\} = \min_{\boldsymbol{\epsilon} \in E} \left\{ h \left( g(\mathcal{A}, \boldsymbol{\mu} + \boldsymbol{\epsilon}), \sum_{i \in s_1} (\mu_i + \epsilon_i), \dots, \sum_{i \in s_T} (\mu_i + \epsilon_i) \right) \right\}$$

$$\begin{aligned}
 &= \min_{w_t, \forall 1 \leq t \leq T} \left\{ \min_{\epsilon \in \prod_{t=1}^T E_t^{w_t} \cap E} \left\{ h \left( g(\mathcal{A}, \boldsymbol{\mu} + \boldsymbol{\epsilon}), \sum_{i \in s_1} (\mu_i + \epsilon_i), \dots, \sum_{i \in s_T} (\mu_i + \epsilon_i) \right) \right\} \right\} \\
 &= \min_{w_t, \forall 1 \leq t \leq T} \left\{ \min_{\epsilon \in \prod_{t=1}^T E_t^{w_t} \cap E} \left\{ h \left( g(\mathcal{A}, \boldsymbol{\mu} + \boldsymbol{\epsilon}), \sum_{i \in s_1} \mu_i + w_1, \dots, \sum_{i \in s_T} \mu_i + w_T \right) \right\} \right\}.
 \end{aligned}$$

Since  $h$  is strictly increasing in  $g(\mathcal{A}, \mathbf{d})$  for fixed  $w_t, \forall t = 1, \dots, T$ , we have  $R^f(\mathcal{A}, U) = \min_{w_t, \forall 1 \leq t \leq T} \left\{ h \left( \min_{\epsilon \in \prod_{t=1}^T E_t^{w_t} \cap E} g(\mathcal{A}, \boldsymbol{\mu} + \boldsymbol{\epsilon}), \sum_{i \in s_1} \mu_i + w_1, \dots, \sum_{i \in s_T} \mu_i + w_T \right) \right\}$ . By Proposition 3,

$$\begin{aligned}
 R^f(\mathcal{A}, U) &= \min_{w_t, \forall 1 \leq t \leq T} \left\{ h \left( \min_{K \in Q, \epsilon \in \prod_{t=1}^T E_t^{w_t} \cap E} \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \epsilon_{s_t, i} \right) \right), \sum_{i \in s_1} \mu_i + w_1, \dots, \sum_{i \in s_T} \mu_i + w_T \right) \right\} \\
 &= \min_{w_t, \forall 1 \leq t \leq T, K \in Q, \epsilon \in \prod_{t=1}^T E_t^{w_t} \cap E} \left\{ h \left( \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \epsilon_{s_t, i} \right) \right), \sum_{i \in s_1} \mu_i + w_1, \dots, \sum_{i \in s_T} \mu_i + w_T \right) \right\} \\
 &= \min_{K \in Q, \epsilon \in E} h \left( \delta_{\boldsymbol{\mu}}^K(\mathcal{A}) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \epsilon_{s_t, i} \right) \right), \sum_{i \in s_1} (\mu_i + \epsilon_i), \sum_{i \in s_2} (\mu_i + \epsilon_i), \dots, \sum_{i \in s_T} (\mu_i + \epsilon_i) \right),
 \end{aligned}$$

which completes the proof of Theorem 1.  $\square$

*Proof of Theorem 2.* (a). Let  $f$  be the deterministic measure function of  $R$  and  $h(\cdot)$  be the function such that  $h(x, y_1, \dots, y_T)$  is strictly increasing in  $x$  for fixed  $\{y_1, \dots, y_T\}$ . For each  $l = 1, 2$ , let  $f(\mathcal{A}_l, \mathbf{d}) = h(g(\mathcal{A}_l, \mathbf{d}), \sum \mathbf{d}_{s_1}, \dots, \sum \mathbf{d}_{s_T})$ . Then by Theorem 1, we have

$$R^f(\mathcal{A}_l, U) = \min_{K \in Q, \epsilon \in E} h \left( \delta_{\boldsymbol{\mu}}^K(\mathcal{A}_l) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \epsilon_{s_t, i} \right) \right), \sum_{i \in s_1} (\mu_i + \epsilon_i), \dots, \sum_{i \in s_T} (\mu_i + \epsilon_i) \right).$$

If  $\delta_{\boldsymbol{\mu}}^K(\mathcal{A}_1) \geq \delta_{\boldsymbol{\mu}}^K(\mathcal{A}_2)$  for all  $K \in Q$ , it then follows from the above equation that  $R(\mathcal{A}_1, U) \geq R(\mathcal{A}_2, U)$ .

(b). Let  $W^{k_1, \dots, k_T} \in Q$  denote the parameter set whose only non-negative entries are at  $(t, 0)$  with value  $k_t$  for each  $1 \leq t \leq T$ . That is,  $W_{t0}^{k_1, \dots, k_T} = k_t$  and  $W_{tr}^{k_1, \dots, k_T} = 0$  for all  $r > 0$ . Suppose there exist some  $\{k_1, \dots, k_T\}$  such that  $\delta_{\boldsymbol{\mu}}^{W^{k_1, \dots, k_T}}(\mathcal{A}_1) < \delta_{\boldsymbol{\mu}}^{W^{k_1, \dots, k_T}}(\mathcal{A}_2)$ . We next construct a part-wise independently symmetric perturbation uncertainty set  $U^*$  such that  $R(\mathcal{A}_1, U^*) < R(\mathcal{A}_2, U^*)$ , which contradicts with the assumption that  $R(\mathcal{A}_1, U) \geq R(\mathcal{A}_2, U)$  for any part-wise independently symmetric perturbation uncertainty set  $U$  and hence would complete the proof. Let  $M = \frac{\max_{K^1, K^2 \in Q} |\delta_{\boldsymbol{\mu}}^{K^1}(\mathcal{A}_2) - \delta_{\boldsymbol{\mu}}^{K^2}(\mathcal{A}_2)|}{\min_{1 \leq t \leq T, 0 \leq r < t} \beta_{tr}}$ . Let  $\boldsymbol{\epsilon}_{s_t}^*$  be a  $|S_t|$ -dimensional vector such that there are exactly  $k_t$  entries with value equal to  $-M$ , and the other  $|S_t| - k_t$  entries are equal to  $M$ . Define  $E^* := \prod_{t=1}^T (\Omega(\boldsymbol{\epsilon}_{s_t}^*))$ , where  $\Omega(\boldsymbol{\epsilon}_{s_t}^*)$  is the set of all vectors that are permutations of  $\boldsymbol{\epsilon}_{s_t}^*$  restricted to product subset  $S_t$ . Let  $U^* = \{\boldsymbol{\mu} + \boldsymbol{\epsilon} | \boldsymbol{\epsilon} \in E^*\}$ . By our choice of  $U^*$  and the definition of  $M$ , we have

$$R(\mathcal{A}_2, U^*) = \min_{K \in Q, \epsilon \in E^*} h \left( \delta_{\boldsymbol{\mu}}^K(\mathcal{A}_2) + \sum_{t=1}^T \left( \sum_{r=0}^{t-1} \beta_{tr} \left( \sum_{i=H_{tr}-k_{tr}+1}^{H_{tr}} \epsilon_{s_t, i} \right) \right), \sum_{i \in s_1} (\mu_i + \epsilon_i), \dots, \sum_{i \in s_T} (\mu_i + \epsilon_i) \right)$$

$$\begin{aligned}
 &= h \left( \delta_{\mu}^{W^{k_1, \dots, k_T}}(\mathcal{A}_2) - \sum_{t=1}^T P_t k_t M, \sum_{i \in s_1} \mu_i + (|S_1| - 2k_1)M, \dots, \sum_{i \in s_T} \mu_i + (|S_T| - 2k_T)M \right) \\
 &> h \left( \delta_{\mu}^{W^{k_1, \dots, k_T}}(\mathcal{A}_1) - \sum_{t=1}^T P_t k_t M, \sum_{i \in s_1} \mu_i + (|S_1| - 2k_1)M, \dots, \sum_{i \in s_T} \mu_i + (|S_T| - 2k_T)M \right) \\
 &\geq R(\mathcal{A}_1, U^*),
 \end{aligned}$$

which completes the proof of part (b).  $\square$

*Proof of Lemma 2.* We analyze problem (5) for the alternate long chain design in three cases. Recall that  $k_{10} = \{b_i \in S_L \mid y_i = P_L\}$ ,  $k_{20} = \{b_i \in S_H \mid y_i = P_H\}$ , and  $k_{21} = \{b_i \in S_H \mid y_i = P_H - P_L\}$ .

**Case 1:**  $k_{20} = 0$ . In this case, the  $y_i$  value associated with each high profit product  $b_i \in S_H$  is equal to either 0 or  $P_H - P_L$ . Consider an optimal solution  $z^*$  to problem (5). For each  $b_i \in S_H$  with  $y_i = 0$ , we have  $z_j^* = P_H$  for its two neighboring plant nodes  $j \in N(b_i, \mathcal{A})$ . By the specific structure of the alternate long chain that each plant is connected to exactly one high profit product and one low profit product, we have  $|\{a_j \in A \mid z_j^* = P_H\}| = 2k_{22}$ . For the rest of high profit products with  $y_i = P_H - P_L$ , the smallest value of  $z_j$  for their neighboring plant nodes that guarantees feasibility of problem (5) is  $z_j^* = P_L$ . Therefore, the optimal objective value of (5) is equal to  $2k_{22}P_H + 2k_{21}P_L$ .

**Case 2:**  $k_{20} = n/2$ . In this case, all the  $n/2$  high profit products have their  $y_i$  value equal to  $P_H$ . Consider an optimal solution  $z^*$  to problem (5). The feasibility requirement for a strictly positive  $z_j^*$  value solely comes from the low profit products whose associated  $y_i$  is equal to zero, in which case its neighboring plant nodes have  $z_j^* = P_L$ . In the alternate long chain design,  $N(b_i, \mathcal{A}) \cap N(b_j, \mathcal{A}) = \emptyset$  for any  $b_i, b_j \in S_L$  such that  $i \neq j$ . Therefore, the optimal objective value of (5) is equal to  $2k_{11}P_L$ .

**Case 3:**  $k_{20} = 1, \dots, n/2 - 1$ . In this case, we can spell out the expression in (6) as follows:

$$\delta^K(\mathcal{A}) = \begin{cases} 2k_{22}P_H + 2k_{21}P_L, & \text{if } k_{10} \geq k_{20} + 1 \\ 2k_{22}P_H + (2k_{21} + 1)P_L, & \text{if } k_{10} = k_{20}, \\ 2k_{22}P_H + (2k_{11} - 2k_{22})P_L, & \text{if } k_{10} \leq k_{20} - 1, \end{cases} \quad (\text{EC.6})$$

We next provide some high-level idea about the analysis of problem (5), and then break down the analysis into three subcases and show that the optimal objective value of (5) is the same as (EC.6).

Notice that with parameter set  $K$  given, minimizing objective function (5) reduces to finding an assignment of the  $y_i$  values to product nodes such that the feasibility requirement on  $z_j$  (specified by  $y_i + z_j \geq p_i$  for each arc  $(a_j, b_i) \in \mathcal{A}$ ) can be achieved by the smallest value possible. It is easy to see that there exists an

optimal solution to (5) such that  $z_j^* \in \{0, P_L, P_H\}$  for any  $1 \leq j \leq n$ . Second, for each one of the  $k_{22}$  high profit products with  $y_i = 0$ , all their distinct  $2k_{22}$  neighboring plant nodes must have  $z_j^* = P_H$ . Finally, for each plant  $j$ , the only case that  $z_j^*$  can take value zero happens when  $y_i = p_i$  for each  $b_i \in N(a_j, \mathcal{AL})$ . Therefore, in order to minimize  $\sum_j z_j$ , it is optimal to assign the  $k_{10}$  low profit products with  $y_i = P_L$  and the  $k_{20}$  high profit products with  $y_i = P_H$  in a consecutive manner to the greatest extent possible so as to maximize the number of plant nodes with  $z_j^* = 0$ . We consider three subcases as follows.

- (a) If  $k_{10} \geq k_{20} + 1$ , then the maximum number of consecutive products with  $y_i = p_i$  is  $2k_{20} + 1$ , and the maximum number of plant nodes with  $z_j^* = 0$  is  $2k_{20}$ . We next show there does exist a feasible solution such that the number of plant nodes with  $z_j^* = 0$  is  $2k_{20}$ , which implies that the number of plant nodes with  $z_j^* = P_L$  is  $n - 2k_{22} - 2k_{20} = 2k_{21}$  and therefore  $\delta^K(\mathcal{AL}) = 2k_{22}P_H + 2k_{21}P_L$ . Consider an alternate long chain  $\mathcal{AL}$  with  $p_1 = p_3 = \dots = p_{n-1} = P_H$  and  $p_2 = p_4 = \dots = p_n = P_L$ . Let  $y_i = 0$  for  $i \in \{1, 3, \dots, 2k_{22} - 1\} \cup \{2, 4, \dots, 2k_{11}\}$ ; let  $y_i = P_L$  for  $i \in \{2k_{11} + 2, \dots, n\}$ ; let  $y_i = P_H - P_L$  for  $i \in \{2k_{22} + 1, \dots, n - 2k_{20} - 1\}$ ; and let  $y_i = P_H$  for  $i \in \{n - 2k_{20} + 1, \dots, n\}$ . Then we have  $z_j = P_H$  for all  $j \in \{1, \dots, 2k_{22}\}$ ,  $z_j = P_L$  for all  $j \in \{2k_{22} + 1, \dots, 2k_{22} + 2k_{21}\}$  and  $z_j = 0$  for all  $j \in \{2k_{22} + 2k_{21} + 1, \dots, n\}$ , and the number of plant nodes with  $z_j = 0$  is  $n - 2k_{22} - 2k_{21} = 2k_{20}$ .
- (b) If  $k_{10} = k_{20}$ , then the maximum number of consecutive products with  $y_i = p_i$  is  $2k_{10}$ , and hence the maximum number of plant nodes with  $z_j^* = 0$  is  $2k_{20} - 1$ . Similar to the analysis in (a), we next show there does exist a feasible solution such that the number of plant nodes with  $z_j^* = 0$  is  $2k_{20} - 1$ , which then implies that the number of plant nodes with  $z_j^* = P_L$  is  $n - 2k_{22} - 2k_{20} + 1 = 2k_{21} + 1$  and hence  $\delta^K(\mathcal{AL}) = 2k_{22}P_H + (2k_{21} + 1)P_L$ . Consider the same design  $\mathcal{AL}$  as in (a). Let  $y_i = 0$  for  $i \in \{1, 3, \dots, 2k_{22} - 1\} \cup \{2, 4, \dots, 2k_{11}\}$ ; let  $y_i = P_L$  for  $i \in \{n - 2k_{10} + 2, \dots, n\}$ ; let  $y_i = P_H - P_L$  for  $i \in \{2k_{22} + 1, \dots, n - 2k_{20} - 1\}$ ; and let  $y_i = P_H$  for  $i \in \{n - 2k_{20} + 1, \dots, n\}$ . Then we have  $z_j = P_H$  for all  $j \in \{1, \dots, 2k_{22}\}$ ,  $z_j = P_L$  for all  $j \in \{2k_{22} + 1, \dots, 2k_{22} + 2k_{21} + 1\}$  and  $z_j = 0$  for all  $j \in \{2k_{22} + 2k_{21} + 2, \dots, n\}$ , in which case the number of plant nodes with  $z_j = 0$  is  $n - 2k_{22} - 2k_{21} - 1 = 2k_{20} - 1$ .
- (c) If  $k_{10} \leq k_{20} - 1$ , then the maximum number of consecutive products with  $y_i = p_i$  is  $2k_{10} + 1$ . Consider the same design  $\mathcal{AL}$  as in (a) and (b). Following a similar analysis, it is straightforward to show that there exist a feasible solution with  $z_j = P_H$  for all  $j \in \{1, \dots, 2k_{22}\}$ ,  $z_j = P_L$  for all  $j \in \{2k_{22} + 1, \dots, 2k_{11}\}$  and  $z_j = 0$  for all  $j \in \{2k_{11} + 2, \dots, n\}$ , which has the maximum possible number  $2k_{10}$  of plants with

$z_j^* = 0$ . It then follows that the number of plants with  $z_j^* = P_L$  is  $n - 2k_{22} - 2k_{10} = 2k_{11} - 2k_{22}$ , which implies  $\delta^K(\mathcal{AL}) = 2k_{22}P_H + (2k_{11} - 2k_{22})P_L$ .  $\square$

*Proof of Proposition 1.* Consider a long chain with  $n/2$  low profit products and  $n/2$  high profit products, where  $n$  is an even integer. We call a consecutive sequence of  $k$  product nodes a *sequence of length  $k$* . We first provide some structural properties of sequences with an odd (cf. Lemma 3(a)) and even (cf. Lemma 3(b)) number of product nodes. We refer the readers to Wang et al. (2020) Appendix A for the proof of Lemma 3.

LEMMA 3. (a) For any  $1 \leq k \leq n/2 - 1$ , there exists a sequence of length  $2k + 1$  with  $k$  high profit products and  $k + 1$  low profit products. (b) For any  $1 \leq k \leq n/2$ , there exists a sequence of length  $2k$  with  $k$  high profit products and  $k$  low profit products.

We next prove Proposition 1, which together with Theorem 2(a) implies Theorem 3. Consider a long chain  $\mathcal{L}_{HL}$  with  $n/2$  high profit products and  $n/2$  low profit products. Consider a fixed  $K = \{k_{10}, k_{20}, k_{21}\}$ . Recall that  $k_{11} = n/2 - k_{10}$  is equal to  $|\{b_i \in S_L \mid y_i = 0\}|$ , and  $k_{22} = n/2 - k_{20} - k_{21}$  is equal to  $|\{b_i \in S_H \mid y_i = 0\}|$ .

**Case 1:**  $k_{20} = 0$ . In this case, since every product is connected to exactly 2 plants, for any optimal solution  $\mathbf{z}^*$  to (5) of design  $\mathcal{L}_{HL}$ , the number of  $z_j^*$  such that  $z_j^* = P_H$  is at most  $2k_{22}$ . The value of the other  $z_j^*$ 's except the above (at most)  $2k_{22}$  nodes is either  $P_L$  or 0 since these plants are not connected to high profit products with  $y_i = 0$ . By Lemma 2, it then follows that  $\delta^K(\mathcal{L}_{HL}) \leq 2k_{22}P_H + (n - 2k_{22})P_L = 2k_{22}P_H + 2k_{21}P_L = \delta^K(\mathcal{AL})$ .

**Case 2:**  $k_{20} = n/2$ . In this case, the  $y_i$  values of all high profit products are equal to  $P_H$  and therefore the feasibility requirement for a strictly positive  $z_j$  value solely comes from the low profit products whose associated  $y_i$  value is zero, in which case  $z_j^* = P_L$ . Since every product node is connected to exactly 2 plant nodes and  $|\{b_i \in S_L \mid y_i = 0\}| = k_{11}$ , we have  $\delta^K(\mathcal{L}_{HL}) \leq 2k_{11}P_L = \delta^K(\mathcal{AL})$ .

**Case 3:**  $k_{20} = 1, \dots, n/2 - 1$ . In this case, the analysis depends on the relationship between  $k_{10}$  and  $k_{20}$ .

- (a) If  $k_{10} \geq k_{20} + 1$ : Since  $k_{20} \leq n/2 - 1$ , by Lemma 3(a), there exists a sequence  $\mathcal{S}_a$  of length  $2k_{20} + 1$  with  $k_{20}$  high profit products and  $k_{20} + 1$  low profit products. Let  $y_i = P_H$  if product  $i$  belongs to  $\mathcal{S}_a$  and  $p_i = P_H$ . Similarly, let  $y_i = P_L$  if product  $i$  belongs to  $\mathcal{S}_a$  and  $p_i = P_L$ . Therefore, for an optimal solution  $\mathbf{z}^*$  to (5) of design  $\mathcal{L}_{HL}$ , the  $2k_{20}$  plants that are solely connected to the above  $2k_{20} + 1$  products in  $\mathcal{S}_a$  all have  $z_j^* = 0$ . This implies that among all the  $n$  plants, the number of plants whose  $z_j^*$  value is equal to zero is at least  $2k_{20}$ . Therefore, the number of plants whose  $z_j^*$  value is equal to either  $P_L$  or  $P_H$  is at most  $n - 2k_{20}$ . Since each product has exactly two neighboring plants, the number of plants with  $z_j^* = P_H$  is at most  $2k_{22}$ . Therefore,  $\delta^K(\mathcal{L}_{HL}) \leq 2k_{22}P_H + (n - 2k_{20} - 2k_{22})P_L = 2k_{22}P_H + 2k_{21}P_L = \delta^K(\mathcal{AL})$ .

- (b) If  $k_{10} = k_{20}$ : Since  $k_{20} \leq n/2$ , by Lemma 3(b), there exists a sequence  $\mathcal{S}_b$  of length  $2k_{20}$  with  $k_{20}$  high profit products and  $k_{20}$  low profit products. Let  $y_i = P_H$  if product  $i$  belongs to  $\mathcal{S}_b$  and  $p_i = P_H$ . Let  $y_i = P_L$  if product  $i$  belongs to  $\mathcal{S}_b$  and  $p_i = P_L$ . Therefore, for an optimal solution  $\mathbf{z}^*$  to (5) of design  $\mathcal{L}_{HL}$ , the  $2k_{20} - 1$  plants that are solely connected to the above  $2k_{20}$  products in  $\mathcal{S}_b$  all have  $z_j^* = 0$ . This implies that among all the  $n$  plants, the number of plants whose  $z_j^*$  value is equal to zero is at least  $2k_{20} - 1$ . Therefore, the number of plants whose  $z_j^*$  value is equal to either  $P_L$  or  $P_H$  is at most  $n - 2k_{20} + 1$ . Since each product has exactly two neighboring plants, the number of plants with  $z_j^* = P_H$  is at most  $2k_{22}$ . It then follows that  $\delta^K(\mathcal{L}_{HL}) \leq 2k_{22}P_H + (n - 2k_{20} - 2k_{22} + 1)P_L = 2k_{22}P_H + (2k_{21} + 1)P_L = \delta^K(\mathcal{A}\mathcal{L})$ .
- (c) If  $k_{10} \leq k_{20} - 1$ : In this case, we have  $k_{10} \leq n/2 - 1$ . By Lemma 3(a), there exists a sequence  $\mathcal{S}_c$  of length  $2k_{10} + 1$  with  $k_{10}$  high profit products and  $k_{10} + 1$  low profit products. Let  $y_i = P_H$  if product  $i$  belongs to  $\mathcal{S}_c$  and  $p_i = P_H$ . Let  $y_i = P_L$  if product  $i$  belongs to  $\mathcal{S}_c$  and  $p_i = P_L$ . Therefore, for an optimal solution  $\mathbf{z}^*$  to (5) of design  $\mathcal{L}_{HL}$ , the  $2k_{10}$  plants that are solely connected to the above  $2k_{10} + 1$  products in  $\mathcal{S}_c$  all have  $z_j^* = 0$ . This implies that among all the  $n$  plants, the number of plants whose  $z_j^*$  value is equal to zero is at least  $2k_{10}$ . Therefore, the number of plants whose  $z_j^*$  value is equal to either  $P_L$  or  $P_H$  is at most  $n - 2k_{10}$ . Since each product has exactly two neighboring plants, the number of plants with  $z_j^* = P_H$  is at most  $2k_{22}$ . Therefore,  $\delta^K(\mathcal{L}_{HL}) \leq 2k_{22}P_H + (n - 2k_{10} - 2k_{22})P_L = 2k_{22}P_H + (2k_{11} - 2k_{22})P_L = \delta^K(\mathcal{A}\mathcal{L})$ .

Combining all the above three cases together with Theorem 2(a) completes the proof.  $\square$

*Proof of Proposition 2.* We show there exist parameter sets  $K = \{k_{10}, k_{20}, k_{21}\}$  and  $K' = \{k'_{10}, k'_{20}, k'_{21}\}$  with  $k_{21} = k'_{21} = 0$  such that  $\delta^K(\mathcal{A}\mathcal{L}) > \delta^K(\mathcal{D}\mathcal{L})$  and  $\delta^{K'}(\mathcal{A}\mathcal{L}) < \delta^{K'}(\mathcal{D}\mathcal{L})$ , which implies Proposition 2 by Theorem 2(b). The disjoint long chain  $\mathcal{D}\mathcal{L}$  consists of two disconnected components, denoted by  $\mathcal{D}\mathcal{L}_H$  and  $\mathcal{D}\mathcal{L}_L$ , where each component is a long chain with size  $n/2$  and the products in component  $\mathcal{D}\mathcal{L}_H$  (resp.  $\mathcal{D}\mathcal{L}_L$ ) have identical margins  $P_H$  (resp.  $P_L$ ). By the definition of DMGI, we have  $\delta^K(\mathcal{D}\mathcal{L}) = \delta^K(\mathcal{D}\mathcal{L}_H) + \delta^K(\mathcal{D}\mathcal{L}_L)$ , where  $\delta^K(\mathcal{D}\mathcal{L}_H)$  and  $\delta^K(\mathcal{D}\mathcal{L}_L)$  are the optimal objective values to problem (5) for designs  $\mathcal{D}\mathcal{L}_H$  and  $\mathcal{D}\mathcal{L}_L$ .

Consider the following two parameter sets  $K$  and  $K'$ :

- $K = \{k_{10}, k_{20}, k_{21}\}$  with  $k_{10} = \frac{n}{2}, k_{20} = k_{21} = 0$ . In this case, we have  $k_{22} = \frac{n}{2}$  and it follows from Lemma 2 that  $\delta^K(\mathcal{A}\mathcal{L}) = nP_H$ . For the disjoint long chain, it is easy to see that  $\delta^K(\mathcal{D}\mathcal{L}_H) = \frac{n}{2}P_H$  and  $\delta^K(\mathcal{D}\mathcal{L}_L) = 0$ , and hence  $\delta^K(\mathcal{D}\mathcal{L}) = \frac{n}{2}P_H$ . Therefore, we have  $\delta^K(\mathcal{A}\mathcal{L}) > \delta^K(\mathcal{D}\mathcal{L})$ .
- $K' = \{k'_{10}, k'_{20}, k'_{21}\}$  with  $k'_{10} = \frac{n}{2} - 1, k'_{20} = \frac{n}{2} - 1, k'_{21} = 0$ . In this case, we have  $k'_{22} = 1$  and by Lemma 2, we have  $\delta^{K'}(\mathcal{A}\mathcal{L}) = 2P_H + (n - 2 - \min(\frac{n}{2} - 1, \frac{n}{2}) - \min(\frac{n}{2} - 1, \frac{n}{2} - 2))P_L = 2P_H + P_L$ . Now we consider

the disjoint long chain. For component  $\mathcal{DL}_L$ , we have  $k'_{11} = \frac{n}{2} - k'_{10} = 1$  and hence  $\delta^{K'}(\mathcal{DL}_L) = (k'_{11} + 1)P_L = 2P_L$ . For component  $\mathcal{DL}_H$ , we have  $k'_{22} = \frac{n}{2} - k'_{20} - k'_{21} = 1$  and  $\delta^{K'}(\mathcal{DL}_H) = (k'_{22} + 1)P_H = 2P_H$ . It then follows that  $\delta^{K'}(\mathcal{DL}) = 2P_H + 2P_L$ , and therefore we have  $\delta^{K'}(\mathcal{AL}) < \delta^{K'}(\mathcal{DL})$ .  $\square$