Appendices to "Procurement Strategies with Unreliable Suppliers under Correlated Random Yields"

The appendices consist of three parts. Appendix A presents additional results that will be used to facilitate the analysis. Appendix B provides the proofs for all the results in the main paper. Appendix C discusses a few results for the ex ante pricing model to confirm the robustness of our results.

Appendix A: Additional Results

The following lemma will be used to prove Lemma 2 and Proposition 2, and thus is presented and proved here as an additional result.

LEMMA A.1. *Consider a random variable $\boldsymbol{\xi} \in [0,1]^2$ with general distribution. Suppose $h(x) = \frac{\mathbf{E}(\xi_2|\xi_1=x)}{x}$ and $\varphi(x)$ is a measurable function. Assume*

$$\eta = \mathbf{E}\xi_1 \mathbf{E}[\varphi(\xi_1)h(\xi_1)] - \mathbf{E}\xi_2 \mathbf{E}\varphi(\xi_1),$$

and all the expectations exist. Let $\bar{\varphi}(x) = \varphi(x)/x$. Then, the following statements hold:

(i) If $\bar{\varphi}$ is decreasing, then $\eta \geq 0$ when h is strictly decreasing, whereas $\eta \leq 0$ when h is strictly increasing.

(ii) If $\bar{\varphi}$ is increasing, then $\eta \geq 0$ when h is strictly increasing, whereas $\eta \leq 0$ when h is strictly decreasing.

(iii) The inequalities are strict if $\bar{\varphi}(\xi_1)$ is not a constant.

PROOF. Let X and Y be two independent copies of ξ_1, then we have:

$$\mathbf{E}\varphi(X)h(X)Y + \mathbf{E}\varphi(Y)h(Y)X = \mathbf{E}Y\mathbf{E}\varphi(X)h(X) + \mathbf{E}X\mathbf{E}\varphi(Y)h(Y) = 2\mathbf{E}\xi_1\mathbf{E}[\varphi(\xi_1)h(\xi_1)],$$

$$\mathbf{E}\varphi(X)Yh(Y) + \mathbf{E}\varphi(Y)Xh(X) = \mathbf{E}\varphi(X)EYh(Y) + \mathbf{E}\varphi(Y)EXh(X) = 2\mathbf{E}\varphi(\xi_1)\mathbf{E}\xi_1 h(\xi_1) = 2\mathbf{E}\varphi(\xi_1)\mathbf{E}\xi_2.$$

Consequently, $\eta \leq 0$ is equivalent to

$$\mathbf{E}\varphi(X)h(X)Y + \mathbf{E}\varphi(Y)h(Y)X \leq \mathbf{E}\varphi(X)Yh(Y) + \mathbf{E}\varphi(Y)Xh(X)$$

$$\Longleftrightarrow \mathbf{E}XY(h(Y) - h(X))(\bar{\varphi}(X) - \bar{\varphi}(Y)) \geq 0.$$

Hence, part (i) and (ii) easily follow. The condition in (iii) means that the last expectation is strictly positive (or negative) even if $\bar{\varphi}(x)$ is not strictly monotonic, which is the desired result. Q.E.D.

Appendix B: Proofs of Statements

PROOF OF LEMMA 1. The linear part $\mathbf{q}^T\mathbf{c}$ automatically satisfies the properties, and thus can be put aside. The joint concavity is straightforward by noting that concavity is preserved by expectation and maximization. Hence, in the following, we prove the differentiability. Let $Q = \mathbf{q}^T\boldsymbol{\xi}$. In scenario $s = f$, we focus on

$$u(\mathbf{q}) := \mathbf{E}\min\{d_0, Q\} = \mathbf{E}\left[Q\mathbf{1}_{\{Q\leq d_0\}} + d_0\mathbf{1}_{\{Q>d_0\}}\right].$$

We have two parts because $Q \leq q := \sum_i q_i$ and $q > d_0$ since the firm will definitely inflate the order when the demand is fixed. Consider a small increase Δq_k in the order quantity:

$$\begin{aligned} u(\mathbf{q}+\Delta q_k) - u(\mathbf{q}) &= \mathbf{E}\left[(Q+\Delta q_k\xi_k)\mathbf{1}_{\{Q+\Delta q_k\xi_k\leq d_0\}} - Q\mathbf{1}_{\{Q\leq d_0\}}\right] \\ &\quad + \mathbf{E}\left[d_0\mathbf{1}_{\{Q+\Delta q_k\xi_k>d_0\}} - d_0\mathbf{1}_{\{Q>d_0\}}\right] \\ &= \mathbf{E}(d_0 - Q)\mathbf{1}_{\{d_0-\Delta q_k\xi_k<Q\leq d_0\}} + \Delta q_k\mathbf{E}\xi_k\mathbf{1}_{\{Q+\Delta q_k\xi_k\leq d_0\}}. \end{aligned}$$

Note that

$$0 < \mathbf{E}(d_0 - Q)\mathbf{1}_{\{d_0-\Delta q_k\xi_k<Q\le d_0\}} = \mathbf{E}(d_0 - Q)\mathbf{1}_{\{d_0-\Delta q_k\xi_k<Q<d_0\}} < \Delta q_k \mathbf{Prob}(d_0 - \Delta q_k\xi_k < Q < d_0).$$

The probability in the last expression approaches to zero as $\Delta q_k \to 0$, no matter the distribution is continuous or discrete. This is because there cannot exist infinitely many point masses in the open interval with a total measure larger than a small $\epsilon > 0$. Therefore,

$$u_k(\mathbf{q}) := \frac{\partial u(\mathbf{q})}{\partial q_k} = \mathbf{E}\xi_k\mathbf{1}_{\{Q\le d_0\}},$$

which shows that $\Pi_f(\mathbf{q})$ is differentiable. Let us further check the differential with respect to q_j:

$$u_k(\mathbf{q}+\Delta q_j) - u_k(\mathbf{q}) = \mathbf{E}\xi_k\mathbf{1}_{\{d_0-\Delta q_j\xi_j<Q\le d_0\}}.$$

Note that, if $\boldsymbol{\xi}$ follows a discrete distribution, then the above may not be zero as $\Delta q_j \to 0$, rendering discontinuity of u_k. However, if it follows a continuous distribution, then so does Q. In this case, let $h_{kj}(x_k, x_j, y)$ be the density function over $[0,1]^2 \times [0, q]$ that characterizes the joint distribution of (ξ_k, ξ_j, Q). Then,

$$\frac{\partial^2 u(\mathbf{q})}{\partial q_k \partial q_j} = \lim_{\Delta q_j \to 0} \frac{1}{\Delta q_j} \int_0^1 \int_0^1 \int_{d_0-\Delta q_j x_j}^{d_0} x_k h_{kj}(x_k, x_j, y) dy dx_j dx_k = \int_0^1 \int_0^1 x_k x_j h_{kj}(x_k, x_j, d_0) dx_k dx_j.$$

In scenario $s = r$, we first solve the pricing problem for $\pi_r(Q)$. Consider the revenue function $r(d) = p(d)d$. We have assumed that it is smooth and concave in d. Let d^* and p^* be the demand and price, respectively, that result in the maximum revenue, r^*. Then, it is direct to verify that the solution to the pricing problem is $\bar{p} = p(Q)$ if $p(Q) > p^*$, and $\bar{p} = p^*$ otherwise; hence,

$$\pi_r(Q) = r(Q)\mathbf{1}_{\{Q<d^*\}} + r^*\mathbf{1}_{\{Q\ge d^*\}}.$$

If $Q \le q < d^*$, then $\mathbf{E}\pi_r(Q) = \mathbf{E}r(Q)$. Since $r(Q)$ is smooth and Q is linear in $\mathbf{q}$, the results follows.

Now, assume that $q \ge d^*$. Apply the same logic as before, and note that, for a small increase in order quantity Δq_k, we have $r(Q + \Delta q_k\xi_k) = r(Q) + \Delta q_k\xi_k r'(Q_0)$ for some $Q_0 \in [Q, Q + \Delta q_k\xi_k]$. Hence,

$$\mathbf{E}\pi_r(\mathbf{q}+\Delta q_k) - \mathbf{E}\pi_r(\mathbf{q}) \;=\; \mathbf{E}\left[(r^* - r(Q))\mathbf{1}_{\{d^*-\Delta q_k\xi_k\le Q\le d^*\}} + \Delta q_k r'(Q_0)\xi_k\mathbf{1}_{\{Q+\Delta q_k\xi_k\le d^*\}}\right].$$

Note that when $d^* - \Delta q_k\xi_k \le Q \le d^*$, $r(d^*) - r(Q) = r'(\hat{Q})\Delta q_k\xi_k$ for some $\hat{Q} \to d^*$ as $\Delta q_k \to 0$. Moreover, $r'(d^*) = 0$ by definition. Hence, we obtain the first order derivative

$$\frac{\partial \mathbf{E}\pi_r(\mathbf{q})}{\partial q_k} = \mathbf{E}\left[r'(Q)\xi_k\mathbf{1}_{\{Q\le d^*\}}\right].$$

Let us continue to check the second derivative by looking at a small increase Δq_j:

$$\begin{aligned}\mathbf{E}r'(Q+\Delta q_j\xi_j)\xi_k\mathbf{1}_{\{Q+\Delta q_j\xi_j\le d^*\}} \;&-\; \mathbf{E}r'(Q)\xi_k\mathbf{1}_{\{Q\le d^*\}} \\ &=\; \mathbf{E}\left[-r'(Q)\xi_k\mathbf{1}_{\{d^*-\Delta q_j\xi_j<Q\le d^*\}} + \Delta q_j\xi_j\xi_k r''(Q_0)\mathbf{1}_{\{Q+\Delta q_j\xi_j\le d^*\}}\right].\end{aligned}$$

Now, since $r'(d^*) - r'(Q) = r''(\hat{Q})\Delta q_j\xi_j$ and $r'(d^*) = 0$, we squeeze the first term and

$$0 < |-\mathbf{E}\left[r'(Q)\xi_k\mathbf{1}_{\{d^*-\Delta q_j\xi_j<Q\le d^*\}}\right]| < \Delta q_j\mathbf{E}\left[|r''(\hat{Q})|\mathbf{1}_{\{d^*-\Delta q_j\xi_j<Q\le d^*\}}\right] \to \Delta q_j|r''(d^*)|\mathbf{Prob}(Q = d^*).$$

If the yield distribution is continuous, then $\mathbf{Prob}(Q=d^*)=0$, and therefore

$$\frac{\partial^2 \Pi_r(\mathbf{q})}{\partial q_k \partial q_j} = \mathbf{E}[r''(Q)\xi_k\xi_j \mathbf{1}_{\{Q\le d^*\}}].$$

As $r(d)$ is smooth, the desired conclusions follow. Q.E.D.

Proof of Lemma 2. Let $h(\xi_1)=\frac{\mathbf{E}(\xi_2|\xi_1)}{\xi_1}$. By definition, we have $\mathbf{E}\xi_1\xi_2=\mathbf{E}\xi_1\mathbf{E}(\xi_2|\xi_1)=\mathbf{E}\xi_1^2 h(\xi_1)$. Then, $\rho<\rho^*=\frac{\sigma_1\mu_2}{\sigma_2\mu_1}\iff \mathbf{Cov}(\xi_1,\xi_2)<\frac{\sigma_1^2\mu_2}{\mu_1}\iff \mathbf{E}\xi_1\mathbf{E}\xi_1\xi_2<\mathbf{E}\xi_2\mathbf{E}\xi_1^2\iff \mathbf{E}\xi_1\mathbf{E}\xi_1^2 h(\xi_1)<\mathbf{E}\xi_2\mathbf{E}\xi_1^2$. Consider $\varphi(x)=x^2$; then the results in (i) and (ii) follow from Lemma A.1(ii) and (iii).

To prove part (iii), we now prove a more general result. We claim that (iii) holds if $\mathbf{E}(\xi_i|\xi_{3-i})=s\xi_{3-i}+m$, $i=1,2$, for some s and m, i.e., the conditional expectation can be written in a linear form. If this is true, then we must necessarily have that $\mathbf{E}(\xi_i|\xi_{3-i})=\mathbf{E}\xi_i+\frac{\mathbf{Cov}(\xi_i,\xi_{3-i})}{\mathbf{Var}(\xi_{3-i})}(\xi_{3-i}-\mathbf{E}\xi_{3-i})$, $i=1,2$. The expression is derived by solving s and m from two equations: $\mathbf{E}(\mathbf{E}(\xi_i|\xi_{3-i}))=\mathbf{E}\xi_i$ and $\mathbf{E}(\xi_{3-i}\mathbf{E}(\xi_i|\xi_{3-i}))=\mathbf{E}\xi_i\xi_{3-i}$. Therefore, the monotonicity of $\frac{\mathbf{E}(\xi_i|\xi_{3-i})}{\xi_{3-i}}$ is completely decided by $m=\mu_i-\frac{\rho\sigma_i\mu_{3-i}}{\sigma_{3-i}}$; note that $m=\mu_1(1-\rho\rho^*)$ when $i=1$, and $m=\mu_2(1-\frac{\rho}{\rho^*})$ when $i=2$. Hence, part (iii) readily follows. To finish the proof, notice that a bivariate normal distribution is just one special example that has linear form conditional expectation. Q.E.D.

Proof of Proposition 1. First, Lemma 1 shows that at optimality we must have

$$p_0\mathbf{E}\boldsymbol{\xi}\mathbf{1}_{\{\mathbf{q}^T\boldsymbol{\xi}\le d_0\}}\le \mathbf{c} \quad \text{for } s=f; \text{ and}$$

$$\mathbf{E}\left[r'(\mathbf{q}^T\boldsymbol{\xi})\boldsymbol{\xi}\mathbf{1}_{\{\mathbf{q}^T\boldsymbol{\xi}\le d^*\}}\right]\le \mathbf{c} \quad \text{for } s=r.$$

Recall that $r(Q)=(a-Q)Q/b$ and $d^*=a/2$. In the two-supplier case, we write the condition generically (suppressing s) as: there exist functions γ_i such that $c_i\ge\gamma_i(\mathbf{q})$ for $i=1,2$ at optimality; moreover, sole sourcing occurs only when one of the inequalities is strict, and the supplier with strict inequality is not selected. Besides, it is straightforward to check that $\gamma_i(\mathbf{q})$ is strictly decreasing in every component of $\mathbf{q}$.

Now, we can define the two threshold functions. The idea is to find the threshold costs at which the firm just turns from dual sourcing to sole sourcing. Let $\mathbf{q}^1=(q_1,0)^T$ and $\mathbf{q}^2=(0,q_2)^T$ for some $q_1>0$ and $q_2>0$; then define the thresholds as

$$C^j=(\gamma_2\circ\gamma_1^{-1})|_{\mathbf{q}=\mathbf{q}^j}, \quad j=1,2.$$

Basically, for every c_1, we first solve for q_j ($j=1$ or 2, depending on the threshold being computed) through γ_1 and then find c_2 using γ_2. For each threshold, the operator on the RHS depends on the order quantity; so it is important to note that threshold j is obtained by letting $\mathbf{q}=\mathbf{q}^j$.

We proceed to prove the properties of $C^j(c_1)$. First, since $\gamma_i(\mathbf{q}^j)$ is strictly decreasing in q_j $(i,j=1,2)$, it is easy to see that $\gamma_2\circ\gamma_1^{-1}$ is increasing; that is, $C^j(c_1)$ is increasing in c_1 for $j=1,2$.

Second, suppose $C^1(\hat{c}_1)<C^2(\hat{c}_1)$ for some $\hat{c}_1$, and set $\hat{c}_2=C^2(\hat{c}_1)$. Let us look at the order quantities under this particular cost pair $(\hat{c}_1,\hat{c}_2)^T$. By the definition of $C^2(\hat{c}_1)$, there exists a $\hat{\mathbf{q}}^2$ such that $\hat{c}_1=\gamma_1(\hat{\mathbf{q}}^2)$ and $\hat{c}_2=C^2(\hat{c}_1)=\gamma_2(\hat{\mathbf{q}}^2)$. So, by KKT condition, $(0,\hat{q}_2)^T$ is the optimal order quantity, where $\hat{q}_2>0$. On the other hand, by definition again, there exists a $\hat{\mathbf{q}}^1$ such that $\hat{c}_1=\gamma_1(\hat{\mathbf{q}}^1)$ and $C^1(\hat{c}_1)=\gamma_2(\hat{\mathbf{q}}^1)$; hence, $C^1(\hat{c}_1)<\hat{c}_2$ means that $\hat{c}_2>\gamma_2(\hat{\mathbf{q}}^1)$. Due to KKT condition, we conclude that $\hat{\mathbf{q}}^1$ is the optimal order quantity with $\hat{q}_2 = 0$. This contradiction proves that $C^1 \geq C^2$ for all c_1. Moreover, from the proof, we know that the strict inequality holds if $q_2 > 0$ with cost pair $(c_1, C^2(c_1))$.

Lastly, the boundary values are straightforward to derive since $\gamma_i(\mathbf{q}^j)$ is strictly decreasing in q_j from $\alpha\mu_i$ to 0 for every $i, j = 1, 2$. Q.E.D.

Proof of Proposition 2. Let us first focus on threshold C_s^1. Based on the definition of the cost thresholds in the proof of Proposition 1 and some necessary manipulation, we can write, for some $\mathbf{q}^1$,

$$c_1 = \gamma_1(\mathbf{q}^1) = \mathbf{E}\varphi_s(\xi_1), \text{ and } C_s^1 = \gamma_2(\mathbf{q}^1) = \mathbf{E}\left[\varphi_s(\xi_1)\frac{\mathbf{E}(\xi_2|\xi_1)}{\xi_1}\right].$$

Furthermore, in scenario f and r, respectively,

$$\varphi_f(\xi_1)/\xi_1 = p_0 \mathbf{1}_{\{q_1\xi_1 \leq d_0\}}, \quad \varphi_r(\xi_1)/\xi_1 = \frac{a - 2q_1\xi_1}{b}\mathbf{1}_{\{q_1\xi_1 \leq a/2\}}.$$

They both decrease in ξ_1 and neither of them is constant. So, by Lemma A.1(i)&(iii),

$$C_s^1 = \mathbf{E}\left[\varphi_s(\xi_1)\frac{\mathbf{E}(\xi_2|\xi_1)}{\xi_1}\right] > \frac{\mu_2}{\mu_1}\mathbf{E}\varphi_s(\xi_1) = \frac{\mu_2}{\mu_1}c_1$$

if $\frac{\mathbf{E}(\xi_2|\xi_1)}{\xi_1}$ is strictly decreasing; and the above inequality is reversed if $\frac{\mathbf{E}(\xi_2|\xi_1)}{\xi_1}$ is strictly increasing. In the similar way, the other part of (i) concerning C_s^2 can be proved. Finally, part (iii) follows directly from Lemma 2, along with (i) and (ii). Q.E.D.

Proof of Lemma 3. It is straightforward calculation to see that $CV(q_1, q_2) < CV(0, q_2)$ is equivalent to

$$\mu_1^2\sigma_2^2\left((\rho^*)^2 - 1\right)q_1^2 + 2\mu_2^2\sigma_1\sigma_2\left(\rho - (\rho^*)^{-1}\right)q_1q_2 < 0,$$

which holds true because $\rho^* < 1$ and $\rho \leq 1 < 1/\rho^*$. Let $m = q_1\mu_1 + q_2\mu_2$ and $\sigma^2 = q_1^2\sigma_1^2 + q_2^2\sigma_2^2 + 2\rho\sigma_1\sigma_2q_1q_2$. Then, again, by direct calculation, we have

$$\frac{\partial CV(q_1, q_2)}{\partial q_2} = -\frac{\sigma}{m^2}\mu_2 + \frac{1}{m\sigma}(\rho\sigma_1\sigma_2q_1 + \sigma_2^2q_2).$$

The above is non-negative if and only if

$$\mu_1(\rho^* - \rho)q_1^2 \leq \mu_2\left(\rho - (\rho^*)^{-1}\right)q_1q_2.$$

Hence, the rest of the proof immediately follows. Q.E.D.

Proof of Proposition 3. By definition, $\mathbf{E}_{\boldsymbol{\xi}}[\kappa(\boldsymbol{\xi})] \leq \mathbf{E}_{\hat{\boldsymbol{\xi}}}[\kappa(\hat{\boldsymbol{\xi}})]$ for any supermodular function $\kappa(\xi_1, \xi_2)$. Recall from the proof of Proposition 1 that the cost thresholds are $C^i = \gamma_2(\mathbf{q}^i)$; let us write them as $C_s^i = \mathbf{E}[\kappa_s(\boldsymbol{\xi}, \mathbf{q}^i)]$ for $i = 1, 2$. In particular, $\kappa_f(\boldsymbol{\xi}, \mathbf{q}^j) = p_0\xi_2\mathbf{1}_{\{q_j\xi_j \leq d_0\}}$ and $\kappa_r(\boldsymbol{\xi}, \mathbf{q}^j) = b^{-1}(a - 2q_j\xi_j)\xi_2\mathbf{1}_{\{q_j\xi_j \leq a/2\}}$ for $j = 1, 2$. Now, to prove (i), we just need to show that $\kappa_s(\boldsymbol{\xi}, \mathbf{q}^1)$ is submodular and $\kappa_s(\boldsymbol{\xi}, \mathbf{q}^2)$ is supermodular for both scenarios $s = \{f, r\}$. This is a straightforward exercise, so we only illustrate with $\kappa_f(\boldsymbol{\xi}, \mathbf{q}^1)$; the rest can be checked similarly. For any $\xi_1 \leq \hat{\xi}_1$ and $\xi_2 \leq \hat{\xi}_2$,

$$\kappa_f(\xi_1, \xi_2) + \kappa_f(\hat{\xi}_1, \hat{\xi}_2) - (\kappa_f(\xi_1, \hat{\xi}_2) + \kappa_f(\hat{\xi}_1, \xi_2)) = \begin{cases} 0 & \text{if } \hat{\xi}_1 \leq \frac{d_0}{q_1} \\ \xi_2 - \hat{\xi}_2 \leq 0 & \text{if } \xi_1 \leq \frac{d_0}{q_1} < \hat{\xi}_1 \\ 0 & \text{otherwise} \end{cases}$$

For (ii), assume first that the firm dual sources from both suppliers. Following the same logic as above, the profit function can be written as $\Pi_s = \mathbf{E}[\psi_s(\boldsymbol{\xi})]$ in scenario s. Specifically, $\psi_f(\boldsymbol{\xi}) = p_0(Q\mathbf{1}_{\{Q\le d_0\}} + d_0\mathbf{1}_{\{Q>d_0\}})$ and $\psi_r(\boldsymbol{\xi}) = r(Q)\mathbf{1}_{\{Q<d^*\}} + r^*\mathbf{1}_{\{Q\ge d^*\}}$. Since these two functions are concave in $Q = \mathbf{q}^T\boldsymbol{\xi}$, which is increasing and linear in $\boldsymbol{\xi}$, we can directly conclude that $\psi_s(\boldsymbol{\xi})$ is submodular; so by the definition of supermodular order, $\Pi_s(\mathbf{q}) \ge \hat{\Pi}_s(\mathbf{q})$. Now, assume that the firm sole sources. Note that being SM comparable implies that the two random variables, $\boldsymbol{\xi}$ and $\hat{\boldsymbol{\xi}}$, have the same marginal distributions. Hence, the same sole sourcing quantity must be derived, and the same profit will be achieved. So, over all possibilities, we have $\Pi_s(\mathbf{q}) \ge \hat{\Pi}_s(\mathbf{q})$ for any $\mathbf{q}$. The optimal profits obviously have the same order. For (iii), we refer the readers to the proof of Proposition 7, where we show this result for general multivariate normal yield distribution. Q.E.D.

Proof of Proposition 4. The KKT conditions are necessary and sufficient, a result following Lemma 1. Hence, the equality holds if and only if the order quantity is positive. In the following, we finish the proof by finding the gradient of the objective function.

Transform the general normal distribution of Q to standard normal: $F(y) = \Phi(\bar{y})$ and $\sigma f(y) = \phi(\bar{y})$, where $\bar{y} = (y-m)/\sigma$. For standard normal distribution, we will use the following identities throughout the proofs.

$$\Phi'(z) = \phi(z), \phi'(z) = -z\phi(z), \int_{-\infty}^{z} t\phi(t)dt = -\phi(z), \int_{-\infty}^{z} t^2\phi(t)dt = \Phi(z) - z\phi(z).$$

Note that $m = \mathbf{q}^T\boldsymbol{\mu}$ and $\sigma^2 = \mathbf{q}^T\boldsymbol{\Sigma}\mathbf{q}$; i.e., they are both functions of $\mathbf{q}$. Hence, for $\boldsymbol{\Sigma} = \{s_{ij}; i,j = 1,2,\cdots,n\}$, we have

$$\frac{\partial m}{\partial q_i} = \mu_i, \ \frac{\partial \sigma}{\partial q_i} = \frac{\sum_j s_{ij}q_j}{\sigma}, \ \frac{\partial \bar{y}}{\partial q_i} = -\frac{1}{\sigma}\left(\mu_i + \bar{y}\frac{\partial \sigma}{\partial q_i}\right).$$

Now, consider the fixed demand model. Using the above relationships, we rewrite the profit function as

$$\Pi_f(\mathbf{q}) = p_0(d_0 - \sigma h(\bar{d}_0)) - \mathbf{q}^T\mathbf{c}, \text{ where } h(t) = \phi(t) + t\Phi(t).$$

Note that $h'(t) = \Phi(t)$, so

$$\frac{\partial h(\bar{d}_0)}{\partial q_i} = \Phi(\bar{d}_0)\frac{\partial \bar{d}_0}{\partial q_i}.$$

Substitute all the partial derivatives, and then we have the gradient of the objective function:

$$\nabla_{\mathbf{q}}\Pi_f = p_0\left(\Phi(\bar{d}_0)\boldsymbol{\mu} - \frac{\phi(\bar{d}_0)}{\sigma}\boldsymbol{\Sigma}\mathbf{q}\right) - \mathbf{c},$$

which proves the result after transforming $\bar{d}_0$ back to d_0.

Next, consider the responsive pricing model with linear demand $d(p) = a - bp$. Let $\bar{d} = (a/2 - m)/\sigma$. Again, use the relationships derived in the beginning, we can rewrite the profit function as

$$\Pi_r(\mathbf{q}) = b^{-1}\left(\frac{a^2}{4} - \sigma^2\bar{d}\phi(\bar{d}) - (\sigma^2 + \frac{a^2}{4} - m(a-m))\Phi(\bar{d})\right) - \mathbf{q}^T\mathbf{c}.$$

Apply chain rules and carefully differentiate the above term by term with respect to q_i. Then, we can write out the gradient of the objective function as

$$\nabla_{\mathbf{q}}\Pi_r = 2b^{-1}\left(\sigma(\bar{d}\Phi(\bar{d}) + \phi(\bar{d}))\boldsymbol{\mu} - \Phi(\bar{d})\boldsymbol{\Sigma}\mathbf{q}\right) - \mathbf{c},$$

which is exactly the desired result after transforming $\bar{d}$ back to $a/2$. Q.E.D.

Proof of Corollary 1. Since the yields are uncorrelated, the matrix $\mathbf{\Sigma}$ is simply diagonal. Then, the unified KKT condition (3) (see the main paper) is $A\mu_k - B\sigma_k^2 q_k - c_k \le 0$ for some optimal quantity $\mathbf{q}$. Hence, for any supplier $i \in J$, its effective cost $c_i/\mu_i = A - \mu_i^{-1} B\sigma_i^2 q_i$, whereas any supplier $k \in J^c$ has effective cost $c_k/\mu_k > A$. Since $B > 0$ and $q_i > 0$, we have $c_k/\mu_k > c_i/\mu_i$. Q.E.D.

Proof of Proposition 5. Apply the unified form of KKT condition (3) (see the main paper) for both scenarios, use the partitioned mean vector and variance-covariance matrix, and note $\mathbf{q}_1 > \mathbf{q}_0 = \mathbf{0}$ to get

$$A_s\boldsymbol{\mu}_1 - B_s\mathbf{\Sigma}_{11}\mathbf{q}_1 - \mathbf{c}_1 = \mathbf{0} \text{ and } A_s\boldsymbol{\mu}_0 - B_s\mathbf{\Sigma}_{01}\mathbf{q}_1 - \mathbf{c}_0 < \mathbf{0}.$$

Recall that $B_s > 0$. Hence, we use the system of equations to solve for $\mathbf{q}_1$ and substitute it to the system of inequalities to have

$$\mathbf{\Sigma}_{01}\mathbf{\Sigma}_{11}^{-1}(A_s\boldsymbol{\mu}_1 - \mathbf{c}_1) > A_s\boldsymbol{\mu}_0 - \mathbf{c}_0.$$

Write the $(n-j)\times j$ matrix $\mathbf{\Sigma}_{01}\mathbf{\Sigma}_{11}^{-1}$ as $\{\tau_{ki}; k = 1,\cdots,n-j,\, i = 1,\cdots,j\}$. Then, $\forall k = 1,\cdots,n-j$,

$$\sum_{i\in J}\tau_{ki}\mu_i\left(A_s - \frac{c_i}{\mu_i}\right) > \mu_{j+k}\left(A_s - \frac{c_{j+k}}{\mu_{j+k}}\right),$$

which, after straightforward rewriting, is equivalent to the inequality (4) (see the main paper). Q.E.D.

Proof of Proposition 6. Suppose $\mathbf{q}$ is the optimal order quantity and $q_k > 0$ for all $k \in J$. Let s_{ij} be the entries of the variance-covariance matrix $\mathbf{\Sigma}$, and define $\tilde{\mathbf{z}}$ as $\tilde{z}_i = z_i\mu_i$. Then, by the unified KKT condition (3) (see the main paper), we have

$$z_l = \mu_l^{-1}B\sum_{k\in J}s_{lk}q_k,\ \forall l \in J; \quad z_t < \mu_t^{-1}B\sum_{k\in J}s_{tk}q_k,\ \forall t \in J^c.$$

Now, if $z_l < \mu_t^{-1}B\sum_{k\in J}s_{tk}q_k$, then we may find appropriate cost parameters to construct z_t such that $z_l < z_t < \mu_t^{-1}B\sum_{k\in J}s_{tk}q_k$ (this is true *unless* $c_l = 0$, which obviously is an extreme case that does not occur under our assumption). Hence, we can see that the KKT still holds and thus $\mathbf{q}$ is still the optimal order; and supplier t, which is less expensive than supplier l, is inactive.

Therefore, we only need to prove that $z_l < \mu_t^{-1}B\sum_{k\in J}s_{tk}q_k$ for some $l \in J$ and $t \in J^c$. Suppose no such pair of suppliers exists, i.e., $z_l \ge \mu_t^{-1}B\sum_{k\in J}s_{tk}q_k$ for all $l \in J$ and $t \in J^c$. Then, we can write the inequalities in matrix form: $z_l\boldsymbol{\mu}_0 \ge \mathbf{\Sigma}_{01}(B\mathbf{q}_1)$ for all $l \in J$. On the other hand, by KKT condition, $\tilde{\mathbf{z}}_1 = \mathbf{\Sigma}_{11}(B\mathbf{q}_1)$; so $B\mathbf{q}_1 = \mathbf{\Sigma}_{11}^{-1}\tilde{\mathbf{z}}_1$. Hence, we have $z_l\boldsymbol{\mu}_0 \ge \mathbf{\Sigma}_{01}(B\mathbf{q}_1) = \mathbf{\Sigma}_{01}\mathbf{\Sigma}_{11}^{-1}\tilde{\mathbf{z}}_1$. In particular, this means that $z_l\mu_t \ge \sum_{i\in J}\tau_{t-j,i}\mu_i z_i$, i.e., $\sum_{i\in J}W_{t-j,i}z_i \le z_l$ for all $l \in J$ and $t \in J^c$, which contradicts our assumption. Q.E.D.

Proof of Proposition 7. Suppose $\boldsymbol{\xi} \sim \mathcal{N}(\boldsymbol{\mu},\mathbf{\Sigma})$ and $\hat{\boldsymbol{\xi}} \sim \mathcal{N}(\hat{\boldsymbol{\mu}},\hat{\mathbf{\Sigma}})$ with the same marginal distribution. By Theorem 4.2 in Muller and Scarsini (2000), we have $\boldsymbol{\xi} \le_{sm} \hat{\boldsymbol{\xi}}$ if $\rho_{ij} \le \hat{\rho}_{ij}$ for any $i \ne j$. In the general form, the firm's profit functions (inside the expectation) under the two scenarios can be written in the same way as described in the proof of Proposition 3, i.e., $\psi_s(\cdot)$; hence they are submodular functions. Thus, by the definition of SM order, $\Pi_s(\mathbf{q}) \ge \hat{\Pi}_s(\mathbf{q})$ and $\Pi_s^* \ge \hat{\Pi}_s^*$; and the desired results follow. Q.E.D.

Proof of Proposition 8. Note that, the random demand is independent of the supply yields, and, thus, the expectations over the demand and the yields, respectively, can interchange order. Therefore, in the following, let us check the concavity of the objective functions in different scenarios (parallel to Lemma 1), write the KKT conditions to obtain the cost thresholds (parallel to Proposition 1), and slightly modify Lemma A.1 to compare the cost thresholds (parallel to Proposition 2).

From the objective functions given in Section 6 (of the main paper), we can easily see that they are concave in Q, and therefore jointly concave in $\mathbf{q}$ since $\lambda Q = (\lambda \mathbf{q})^T \boldsymbol{\xi}$ for any λ.

Now that concavity is proved, we can use KKT condition to obtain the cost thresholds in exactly the same way as in the proof of Proposition 1: write

$$\mathbf{E}_D \mathbf{E}_{\boldsymbol{\xi}} \left[p_0 \boldsymbol{\xi} \mathbf{1}_{\{\mathbf{q}^T \boldsymbol{\xi} \le D\}} \right] \le \mathbf{c} \quad \text{for } s = f; \text{ and } \quad \mathbf{E}_D \mathbf{E}_{\boldsymbol{\xi}} \left[\frac{D - 2Q}{b} \boldsymbol{\xi} \mathbf{1}_{\{Q \le D/2\}} \right] \le \mathbf{c} \quad \text{for } s = r.$$

In both scenarios, the condition can be generically written as $c_i \ge \bar{\gamma}_i(\mathbf{q}) = \mathbf{E}_D \gamma_i(\mathbf{q}, D)$ $(i = 1, 2)$ in the two-supplier model. The inequality is strict if and only if the supplier is not selected. It is important to note that, for any realization $D = d$, the functions $\gamma_i(\mathbf{q}, d)$ have exactly the same properties as before; i.e., they are strictly decreasing in every component of $\mathbf{q}$. Moreover, after taking expectation with respect to D, the monotonicity is preserved ($\mathbf{E}[D] < +\infty$ by assumption). Therefore, we can similarly define the cost thresholds as $\bar{C}^j = (\bar{\gamma}_2 \circ \bar{\gamma}_1^{-1})|_{\mathbf{q} = \mathbf{q}^j}$ $(j = 1, 2)$. The properties of $\bar{C}^j$ can thus be derived in the same way as before.

To compare the cost thresholds, we simply follow the proof of Proposition 2. Write

$$c_1 = \mathbf{E}_D \gamma_1(\mathbf{q}^1, D) = \mathbf{E}_D \mathbf{E}_{\boldsymbol{\xi}} \varphi_s(\xi_1, D), \text{ and } \bar{C}_s^1 = \mathbf{E}_D \gamma_2(\mathbf{q}^1, D) = \mathbf{E}_D \mathbf{E}_{\boldsymbol{\xi}} \left[\varphi_s(\xi_1, D) \frac{\mathbf{E}(\xi_2 | \xi_1)}{\xi_1} \right],$$

where

$$\varphi_f(\xi_1, D)/\xi_1 = p_0 \mathbf{1}_{\{q_1 \xi_1 \le D\}}, \quad \varphi_r(\xi_1, D)/\xi_1 = \frac{D - 2q_1\xi_1}{b} \mathbf{1}_{\{q_1 \xi_1 \le D/2\}}.$$

For any demand realization, the above two functions are decreasing in ξ_1. Now, it suffices to confirm a modified version of Lemma A.1. In particular, replace η with

$$\eta(d) = \mathbf{E}_{\boldsymbol{\xi}} \xi_1 \mathbf{E}_{\boldsymbol{\xi}} [\varphi(\xi_1, d) h(\xi_1)] - \mathbf{E}_{\boldsymbol{\xi}} \xi_2 \mathbf{E}_{\boldsymbol{\xi}} \varphi(\xi_1, d).$$

Then, the Lemma is straightforwardly generalized to include any demand realization. Finally, $\eta(d) > (<) 0$ for all possible d implies that $\mathbf{E}_D[\eta(D)] > (<) 0$; thus the comparison results are preserved after taking $\mathbf{E}_D$.

Hence, we have proved that, under demand uncertainty, our main results for the two-supplier model with deterministic demand, i.e., Propositions 1 and 2, still hold. Q.E.D.

Appendix C: The Ex ante Pricing Model

In this appendix, we briefly discuss the supplier selection problem under the *ex ante pricing* scheme, in which the firm faces a one-stage problem of setting price and order quantities simultaneously before yield realizations. The objective function Π_a (the subscript "a" denotes the ex $a$nte pricing) is given by

$$\Pi_a(p, \mathbf{q}) = p \mathbf{E}[\min\{d(p), \mathbf{q}^T \boldsymbol{\xi}\}] - \mathbf{q}^T \mathbf{c}. \tag{1}$$

We adopt the sequential optimization approach to tackle problem (1). For any give p, the price-induced demand is $d(p)$ and the firm's optimal supplier selection decision is completely characterized in the fixed demand model. Moreover, Lemma 1 still holds in this scenario. Back to the pricing decision, in order to use KKT as a necessary and sufficient condition, the firm's expected profit as a function of p must be well-behaved. Specifically, we need to establish its global unimodality, which normally is not necessarily guaranteed. So, the KKT condition can only be used as a necessary condition. If certain conditions can be identified to guarantee the unimodality (we remark that this is a technically challenging problem), then we are able to use KKT as a necessary and sufficient condition. Assuming multivariate normally distributed yield, we may write the optimality condition in the same unified way as in (3) (see the main paper):

$$A_a(p^*, \mathbf{q}^*)\boldsymbol{\mu} - B_a(p^*, \mathbf{q}^*)\boldsymbol{\Sigma}\mathbf{q}^* - \mathbf{c} \leq \mathbf{0}.$$

In this case, all the previous results in Section 5 (of the main paper) continues to hold.

More often, however, we may only conclude the relevant results when fixing the price (before price optimization). In this case, we can still partially verify the previously derived insights concerning the optimal supplier selection. The following corollary serves as an illustrating example in this regard.

Corollary C.1. *Assume $(\xi_1, \xi_2)^T$ follows a bivariate normal distribution. In the ex ante pricing model, the following statements hold:*

(i) If $\rho < \rho^*$, then the firm never sole sources from supplier 1 [supplier 2] when $\frac{c_1}{\mu_1} \geq \frac{c_2}{\mu_2}$ [$\frac{c_2}{\mu_2} \geq \frac{c_1}{\mu_1}$].

(ii) If $\rho > \rho^*$, then the firm sole sources from supplier 1 when $\frac{c_2}{\mu_2} \geq \frac{c_1}{\mu_1}$.

Proof.Notice that, for any fixed price, Proposition 2 holds true. Then, the diagonal line $\frac{c_1}{\mu_1} = \frac{c_2}{\mu_2}$ is a uniform bound for the cost thresholds. Hence, after optimizing over price, we may use the diagonal line to infer the firm's supplier selection decisions. Q.E.D.

Basically, Corollary C.1(i) confirms that the ex ante pricing firm never sole sources from the more expensive supplier alone when the yields are either negatively or weakly positively correlated; put differently, the cost ranking alone can qualify a supplier in this case. Part (ii) indicates that when yields are highly positively correlated, the firm sources entirely from supplier 1 when it has lower effective cost (i.e., $\frac{c_1}{\mu_1} \leq \frac{c_2}{\mu_2}$). Although it is likely, due to function continuity, that the sole sourcing would continue when we cross the diagonal line for just a little bit, a rigorous and complete characterization about the sourcing decision is a challenging exercise that requires additional technical conditions (see Kouvelis et al. 2017, for detailed discussions).

Nevertheless, we have numerically observed that the firm may indeed sole source from supplier 1 at optimality even when it is more expensive. Consider the same numerical setting studied in Section 4.2 (of the main paper). Assume that $\boldsymbol{\xi}$ follow a bivariate normal distribution with $\boldsymbol{\mu} = (0.5, 0.6)^T$, $\sigma_1 = 0.06$, and $\sigma_2 = 0.12$. Let the correlation coefficient $\rho = 0.85 > \rho^* = 0.6$. We use $d(p) = 1 - p$ and pick cost pair $\mathbf{c} = (0.2, 0.235)^T$. It is direct to verify that supplier 1 has a higher effective cost, i.e., $c_1/\mu_1 > c_2/\mu_2$. Moreover, the optimal decisions in this setting are $(p^*, q_1^*, q_2^*) \approx (0.707, 0.567, 0)$, showing that the firm optimally sole sources from the more expensive but more reliable supplier 1.

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