

Online Supplement

Hanlin Liu (liuhl@sustech.edu.cn), Yimin Yu (yiminyu@cityu.edu.hk), Saif Benjaafar
(saif@umn.edu), Huihui Wang (huihwang@shu.edu.cn)

Appendix A: Main Proofs

A.1. Results and Proofs for Section 4

Since $H_j(m_j) = h \frac{m_j}{\mu_j - m_j}$ is convex in m_j for each $j \in F(S)$, and $c_{ij}\alpha_{ij}\lambda_i$ is linear in α_{ij} for each $i \in D(S), j \in F(S)$, it can be verified that the objective function is convex in the decision variables m_j and α_{ij} for all $i \in D(S), j \in F(S)$. Since constraints (3)-(5) are affine and constraints (6)-(7) are convex, the DA problem in (2)-(7) is a convex optimization problem.

Next, we present the Lagrangian function for the DA problem. Let $\beta_j \leq 0, \xi_i \leq 0, \theta_{ij} \leq 0, \gamma_j \geq 0$ for $i \in D(S), j \in F(S)$ be the Lagrangian multipliers associated with constraints (3)-(6), respectively. Define vectors $\beta = (\beta_j)_{j \in F(S)}, \xi = (\xi_i)_{i \in D(S)}, \gamma = (\gamma_j)_{j \in F(S)}$, and matrix $\theta = [\theta_{ij}]_{i \in D(S), j \in F(S)}$. The Lagrangian function for the DA problem in (2)-(7) for coalition S is

$$\begin{aligned} & \mathcal{L}_S(m_S, \alpha_S, \beta, \xi, \theta, \gamma) \\ &= \sum_{j \in F(S)} H_j(m_j) + \sum_{i \in D(S)} \sum_{j \in F(S)} c_{ij} \alpha_{ij} \lambda_i + \sum_{j \in F(S)} (m_j - \sum_{i \in D(S)} \alpha_{ij} \lambda_i) \beta_j + \sum_{i \in D(S)} (\sum_{j \in F(S)} \alpha_{ij} - 1) \xi_i \\ &+ \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij} \theta_{ij} + \sum_{j \in F(S)} (\frac{1}{\mu_j - m_j} - s_j) \gamma_j \\ &= \sum_{j \in F(S)} [H_j(m_j) + m_j \beta_j + (\frac{1}{\mu_j - m_j} - s_j) \gamma_j] + \sum_{i \in D(S)} \sum_{j \in F(S)} [(c_{ij} - \beta_j) \lambda_i + \xi_i + \theta_{ij}] \alpha_{ij} - \sum_{i \in D(S)} \xi_i. \end{aligned} \quad (\text{A.1})$$

The Karush-Kuhn-Tucker (KKT) conditions for the DA problem are

$$\alpha_{ij} \geq 0, \alpha_{ij} \theta_{ij} = 0, (\frac{1}{\mu_j - m_j} - s_j) \gamma_j = 0, \forall i \in D(S), j \in F(S), \quad (\text{A.2})$$

$$H'_j(m_j) + \beta_j + \frac{1}{(\mu_j - m_j)^2} \gamma_j = 0, \forall j \in F(S), \quad (\text{A.3})$$

$$c_{ij} \lambda_i - \lambda_i \beta_j + \xi_i + \theta_{ij} = 0, \forall i \in D(S), j \in F(S), \quad (\text{A.4})$$

$$\beta_j \leq 0, \xi_i \leq 0, \theta_{ij} \leq 0, \gamma_j \geq 0, \alpha_{ij} \geq 0, \forall i \in D(S), j \in F(S). \quad (\text{A.5})$$

Noting that there is always an interior point for every inequality constraint in (3)-(6) in our optimization problem, i.e., the Slater condition holds, as each firm on its own can fulfill its service level requirements.

Proof of Proposition 1. Since the DA problem in (2)-(7) is a convex optimization problem with affine and convex constraints, KKT conditions are both necessary and sufficient for its optimality (Boyd and Vandenberghe 2004). Thus, if $(m_S^*, \alpha_S^*, \beta_S^*, \xi_S^*, \theta_S^*, \gamma_S^*)$ is a solution to the KKT conditions in (A.2)-(A.5), then (m_S^*, α_S^*) is an optimal solution to the DA problem.

(i) If $(\alpha_S^*)_{ij}, (\alpha_S^*)_{il} > 0$, it follows from (A.2) that $(\theta_S^*)_{ij} = (\theta_S^*)_{il} = 0$ and hence from (A.4) that $(\beta_S^*)_j = c_{ij} + \frac{(\xi_S^*)_i}{\lambda_i} = c_{il} + \frac{(\xi_S^*)_i}{\lambda_i}$. It also follows from (A.2) that $\frac{1}{(\mu_l - (m_S^*)_{il})^2} (\gamma_S^*)_l = s_l^2 (\gamma_S^*)_l$ and $\frac{1}{(\mu_j - (m_S^*)_{ij})^2} (\gamma_S^*)_j = s_j^2 (\gamma_S^*)_j$. Thus, it follows from (A.3) that $H'_l((m_S^*)_{il}) + s_l^2 (\gamma_S^*)_l + c_{il} = H'_j((m_S^*)_{ij}) + s_j^2 (\gamma_S^*)_j + c_{ij}$, that is, $\Gamma_{il}((m_S^*)_{il}) = \Gamma_{ij}((m_S^*)_{ij})$.

(ii) If $(\alpha_S^*)_{ij} = 0, (\alpha_S^*)_{il} > 0$, it follows from (A.2) that $(\theta_S^*)_{il} = 0$ and $(\theta_S^*)_{ij} \leq 0$. Similarly, it follows from (A.2)-(A.4) that $H'_l((m_S^*)_{il}) + s_l^2 (\gamma_S^*)_l + c_{il} \leq H'_j((m_S^*)_{ij}) + s_j^2 (\gamma_S^*)_j + c_{ij}$, that is, $\Gamma_{il}((m_S^*)_{il}) \leq \Gamma_{ij}((m_S^*)_{ij})$.

(iii) This result is directly implied by (i) and (ii). \blacksquare

Proof of Theorem 1. Note that in the optimization problem in (9), no constraint is imposed on the decision variable α_N . Therefore, at the optimal solution (α^*, m^*) and based on the KKT conditions for the DA problem in (2)-(7) when $S = N$, we must have that, for all $i \in D(N), j \in F(N)$,

$$H'_j(m_j^*) + \beta_j^* + \frac{1}{(\mu_j - m_j^*)^2} \gamma_j^* = 0, \quad (\text{A.6})$$

$$(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^* = 0, -\alpha_{ij}^* \theta_{ij}^* = 0, (\frac{1}{\mu_j - m_j^*} - s_j) \gamma_j^* = 0. \quad (\text{A.7})$$

Therefore, according to (9), we have

$$\sum_{k \in N} \phi_k = \sum_{k \in N} [- \sum_{i \in D(k)} \xi_i^* + \sum_{j \in F(k)} [H_j(m_j^*) + m_j^* \beta_j^*]]$$

$$\begin{aligned}
&= \sum_{j \in F(N)} [H_j(m_j^*) + m_j^* \beta_j^*] - \sum_{i \in D(N)} \xi_i^* \\
&= \sum_{j \in F(N)} [H_j(m_j^*) + m_j^* \beta_j^* + (\frac{1}{\mu_j - m_j^*} - s_j) \gamma_j^*] + \sum_{i \in D(N)} \sum_{j \in F(N)} [(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^*] \alpha_{ij}^* - \sum_{i \in D(N)} \xi_i^* \\
&= \min_{m_j < \mu_j, j \in F(N), \alpha_N} \mathcal{L}_N(m_N, \alpha_N, \beta^*, \xi^*, \theta^*, \gamma^*) = z(N).
\end{aligned}$$

On the other hand, for any $S \subset N$, we have

$$\begin{aligned}
\sum_{k \in S} \phi_k &= \sum_{k \in S} [- \sum_{i \in D(k)} \xi_i^* + \sum_{j \in F(k)} [H_j(m_j^*) + m_j^* \beta_j^*]] = \sum_{j \in F(S)} [H_j(m_j^*) + m_j^* \beta_j^*] - \sum_{i \in D(S)} \xi_i^* \\
&= \sum_{j \in F(S)} [H_j(m_j^*) + m_j^* \beta_j^* + (\frac{1}{\mu_j - m_j^*} - s_j) \gamma_j^*] + \sum_{i \in D(S)} \sum_{j \in F(S)} [(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^*] \alpha_{ij}^* - \sum_{i \in D(S)} \xi_i^* \\
&= \sum_{j \in F(S)} \min_{m_j < \mu_j} [H_j(m_j) + m_j \beta_j^* + (\frac{1}{\mu_j - m_j} - s_j) \gamma_j^*] + \sum_{i \in D(S)} \sum_{j \in F(S)} [(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^*] \alpha_{ij}^* - \sum_{i \in D(S)} \xi_i^* \\
&= \min_{m_j < \mu_j, j \in F(S), \alpha_S} \mathcal{L}_S(m_S, \alpha_S, \beta^*, \xi^*, \theta^*, \gamma^*) = \pi_S(\beta^*, \xi^*, \theta^*, \gamma^*) \leq z(S).
\end{aligned}$$

The second equality is due to (A.7), the third equality is due to (A.6), the fourth equality is due to (A.7), i.e., $(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^* = 0$, and the last inequality is because $(\beta^*, \xi^*, \theta^*, \gamma^*)$ is a feasible solution for the optimization problem in (8).

It follows that ϕ is in the core of the cooperative game (N, z) . Furthermore, it is clear that ϕ does not depend on α^* directly as it is determined by $m^*, \beta^*, \xi^*, \theta^*, \gamma^*$ only. This completes the proof. \blacksquare

Proof of Theorem 2. First, we show that, if $(m^*, \alpha^*, \beta^*, \xi^*, \theta^*, \gamma^*)$ is an optimal solution to the DA problem, then $(m^*, \alpha^*, \beta^*, \xi^*)$ is a market equilibrium. It goes as follows.

According to (9), we have

$$\begin{aligned}
z(N) &= \mathcal{L}_N(m^*, \alpha^*, \beta^*, \xi^*, \theta^*, \gamma^*) \\
&= \sum_{j \in F(N)} \left[\min_{m_j < \mu_j} [H_j(m_j) + m_j \beta_j^* + (\frac{1}{\mu_j - m_j} - s_j) \gamma_j^*] \right] \\
&\quad + \min_{\alpha_N} \left[\sum_{i \in D(N)} \sum_{j \in F(N)} [(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^*] \alpha_{ij} \right] - \sum_{i \in D(N)} \xi_i^* \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j \in F(N)} \left[\min_{m_j < \mu_j} [H_j(m_j) + m_j \beta_j^* + (\frac{1}{\mu_j - m_j} - s_j) \gamma_j^*] - \sum_{i \in D(N)} \xi_i^* \right] \\
&= \sum_{k \in N} \left\{ \sum_{j \in F(k)} \left[\min_{m_j < \mu_j} [H_j(m_j) + m_j \beta_j^* + (\frac{1}{\mu_j - m_j} - s_j) \gamma_j^*] \right] - \sum_{i \in D(k)} \xi_i^* \right\}. \tag{A.9}
\end{aligned}$$

From (A.9), it implies that firm k 's optimization problem is given as

$$\phi_k = \sum_{j \in F(k)} \left[\min_{m_j < \mu_j} [H_j(m_j) + m_j \beta_j^* + (\frac{1}{\mu_j - m_j} - s_j) \gamma_j^*] \right] - \sum_{i \in D(k)} \xi_i^*,$$

which is exactly the same as that in (13). The reason why the demand allocation problem in (A.8) is equivalent to that in (A.9), i.e.,

$$\min_{\alpha_N} \left[\sum_{i \in D(N)} \sum_{j \in F(N)} [(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^*] \alpha_{ij} \right] = \sum_{k \in N} \min_{\alpha_k} \left[\sum_{i \in D(k)} \sum_{j \in F(N)} [(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^*] \alpha_{ij} \right],$$

is given as follows: For each $k \in N$, the optimization problem $\min_{\alpha_k} \left[\sum_{i \in D(k)} \sum_{j \in F(N)} [(c_{ij} - \beta_j^*) \lambda_i + \xi_i^* + \theta_{ij}^*] \alpha_{ij} \right]$ is equivalent to the following optimization problem, i.e.,

$$\begin{aligned}
&\min_{\alpha_k} \sum_{i \in D(k)} \left[\sum_{j \in F(N)} (c_{ij} - \beta_j^* + \xi_i^* / \lambda_i) \alpha_{ij} \lambda_i \right] \\
&\text{subject to} \quad \sum_{j \in F(N)} \alpha_{ij} = 1, \forall i \in D(k), \\
&\quad \alpha_{ij} \geq 0, \forall i \in D(k), j \in F(N),
\end{aligned}$$

which is due to the facts that, under the optimal solution $(m^*, \alpha^*, \beta^*, \xi^*, \theta^*, \gamma^*)$ of the DA, we must have

$$m_j^* = \sum_{i \in D(N)} \lambda_i \alpha_{ij}^*, \forall j \in F(N) \text{ and } \sum_{j \in F(N)} \alpha_{ij}^* = 1, \forall i \in D(N).$$

It follows that $(m^*, \alpha^*, \beta^*, \xi^*)$ must be a market equilibrium.

Next, we show that, if $(\bar{m}, \bar{\alpha}, \bar{\beta}, \bar{\xi})$ is a market equilibrium, then $(\bar{m}, \bar{\alpha})$ is an optimal solution to the DA problem. It goes as follows.

Under the market clearing condition, we must have

$$\bar{m}_j = \sum_{i \in D(N)} \lambda_i \bar{\alpha}_{ij}, \forall j \in F(N) \text{ and } \sum_{j \in F(N)} \bar{\alpha}_{ij} = 1, \forall i \in D(N).$$

Under market clearing prices, firm k 's optimization problem is given as

$$\begin{aligned} (\bar{m}_j, \bar{\alpha}_{ij})_{i \in D(k), j \in F(N)} \in \arg \min_{\alpha_k, m_j, j \in F(k)} & - \sum_{i \in D(k)} \bar{\xi}_i + \sum_{j \in F(k)} [H_j(m_j) + m_j \bar{\beta}_j] + \sum_{i \in D(k)} \sum_{j \in F(N)} (c_{ij} - \bar{\beta}_j + \bar{\xi}_i / \lambda_i) \alpha_{ij} \lambda_i \\ \text{subject to } & \alpha_{ij} \geq 0, \forall i \in D(k), j \in F(N), \\ & \frac{1}{\mu_j - m_j} \leq s_j, \forall j \in F(k), \\ & \text{and } m_j < \mu_j, \forall j \in F(k). \end{aligned}$$

(Notice that the constraint $\sum_{j \in F(N)} \alpha_{ij} = 1$ for each $i \in D(k)$ is ignored in the above optimization as we already require that under the optimal solution $\sum_{j \in F(N)} \bar{\alpha}_{ij} = 1$ for each $i \in D(k)$.) Let $\bar{\theta}_{ij} \leq 0$ for $i \in D(k), j \in F(N)$ be the optimal Lagrangian multiplier associated with the constraint $\alpha_{ij} \geq 0, i \in D(k), j \in F(N)$ and $\bar{\gamma}_j \geq 0$ for $j \in F(k)$ the optimal Lagrangian multiplier associated with the constraint $\frac{1}{\mu_j - m_j} \leq s_j, j \in F(k)$. Then, the KKT conditions for firm k 's optimization problem are given as

$$\begin{aligned} \bar{m}_j &= \sum_{i \in D(N)} \lambda_i \bar{\alpha}_{ij}, j \in F(k); \sum_{j \in F(N)} \bar{\alpha}_{ij} = 1, i \in D(k); \bar{\alpha}_{ij} \bar{\theta}_{ij} = 0, i \in D(k), j \in F(N); \\ & \left(\frac{1}{\mu_j - \bar{m}_j} - s_j \right) \bar{\gamma}_j = 0, j \in F(k); (c_{ij} - \bar{\beta}_j) \lambda_i + \bar{\xi}_i + \bar{\theta}_{ij} = 0, i \in D(k), j \in F(N); \\ & H'_j(\bar{m}_j) + \bar{\beta}_j + \frac{1}{(\mu_j - \bar{m}_j)^2} \bar{\gamma}_j = 0, j \in F(k). \end{aligned}$$

So the optimality conditions for the market equilibrium are the aggregation of all firms' KKT conditions, i.e.,

$$\begin{aligned} \bar{m}_j &= \sum_{i \in D(N)} \lambda_i \bar{\alpha}_{ij}, \sum_{j \in F(N)} \bar{\alpha}_{ij} = 1, \bar{\alpha}_{ij} \bar{\theta}_{ij} = 0, \\ & \left(\frac{1}{\mu_j - \bar{m}_j} - s_j \right) \bar{\gamma}_j = 0, (c_{ij} - \bar{\beta}_j) \lambda_i + \bar{\xi}_i + \bar{\theta}_{ij} = 0, \\ & H'_j(\bar{m}_j) + \bar{\beta}_j + \frac{1}{(\mu_j - \bar{m}_j)^2} \bar{\gamma}_j = 0, \\ & \bar{\beta}_j \leq 0, \bar{\xi}_i \leq 0, \bar{\theta}_{ij} \leq 0, \bar{\gamma}_j \geq 0, \forall i \in D(N), j \in F(N). \end{aligned}$$

Since the above optimality conditions are exactly the same as the KKT conditions for the DA problem in (A.2)-(A.5) with $S = N$, it follows that $(\bar{m}, \bar{\alpha})$ is an optimal solution to the DA problem. This completes the proof. \blacksquare

A.2. Results and Proofs for Section 5

Proof of Lemma 1. Given m_j and without the service level requirement, the optimal capacity decision for facility j can be solved by taking the first-order condition with respect to μ_j in (14), i.e., $-h \frac{m_j}{(\mu_j - m_j)^2} + K_j = 0$, it follows that $\mu_j = m_j + \sqrt{\frac{hm_j}{K_j}}$.

As the service level requirements $\frac{1}{\mu_j - m_j} \leq s_j$ for all $j \in F(S)$, that is, $\mu_j \geq m_j + \frac{1}{s_j}$. Consequently, the optimal capacity for facility j is

$$\mu_j^* = m_j + \max \left\{ \sqrt{\frac{hm_j}{K_j}}, \frac{1}{s_j} \right\} = \begin{cases} m_j + \frac{1}{s_j} & \text{for } m_j \in [0, m_j^o], \\ m_j + \sqrt{\frac{hm_j}{K_j}} & \text{for } m_j \in (m_j^o, +\infty), \end{cases}$$

where $m_j^o = \frac{K_j}{hs_j^2}$, which is not convex in m_j . And, the total cost under the optimal capacity for facility j is

$$\eta_j^*(m_j) = K_j [m_j + \max \left\{ \sqrt{\frac{hm_j}{K_j}}, \frac{1}{s_j} \right\}] + \frac{hm_j}{\max \left\{ \sqrt{\frac{hm_j}{K_j}}, \frac{1}{s_j} \right\}} = \begin{cases} (K_j + hs_j)m_j + \frac{K_j}{s_j} & \text{for } m_j \in [0, m_j^o], \\ K_j m_j + 2\sqrt{hK_j m_j} & \text{for } m_j \in (m_j^o, +\infty). \end{cases}$$

Notice that $\eta_j^*(m_j)$ is a linear function when $m_j \in [0, m_j^o]$ and $\eta_j^*(m_j)$ is also concave when $m_j \in (m_j^o, +\infty)$. Moreover, the left derivative of $\eta_j^*(m_j)$ at m_j^o is $\eta_j^{*'}(m_j)|_{m_j=m_j^o-} = K_j + hs_j$ and $\eta_j^{*'}(m_j)|_{m_j=m_j^o+} = K_j + hs_j$. Hence, $\eta_j^*(m_j)$ is differentiable at m_j^o and continuously differentiable. Then $\eta_j^*(m_j)$ is a continuously differentiable and increasing concave function. \blacksquare

In the following, we derive the dual problem of the problem in (17). First, as $f_j(\cdot)$ is continuously differentiable per Lemma 1, let $f'_j(\cdot)$ be the derivative of $f_j(\cdot)$. According to the concavity of $f_j(\cdot)$, we have the following result:

$$f_j(m_j) = \min_{\hat{m}_j \geq 0} [f_j(\hat{m}_j) + f'_j(\hat{m}_j)(m_j - \hat{m}_j)], \forall j \in F(S).$$

Consequently,

$$\begin{aligned} z(S) &= \min_{m_S \in \mathcal{M}_S} \min_{\hat{m}_S \in \mathcal{M}_S} \left\{ \sum_{j \in F(S)} [f_j(\hat{m}_j) - f'_j(\hat{m}_j)\hat{m}_j + f'_j(\hat{m}_j)m_j + G(m_S, S)] + \sum_{j \in F(S)} \frac{K_j}{s_j} \right\} \\ &= \min_{\hat{m}_S \in \mathcal{M}_S} \left\{ \sum_{j \in F(S)} [f_j(\hat{m}_j) - f'_j(\hat{m}_j)\hat{m}_j] + \min_{m_S \in \mathcal{M}_S} \left\{ \sum_{j \in F(S)} f'_j(\hat{m}_j)m_j + G(m_S, S) \right\} \right\} + \sum_{j \in F(S)} \frac{K_j}{s_j}. \end{aligned} \quad (\text{A.10})$$

Proof of Lemma 2. Notice that we can equivalently reformulate the optimization problem in (A.10) as follows:

$$z(S) = \max z + \sum_{j \in F(S)} \frac{K_j}{s_j} \quad (\text{A.11})$$

$$\text{subject to } z \leq \sum_{j \in F(S)} [f_j(\hat{m}_j) - f'_j(\hat{m}_j)\hat{m}_j] + \min_{m_S \in \mathcal{M}_S} \left\{ \sum_{j \in F(S)} f'_j(\hat{m}_j)m_j + G(m_S, S) \right\}, \forall \hat{m}_S \in \mathcal{M}_S. \quad (\text{A.12})$$

For any given $\hat{m}_S \in \mathcal{M}_S$, the minimization problem (the primal problem)

$$\min_{m_S \in \mathcal{M}_S} \left\{ \sum_{j \in F(S)} f'_j(\hat{m}_j)m_j + G(m_S, S) \right\},$$

equals to the following optimization problem

$$\min_{m_S, \alpha_S} \left\{ \sum_{j \in F(S)} f'_j(\hat{m}_j)m_j + \sum_{j \in F(S)} c_{ij}\alpha_{ij}\lambda_i \right\} \quad (\text{A.13})$$

$$\text{subject to } m_j \geq \sum_{i \in D(S)} \alpha_{ij}\lambda_i, \forall j \in F(S), \quad (\text{A.14})$$

$$\sum_{j \in F(S)} \alpha_{ij} \geq 1, \forall i \in D(S), \quad (\text{A.15})$$

$$\alpha_{ij} \geq 0, \forall i \in D(S), j \in F(S), \quad (\text{A.16})$$

which is a convex optimization (linear programming) problem. Next, consider the dual problem of (A.13). More specifically, let $\beta_j \leq 0, \xi_i \leq 0, \theta_{ij} \leq 0$ for $i \in D(S), j \in F(S)$ be the Lagrangian multipliers associated with constraints (A.14)-(A.16), respectively. Then, the Lagrangian function for the primal problem in (A.13) is

$$\begin{aligned} \mathcal{L}_S(m_S, \alpha_S, \beta, \xi, \theta) &= \sum_{j \in F(S)} f'_j(\hat{m}_j)m_j + \sum_{i \in D(S)} \sum_{j \in F(S)} c_{ij}\alpha_{ij}\lambda_i \\ &\quad + \sum_{j \in F(S)} \beta_j(m_j - \sum_{i \in D(S)} \alpha_{ij}\lambda_i) + \sum_{i \in D(S)} \xi_i \left(\sum_{j \in F(S)} \alpha_{ij} - 1 \right) + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij}\theta_{ij} \\ &= \sum_{j \in F(S)} [f'_j(\hat{m}_j) + \beta_j]m_j + \sum_{i \in D(S)} \sum_{j \in F(S)} [(c_{ij} - \beta_j)\lambda_i + \xi_i + \theta_{ij}]\alpha_{ij} - \sum_{i \in D(S)} \xi_i. \end{aligned}$$

The corresponding KKT conditions for the minimization problem are

$$\begin{aligned} f'_j(\hat{m}_j) + \beta_j &= 0, \forall j \in F(S), \\ (c_{ij} - \beta_j)\lambda_i + \xi_i + \theta_{ij} &= 0, \forall i \in D(S), j \in F(S), \end{aligned}$$

and the dual problem of the problem in (A.13) is

$$\begin{aligned} \max & - \sum_{i \in D(S)} \xi_i \\ \text{subject to } & f'_j(\hat{m}_j) + \beta_j \geq 0, \forall j \in F(S), \\ & (c_{ij} - \beta_j)\lambda_i + \xi_i + \theta_{ij} \geq 0, \forall i \in D(S), j \in F(S), \\ & \beta_j \leq 0, \xi_i \leq 0, \theta_{ij} \leq 0, \forall i \in D(S), j \in F(S), \end{aligned}$$

which is again equivalent to (notice that in the above dual problem, as $\theta_{ij} \leq 0$, it is optimal to set $\theta_{ij}^* = 0$ for all $i \in D(S), j \in F(S)$)

$$\begin{aligned} \max & - \sum_{i \in D(S)} \xi_i \quad (\text{A.17}) \\ \text{subject to } & f'_j(\hat{m}_j) + \beta_j \geq 0, \forall j \in F(S), \\ & (c_{ij} - \beta_j)\lambda_i + \xi_i \geq 0, \forall i \in D(S), j \in F(S), \\ & \beta_j \leq 0, \xi_i \leq 0, \forall i \in D(S), j \in F(S). \end{aligned}$$

It is clear that the optimal solution of the dual problem depends on \hat{m}_S . Thus, based on the above result, we reformulate the optimization problem in (A.11) by substituting (A.13) with (A.17):

$$z(S) = \max z + \sum_{j \in F(S)} \frac{K_j}{s_j} \quad (\text{A.18})$$

$$\begin{aligned} \text{subject to } z &\leq \sum_{j \in F(S)} [f_j(\hat{m}_j) - f'_j(\hat{m}_j)\hat{m}_j] - \sum_{i \in D(S)} \xi_i(\hat{m}_S), \forall \hat{m}_S \in \mathcal{M}_S, \\ f'_j(\hat{m}_j) + \beta_j(\hat{m}_S) &\geq 0, \forall j \in F(S), \hat{m}_S \in \mathcal{M}_S, \\ [c_{ij} - \beta_j(\hat{m}_S)]\lambda_i + \xi_i(\hat{m}_S) &\geq 0, \forall i \in D(S), j \in F(S), \hat{m}_S \in \mathcal{M}_S, \\ \beta_j(\hat{m}_S) &\leq 0, \xi_i(\hat{m}_S) \leq 0, \forall i \in D(S), j \in F(S), \hat{m}_S \in \mathcal{M}_S. \end{aligned} \quad (\text{A.19})$$

Furthermore, for any given \hat{m}_S , we can define new variables $\rho_i(\hat{m}_S) \leq 0, i \in D(S)$ so that

$$\sum_{j \in F(S)} [f_j(\hat{m}_j) - f'_j(\hat{m}_j)\hat{m}_j] \geq - \sum_{i \in D(S)} \rho_i(\hat{m}_S). \quad (\text{A.20})$$

This is possible as $f_j(\hat{m}_j) - f'_j(\hat{m}_j)\hat{m}_j \geq 0$ for all $j \in F(S)$. By substituting the inequality (A.20) into the constraint (A.19), we have

$$z \leq - \sum_{i \in D(S)} [\rho_i(\hat{m}_S) + \xi_i(\hat{m}_S)], \forall \hat{m}_S \in \mathcal{M}_S. \quad (\text{A.21})$$

We can further define $\pi_i, i \in D(S)$ so that $0 \leq \pi_i \leq -\rho_i(\hat{m}_S) - \xi_i(\hat{m}_S)$ for all $i \in D(S), \hat{m}_S \in \mathcal{M}_S$. Then, the constraint in (A.21) becomes

$$z \leq \sum_{i \in D(S)} \pi_i, \forall \hat{m}_S \in \mathcal{M}_S.$$

Proof of Theorem 3. By definition, it is easy to check

$$\sum_{k \in N} \phi_k = \sum_{k \in N} \left(\sum_{i \in D(k)} \pi_i^* + \sum_{j \in F(k)} \frac{K_j}{s_j} \right) = \sum_{i \in D(N)} \pi_i^* + \sum_{j \in F(k)} \frac{K_j}{s_j} = z(N).$$

For any optimal solution $(\pi^*, \rho^*(m_N), \beta^*(m_N), \xi^*(m_N))$ to the dual problem in (18) with $S = N$, we can construct a feasible solution $(\pi^S, \rho^S(m_S), \beta^S(m_S), \xi^S(m_S))$ to (18) with coalition S as follows:

$$\begin{aligned} \pi_i^S &:= \pi_i^*, \quad \forall i \in D(S), \\ \rho_i^S(m_S) &:= \rho_i^*(T^S(m_S)), \quad \forall i \in D(S), \\ \beta_j^S(m_S) &:= \beta_j^*(T^S(m_S)), \quad \forall j \in F(S), \\ \xi_i^S(m_S) &:= \xi_i^*(T^S(m_S)), \quad \forall i \in D(S) \end{aligned}$$

in which $T^S(\cdot)$ is a function from \mathcal{M}_S to \mathcal{M}_N such that:

(1) If $\sum_{j \in F(S)} m_j \geq \sum_{i \in D(N)} \lambda_i$, then

$$T_j^S(m_S) := \begin{cases} m_j & \text{if } j \in F(S) \\ 0 & \text{others} \end{cases}$$

(2) If $\sum_{i \in D(S)} \lambda_i \leq \sum_{j \in F(S)} m_j < \sum_{i \in D(N)} \lambda_i$, then

$$T_j^S(m_S) := \begin{cases} m_j & \text{if } j \in F(S) \\ \frac{\sum_{i \in D(N)} \lambda_i - \sum_{j \in F(S)} m_j}{\sum_{j \in F(N) \setminus F(S)} \frac{K_j}{hs_j^2}} \cdot \frac{K_j}{hs_j^2} & \text{others.} \end{cases}$$

When $\sum_{j \in F(S)} m_j \geq \sum_{i \in D(N)} \lambda_i$, it is easy to check $(\pi^S, \rho^S(m_S), \beta^S(m_S), \xi^S(m_S))$ satisfies constraints (19)-(24).

When $\sum_{i \in D(S)} \lambda_i \leq \sum_{j \in F(S)} m_j < \sum_{i \in D(N)} \lambda_i$, we can see that $-\sum_{i \in D(N)} \rho_i^*(T^S(m_S)) \leq \sum_{j \in F(N)} [f_j(T_j^S(m_S)) - f'_j(T_j^S(m_S))T_j^S(m_S)]$ as $T^S(m_S) \in \mathcal{M}_S$. For any $j \in F(N) \setminus F(S)$, we have

$$\frac{\sum_{i \in D(N)} \lambda_i - \sum_{j \in F(S)} m_j}{\sum_{j \in F(N) \setminus F(S)} \frac{K_j}{hs_j^2}} \leq \frac{\sum_{i \in D(N) \setminus D(S)} \lambda_i}{\sum_{j \in F(N) \setminus F(S)} \frac{K_j}{hs_j^2}} \leq 1$$

in which the last inequality holds due to our assumption $\sum_{i \in D(N \setminus S)} \lambda_i \leq \sum_{j \in F(N \setminus S)} \frac{K_j}{hs_j^2}$. Therefore, we have $T_j^S(m_S) \leq \frac{K_j}{hs_j^2}$ for any $j \in F(N) \setminus F(S)$. Then constraint (24) is satisfied due to the following inequality:

$$\begin{aligned} - \sum_{i \in D(S)} \rho_i^S(m_S) &\leq - \sum_{i \in D(N)} \rho_i^*(T^S(m_S)) \leq \sum_{j \in F(N)} [f_j(T_j^S(m_S)) - f'_j(T_j^S(m_S))T_j^S(m_S)] \\ &= \sum_{j \in F(S)} [f_j(m_j) - f'_j(m_j)m_j] \end{aligned}$$

in which the last equality holds due to the fact that $f_j(m_j)$ is a linear function when $m_j \leq \frac{K_j}{h_{s_j^2}}$.

Till now, we have verified that $(\pi^S, \rho^S(m_S), \beta^S(m_S), \xi^S(m_S))$ is a feasible solution to problem (18). By definition,

$$\sum_{k \in S} \phi_k = \sum_{k \in S} \left(\sum_{i \in D(k)} \pi_i^* + \sum_{j \in F(k)} \frac{K_j}{s_j} \right) = \sum_{i \in D(S)} \pi_i^S + \sum_{j \in F(S)} \frac{K_j}{s_j} \leq z(S).$$

A.3. Results and Proofs for Section 6

Consider a coalition $S \subseteq N$, let L_{ij} be the expected number of customers from demand source i at facility j for $i \in D(S), j \in F(S)$. Based on Coffman and Mitrani (1980), given demand allocation matrix $\alpha_S = [\alpha_{ij}]_{i \in D(S), j \in F(S)}$ for coalition S , the feasible region for the vector $(L_{ij})_{i \in D(S)}$ of facility j under any mixed preemptive priority policy can be described by the following polyhedron:

$$\begin{aligned} \sum_{i \in D(T)} L_{ij} &\geq \frac{\sum_{i \in D(T)} \lambda_i \alpha_{ij}}{\mu_j - \sum_{i \in D(T)} \lambda_i \alpha_{ij}}, \forall T \subseteq S, \\ \sum_{i \in D(S)} L_{ij} &= \frac{\sum_{i \in D(S)} \lambda_i \alpha_{ij}}{\mu_j - \sum_{i \in D(S)} \lambda_i \alpha_{ij}}, \end{aligned}$$

and $L_{ij} \geq 0$ for all $i \in D(S)$. In fact, the achievable region has the *polymatroid* structure (the reader can refer to the proof of Proposition 1 in Yu et al. 2015 for a more detailed discussion on the polymatroid).

Let $\zeta_j(T) \leq 0, \beta_j(T) \leq 0, \gamma_{ij} \geq 0, \xi_i \leq 0, \theta_{ij} \leq 0$, for $i \in D(S), j \in F(S)$ be the Lagrangian multipliers associated with constraints (27)-(29), (4) and (5), respectively. Define vectors $\zeta = (\zeta_j(T))_{j \in F(T), T \subseteq S}, \beta = (\beta_j(T))_{j \in F(T), T \subseteq S}, \xi = (\xi_i)_{i \in D(S)}$ and matrices $\theta = [\theta_{ij}]_{i \in D(S), j \in F(S)}, \gamma = [\gamma_{ij}]_{i \in D(S), j \in F(S)}$. Then, the Lagrangian function $\mathcal{L}_S(L_S, m_S(T), \alpha_S, \zeta, \beta, \xi, \theta, \gamma)$ for the PDA problem for coalition S is given as

$$\begin{aligned} &\mathcal{L}_S(L_S, m_S(T), \alpha_S, \zeta, \beta, \xi, \theta, \gamma) \\ &= \sum_{i \in D(S)} \sum_{j \in F(S)} [h_i L_{ij} + \lambda_i \alpha_{ij} c_{ij}] + \sum_{T \subseteq S} \sum_{j \in F(T)} \left[\sum_{i \in D(T)} L_{ij} - \frac{m_j(T)}{\mu_j - m_j(T)} \right] \zeta_j(T) + \sum_{i \in D(S)} \left(\sum_{j \in F(S)} \alpha_{ij} - 1 \right) \xi_i \\ &\quad + \sum_{T \subseteq S} \sum_{j \in F(T)} (m_j(T) - \sum_{i \in D(T)} \alpha_{ij} \lambda_i) \beta_j(T) + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij} \theta_{ij} + \sum_{i \in D(S)} \sum_{j \in F(S)} (L_{ij} - \lambda_i \alpha_{ij} s_{ij}) \gamma_{ij} \\ &= \sum_{i \in D(S)} \sum_{j \in F(S)} L_{ij} [h_i + \gamma_{ij} + \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \zeta_j(T)] + \sum_{T \subseteq S} \sum_{j \in F(T)} [m_j(T) \beta_j(T) - \frac{m_j(T)}{\mu_j - m_j(T)} \zeta_j(T)] \\ &\quad + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij} [\theta_{ij} + \xi_i + \lambda_i (c_{ij} - s_{ij} \gamma_{ij}) - \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j(T)] - \sum_{i \in D(S)} \xi_i. \end{aligned} \quad (\text{A.22})$$

The corresponding KKT conditions of the PDA problem are given as

$$h_i + \gamma_{ij} + \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \zeta_j(T) = 0, \forall i \in D(S), j \in F(S), \quad (\text{A.23})$$

$$\beta_j(T) - \frac{\mu_j}{(\mu_j - m_j(T))^2} \zeta_j(T) = 0, \forall j \in F(T), T \subseteq S, \quad (\text{A.24})$$

$$\theta_{ij} + \xi_i + \lambda_i (c_{ij} - s_{ij} \gamma_{ij}) - \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j(T) = 0, \forall i \in D(S), j \in F(S), \quad (\text{A.25})$$

$$\left[\sum_{i \in D(T)} L_{ij} - \frac{m_j(T)}{\mu_j - m_j(T)} \right] \zeta_j(T) = 0, [m_j(T) - \sum_{i \in D(T)} \lambda_i \alpha_{ij}] \beta_j(T) = 0, \forall j \in F(T), T \subseteq S, \quad (\text{A.26})$$

$$\left(\sum_{j \in F(S)} \alpha_{ij} - 1 \right) \xi_i = 0, \alpha_{ij} \theta_{ij} = 0, \alpha_{ij} \geq 0, \forall i \in D(S), j \in F(S), \quad (\text{A.27})$$

$$(L_{ij} - \lambda_i \alpha_{ij} s_{ij}) \gamma_{ij} = 0, \forall i \in D(S), j \in F(S), \quad (\text{A.28})$$

$$\zeta_j(T) \leq 0, \beta_j(T) \leq 0, \xi_i \leq 0, \theta_{ij} \leq 0, \gamma_{ij} \geq 0, \forall i \in D(S), j \in F(S), T \subseteq S. \quad (\text{A.29})$$

In the following lemma, we provide conditions under which the optimal policy is a strict priority policy. Let γ_{ij}^S be the optimal Lagrangian multiplier for the service level requirement in (29). The dual problem (30)-(31) implies the following result for each facility j .

PROPOSITION 1 (Yu et al. 2015). *For any permutation $\pi_j = (\pi_{1,j}, \dots, \pi_{|D(S)|,j})$ of the index set $D(S)$ for any $j \in F(S)$ such that $h_{\pi_{1,j}} + \gamma_{\pi_{1,j}}^S > h_{\pi_{2,j}} + \gamma_{\pi_{2,j}}^S > \dots > h_{\pi_{|D(S)|,j}} + \gamma_{\pi_{|D(S)|,j}}^S$, then a strict priority policy is optimal for facility j and demand source $\pi_{1,j}$ has a higher priority over demand source $\pi_{k,j}$ for any $l < k$.*

If there exists a subset A such that $h_{lj} + \gamma_{lj}^S = h_{kj} + \gamma_{kj}^S$ for all $l, k \in A$ and some $A \subseteq D(S)$, then a strict priority policy may not be optimal for facility j and the optimal policy may be a mixed priority policy. It is interesting to note the $c\mu$ rule may not be optimal and the optimal priority order is determined by the values $h_{ij} + \gamma_{ij}^S, i \in D(S), j \in F(S)$.

Proof of Theorem 4. Note that $(\alpha^*, m^*(T), L^*)$ is the optimal solution for the primal problem in (26) with $S = N$ and $(\zeta^*, \beta^*, \xi^*, \theta^*, \gamma^*)$ the optimal Lagrangian multipliers for the dual problem in (31) with $S = N$. Based on the KKT conditions for the PDA problem when $S = N$, we must have that, for all $i \in D(N), j \in F(N)$,

$$h_i + \gamma_{ij}^* + \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \zeta_j^*(T) = 0, \quad (\text{A.30})$$

$$\beta_j^*(T) - \frac{\mu_j}{(\mu_j - m_j^*(T))^2} \zeta_j^*(T) = 0, \quad \forall j \in F(T), T \subseteq N, \quad (\text{A.31})$$

$$\theta_{ij}^* + \xi_i^* + \lambda_i(c_{ij} - s_{ij}\gamma_{ij}^*) - \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j^*(T) = 0, \quad (\text{A.32})$$

$$\left[\sum_{i \in D(T)} L_{ij}^* - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \right] \zeta_j^*(T) = 0, [m_j^*(T) - \sum_{i \in D(T)} \lambda_i \alpha_{ij}^*] \beta_j^*(T) = 0, \quad \forall j \in F(T), T \subseteq N. \quad (\text{A.33})$$

Therefore, we have

$$\begin{aligned} \sum_{k \in N} \phi_k &= \sum_{k \in N} \left[- \sum_{i \in D(k)} \xi_i^* + \sum_{j \in F(k)} \sum_{T \subseteq N: F(T) \ni j} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \right] \\ &= \sum_{T \subseteq N} \sum_{j \in F(T)} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] - \sum_{i \in D(N)} \xi_i^* \\ &= \sum_{i \in D(N)} \sum_{j \in F(N)} L_{ij}^* [h_i + \gamma_{ij}^* + \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \zeta_j^*(T)] + \sum_{T \subseteq N} \sum_{j \in F(T)} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \\ &\quad + \sum_{i \in D(N)} \sum_{j \in F(N)} \alpha_{ij}^* [\theta_{ij}^* + \xi_i^* + \lambda_i(c_{ij} - s_{ij}\gamma_{ij}^*)] - \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j^*(T) - \sum_{i \in D(N)} \xi_i^* \\ &= \mathcal{L}_N(L^*, m(T)^*, \alpha^*, \zeta^*, \beta^*, \xi^*, \theta^*, \gamma^*) = z(N). \end{aligned}$$

On the other hand, for any $S \subset N$, we have

$$\begin{aligned} &\sum_{k \in S} \phi_k \\ &= \sum_{k \in S} \left[- \sum_{i \in D(k)} \xi_i^* + \sum_{j \in F(k)} \sum_{T \subseteq N: F(T) \ni j} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \right] \\ &= \sum_{j \in F(S)} \sum_{T \subseteq N: F(T) \ni j} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] - \sum_{i \in D(S)} \xi_i^* \\ &= \sum_{i \in D(S)} \sum_{j \in F(S)} L_{ij}^* [h_i + \gamma_{ij}^* + \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \zeta_j^*(T)] + \sum_{j \in F(S)} \sum_{T \subseteq N: F(T) \ni j} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \\ &\quad + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij}^* [\theta_{ij}^* + \xi_i^* + \lambda_i(c_{ij} - s_{ij}\gamma_{ij}^*)] - \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j^*(T) - \sum_{i \in D(S)} \xi_i^* \\ &= \min_{L_S, \alpha_S} \left\{ \sum_{i \in D(S)} \sum_{j \in F(S)} L_{ij} [h_i + \gamma_{ij}^* + \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \zeta_j^*(T)] + \sum_{j \in F(S)} \sum_{T \subseteq N: F(T) \ni j} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \right. \\ &\quad \left. + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij} [\theta_{ij}^* + \xi_i^* + \lambda_i(c_{ij} - s_{ij}\gamma_{ij}^*)] - \sum_{T \subseteq N: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j^*(T) - \sum_{i \in D(S)} \xi_i^* \right\} \\ &= \min_{L_S, \alpha_S} \left\{ \sum_{i \in D(S)} \sum_{j \in F(S)} L_{ij} [h_i + \gamma_{ij}^* + \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \zeta_j^*(T)] + \sum_{j \in F(S)} \sum_{T \subseteq S: F(T) \ni j} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \right. \\ &\quad \left. + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij} [\theta_{ij}^* + \xi_i^* + \lambda_i(c_{ij} - s_{ij}\gamma_{ij}^*)] - \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j^*(T) - \sum_{i \in D(S)} \xi_i^* \right. \\ &\quad \left. + \sum_{j \in F(S)} \sum_{T \subseteq N, T \not\subseteq S: F(T) \ni j} \left\{ \left[\sum_{i \in D(S)} L_{ij} - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \right] \zeta_j^*(T) + [m_j^*(T) - \sum_{i \in D(S)} \lambda_i \alpha_{ij}] \beta_j^*(T) \right\} \right\}, \end{aligned}$$

where the third and fourth equalities are due to (A.32) and (A.33), and the last equality is due to the rearranging of the terms. Notice that

$$\begin{aligned} \Delta(m^*) &\equiv \sum_{j \in F(S)} \sum_{T \subseteq N, T \not\subseteq S: F(T) \ni j} \left\{ \left[\sum_{i \in D(S)} L_{ij} - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \right] \zeta_j^*(T) + [m_j^*(T) - \sum_{i \in D(S)} \lambda_i \alpha_{ij}] \beta_j^*(T) \right\} \\ &= \sum_{T \subseteq N, T \not\subseteq S} \sum_{j \in F(T \cap S)} \left\{ \left[\sum_{i \in D(S)} L_{ij} - \frac{m_j^*(S)}{\mu_j - m_j^*(S)} \right] \zeta_j^*(T) + [m_j^*(S) - \sum_{i \in D(S)} \lambda_i \alpha_{ij}] \beta_j^*(T) \right. \\ &\quad \left. + \left[\frac{m_j^*(S)}{\mu_j - m_j^*(S)} - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \right] \zeta_j^*(T) + [m_j^*(T) - m_j^*(S)] \beta_j^*(T) \right\} \leq 0, \end{aligned}$$

where the inequality is due to: (i) $[\sum_{i \in D(S)} L_{ij} - \frac{m_j^*(S)}{\mu_j - m_j^*(S)}] \zeta_j^*(T) \leq 0$ based on (27) and $\zeta_j^*(T) \leq 0$; (ii) $[m_j^*(S) - \sum_{i \in D(S)} \lambda_i \alpha_{ij}] \beta_j^*(T) \leq 0$ based on (28) and $\beta_j^*(T) \leq 0$; and (iii) by using (A.31), $[\frac{m_j^*(S)}{\mu_j - m_j^*(S)} - \frac{m_j^*(T)}{\mu_j - m_j^*(T)}] \zeta_j^*(T) + [m_j^*(T) - m_j^*(S)] \beta_j^*(T) = [\frac{m_j^*(S)}{\mu_j - m_j^*(S)} - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} + \frac{\mu_j (m_j^*(T) - m_j^*(S))}{(\mu_j - m_j^*(T))^2}] \zeta_j^*(T) = \frac{\mu_j [m_j^*(T) - m_j^*(S)]^2}{(\mu_j - m_j^*(S)) [\mu_j - m_j^*(T)]^2} \zeta_j^*(T) \leq 0$ as $\zeta_j^*(T) \leq 0$ and $\mu_j - m_j^*(S) > 0$.

Consequently,

$$\begin{aligned}
& \sum_{k \in S} \phi_k \\
& \leq \min_{L_S, \alpha_S} \left\{ \sum_{i \in D(S)} \sum_{j \in F(S)} L_{ij} [h_i + \gamma_{ij}^* + \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \zeta_j^*(T)] + \sum_{j \in F(S)} \sum_{T \subseteq S: F(T) \ni j} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \right. \\
& \quad \left. + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij} [\theta_{ij}^* + \xi_i^* + \lambda_i (c_{ij} - s_{ij} \gamma_{ij}^*) - \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j^*(T)] - \sum_{i \in D(S)} \xi_i^* \right\} \\
& = \min_{L_S, m_j(S) < \mu_j, j \in F(S), \alpha_S} \left\{ \sum_{i \in D(S)} \sum_{j \in F(S)} L_{ij} [h_i + \gamma_{ij}^* + \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \zeta_j^*(T)] \right. \\
& \quad \left. + \sum_{T \subseteq S} \sum_{j \in F(T)} [m_j^*(T) \beta_j^*(T) - \frac{m_j^*(T)}{\mu_j - m_j^*(T)} \zeta_j^*(T)] \right. \\
& \quad \left. + \sum_{i \in D(S)} \sum_{j \in F(S)} \alpha_{ij} [\theta_{ij}^* + \xi_i^* + \lambda_i (c_{ij} - s_{ij} \gamma_{ij}^*) - \sum_{T \subseteq S: D(T) \ni i, F(T) \ni j} \lambda_i \beta_j^*(T)] - \sum_{i \in D(S)} \xi_i^* \right\} \\
& = \min_{L_S, m_j(S) < \mu_j, j \in F(S), \alpha_S} \mathcal{L}_S(L_S, m_S(T), \alpha_S, \zeta^*, \beta^*, \xi^*, \theta^*, \gamma^*) \\
& = \pi_S(\zeta^*, \beta^*, \xi^*, \theta^*, \gamma^*) \leq z(S).
\end{aligned}$$

The first inequality is because $\Delta(m^*) \leq 0$. The first equality is due to the first-order optimality conditions (A.31). The last inequality above is because $(\zeta^*, \beta^*, \xi^*, \theta^*, \gamma^*)$ is a feasible solution for the optimization problem in (31). It follows that ϕ is in the core of the cooperative game (N, z) for the PDA problem. \blacksquare

Appendix B: Extensions

This part can be found in the unabridged version of the paper available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3654046

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