Optimal Intervention Policies for TJR Postoperative Care Process: Online Supplement

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Lemma 1. Given Assumption 4.5, for any $f(s)$ that is nondecreasing in s, the following holds:

$$\sum_{s'\in\mathbb{S}} p_t(s'|s)f(s') \leq \sum_{s'\in\mathbb{S}} p_{t+1}(s'|s)f(s').$$

Proof of Lemma 1. By the definition of probability function, the following holds true:

$\sum_{s'=0}^{S} p_t(s'|s) = \sum_{s'=0}^{S} p_{t+1}(s'|s) = 1.$

Let $A = \sum_{s'=0}^{S} p_{t+1}(s'|s) - \sum_{s'=0}^{S} p_t(s'|s) = 0$. Then,

$$\begin{aligned}
A &= \Big\{ \sum_{s'=0}^{S} p_{t+1}(s'|s) - \sum_{s'=0}^{S} p_t(s'|s) \Big\} f(0) \\
&\leq p_{t+1}(0|s)f(0) - p_t(0|s)f(0) + \Big\{ \sum_{s'=1}^{S} p_{t+1}(s'|s) - \sum_{s'=1}^{S} p_t(s'|s) \Big\} f(1) \\
&\leq \sum_{s'=0}^{1} p_{t+1}(s'|s)f(s') - \sum_{s'=0}^{1} p_t(s'|s)f(s') + \Big\{ \sum_{s'=2}^{S} p_{t+1}(s'|s) - \sum_{s'=2}^{S} p_t(s'|s) \Big\} f(2),
\end{aligned}$$

since $f(s)$ is a nondecreasing function in s.

Continue in a similar fashion until we arrive at the following:

$A \leq \sum_{s'=0}^{S} p_{t+1}(s'|s)f(s') - \sum_{s'=0}^{S} p_t(s'|s)f(s').$

Since by definition $A = 0$, we have

$0 \leq \sum_{s'=0}^{S} p_{t+1}(s'|s)f(s') - \sum_{s'=0}^{S} p_t(s'|s)f(s').$

Hence, $\sum_{s'=0}^{S} p_t(s'|s)f(s') \leq \sum_{s'=0}^{S} p_{t+1}(s'|s)f(s').$

Proposition 2. Given Assumptions 4.1-4.5, $v_t(s_t)$ is nondecreasing in t for all $s_t \in \mathbb{S}$.

Proof of Proposition 2. We use backward induction to prove the proposition.

Step 1: Show true for $t = T$.

$v_T(s) = r_T(s,a) \geq 0 \geq v_{T-1}(s)$.

For $t \in \{0, 1, ..., T-1\}$, $v_t(s)$ only consists of costs that are all negative values, hence $v_t(s) \leq 0$ always holds true.

Step 2: Suppose true for $t = k+1$.

$v_{k+1}(s) \geq v_k(s)$.

Step 3: Show true for $t = k$.

$$v_k(s) = max\Big\{r_k(s,a) + \textstyle\sum_{s' \in S} p_k^a(s'|s) v_{k+1}(s')\Big\}$$

$$\geq max\Big\{r_{k-1}(s,a) + \textstyle\sum_{s' \in S} p_{k-1}^a(s'|s) v_{k+1}(s')\Big\} \quad \text{(P2.1)}$$

$$\geq max\Big\{r_{k-1}(s,a) + \textstyle\sum_{s' \in S} p_{k-1}^a(s'|s) v_k(s')\Big\} \quad \text{(P2.2)}$$

$$= v_{k-1}(s),$$

where (P2.1) holds due to Assumption 4.4, Proposition 1 and Lemma 1, while (P2.2) holds true due to induction hypothesis that $v_{k+1}(s) \geq v_k(s)$. Hence, $v_{t+1}(s) \geq v_t(s)$ holds true for all t.