Online Appendix: Pre-Positioning and Local-Purchasing for Emergency Operations Under Budget and Supply Uncertainty

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Proofs of Proposition 1 and Proposition 2. Follows the proof of Proposition 5 with $t_x = \infty$ and $n = 1$. □

Lemma A1. *If D and Q share the same marginal density function that is symmetric, then the density function of* $(D - \mu_D) - (Q - \mu_Q)$ *is symmetric about mean zero*.

Proof. We write D and Q in terms of corresponding standard random variables, $D = \mu_D + \sigma_D Z_D$, $Q = \mu_Q + \sigma_Q Z_Q$. Note that $(D - \mu_D) - (Q - \mu_Q) = \sigma_D Z_D - \sigma_Q Z_Q$. The marginal and conditional density functions of (Z_D, Z_Q) satisfy $\phi_{Z_D}(\cdot) = \phi_{Z_Q}(\cdot) = \phi(\cdot)$ with $\phi(z) = \phi(-z)$, $\phi_{Z_D|z_Q}(x \mid y) = \phi_{Z_D|z_Q}(-x \mid -y)$, $\phi_{Z_Q|z_D}(x \mid y) = \phi_{Z_Q|z_D}(-x \mid -y)$ for all x and y (due to symmetry). Let $\Omega(x) = \{z_D, z_Q \mid \sigma_D z_D - \sigma_Q z_Q \ge x\}$. Then

$$P\left[\sigma_D Z_D - \sigma_Q Z_Q \ge x\right] = \iint_{\Omega(x)} \phi_{Z_D|z_Q}\left(z_D \mid z_Q\right)\phi\left(z_Q\right) dz_D dz_Q = \iint_{\Omega(x)} \phi_{Z_D|z_Q}\left(-z_D \mid -z_Q\right)\phi\left(-z_Q\right) dz_D dz_Q$$

$$= \iint_{\{z_D, z_Q \mid \sigma_D z_D - \sigma_Q z_Q \le -x\}} \phi_{Z_D|z_Q}\left(z_D \mid z_Q\right)\phi\left(z_Q\right) dz_D dz_Q$$

(because $\sigma_D(-z_D) - \sigma_Q(-z_Q) \le -x$ for all $(z_D, z_Q) \in \Omega(x)$)

$$= P\left[\sigma_D Z_D - \sigma_Q Z_Q \le -x\right]. \ \square$$

Proof of Proposition 3. Since $b > \bar{b}$, it follows from Proposition 2 that $C(x) = C_a(x)$. To consider the effect of changes in unit cost, we augment expected cost to include parameter c:

$$C(x) = C_a(x) = c\left[\alpha\mu_D + i\mu_T x + (1-\alpha)E\left[S(x)\right] + (v/c - 1)E\left[\left(S(x) - x\right)^+\right]\right]$$

$$x^* = \min\left\{(x_+)^+, b/c\right\} = \min\left\{\left(\bar{F}_{D-Q}^{-1}\left(\beta^*\right)\right)^+, b/c\right\} = \min\left\{\left(\bar{F}_{D-Q}^{-1}\left(\frac{i\mu_T}{v/c-1}\right)\right)^+, b/c\right\} \qquad \text{(i)}$$

(see propositions 1 and 2). Parts (a) – (c). From (i) it follows that x^* is unaffected by changes in α, γ, μ_R, σ_R, and σ_T; is increasing in $1/\mu_T$, v, and b; is decreasing in i and c. To clarify the effects of increasing μ_D and μ_Q, we write in terms of standard random variables, $D = \mu_D + \sigma_D Z_D$, $Q = \mu_Q + \sigma_Q Z_Q$. Let $\Delta = \mu_D - \mu_Q$, which is increasing in μ_D and decreasing in μ_Q. For the sign of $\frac{d(x_+)^+}{d\Delta}$. It is sufficient to analyze the case of $P[D - Q] > \frac{i\mu_T}{v/c-1}$. (If $P\left[D - Q > 0\right] < \frac{i\mu_T}{v/c-1}$, then $\frac{d(x_+)^+}{d\Delta} = 0$. If $P\left[D - Q > 0\right] = \frac{i\mu_T}{v/c-1}$, then either the left or right derivative is zero, with the nonzero directional derivative matching the sign of $\frac{d(x_+)^+}{d\Delta}$ for the case of $P\left[D - Q > 0\right] > \frac{i\mu_T}{v/c-1}$.) Note that

$$1-\frac{i\mu_T}{v/c-1}=P\left[\left(\Delta+\sigma_D Z_D-\sigma_Q Z_Q\right)^+\le x_+\right]=P\left[\sigma_D Z_D-\sigma_Q Z_Q\le x_+-\Delta\right]. \quad \text{(ii)}$$

Thus, to maintain the above identity, any increase (decrease) in Δ must be offset by an increase (decrease) in x_+. Therefore, $\frac{d\left(x_+\right)^+}{d\Delta}\ge 0$, i.e., x^* is increasing in μ_D and decreasing in μ_Q. Part (*d*). Rewrite the right-hand side of (ii) as $P\left[k\times\left(\sigma_D Z_D-\sigma_Q Z_Q\right)\le x_+-\Delta\right]=P\left[\sigma_D Z_D-\sigma_Q Z_Q\le\frac{x_+-\Delta}{k}\right]$ where k = 1. Thus, $\frac{x_+-\Delta}{k}$ is decreasing (increasing) in proportionality factor k if $x_+ > \mu_D - \mu_Q$ ($x_+ < \mu_D - \mu_Q$). Therefore, x_+ (and x^*) is increasing (decreasing) in k if $x_+ > \mu_D - \mu_Q$ ($x_+ < \mu_D - \mu_Q$). Part (*e*). Due to Lemma A1, $\zeta = \sigma_D Z_D - \sigma_Q Z_Q$ is a symmetric random variable about mean zero. Recall that (Z_D, Z_Q) correlation (and covariance) can be expressed as $\iint z_D z_Q \phi_{Z_D|z_Q}\left(z_D \mid z_Q\right)\phi_{Z_Q}\left(z_Q\right)dz_D dz_Q$ = $\iint z_D z_Q \phi_{Z_Q|z_D}\left(z_Q \mid z_D\right)\phi_{Z_D}\left(z_D\right)dz_Q dz_D$ (which is also the value of the (D, Q) correlation coefficient). If we keep the marginal distributions fixed (i.e., other things equal), we see that an increase in negative correlation, corresponds to expansions (compressions) of the density $\phi\left(z_D \mid z_Q\right)$ for realizations where z_D is small (large) and z_Q is large (small). The consequent effect is an increase in the tails and a reduction in the mid-region of the symmetric density function (about mean zero) of random variable ζ. Thus, if $x_+ - \Delta > 0$ ($x_+ - \Delta < 0$), an increase in negative correlation leads to an decrease (increase) in $P\left[\sigma_D Z_D-\sigma_Q Z_Q\le x_+-\Delta\right]$, and from (ii) it follows that x_+ (and x^*) is increasing (decreasing) in negative correlation. Similarly, an increase in σ_D or σ_Q increases the variance of ζ, i.e., the tails of the symmetric density function of ζ expand and the mid-region shrinks. Therefore, if $x_+ - \Delta > 0$ ($x_+ - \Delta < 0$), an increase in σ_D or σ_Q leads to a decrease (increase) in $P\left[\sigma_D Z_D-\sigma_Q Z_Q\le x_+-\Delta\right]$, and thus from (ii) it follows that x_+ (and x^*) is increasing (decreasing) in σ_D or σ_Q. □

Proof of Proposition 4. In the following analysis, we assume $x^o > 0$. This assumption does not affect our conclusions, e.g., if $(x^o)^+ = 0$, then $(x^o)^+$ remains unchanged as a parameter changes over some interval. Outside of this interval, the directional effects we obtain for the case of $x^o > 0$ apply. As in the proof of Proposition 3, we augment relevant expressions to include parameter c in order to analyze the effect of changes in unit cost: $\hat{b}\left(x,R,T\right) = b + \gamma T + R - cx$, $S(x) = \left(D-\min\left\{Q,\frac{\hat{b}\left(x,R,T\right)}{\alpha c}\right\}\right)^+$, $C(x)$ = $c\left[\alpha\mu_D+i\mu_T x+\left(1-\alpha\right)E\left[S\left(x\right)\right]+\left(\frac{v}{c}-1\right)E\left[\left(S\left(x\right)-x\right)^+\right]\right]$. We write ($D$, Q, R, T) in terms of standard random variables: $D = \mu_D + \sigma_D Z_D$, $Q = \mu_Q + \sigma_Q Z_Q$, $R = \mu_R + \sigma_R Z_R$, $T = \mu_T + \sigma_T Z_T$ with $E[Z_j] = 0$, $V[Z_j] = 1$,

$j \in \{D, Q, R, T\}$, pdf ϕ_{DQR} of (Z_D, Z_Q, Z_R), and pdf ϕ_T of Z_T. Let (z_D, z_Q, z_R, z_T) denote a realization of (Z_D, Z_Q, Z_R, Z_T). Then $C'(x) = cm(x) = c[m_c(x) - m_s(x)]$, $m_c(x) = i\mu_T + \frac{1-\alpha}{\alpha}\left(P\left[\Omega_1(x)\right] + \left(\frac{v}{c}-1\right)P\left[\Omega_2(x)\right]\right)$,

$m_s(x) = \left(\frac{v}{c}-1\right)P\left[\Omega_3(x)\right]$ and the sign of $x^{o\prime}(u)$ for $u \in \{\alpha, b, c, i, v, \mu_D, \mu_Q, \mu_Q, \mu_T, \mu_R, \sigma_D, \sigma_Q, \sigma_T, \sigma_R\}$ is the opposite of the sign of $\frac{\partial m(x^o)}{\partial u}$ (follows from the implicit function theorem and the convexity of $C(x)$). Parameters i and v are not included in Ω_i, It is clear that $\frac{\partial m(x^o)}{\partial i} = \mu_T > 0$, which implies x^* is decreasing in i. Note that $P[\Omega_1] \geq P[\Omega_2]$ (because $\Omega_2 \subset \Omega_1$),

$$0 = m(x^o) = i\mu_T + \frac{1-\alpha}{\alpha}\left(P\left[\Omega_1(x^o)\right] + \left(\frac{v}{c}-1\right)P\left[\Omega_2(x^o)\right]\right) - \left(\frac{v}{c}-1\right)P\left[\Omega_3(x^o)\right]$$

$$= \frac{v}{c}\left(\frac{1-\alpha}{\alpha}P\left[\Omega_2(x^o)\right] - P\left[\Omega_3(x^o)\right]\right) + i\mu_T + \frac{1-\alpha}{\alpha}\left(P\left[\Omega_1(x^o)\right] - P\left[\Omega_2(x^o)\right]\right) + P\left[\Omega_3(x^o)\right]$$

$$\geq \frac{v}{c}\left(\frac{1-\alpha}{\alpha}P\left[\Omega_2(x^o)\right] - P\left[\Omega_3(x^o)\right]\right), \tag{iii}$$

which implies $\frac{\partial m(x^o)}{\partial v} \leq 0$. Therefore, x^* is increasing in v. Note that

$$\frac{\partial P\left[\Omega_1(x^o)\right]}{\partial u} \leq 0 \text{ for } u \in \{b, \gamma, \mu_R, \mu_T\},\ \frac{\partial P\left[\Omega_1(x^o)\right]}{\partial u} \geq 0 \text{ for } u \in \{\alpha, c, \mu_D, \mu_Q\} \tag{iv}$$

$$\frac{\partial P\left[\Omega_2(x^o)\right]}{\partial u} \leq 0 \text{ for } u \in \{b, \gamma, \mu_R, \mu_T\},\ \frac{\partial P\left[\Omega_2(x^o)\right]}{\partial u} \geq 0 \text{ for } u \in \{\alpha, c, \mu_D, \mu_Q\} \tag{v}$$

$$\frac{\partial P\left[\Omega_3(x^o)\right]}{\partial u} \geq 0 \text{ for } u \in \{b, \gamma, \mu_D, \mu_R, \mu_T\},\ \frac{\partial P\left[\Omega_3(x^o)\right]}{\partial u} \leq 0 \text{ for } u \in \{\alpha, c, \mu_Q\}. \tag{vi}$$

$$\frac{\partial m(x^o)}{\partial u} = \frac{\partial}{\partial u}\left[i\mu_T + \left(\frac{1}{\alpha}-1\right)\begin{pmatrix} P\left[\Omega_1(x^o)\right] + \\ \left(\frac{v}{c}-1\right)P\left[\Omega_2(x^o)\right] \end{pmatrix} - \left(\frac{v}{c}-1\right)P\left[\Omega_3(x^o)\right]\right] \tag{vii}$$

is nonpositive for $u \in \{b, \gamma, \mu_R\}$ (follows from (iv) – (vi)). Accordingly, $x^* = \min\{x^o, b/c\}$ is increasing in b, γ, and μ_R. Similarly, (vii) is nonnegative for $u = \mu_Q$. Applying (iv) – (vi) and noting inequality (iii), it follows that (vii) is nonnegative for $u = c$. Accordingly, i.e., $x^* = \min\{x^o, b/c\}$ is decreasing in c and μ_Q. The sign of (vii) is ambiguous for $u \in \{\alpha, \mu_D, \mu_T\}$, i.e., there are some elements of the $m(x^o)$ function that are increasing in these parameters and some that are decreasing. However, if $i = 0$, then it is clear that

$\partial m(x^{o}) / \partial \mu_T \le 0$, and thus x^* is increasing in μ_T (and decreasing in $1/\mu_T$). Lastly, we consider $\partial m(x^{o}) / \partial u$ for $u \in \{\sigma_D, \sigma_Q, \sigma_R, \sigma_T\}$. Note that the sign of $\sigma_D z_D$ depends on the sign of realization z_D of Z_D, which can be negative and can be positive (i.e., recall $E[Z_D] = 0$). As a consequence, just as in Proposition 3, the value of $\partial m(x^{o}) / \partial \sigma_D$ can be positive or negative depending on the parameter values and distribution functions. The same arguments apply to σ_Q, σ_R, and σ_T. □

Proof of Proposition 5. Part (*a*). $\bar{C}_n(x, y_n)$ can be expressed as the sum of two univariate functions: $H_n(y_n)$ = cost of holding x_1 in period 1 + … + cost of holding x_{n-1} in period $n-1 = it_x y_n$, $K_n(x)$ = expected cost of holding x in period n and cost of disaster given that the disaster occurs in period n = $i\mu_n x + (1-\alpha)E[S_n(x)] + (v-1)E(S_n(x)-x)^+$. Define J_n^* = optimal expected cost-to-go excluding holding cost of past prepo levels and given the disaster occurs in period n or later, and note that J_n^* = $C_n^*(y_n) - H_n(y_n) = C_n^*(y_n) - it_x y_n$, which is independent of y_n (i.e., y_n only affects the holding cost over past periods). Substituting the above into (6),

$$C_n^*(y_n) = \min_{x \le b_n} \left\{ \begin{array}{l} [H_n(y_n) + K_n(x)] P[T \in [(n-1)t_x, nt_x) \mid T \ge (n-1)t_x] \\ + [H_{n+1}(x+y_n) + J_{n+1}^*] P[T \ge nt_x \mid T \ge (n-1)t_x] \end{array} \right\}$$

$$= \left(\begin{array}{l} \min_{x \le b_n} \left\{ \begin{array}{l} it_x y_n + i\mu_n x + (1-\alpha)E[S_n(x)] \\ + (v-1)E(S_n(x)-x)^+ \\ + it_x(x+y_n) \dfrac{P[T \ge nt_x \mid T \ge (n-1)t_x]}{P[T \in [(n-1)t_x, nt_x) \mid T \ge (n-1)t_x]} \end{array} \right\} \\ \times P[T \in [(n-1)t_x, nt_x) \mid T \ge (n-1)t_x] + J_{n+1}^* P[T \ge nt_x \mid T \ge (n-1)t_x] \end{array} \right).$$

From the above expression, it follows that $x_n^* = \arg\min_{x \le b_n} \hat{C}_n(x, y_n)$ where

$$\hat{C}_n(x, y_n) = \left(\begin{array}{l} it_x y_n + i\mu_n x + (1-\alpha)E[S_n(x)] + (v-1)E(S_n(x)-x)^+ \\ + it_x(x+y_n) \dfrac{P[T \ge nt_x \mid T \ge (n-1)t_x]}{P[T \in [(n-1)t_x, nt_x) \mid T \ge (n-1)t_x]} \end{array} \right).$$

$$\frac{dE[S_n(x)]}{dx} = \frac{1}{\alpha} P[\Omega_1(x)]$$

$$\frac{dE(S_n(x)-x)^+}{dx} = \frac{1-\alpha}{\alpha} P[\Omega_2(x)] - P[\Omega_3(x)]$$

$$\frac{d^2 E[S_n(x)]}{dx^2} = \frac{1}{\alpha} \frac{d}{dx} P\left[D > \frac{b_n + \gamma T_n + R - x}{\alpha},\ Q > \frac{b_n + \gamma T_n + R - x}{\alpha} \right] \ge 0$$

$$\frac{d^2E\left(S_n\left(x\right)-x\right)^+}{dx^2}=\frac{1-\alpha}{\alpha}\frac{d}{dx}P\begin{bmatrix} D>\frac{b_n+\gamma T_n+R}{\alpha}-\left(\frac{1-\alpha}{\alpha}\right)x, \\ Q>\frac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix}-\frac{d}{dx}P\begin{bmatrix} D>Q+x, \\ Q\le\frac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix}\ge 0$$

$$p_n=P\left[T\in\left[\left(n-1\right)t_x,nt_x\right)\mid T\ge\left(n-1\right)t_x\right].$$

Therefore, $\hat{C}_n\left(x,y_n\right)$ is convex in x, and the unconstrained optimal prepo satisfies $0=\frac{\partial\hat{C}_n\left(x^{\circ},y_n\right)}{\partial x}=$ $m_c\left(x\right)-m_s\left(x\right)=m\left(x\right)$, $m_c\left(x\right)=i\left(\mu_n+t_x\frac{1-p_n}{p_n}\right)+\frac{1-\alpha}{\alpha}\left(P\left[\Omega_1\left(x\right)\right]+\left(v-1\right)P\left[\Omega_2\left(x\right)\right]\right)$, $m_s\left(x\right)=$ $\left(v-1\right)P\left[\Omega_3\left(x\right)\right]$. Thus, (7) follows from KKT conditions. To prove the upper bound, note that

$$m_c\left(x\right)=i\left(\mu_n+t_x\frac{1-p_n}{p_n}\right)+\left(\frac{1-\alpha}{\alpha}\right)\left(P\left[\Omega_1\left(x\right)\right]+\left(v-1\right)P\left[\Omega_2\left(x\right)\right]\right)\ge i\left(\mu_n+t_x\frac{1-p_n}{p_n}\right)=\underline{m}_c\left(x\right)$$

$$m_s\left(x\right)=\left(v-1\right)P\left[D>Q+x,Q\le\frac{b_n+\gamma T_n+R-x}{\alpha}\right]\le\left(v-1\right)\overline{F}_{D-Q}\left(x\right)=\overline{m}_s\left(x\right).$$

$$0=\frac{\partial\hat{C}_n\left(x^{\circ},y_n\right)}{\partial x}=m\left(x^{\circ}\right)=m_c\left(x^{\circ}\right)-m_s\left(x^{\circ}\right)\ge\underline{m}_c\left(x^{\circ}\right)-\overline{m}_s\left(x^{\circ}\right)=\underline{m}\left(x^{\circ}\right),$$

which implies $x_+=\underline{m}^{-1}\left(0\right)\ge m^{-1}\left(0\right)=x^{\circ}$, and thus $x_n^*\le\overline{x}_n^*$. Part (*b*). From the above proof, we see that $\overline{x}_n^*$ is the optimal prepo for an alternative model where the local budget constraint is ignored, or equivalently, the local budget is infinity. That is, under an infinite local budget, $P\left[\Omega_1\left(x\right)\right]=P\left[\Omega_2\left(x\right)\right]=0$, $P\left[\Omega_3\left(x\right)\right]$ $P\left[D>Q+x\right]=\overline{F}_{D-Q}\left(x\right)$. Note that $\max\left\{\alpha\min\left\{d,q\right\}-\gamma t-r:\left(d,q,r,t\right)\in\Omega\right\}=$ $\max\left\{\alpha\min\left\{d,q\right\}-r:\left(d,q,r\right)\in\Omega\right\}-\gamma\underline{T}_n$ is an upper limit on the funds needed for local spend. Thus, if the budget at the beginning of time n is $\overline{b}_n=\max\left\{\alpha\min\left\{d,q\right\}-r:\left(d,q,r\right)\in\Omega\right\}-\gamma\underline{T}_n+\left(x_+\right)^+$ or more (i.e., $b_n\ge\overline{b}_n$), then the local spend constraint is assured to be not binding, which implies that the optimal solution obtained when ignoring the local spend constraint is optimal for the true problem, i.e., $x_n^*=\overline{x}_n^*$.

Part (*c*). Note that

$$P\begin{bmatrix} D>\frac{b_n+\gamma T_n+R-x}{\alpha}, \\ Q>\frac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix}\le P\left[D>\frac{b_n+\gamma T_n+R-x}{\alpha}\right]P\left[Q>\frac{b_n+\gamma T_n+R-x}{\alpha}\right]$$

$$\le\overline{F}_Q\left(\frac{b_n-x}{\alpha}\right)\overline{F}_D\left(\frac{b_n-x}{\alpha}\right)$$

$$P\begin{bmatrix} D > \frac{b_n+\gamma T_n+R}{\alpha} - \left(\frac{1-\alpha}{\alpha}\right)x, \\ Q > \frac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix} \le P\left[D > \frac{b_n+\gamma T_n+R-x}{\alpha}+x\right] P\left[Q > \frac{b_n+\gamma T_n+R-x}{\alpha}\right]$$

$$\le \bar{F}_Q\left(\frac{b_n-x}{\alpha}\right)\bar{F}_D\left(\frac{b_n-x}{\alpha}+x\right)$$

$$P\begin{bmatrix} D > Q+x, \\ Q \le \frac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix} \ge P\left[D > Q+x\ \right] P\left[Q \le \frac{b_n+\gamma T_n+R-x}{\alpha}\right] \ge \bar{F}_{D-Q}(x) F_Q\left(\frac{b_n-x}{\alpha}\right)$$

where the first inequality in each case follows from Assumption 1, and the second inequality follows from nonnegativity of T_n and R. Therefore,

$$m_c(x) = i\left(\mu_n + t_x\frac{1-p_n}{p_n}\right) + \left(\frac{1-\alpha}{\alpha}\right)\left(\begin{array}{l} P\begin{bmatrix} D > \frac{b_n+\gamma T_n+R-x}{\alpha}, \\ Q > \frac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix} \\ +(v-1)P\begin{bmatrix} D > \frac{b_n+\gamma T_n+R}{\alpha} - \left(\frac{1-\alpha}{\alpha}\right)x, \\ Q > \frac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix} \end{array}\right)$$

$$\le i\left(\mu_n + t_x\frac{1-p_n}{p_n}\right) + \left(\frac{1-\alpha}{\alpha}\right)\bar{F}_Q\left(\frac{b_n-x}{\alpha}\right)\left(\begin{array}{l} \bar{F}_D\left(\frac{b_n-x}{\alpha}\right) \\ +(v-1)\bar{F}_D\left(\frac{b_n-x}{\alpha}+x\right) \end{array}\right) = \bar{m}_c(x)$$

$$m_s(x) = (v-1)P\left[D > Q+x, Q \le \frac{b_n+\gamma T_n+R-x}{\alpha}\right] \ge (v-1)\bar{F}_{D-Q}(x)F_Q\left(\frac{b_n-x}{\alpha}\right) = \underline{m}_s(x).$$

Therefore, $0 = \frac{\partial \hat{C}_n(x^{o}, y_n)}{\partial x} = m(x^{o}) = m_c(x^{o}) - m_s(x^{o}) \le \bar{m}_c(x^{o}) - \underline{m}_s(x^{o}) = \bar{m}(x^{o})$, which implies $x_- = \bar{m}^{-1}(0) \le m^{-1}(0) = x^{o}$, and thus $x_n^* \ge \underline{x}_n^*$. Part ($d$). Note that $x_n^* = \min\left\{(x^{o})^+, b_n\right\} \ge \underline{x}_{n-} = \min\left\{(x_-)^+, b_n\right\}$, $x^{o} \ge x_-$ and $(x^{o})^+ \ge (x_-)^+$. Therefore, if $\underline{x}_{n-} = b_n$, then $x_n^* = \min\left\{(x^{o})^+, b_n\right\} = b_n = \underline{x}_{n-}$. Part ($e$). The inequalities are proved above in Part (c). □

Proof of Lemma 1. $\mu_n^+ = E\left[T-(n-1)t_x \mid (n-1)t_x \le T < nt_x\right]p_n + E\left[T-(n-1)t_x \mid T \ge nt_x\right](1-p_n) =$ $\mu_n p_n + \left(t_x + E\left[T - nt_x \mid T \ge nt_x\right]\right)(1-p_n) = \mu_n p_n + \left(\mu_n^+ + t_x - \left(\mu_n^+ - \mu_{n+1}^+\right)\right)(1-p_n) =$ $\mu_n + \left(t_x - \left(\mu_n^+ - \mu_{n+1}^+\right)\right)\frac{1-p_n}{p_n}$ (we move μ_n^+ to the left and divide by p_n). Rearranging yields Lemma 1. □

Proof of Corollary 1. Due to the memoryless property of an exponential random variable, $\mu_n^+ =$ $E\left[T-(n-1)t_x \mid T-(n-1)t_x \ge 0\right] = E\left[T-nt_x \mid T-nt_x \ge 0\right] = \mu_{n+1}^+$, $\mu_n^+ = E\left[T-(n-1)t_x \mid T-(n-1)t_x \ge 0\right] =$ $E\left[T \mid T \ge 0\right] = \mu_1^+ = \mu_T$. Therefore, from Lemma 1, $\mu_n + t_x \dfrac{1-p_n}{p_n} = \mu_T$. □

Proof of Proposition 6. Part (*a*). $x^* = \min\left\{\left(m^{-1}(0)\right)^+, b\right\}$, $m(x) = m_c(x) - m_s(x)$, $x_n^* = \min\left\{\left(m^{-1}(0)\right)^+, b_n\right\}$, $m(x) = m_c(x) - m_s(x)$, $b_n = b + \gamma(n-1)t_x$,

$$m_c(x) = i\mu_T + \frac{1-\alpha}{\alpha}\left(P\begin{bmatrix} D > \dfrac{b+\gamma T+R-x}{\alpha}, \\ Q > \dfrac{b+\gamma T+R-x}{\alpha} \end{bmatrix} + (v-1)P\begin{bmatrix} D > \dfrac{b+\gamma T+R-x}{\alpha}+x, \\ Q > \dfrac{b+\gamma T+R-x}{\alpha} \end{bmatrix}\right)$$

$$m_s(x) = (v-1)P\left[D > Q+x, Q \le \frac{b+\gamma T+R-x}{\alpha}\right]$$

$$m_c(x) = i\mu_T + \frac{1-\alpha}{\alpha}\left(P\begin{bmatrix} D > \dfrac{b_n+\gamma T_n+R-x}{\alpha}, \\ Q > \dfrac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix} + (v-1)P\begin{bmatrix} D > \dfrac{b_n+\gamma T_n+R-x}{\alpha}+x, \\ Q > \dfrac{b_n+\gamma T_n+R-x}{\alpha} \end{bmatrix}\right)$$

$$m_s(x) = (v-1)P\left[D > Q+x, Q \le \frac{b_n+\gamma T_n+R-x}{\alpha}\right].$$

Note that $f_{T_n}(t) = \dfrac{f_T(t)}{F_T(t_x)} \ge f_T(t)$ and $b_1 = b$. Thus, for any realization (d, q, r) of (D, Q, R) we have

$$P\begin{bmatrix} d > \dfrac{b_1+\gamma T_1+r-x}{\alpha}, \\ q > \dfrac{b_1+\gamma T_1+r-x}{\alpha} \end{bmatrix} \ge P\begin{bmatrix} d > \dfrac{b+\gamma T+r-x}{\alpha}, \\ q > \dfrac{b+\gamma T+r-x}{\alpha} \end{bmatrix},\ P\begin{bmatrix} d > \dfrac{b_1+\gamma T_1+r-x}{\alpha}+x, \\ q > \dfrac{b_1+\gamma T_1+r-x}{\alpha} \end{bmatrix} \ge$$

$$P\begin{bmatrix} d > \dfrac{b+\gamma T+r-x}{\alpha}+x, \\ q > \dfrac{b+\gamma T+r-x}{\alpha} \end{bmatrix},\ P\left[q \le \frac{b_1+\gamma T_1+r-x}{\alpha}\right] \le P\left[q \le \frac{b+\gamma T+r-x}{\alpha}\right].$$ Therefore, for $n = 1$,

$m_c^{\text{dynamic}}(x) \ge m_c^{\text{static}}(x)$ and $m_s^{\text{dynamic}}(x) \le m_s^{\text{static}}(x)$, which implies $x_1^* \le x^*$. Note that the probability distribution of T_1 and T_n for any n are identical. As noted after Table 4, $b_n \le b_{n+1}$ for all n. Therefore, for the dynamic model, m_c is decreasing in n and m_s is increasing in n, which implies $x_n^* \le x_{n+1}^*$ for all n. Part (*b*). If $\gamma = 0$, then $b = b_n$ and $\hat{b}(x,R,T) = \hat{b}_n(x,R,T_n)$ for any n, i.e., there is no prepo budget effect and no local budget effect. From Corollary 1, there is no marginal holding effect. Therefore, $x^* = x_n^*$ for all n. Part (*c*). Since T is an exponential random variable, we have $\underline{T} = \underline{T}_n = 0$, and thus $\bar{b} =$

$\max\{\alpha\min\{d,q\}-r:(d,q,r)\in\Omega\}+\left(\bar{F}_{D-Q}^{-1}\left(\beta^*\right)\right)^+ = \bar{b}_n$ for all n. If $b \geq \bar{b}$, then from Proposition 2, x^* = $\bar{x}^* = \left(\bar{F}_{D-Q}^{-1}\left(\beta^*\right)\right)^+$. Furthermore, from $b = b_1 \leq b_2 \leq b_3 \leq \ldots$, it follows that $b_n \geq \bar{b} = \bar{b}_n$ for all n. Then from Proposition 5, $x_n^* = \bar{x}_n^* = \left(\bar{F}_{D-Q}^{-1}\left(\beta^*\right)\right)^+ = x^*$ for all n. Part (*d*). $\bar{m}_c(x) = i\mu_T +$ $\frac{1-\alpha}{\alpha}\bar{F}_Q\left(\frac{b-x}{\alpha}\right)\left(\bar{F}_D\left(\frac{b-x}{\alpha}\right)+(v-1)\bar{F}_D\left(\frac{b-x}{\alpha}+x\right)\right)$, $\underline{m}_s(x) = (v-1)\bar{F}_{D-Q}(x)F_Q\left(\frac{b-x}{\alpha}\right)$, $\bar{m}_c(x) =$ $i\mu_T + \frac{1-\alpha}{\alpha}\bar{F}_Q\left(\frac{b_n-x}{\alpha}\right)\left(\bar{F}_D\left(\frac{b_n-x}{\alpha}\right)+(v-1)\bar{F}_D\left(\frac{b_n-x}{\alpha}+x\right)\right)$ which is decreasing in n (because $b_n < b_{n+1}$ for all n; the inequality is strict because $\gamma > 0$), $\underline{m}_s(x) = (v-1)\bar{F}_{D-Q}(x)F_Q\left(\frac{b_n-x}{\alpha}\right)$ which is increasing in n (because $b_n < b_{n+1}$ for all n). Therefore, $x_- = \bar{m}^{-1}(0)$ is increasing in n and $\underline{x}_n^* = \min\left\{\left(\bar{m}^{-1}(0)\right)^+, b_n\right\}$ is increasing in n. Furthermore, since $b_1 = b$, the marginal cost and savings functions are identical for the static model and the dynamic model when $n = 1$, and thus,

$$\underline{x}^* = \underline{x}_1^* \leq \underline{x}_2^* \leq \underline{x}_3^* \cdots \qquad \text{(viii)}$$

Therefore, if $\underline{x}^* = b$ and $\underline{x}_1^* = b = b_1$, and it follows from propositions 2 and 5 that $x^* = x_1^* = b$. From Proposition 5, $x_n^* \geq \underline{x}_n^*$ for all n, which in conjunction with (viii) implies $x^* \leq x_n^*$ for all n. □

Proof of Proposition 7. If $\gamma = 0$, then the result follows from Proposition 6(b). Suppose $\gamma > 0$. Given exponential T and prepo budget b, the marginal cost and savings functions are $m_c(x) =$ $i\mu_T + \frac{1-\alpha}{\alpha}\left(P\left[\Omega_1(x)\right]+(v-1)P\left[\Omega_2(x)\right]\right)$, $m_s(x) = (v-1)P\left[\Omega_3(x)\right]$ where Ω = support of (D, Q, R, T_1) and T_1 = random time to disaster given that the disaster occurs within the review period of length t_x. The cdf of T_1 is $F_{T_1}(x) = \max\left\{0, \min\left\{\left(1-e^{\frac{-x}{\mu_T}}\right)\left(1-e^{\frac{-t_x}{\mu_T}}\right)^{-1}, 1\right\}\right\}$. Consider review periods $t_x^a < t_x^b$ with corresponding conditional random times to disaster, T_1^a and T_1^b. Note that T_1^b has first-order stochastic dominance over T_1^a, i.e., $1-F_{T_1^b}(x) \geq 1-F_{T_1^a}(x)$ for all x. Therefore $P\left[\Omega_1\left(x\,|\,T_1^b\right)\right] \leq P\left[\Omega_1\left(x\,|\,T_1^a\right)\right]$, $P\left[\Omega_2\left(x\,|\,T_1^b\right)\right] \leq P\left[\Omega_2\left(x\,|\,T_1^a\right)\right]$, $P\left[\Omega_3\left(x\,|\,T_1^b\right)\right] \geq P\left[\Omega_3\left(x\,|\,T_1^a\right)\right]$, which implies $m_c\left(x\,|\,T_1^b\right) \leq m_c\left(x\,|\,T_1^a\right)$, $m_s\left(x\,|\,T_1^b\right) \geq m_s\left(x\,|\,T_1^a\right)$, and thus $x_{1|T_1^a}^* \leq x_{1|T_1^b}^*$. □