Online Appendix for *Intraday Scheduling with Patient Re-entries and Variability in Behaviors*

Appendix A: Proofs and other Technical Results

Proof of Proposition 3. Notice that

$$k \log \mathbb{E}\left[\exp\left(\frac{\sum_{s=0}^{t} z^{t,s} - C}{k}\right)\right] = k \log \mathbb{E}\left[\exp\left(\frac{\sum_{s=0}^{t} z^{t,s}}{k}\right)\right] - C.$$

Therefore, we just need to reformulate $k \log \mathbb{E}\left[\exp\left(\frac{\sum_{s=0}^{t} z^{t,s}}{k}\right)\right]$. Recall that by Proposition 1, the state variable $z^{t,s}$ can be written as $z^{t,s} \sim \text{Bin}\left(z^{t-s,0}, \hat{h}^{t,s}\right)$. Evaluating its moment generating function and using $z^{t-s,0} = \sum_{\tau=0}^{t-s} p^{t-s,\tau}$,

$$\mathbb{E}\big[\exp\left(z^{t,s}/k\right)\big] = \exp\left(\sum_{\tau=0}^{t-s} p^{t-s,\tau} \log\left(1 - \hat{h}^{t,s} + \hat{h}^{t,s} \exp\left(\frac{1}{k}\right)\right)\right).$$

Notice that $z^{t,s} \sim \text{Bin}\left(z^{t-s,0}, \hat{h}^{t,s}\right)$, which also indicates that for a fixed time t, the state variables $z^{t,s}$ are *independent* across $s \in \mathcal{T}$. Thus, the complete reformulation can be written as

$$\begin{aligned}
&k \log \mathbb{E}\left[\exp\left(\frac{\sum_{s=0}^{t} z^{t,s}}{k}\right)\right] = \sum_{s=0}^{t} k \log \mathbb{E}\left[\exp\left(\frac{z^{t,s}}{k}\right)\right] \\
=&k \log \mathbb{E}\left[\exp\left(\frac{z^{t,0}}{k}\right)\right] + \sum_{s=1}^{t} k \log \mathbb{E}\left[\exp\left(\frac{z^{t,s}}{k}\right) \Big| z^{t-s,0}\right] \\
=&\sum_{\tau=0}^{t} p^{t,\tau} + \sum_{s=1}^{t} k \log\left[\exp\left(\sum_{\tau=0}^{t-s} p^{t-s,\tau} \log\left(1 - \hat{h}^{t,s} + \hat{h}^{t,s} \exp\left(\frac{1}{k}\right)\right)\right)\right] = \sum_{s=0}^{t} \beta^{t,s} \sum_{\tau=0}^{t-s} p^{t-s,\tau}
\end{aligned}$$

$\square$

Proof of Proposition 4. Recall from (2) that $y^{t,s} = y^{t-s,0} - \sum_{\tau=0}^{s-1} p^{t-\tau,s-\tau}$. In addition, for any time t, $y^{t-s,0} = x_{t-s} + \lambda^{t-s}$ are assumed to be independent across s. As such, we can derive (14):

$$\begin{aligned}
&k \log \mathbb{E}\left[\exp\left(\frac{\sum_{s=0}^{t} r(s) y^{t,s}}{k}\right)\right] \\
=&k \log \mathbb{E}\left[\exp\left(\sum_{s=0}^{t} \frac{r(s)}{k}\left(y^{t-s,0} - \sum_{\tau=0}^{s-1} p^{t-\tau,s-\tau}\right)\right)\right] \\
=&\sum_{s=0}^{t} k \log \mathbb{E}\left[\exp\left(\frac{r(s)}{k} y^{t-s,0}\right)\right] - \sum_{s=0}^{t} k \log\left[\exp\left(\frac{r(s)}{k} \sum_{\tau=0}^{s-1} p^{t-\tau,s-\tau}\right)\right] \\
=&\sum_{s=0}^{t} r(s) x_{t-s} + \sum_{s=0}^{t} k \log \mathbb{E}\left[\exp\left(\lambda_{t-s} r(s)/k\right)\right] - \sum_{s=0}^{t} r(s) \sum_{\tau=0}^{s-1} p^{t-\tau,s-\tau} \\
=&\sum_{s=0}^{t} r(s)\left(x_{t-s} - \sum_{\tau=0}^{s-1} p^{t-\tau,s-\tau}\right) + \sum_{s=0}^{t} k \log \mathbb{E}\left[\exp\left(\lambda_{t-s} r(s)/k\right)\right]
\end{aligned}$$

$\square$

Proof of Proposition 5. The reformulation follows from the fact that $p^{t,s} - y^{t-1,s-1} = -y^{t,s}$.

$$\begin{aligned}
&k\log\mathbb{E}\left[\exp\left(\frac{p^{t,s}-y^{t-1,s-1}}{k}\right)\right]\\
=&k\log\mathbb{E}\left[\exp\left(\frac{\sum_{\tau=0}^{s-1}p^{t-\tau,s-\tau}-y^{t-s,0}}{k}\right)\right]\\
=&\sum_{\tau=0}^{s-1}p^{t-\tau,s-\tau}-x_{t-s}+k\log\mathbb{E}\left[\exp\left(-\lambda_{t-s}/k\right)\right]
\end{aligned}$$

□

Proof of Proposition 6. For any patient ℓ that has been in service since time $t-s$, by conditioning on $b_{\ell,t-s}$, we have $\mathbb{P}\Big(\mathbb{1}(\text{patient } \ell \text{ stays till time } t)=1\Big)=\hat{h}_1^{t,s}$, where

$$\hat{h}_1^{t,s} \triangleq q\prod_{\tau=1}^{s}(1-h_1^{t-\tau,s-\tau})+(1-q)\prod_{\tau=1}^{s}(1-h_0^{t-\tau,s-\tau}).$$

Therefore, we can write

$$\mathbb{1}(b_{\ell,t-s}=1,\text{ patient } \ell \text{ stays till time } t)+\mathbb{1}(b_{\ell,t-s}=0,\text{ patient } \ell \text{ stays till time } t)\sim\text{Bernoulli}(\hat{h}_1^{t,s}).$$

Because random variables on different patients are independent, equation (19) is a sum of $z_1^{t-s,0}$ identical and independent Bernoulli random variables, resulting in $z_1^{t,s}\sim\text{Bin}(z_1^{t-s,0},\hat{h}_1^{t,s})$.

To prove b), first notice that the first consultation service times are *i.i.d.* for all patients requiring X-ray examination. In addition, by definition, we have $\mathbb{P}\Big(u_\ell^{t-1,s}=1\Big|b_{\ell,t-s-1}=1\Big)=h_1^{t-1,s}\prod_{\tau=1}^{s}(1-h_1^{t-1-\tau,s-\tau})$ for all $t\in[T], s=0,\dots,t-1, \ell\in[z^{t-s-1}]$. Therefore, $\mathbb{P}\Big(b_{\ell,t-s-1}=1,u_\ell^{t-1,s}=1\Big)=\bar{h}_1^{t-1,s}$, which indicates $\mathbb{1}\left(b_{\ell,t-s-1}=1,u_\ell^{t-1,s}=1\right)\sim\text{Bernoulli}\left(\bar{h}_1^{t-1,s}\right)$. It now follows from independence.

Part c) easily follows from the definition of $y_3^{t,0}$ and Proposition 1. □

Proof of Proposition 7. In the proof of Proposition 4, we have argued that for any time t, the state variables $y_1^{t,s}$ are independent for $s\in\mathcal{T}$. However, it is not true for $y_2^{t,s}$ for $s\in\mathcal{T}$; hence, we cannot use the same technique in Proposition 4. Nevertheless, we will show that $\sum_{s=0}^{t}r(s)y_2^{t,s}$ can still be evaluated efficiently. By Proposition 6, $y_2^{t,0}\sim\sum_{s=0}^{t-1}\text{Bin}\left(z_1^{t-s-1,0},\bar{h}_1^{t-1,s}\right)$. Then, for any time t, the distribution of $\sum_{s=0}^{t}y_2^{t-s,0}$ can be written explicitly as

$$\sum_{s=0}^{t}y_2^{t-s,0}\sim\sum_{s=0}^{t}\sum_{\tau=0}^{t-s-1}\text{Bin}(z_1^{t-s-\tau-1,0},\bar{h}_1^{t-s-1,\tau}). \tag{1}$$

We now represent the binomial random variable as a sum of Bernoulli random variables. Let $L_\ell^{t,s}\sim\text{Bernoulli}(\bar{h}_1^{t,s})$ indicating whether a patient ℓ who has been in service for s periods at time t will be routed to X-ray at time $t+1$. Then, we rewrite (29) as

$$\begin{aligned}
\sum_{s=0}^{t}y_2^{t-s,0}=&\sum_{s=0}^{t}\sum_{\tau=0}^{t-s-1}\sum_{\ell=1}^{z_1^{t-s-\tau-1,0}}L_\ell^{t-s-1,\tau} &(2)\\
=&\sum_{\bar{\tau}=0}^{t-1}\sum_{\ell_{\bar{\tau}}=1}^{z^{t-\bar{\tau}-1,0}}\left(\sum_{s=0}^{\bar{\tau}}L_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s}\right), &(3)
\end{aligned}$$

via a change in the order of summation and letting $\bar{\tau}=s+\tau$. The first summation follows because $s\in\{0,\dots,t-1\}$ and $\tau\in\{0,\dots,t-s-1\}$. We do not need to consider the term where $s=t$ because it equals to zero. The smallest possible value for $\bar{\tau}$ is 0, which happens when $s=0$ and $\tau=0$. Similarly, the largest value for $\bar{\tau}$ is $t-1$, which happens when $s\in\{0,\dots,t-1\}$ and $\tau=t-s-1$. For any $\bar{\tau}\in\{0,\dots,t-1\}$, we know s can take any value in $\{0,\dots,\bar{\tau}\}$ while τ takes value $\bar{\tau}-s$. In equation (31), the most inner summations $\sum_{s=0}^{\bar{\tau}} L_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s}$ are independent for all $\bar{\tau}$ and $\ell_{\bar{\tau}}$, because they correspond to some random events of different patients. As such, the following reformulation is valid,

$$\begin{aligned}
&k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{t} r(s)y_2^{t,s}/k\right)\right]\\
=&k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{t} r(s)y_2^{t-s,0}/k\right)\right]-\sum_{s=0}^{t} r(s)\sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau}\\
=&\sum_{\bar{\tau}=0}^{t-1}\sum_{\ell_{\bar{\tau}}=1}^{z^{t-\bar{\tau}-1,0}} k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{\bar{\tau}} r(s)L_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s}/k\right)\right]-\sum_{s=0}^{t} r(s)\sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau}\\
=&\sum_{\bar{\tau}=0}^{t-1}\sum_{\tau=0}^{t-\bar{\tau}-1} p_1^{t-\bar{\tau}-1,\tau}\eta_2^{t,\bar{\tau}}-\sum_{s=0}^{t} r(s)\sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau},
\end{aligned}$$

where equality follows because $\boldsymbol{\eta}_2$ are constants and $z^{t-\bar{\tau}-1,0}=\sum_{\tau=0}^{t-\bar{\tau}-1} p_1^{t-\bar{\tau}-1,\tau}$. $\eta_2^{t,\bar{\tau}}$ are defined as

$$\eta_2^{t,\bar{\tau}} \triangleq k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{\bar{\tau}} r(s)L_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s}/k\right)\right],\ \forall t\in[T],\bar{\tau}=0,1,\dots,t-1,$$

which evaluates to give the expression in the statement of the Proposition. □

Proof of Proposition 9. First, state variables $\boldsymbol{z}_j$ are independent over $j=1,\dots,3$. Therefore,

$$k\log\mathbb{E}\left[\exp\left(\sum_{j=1}^{3}\sum_{t=T+1}^{\bar{T}}\sum_{s=1}^{t} z_j^{t,s}/k\right)\right]=\sum_{j=1}^{3} k\log\mathbb{E}\left[\exp\left(\sum_{t=T+1}^{\bar{T}}\sum_{s=1}^{t}\mathrm{Bin}\left(z_j^{t-s,0},\hat{h}_j^{t,s}\right)/k\right)\right].$$

The binomial random variables in two inner summations are dependent. We will use a change of variable to identify the cohorts of different patients. We write

$$\begin{aligned}
&\sum_{j=1}^{3} k\log\mathbb{E}\left[\exp\left(\sum_{t=T+1}^{\bar{T}}\sum_{s=1}^{t}\mathrm{Bin}\left(z_j^{t-s,0},\hat{h}_j^{t,s}\right)/k\right)\right]\\
=&\sum_{j=1}^{3} k\log\mathbb{E}\left[\exp\left(\sum_{\bar{t}=0}^{T}\sum_{t=T+1}^{\bar{T}}\mathrm{Bin}\left(z_j^{\bar{t},0},\hat{h}_j^{t,\bar{t}+t}\right)/k\right)\right]\\
=&\sum_{j=1}^{3}\sum_{\bar{t}=0}^{T} k\log\mathbb{E}\left[\exp\left(\sum_{t=T+1}^{\bar{T}}\mathrm{Bin}\left(z_j^{\bar{t},0},\hat{h}_j^{t,\bar{t}+t}\right)/k\right)\right],
\end{aligned}$$

where the last equation follows from the independence of patients in different cohorts. The most inner summation is a summation of dependent random variables (*e.g.,* this summation cannot be greater than $z_j^{\bar{t},0}$). Then, similar to the proof of Proposition 7, we can write

$$\sum_{t=T+1}^{\bar{T}} \mathrm{Bin}\left(z_j^{\bar{t},0}, \hat{h}_j^{t,\bar{t}+t}\right) = \sum_{\ell_{\bar{t}}=1}^{z_j^{\bar{t},0}} \sum_{t=T+1}^{\bar{T}} \mathbb{1}(\text{patient } \ell_{\bar{t}} \text{ stays until time } t).$$

Then, by the independence of patients

$$\begin{aligned}
&\sum_{j=1}^{3}\sum_{\bar{t}=0}^{T} k\log\mathbb{E}\left[\exp\left(\sum_{t=T+1}^{\bar{T}} \mathrm{Bin}\left(z_j^{\bar{t},0}, \hat{h}_j^{t,\bar{t}+t}\right)/k\right)\right] \\
=&\sum_{j=1}^{3}\sum_{\bar{t}=0}^{T}\sum_{\ell_{\bar{t}}=1}^{z_j^{\bar{t},0}} k\log\mathbb{E}\left[\exp\left(\sum_{t=T+1}^{\bar{T}} \mathbb{1}(\text{patient } \ell_{\bar{t}} \text{ stays till time } t)/k\right)\right] \\
=&\sum_{j=1}^{3}\sum_{\bar{t}=0}^{T}\sum_{\tau_{\bar{t}}=1}^{\bar{t}} p_j^{\bar{t},\tau}\phi_j^{\bar{t}},
\end{aligned}$$

where we define

$$\begin{aligned}
\phi_j^{\bar{t}} \triangleq& k\log\mathbb{E}\left[\exp\left(\sum_{t=T+1}^{\bar{T}} \mathbb{1}(\text{patient } \ell_{\bar{t}} \text{ stays till time } t)/k\right)\right] \\
=& k\log\left(\sum_{t=T+1}^{\bar{T}} \hat{h}_j^{t,\bar{t}+t}\exp(1/k)\right)
\end{aligned}$$

The final reformulation is affine. ☐

Proposition 10. *For any $t\in[T]$, the term $k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{t} z_1^{t,s}/k\right)\right]$ can be reduced to affine constraints in push variables $\boldsymbol{p}_1$. More specifically, we have:*

$$k\log\mathbb{E}\left[\exp\left(\frac{\sum_{s=0}^{t} z_1^{t,s}}{k}\right)\right] = \sum_{\tau=0}^{t} p_1^{t,\tau} + \sum_{s=1}^{t} k\delta^{t,s}\sum_{\tau=0}^{t-s} p_1^{t-s,\tau}, \tag{4}$$

where constants

$$\delta^{t,s} \triangleq \log\left(1-\hat{h}_1^{t,s}+\hat{h}_1^{t,s}\exp\left(\frac{1}{k}\right)\right) \quad \forall t\in[T], s\in[t]$$

can be calculated from primitive data.

Proof of Proposition 10. This follows from Proposition 6, and the proof of Proposition 3. ☐

Proposition 11. *For any $t\in[T], s\in\{0,\dots,t\}$, the term $k\log\mathbb{E}\left[\exp\left(y_2^{t,s}/k\right)\right]$ can be reduced to an affine function in decision variables $\boldsymbol{p}_1,\boldsymbol{p}_2$. More specifically, we have:*

$$k\log\mathbb{E}\left[\exp\left(y_2^{t,s}/k\right)\right] = \sum_{\tau=0}^{t-s-1} \delta_2^{t-s-1,\tau}\left(\sum_{\tau_2=0}^{t-s-1-\tau} p_1^{t-s-1-\tau,\tau_2}\right) - \sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau} \tag{5}$$

where $\delta_2^{t,s}$ are constants that can be calculated from data for all $t \in [T], s = 0, \ldots, t$:

$$\delta_2^{t,s} \triangleq k \log\left(1 - \bar{h}_1^{t,s} + \bar{h}_1^{t,s} \exp(1/k)\right).$$

Proof of Proposition 11. By Proposition 6, we have $y_2^{t,0} \sim \sum_{s=0}^{t-1} \mathrm{Bin}\left(z_1^{t-s-1,0}, \bar{h}_1^{t-1,s}\right)$, where $\bar{h}_1^{t-1,s} \triangleq q h_1^{t-1,s} \prod_{\tau=1}^{s}(1 - h_1^{t-1-\tau,s-\tau})$ is as defined in Proposition 6. For simplicity, we define $\bar{h}_1^{t,0} \triangleq q h_1^{t-1,0}$ for all $t \in \mathcal{T}$. For any fixed time t, the state variables $z_1^{t,s}$ are independent for $s \in \mathcal{T}$. Then,

$$\begin{aligned}
&k \log \mathbb{E}\left[\exp\left(y_2^{t,s}/k\right)\right] \\
=& k \log \mathbb{E}\left[\exp\left(y_2^{t-s,0}/k\right)\right] - \sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau} \\
=& \sum_{\tau=0}^{t-s-1} k \log \mathbb{E}\left[\exp\left(\mathrm{Bin}\left(z^{t-s-1-\tau,0}, \bar{h}_1^{t-s-1,\tau}\right)/k\right)\right] - \sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau} \\
=& \sum_{\tau=0}^{t-s-1} k z^{t-s-1-\tau,0} \log\left(1 - \bar{h}_1^{t-s-1,\tau} + \bar{h}_1^{t-s-1,\tau} \exp(1/k)\right) - \sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau} \\
=& \sum_{\tau=0}^{t-s-1} \delta_2^{t-s-1,\tau} \left(\sum_{\tau_2=0}^{t-s-1-\tau} p_1^{t-s-1-\tau,\tau_2}\right) - \sum_{\tau=0}^{s-1} p_2^{t-\tau,s-\tau},
\end{aligned}$$

where we define the constants $\delta_2^{t,s} \triangleq k \log\left(1 - \bar{h}_1^{t,s} + \bar{h}_1^{t,s} \exp(1/k)\right)$ for all $t \in [T], s = 0, \ldots, t$. The last expression is the final reformulation, and it is affine. □

Proposition 12. *For any $t \in [T]$, the term $k \log \mathbb{E}\left[\exp\left(\sum_{s=0}^{t} z_2^{t,s}/k\right)\right]$ can be reduced to an affine function in decision variables $\boldsymbol{p}_x$. More specifically, we have:*

$$k \log \mathbb{E}\left[\exp\left(\sum_{s=0}^{t} z_2^{t,s}/k\right)\right] = \sum_{s=0}^{t} p_2^{t,s} + \sum_{s=1}^{t} k \beta_2^{t,s} \sum_{\tau=0}^{t-s} p_2^{t-s,\tau}, \tag{6}$$

where $\beta_2^{t,s}$ are constants that can be calculated from data for all $t \in [T], s \in [t]$:

$$\beta_2^{t,s} \triangleq \log\left(1 - \hat{h}_2^{t,s} + \hat{h}_2^{t,s} \exp\left(\frac{1}{k}\right)\right).$$

Proof of Proposition 12. The result follows from the proof of Proposition 3. □

Proposition 13. *For any $t \in [T], s \in \{0, \ldots, t\}$, the term $k \log \mathbb{E}\left[\exp\left(y_3^{t,s}/k\right)\right]$ can be reduced to an affine function in decision variables $\boldsymbol{p}_2, \boldsymbol{p}_3$. More specifically, we have:*

$$k \log \mathbb{E}\left[\exp\left(y_3^{t,s}/k\right)\right] = \sum_{\tau=0}^{t-s-1} \delta_3^{t-s-1,\tau} \left(\sum_{\tau_2=0}^{t-s-1-\tau} p_2^{t-s-1-\tau,\tau_2}\right) - \sum_{\tau=0}^{s-1} p_3^{t-\tau,s-\tau}, \tag{7}$$

where $\delta_3^{t,s}$ are constants for all $t \in [T], s = 0, \ldots, t$:

$$\delta_3^{t,s} \triangleq k \log\left(1 - h_2^{t,s} \hat{h}_2^{t,s} + h_2^{t,s} \hat{h}_2^{t,s} \exp(1/k)\right).$$

Proof of Proposition 13. This follows from the proof of Proposition 11.

By Proposition 6, we have $y_3^{t,0} \sim \sum_{s=0}^{t-1} \mathrm{Bin}(z_2^{t-s-1,0}, h_2^{t-1,s}\hat{h}_2^{t-1,s})$. For simplicity, we define $\hat{h}_2^{t,0} \triangleq 1$ for all $t \in \mathcal{T}$. For any fixed time t, the state variables $z_2^{t,s}$ are independent for $s \in \mathcal{T}$. Then,

$$\begin{aligned}
& k \log \mathbb{E}\left[\exp\left(y_3^{t,s}/k\right)\right] \\
=& k \log \mathbb{E}\left[\exp\left(y_3^{t-s,0}/k\right)\right] - \sum_{\tau=0}^{s-1} p_3^{t-\tau,s-\tau} \\
=& \sum_{\tau=0}^{t-s-1} \delta_3^{t-s-1,\tau} \left(\sum_{\tau_2=0}^{t-s-1-\tau} p_2^{t-s-1-\tau,\tau_2} \right) - \sum_{\tau=0}^{s-1} p_3^{t-\tau,s-\tau},
\end{aligned}$$

where we define constants $\delta_3^{t,s} \triangleq k \log\left(1 - h_2^{t,s}\hat{h}_2^{t,s} + h_2^{t,s}\hat{h}_2^{t,s}\exp(1/k)\right)$ for all $t \in [T], s = 0, \ldots, t$. The last expression is the final reformulation, and it is affine. ☐

Proposition 14. *For any $t \in [T]$, the term $k \log \mathbb{E}\left[\exp\left(\sum_{s=0}^{t} r(s) y_3^{t,s}/k\right)\right]$ can be reduced to an affine function in decision variables $\boldsymbol{p}, \boldsymbol{p}_2$. More specifically, we have:*

$$k \log \mathbb{E}\left[\exp\left(\sum_{s=0}^{t} r(s) y_3^{t,s}/k\right)\right] = \sum_{\bar{\tau}=0}^{t-1} \sum_{\tau=0}^{t-\bar{\tau}-1} p_2^{t-\bar{\tau}-1,\tau} \eta_3^{t,\bar{\tau}} - \sum_{s=0}^{t} r(s) \sum_{\tau=0}^{s-1} p_3^{t-\tau,s-\tau} \tag{8}$$

where $\eta_3^{t,s}$ are constants that can be calculated from data:

$$\eta_3^{t,\bar{\tau}} \triangleq k \log\left(1 + \sum_{s=0}^{\bar{\tau}} h_2^{t-s-1,\bar{\tau}-s} \hat{h}_2^{t-s-1,\bar{\tau}-s} \left(\exp\left(\frac{r(s)}{k}\right) - 1\right)\right), \ \forall t \in [T], \bar{\tau} = 0, 1, \ldots, t-1.$$

Proof of Proposition 14. This follows directly from the proof of Proposition 7.

For any time t, the distribution of $\sum_{s=0}^{t} y_3^{t-s,0}$ can be written as

$$\sum_{s=0}^{t} y_3^{t-s,0} \sim \sum_{s=0}^{t} \sum_{\tau=0}^{t-s-1} \mathrm{Bin}(z_2^{t-s-\tau-1,0}, h_2^{t-s-1,\tau}\hat{h}_2^{t-s-1,\tau}). \tag{9}$$

We define the random variable $M_\ell^{t,s} \sim \mathrm{Bernoulli}(h_2^{t,s}\hat{h}_2^{t,s})$, which indicates whether a patient ℓ who has been in X-ray service for s periods at time t will be routed back to the consultation doctor at time $t+1$. Then, we rewrite (37) as:

$$\sum_{s=0}^{t} y_3^{t-s,0} = \sum_{\bar{\tau}=0}^{t-1} \sum_{\ell_{\bar{\tau}}=1}^{z_2^{t-\bar{\tau}-1,0}} \left(\sum_{s=0}^{\bar{\tau}} M_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s} \right). \tag{10}$$

In equation (38), the most inner summations $\sum_{s=0}^{\bar{\tau}} M_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s}$ are independent for all $\bar{\tau}$ and $\ell_{\bar{\tau}}$, because they correspond to some random events of different patients and different patients are clearly independent. Then,

$$k \log \mathbb{E}\left[\exp\left(\sum_{s=0}^{t} r(s) y_3^{t,s}/k\right)\right]$$

$$
\begin{aligned}
&= k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{t} r(s) y_3^{t-s,0}/k\right)\right] - \sum_{s=0}^{t} r(s)\sum_{\tau=0}^{s-1} p_3^{t-\tau,s-\tau}\\
&= \sum_{\bar{\tau}=0}^{t-1}\sum_{\ell_{\bar{\tau}}=1}^{z_2^{t-\bar{\tau}-1,0}} k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{\bar{\tau}} r(s) M_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s}/k\right)\right] - \sum_{s=0}^{t} r(s)\sum_{\tau=0}^{s-1} p_3^{t-\tau,s-\tau}\\
&= \sum_{\bar{\tau}=0}^{t-1}\sum_{\tau=0}^{t-\bar{\tau}-1} p_2^{t-\bar{\tau}-1,\tau}\eta_3^{t,\bar{\tau}} - \sum_{s=0}^{t} r(s)\sum_{\tau=0}^{s-1} p_3^{t-\tau,s-\tau}
\end{aligned}
$$

where we define the constants $\eta_3^{t,\bar{\tau}} \triangleq k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{\bar{\tau}} r(s) M_{\ell_{\bar{\tau}}}^{t-s-1,\bar{\tau}-s}/k\right)\right]$ for all $t\in[T], \bar{\tau} = 0,\dots,t-1$, which can be calculated and its final expression is as stated in the Proposition. $\square$

PROPOSITION 15. *For any $t\in\mathcal{T}$, the term $k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{t} z_3^{t,s}/k\right)\right]$ can be reduced to an affine function in decision variables $\boldsymbol{p}_3$. More specifically, we have:*

$$
k\log\mathbb{E}\left[\exp\left(\sum_{s=0}^{t} z_3^{t,s}/k\right)\right] = \sum_{s=0}^{t} p_3^{t,s} + \sum_{s=1}^{t} k\beta_3^{t,s}\sum_{\tau=0}^{t-s} p_3^{t-s,\tau}, \tag{11}
$$

where $\beta_3^{t,s}$ are constants that can be calculated from data for $t\in[T], s\in[t]$:

$$
\beta_3^{t,s} \triangleq \log\left(1-\hat{h}_3^{t,s}+\hat{h}_3^{t,s}\exp\left(\frac{1}{k}\right)\right).
$$

Proof of Proposition 15. The proof follows directly from the proof of Proposition 3. $\square$

Appendix B: Omitted Result Tables for Experiments 2–5

Table 6 **Performance comparison (Experiment 2): varying probability γ**

		Metrics (mins)		
		Total waiting	Overtime	Max. instantaneous waiting
$\gamma = 0.90$	Equal-interval	168.6 (**10.1%**)	28.3 (**10.6%**)	79.9 (**15.1%**)
	SAA	157.6 (**2.9%**)	26.6 (**0.4%**)	72.0 (**3.7%**)
	Ours (± s.d.)	153.1 (± 0.62)	26.5 (± 0.11)	69.4 (± 0.29)
$\gamma = 0.85$	Equal-interval	149.8 (**11.5%**)	25.5 (**7.1%**)	77.8 (**14.4%**)
	SAA	139.0 (**3.4%**)	24.0 (**0.8%**)	70.6 (**3.8%**)
	Ours (± s.d.)	134.4 (± 0.56)	23.8 (± 0.1)	68.0 (± 0.28)
$\gamma = 0.80$	Equal-interval	128.4 (**14.2%**)	22.5 (**6.1%**)	74.8 (**14.2%**)
	SAA	118.4 (**5.3%**)	21.3 (**0.5%**)	68.7 (**4.9%**)
	Ours (± s.d.)	112.4 (± 0.51)	21.2 (± 0.1)	65.5 (± 0.27)

Table 7 **Performance comparison (Experiment 3): varying rate α and probability γ**

		Metrics (mins)		
		Total waiting	Overtime	Max. instantaneous waiting
$\alpha = 0.70$	Equal-interval	149.5 (**12.2%**)	22.9 (**7.5%**)	75.3 (**14.8%**)
$\gamma = 0.85$	SAA	137.5 (**3.2%**)	21.5 (**0.9%**)	68.2 (**4.0%**)
	Ours (± s.d.)	133.2 (± 0.58)	21.3 (± 0.09)	65.6 (± 0.27)
$\alpha = 0.80$	Equal-interval	158.4 (**12.5%**)	23.7 (**7.7%**)	79.4 (**15.1%**)
$\gamma = 0.85$	SAA	145.5 (**3.3%**)	22.2 (**0.9%**)	71.9 (**4.2%**)
	Ours (± s.d.)	140.8 (± 0.60)	22.0 (± 0.1)	69.0 (± 0.28)
$\alpha = 0.85$	Equal-interval	139.0 (**13.5%**)	21.2 (**7.6%**)	76.9 (**13.6%**)
$\gamma = 0.8$	SAA	127.0 (**3.7%**)	19.8 (**0.5%**)	70.2 (**3.7%**)
	Ours (± s.d.)	122.5 (± 0.55)	19.7 (± 0.09)	67.7 (± 0.27)

Table 8 **Performance comparison (Experiment 4): varying number of Type A n_A and X-ray probability q**

		Metrics (mins)		
		Total waiting	Overtime	Max. instantaneous waiting
$n_A = 4$	Equal-interval	168.6 (**13.9%**)	22.3 (**3.2%**)	66.9 (**20.8%**)
$q = 0.7$	SAA	152.2 (**2.8%**)	21.5 (**-0.5%**)	57.5 (**3.8%**)
	Ours (± s.d.)	148.0 (± 0.56)	21.6 (± 0.09)	55.4 (± 0.23)
$n_A = 4$	Equal-interval	184.1 (**18.6%**)	23.7 (**4.4%**)	73.4 (**25.9%**)
$q = 0.9$	SAA	166.6 (**7.3%**)	22.9 (**0.9%**)	63.3 (**8.6%**)
	Ours (± s.d.)	155.2 (± 0.57)	22.7 (± 0.10)	58.3 (± 0.24)
$n_A = 3$	Equal-interval	169.4 (**13.4%**)	21.7 (**4.8%**)	66.5 (**19.4%**)
$q = 0.9$	SAA	155.5 (**4.1%**)	21.4 (**3.4%**)	58.6 (**5.2%**)
	Ours (± s.d.)	149.4 (± 0.58)	20.7 (± 0.09)	55.7 (± 0.24)
$n_A = 3$	Equal-interval	175.8 (**15.6%**)	22.3 (**4.2%**)	69.0 (**20.4%**)
$q = 1.0$	SAA	159.9 (**5.1%**)	21.6 (**0.9%**)	60.0 (**4.7%**)
	Ours (± s.d.)	152.1 (± 0.58)	21.4 (± 0.09)	57.3 (± 0.25)

Table 9 **Performance comparison (Experiment 5): varying maximum earliness D**

		Metrics (mins)		
		Total waiting	Overtime	Max. instantaneous waiting
$D = 2$	Equal-interval	173.7 (**16.0%**)	22.1 (**4.2%**)	67.8 (**22.4%**)
	SAA	158.5 (**5.8%**)	21.2 (**0%**)	59.5 (**7.4%**)
	Ours (± s.d.)	149.8 (± 0.58)	21.2 (± 0.09)	55.4 (± 0.24)
$D = 3$	Equal-interval	172.5 (**14.8%**)	21.9 (**3.8%**)	67.2 (**21.1%**)
	SAA	158.6 (**5.5%**)	21.1 (**0%**)	59.4 (**7.0%**)
	Ours (± s.d.)	150.3 (± 0.58)	21.1 (± 0.09)	55.5 (± 0.24)
$D = 5$	Equal-interval	170.2 (**14.6%**)	21.7 (**1.8%**)	66.2 (**19.7%**)
	SAA	159.1 (**7.1%**)	21.1 (**-0.9%**)	59.2 (**7.1%**)
	Ours (± s.d.)	148.5 (± 0.56)	21.3 (± 0.09)	55.3 (± 0.24)