Online Appendix

EC.1. Proof of Lemma 1

(i) If an optimal price is an interior point, then it must satisfy the first-order condition $\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}} = 0$. From (4a-4b), we can write $\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}}$ as follows:

$$\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}} = \frac{\partial \alpha_j(p_{tj})}{\partial p_{tj}} \cdot \left(p_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)\right) + \alpha_j(p_{tj}), \quad \text{for } j \in [J]. \tag{EC.1}$$

Thus, we have $\left(\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}}\right)_{p_{tj}=p^*_{tj}} = 0$ which can be written as

$$\left(\frac{\partial \alpha_j(p_{tj})}{\partial p_{tj}}\right)_{p_{tj}=p^*_{tj}} \cdot \left(p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)\right) + \alpha_j(p^*_{tj}) = 0 \tag{EC.2}$$

$$p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) = -\frac{\alpha_j(p^*_{tj})}{\left(\frac{\partial \alpha_j(p_{tj})}{\partial p_{tj}}\right)_{p_{tj}=p^*_{tj}}}. \tag{EC.3}$$

Note that the Mills ratio of a continuous random variable X is the function $m_X(x) := \frac{1-F_X(x)}{f_X(x)}$, where $f_X(x)$ is the probability density function and $F_X(x)$ is the cumulative distribution function. Note also that $f_X(x) = \frac{\partial F_X(x)}{\partial x}$. In our setting $\alpha_j(p_{tj}) := 1 - F_{X_j}(p_{tj})$ is the complementary cumulative distribution function of random valuation of the j^{th} product X_j at price p_{tj}, and $\frac{\partial \alpha_j(p_{tj})}{\partial p_{tj}} = -f_{X_j}(p_{tj})$. Multiplying the numerator and denominator of the RHS of (EC.3) by (-1) we have

$$p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) = m_{X_j}(p^*_{tj}), \tag{EC.4}$$

resulting in the optimal price formulation given in the theorem.

(ii) First note that if $y_{tj} = 0$ for any $j \in [J]$, then according to Assumption A2, $p^*_{tj} = \bar{p}_j$. If $y_{tj} > 0$ and the optimal solution is an interior point, the first-order condition given in (EC.4) is sufficient. Note that the LHS of the Equation (EC.4) is increasing in p_{tj} with a non-positive $y-$intercept (since $\nu^{\mathrm{I}}_{tj}(y_t) + \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) \geq 0$). In addition, the RHS $m_{X_j}(p_{tj})$ has a non-negative $y-$intercept. Thus, assuming $m_{X_j}(p_{tj})$ to be non-increasing in p_{tj} (i.e., the assumption stated in the theorem) guarantees that the Equation (EC.4) has a unique solution p^*_{tj}, and this completes the proof. □

EC.2. Proof of Lemma 2

(i) We prove the quasi-concavity of $v_{tj}(y_t,p_{tj})$ in p_{tj} for $j \in [J-1]$ (we omit the proof for $j = J$ as it is similar to that of $j \in [J-1]$). To this end, we show that for all $j \in [J-1]$, there exists a price point $\tilde{p}_{tj}$ such that $v_{tj}(y_t,p_{tj})$ is non-decreasing on $\{p_{tj} \geq 0 : p_{tj} \leq \tilde{p}_{tj}\}$, and non-increasing on $\{p_{tj} \geq 0 : p_{tj} \geq \tilde{p}_{tj}\}$—that is, $\left(\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}}\right)_{p_{tj}\geq 0:\, p_{tj}\leq \tilde{p}_{tj}} \geq 0$ and $\left(\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}}\right)_{p_{tj}\geq 0:\, p_{tj}\geq \tilde{p}_{tj}} \leq 0$.

Proof of quasi-concavity under logistic distribution

Incorporating $\alpha_j(p_{tj})$ under the logistic rate from (7) into $v_{tj}(y_t,p_{tj})$ given in (4a), we have

$$v_{tj}(y_t,p_{tj}) = \left(\frac{\mathrm{e}^{q_j-p_{tj}}}{1+\mathrm{e}^{q_j-p_{tj}}}\right)\cdot\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)+\big(\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)+\rho^*_{t+1}(y_t)\big),\quad j\in[J-1]$$

whose first-order derivative is

$$\begin{aligned}\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}} &= \frac{\mathrm{e}^{q_j-p_{tj}}}{1+\mathrm{e}^{q_j-p_{tj}}}+\frac{-\mathrm{e}^{q_j-p_{tj}}\cdot(1+\mathrm{e}^{q_j-p_{tj}})+\mathrm{e}^{2(q_j-p_{tj})}}{\left(1+\mathrm{e}^{q_j-p_{tj}}\right)^2}\cdot\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)\\ &= \frac{\mathrm{e}^{q_j-p_{tj}}\cdot(1+\mathrm{e}^{q_j-p_{tj}})-\mathrm{e}^{q_j-p_{tj}}\cdot\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)}{\left(1+\mathrm{e}^{q_j-p_{tj}}\right)^2}\\ &= \frac{\mathrm{e}^{q_j-p_{tj}}\cdot\big(1-p_{tj}+\nu^{\mathrm{I}}_{tj}(y_t)+\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)+\mathrm{e}^{2q_j-2p_{tj}}}{\left(1+\mathrm{e}^{q_j-p_{tj}}\right)^2}\\ &= -\mathrm{e}^{q_j}\cdot\frac{\mathrm{e}^{p_{tj}}\cdot\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)-1\big)-\mathrm{e}^{q_j}}{\left(\mathrm{e}^{p_{tj}}+\mathrm{e}^{q_j}\right)^2} = -\mathrm{e}^{q_j}\cdot\frac{K(y_t,p_{tj})-\mathrm{e}^{q_j}}{\left(\mathrm{e}^{p_{tj}}+\mathrm{e}^{q_j}\right)^2},\end{aligned}$$

where $K(y_t,p_{tj}):=\mathrm{e}^{p_{tj}}\cdot\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)-1\big)$. Note that, (1) for any $0\le p_{tj}\le\dot{p}_{tj}:=\nu^{\mathrm{I}}_{tj}(y_t)+\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)$, we have $K(y_t,p_{tj})<0$, $K(y_t,p_{tj})-\mathrm{e}^{q_j}<0$, and $\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}>0$, accordingly; (2) in addition, $\frac{\partial\, K(y_t,p_{tj})}{\partial\, p_{tj}}=\mathrm{e}^{p_{tj}}\cdot(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t))=\mathrm{e}^{p_{tj}}\cdot(p_{tj}-\dot{p}_{tj})$ indicates that $K(y_t,p_{tj})$ is strictly increasing on $p_{tj}>\dot{p}_{tj}$. This implies that there exists a price point $\hat{p}_{tj}>\dot{p}_{tj}$ at which $K(y_t,p_{tj})$ and e^{q_j} intercept resulting in $\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}=0$, while at $\dot{p}_{tj}\le p_{tj}<\hat{p}_{tj}$ ($p_{tj}>\hat{p}_{tj}$) we have $K(y_t,p_{tj})-\mathrm{e}^{q_j}<0$ ($K(y_t,p_{tj})-\mathrm{e}^{q_j}>0$), and thus $\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}>0$ ($\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}<0$). Considering (1) and (2), we can conclude that there exists a price point $\tilde{p}_{tj}=\hat{p}_{tj}$ where $\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}\ge 0$ for all $p_{tj}\le\tilde{p}_{tj}$, and $\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}\le 0$ for all $p_{tj}\ge\tilde{p}_{tj}$, which proves our claim.

Proof of quasi-concavity under exponential distribution

Invoking $\alpha_j(p_{tj})$ under exponential rate from (7), we have

$$v_{tj}(y_t,p_{tj}) = e^{-\frac{p_{tj}}{q_j}}\cdot\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)+\big(\zeta_j\cdot\nu^{\mathrm{I}}_{tj}(y_t)+\rho^*_{t+1}(y_t)\big),\quad j\in[J-1] \tag{EC.5}$$

whose first derivative with respect to p_{tj} is

$$\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}} = -\frac{1}{q_j}\cdot e^{-\frac{p_{tj}}{q_j}}\cdot\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)+e^{-\frac{p_{tj}}{q_j}} \tag{EC.6}$$

$$= -e^{-\frac{p_{tj}}{q_j}}\cdot\frac{\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)-q_j}{q_j}. \tag{EC.7}$$

Thus, we have $\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}\ge 0$ ($\frac{\partial\, v_{tj}(y_t,p_{tj})}{\partial\, p_{tj}}\le 0$) when $\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)-q_j\le 0$ ($\big(p_{tj}-\nu^{\mathrm{I}}_{tj}(y_t)-\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)\big)-q_j\ge 0$), or equivalently when $p_{tj}\le q_j+\nu^{\mathrm{I}}_{tj}(y_t)+\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)$ ($p_{tj}\ge q_j+\nu^{\mathrm{I}}_{tj}(y_t)+\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)$). This indicates the existence of the price point $\tilde{p}_{tj}=q_j+\nu^{\mathrm{I}}_{tj}(y_t)+\zeta_j\cdot\nu^{\mathrm{S}}_{tj}(y_t)$, and the proof of quasi-concavity follows.

Proof of concavity under linear distribution

Invoking $\alpha_j(p_{tj})$ under linear rate from (7), we have

$$\frac{\partial v_{tj}(y_t, p_{tj})}{\partial p_{tj}} = 1 - \frac{p_{tj}}{q_j} - \frac{p_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)}{q_j}, \qquad j \in [J-1], \tag{EC.8}$$

and $\frac{\partial^2 v_{tj}(y_t,p_{tj})}{\partial p_{tj}^2} = -2 \cdot \dfrac{1}{q_j} \leq 0$, $j \in [J-1]$, accordingly. Also,

$$\frac{\partial v_{tJ}(y_t, p_{tJ})}{\partial p_{tJ}} = 1 - \frac{p_{tJ}}{q_J} - \frac{p_{tJ} - \nu^{\mathrm{I}}_{tJ}}{q_J}, \tag{EC.9}$$

and $\frac{\partial^2 v_{tJ}(y_t,p_{tJ})}{\partial p_{tJ}^2} = -2 \cdot \dfrac{1}{q_J} \leq 0$, accordingly, implying that $v_{tj}(y_t, p_{tj})$ is concave in p_{tj}.

(ii) Due to the unimodality property shown in part (i), the first-order condition satisfies the optimal price.

Proof of optimal price under logistic distribution

We can write

$$\frac{\partial v_{tj}(y_t, p_{tj})}{\partial p_{tj}} = -\mathrm{e}^{q_j} \cdot \frac{\mathrm{e}^{p_{tj}} \cdot (p_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1) - \mathrm{e}^{q_j}}{\left(\mathrm{e}^{p_{tj}} + \mathrm{e}^{q_j}\right)^2}, \qquad j \in [J-1], \tag{EC.10}$$

$$\frac{\partial v_{tJ}(y_t, p_{tJ})}{\partial p_{tJ}} = -\mathrm{e}^{q_J} \cdot \frac{\mathrm{e}^{p_{tJ}} \cdot (p_{tJ} - \nu^{\mathrm{I}}_{tJ}(y_t) - 1) - \mathrm{e}^{q_J}}{\left(\mathrm{e}^{p_{tJ}} + \mathrm{e}^{q_J}\right)^2}. \tag{EC.11}$$

Let $W(\cdot)$ represent the principal branch of the Lambert's W function. The optimal price of product j $\forall j \in [J]$ can be derived as follows:

$$\left(\frac{\partial v_{tj}(y_t, p_{tj})}{\partial p_{tj}}\right)_{p_{tj}=p^*_{tj}} = 0, \tag{EC.12}$$

$$\left(p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1\right) \cdot \mathrm{e}^{p^*_{tj}} = \mathrm{e}^{q_j}, \tag{EC.13}$$

$$\left(p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1\right) \cdot \mathrm{e}^{p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1} = \mathrm{e}^{q_j - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1}, \tag{EC.14}$$

$$W\left(\left(p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1\right) \cdot \mathrm{e}^{p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1}\right) = W\left(\mathrm{e}^{q_j - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1}\right), \tag{EC.15}$$

$$p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1 = W\left(\mathrm{e}^{q_j - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1}\right), \tag{EC.16}$$

resulting in $p^*_{tj} = 1 + \nu^{\mathrm{I}}_{tj}(y_t) + \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) + W(\mathrm{e}^{q_j - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t) - 1})$, where by definition we have $\nu^{\mathrm{S}}_{tJ}(y_t) := 0$. Also, (EC.16) is by the following property of the Lambert's W function $W(ze^z) = z$.

Proof of optimal price under exponential distribution

The first order conditions can be written as follows:

$$\frac{\partial v_{tj}(y_t, p_{tj})}{\partial p_{tj}} = -e^{-\frac{p_{tj}}{q_j}} \cdot \frac{\left(p_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)\right) - q_j}{q_j}, \qquad j \in [J-1], \tag{EC.17}$$

$$\frac{\partial v_{tJ}(y_t, p_{tJ})}{\partial p_{tJ}} = -e^{-\frac{p_{tJ}}{q_J}} \cdot \frac{\left(p_{tJ} - \nu^{\mathrm{I}}_{tJ}(y_t)\right) - q_J}{q_J}. \tag{EC.18}$$

To find the optimal price we use the first order condition $\left(\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}}\right)_{p_{tj}=p^*_{tj}} = 0$ as $(p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)) - q_j = 0$, which gives $p^*_{tj} = q_j + \nu^{\mathrm{I}}_{tj}(y_t) + \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)$, where by definition we have $\nu^{\mathrm{S}}_{tJ}(y_t) := 0$.

Proof of optimal price under linear distribution

The optimal price can be determined by solving $\frac{\partial r_{tj}(y_t,p_{tj})}{\partial p_{jt}} = 0$ given in (EC.8) and (EC.9) as $\left(\frac{\partial v_{tj}(y_t,p_{tj})}{\partial p_{tj}}\right)_{p_{tj}=p^*_{tj}} = 0$, yielding $1 - \dfrac{p^*_{tj}}{q_j} - \dfrac{p^*_{tj} - \nu^{\mathrm{I}}_{tj}(y_t) - \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)}{q_j} = 0$. Then, $p^*_{tj} = \dfrac{1}{2} \cdot \big(q_j + \nu^{\mathrm{I}}_{tj}(y_t) + \zeta_j \cdot \nu^{\mathrm{S}}_{tj}(y_t)\big)$, where by definition we have $\nu^{\mathrm{S}}_{tJ}(y_t) := 0$. This concludes the proof. $\square$

EC.3. Proof of Lemma EC.4

The proof follows by plugging the optimal price formulation given in (8), (9), and (10) into $p^*_{tj}(y_t) < p^*_{t+1,j}(y_{t+1})$ and simplifying the inequality. $\square$

EC.4. Proof of Lemma 3

Throughout this section, we will fix t. We prove Lemma 3 using a reformulation of $\rho^{\mathsf{D}}_t(y_t)$ for a fixed t. First, note that

$$\rho^{\mathsf{D}}_t(y_t) = \max_{(\alpha_{t'j})_{j=1}^J} \sum_{t'=t}^{T} \sum_{j=1}^{J} \sigma_j(\alpha_{t'1}, \ldots, \alpha_{t'J}) r_j(\alpha_{t'j}) \tag{EC.19a}$$

$$\text{s.t.} \sum_{t'=t}^{T} \sigma_j(\alpha_{t'1}, \ldots, \alpha_{t'J}) \alpha_{t'j} \le y_{tj}, \quad \forall j \in [J] \tag{EC.19b}$$

$$\alpha_{t'j} \in \Omega_\alpha, \quad \forall t' = t, \ldots, T,\ j \in [J] \tag{EC.19c}$$

Per our discussions in Section 5, σ_j can be interpreted as *customer arrival rate* to product j. Let $\sigma_{t'j} = \sigma_j(\alpha_{t'})$. By definition, for any $j \in [J]$, we have $\sigma_{t',j+1} = \sigma_{t'j}\zeta_j - \zeta_j \sigma_{t'j}\alpha_{t'j}$, $\forall t' = t, \ldots, T$. Thus, customer arrival rates $\sigma_{j+1} := (\sigma_{t',j+1})_{t'=t}^T$ to product $j+1$ in periods t' to T are completely determined by the previous product's arrival rates $\sigma_j := (\sigma_{t'j})_{t'=t}^T$ and conditional purchase rates $\alpha_j := (\alpha_{t'j})_{t'=t}^T$. This suggests that we re-write (EC.19) as a DP whose state is the vector of arrival rates and remaining inventory levels.

Let $R^{\mathsf{D}}_{tj}(\sigma_j, z)$ be the deterministic revenue-to-go from product j in periods t to T if the customer starts at product j in periods t to T with arrival rates $\sigma_j = (\sigma_{t'j})$ and given the remaining inventory levels $z = (z_j)$. For any $j \in [J]$, we have the following DP recursion:

$$R^{\mathsf{D}}_{tj}(\sigma_j, z) = \max_{\alpha_{t'j}} \sum_{t'=t}^{T} \sigma_{t'j} r_j(\alpha_{t'j}) + R^{\mathsf{D}}_{t,j+1}\left(\sigma_j\zeta_j - \zeta_j \sigma_j \circ \alpha_j, z - \left(\sum_{t'=t}^{T} \sigma_{t'j}\alpha_{t'j}\right) e_j\right) \tag{EC.20a}$$

$$\text{s.t.} \sum_{t'=t}^{T} \sigma_{t'j}\alpha_{t'j} \le z_j \tag{EC.20b}$$

$$\alpha_{t'j} \in \Omega_\alpha, \quad \forall t' = t, \ldots, T \tag{EC.20c}$$

where $R_{t,J+1}(\cdot,\cdot)=0$. (The operator $\circ$ in the objective of (EC.20a) denotes elementwise multiplication betweeen two vectors.) Let Λ be the $(T-t+1)$-dimensional vector whose elements are all equal to λ. By definition, $\rho_t^{\mathsf{D}}(y_t)=R_{t1}^{\mathsf{D}}(\Lambda,y_t)$. Thus, to prove that $\rho_t^D(y_t)$ is jointly concave, it is sufficient that we prove the following lemma.

LEMMA EC.1. *$R_{tj}^D(\sigma_j,z)$ is jointly concave in (σ_j,z) for all $j=1,\dots,J$.*

PROOF. Fix j. Consider any $\sigma_j^{(1)},\sigma_j^{(2)},z^{(1)},z^{(2)}$ and $b\in[0,1]$. Let $\alpha_j^{(1)}$ and $\alpha_j^{(2)}$ be the optimal solutions of (EC.20a) under $(\sigma_j^{(1)},z^{(1)})$ and $(\sigma_j^{(2)},z^{(2)})$, respectively. We will prove the joint concavity of $R_{tj}^D(\cdot,\cdot)$ by showing that $b\cdot R_{tj}^{\mathsf{D}}(\sigma_j^{(1)},z^{(1)})+(1-b)\cdot R_{tj}^{\mathsf{D}}(\sigma_j^{(2)},z^{(2)})\le R_{tj}^{\mathsf{D}}(\bar\sigma_j,\bar z)$, where $(\bar\sigma_j,\bar z):=b\cdot(\sigma_j^{(1)},z^{(1)})+(1-b)\cdot(\sigma_j^{(2)},z^{(2)})$. To aid in our proof, we define $\bar\alpha_j=(\bar\alpha_{t'j})$, where

$$\bar\alpha_{t'j}:=\frac{b\sigma_{t'j}^{(1)}}{\bar\sigma_{t'j}}\cdot\alpha_{t'j}^{(1)}+\frac{(1-b)\sigma_{t'j}^{(2)}}{\bar\sigma_{t'j}}\cdot\alpha_{t'j}^{(2)},\quad \text{for } t'=t,t+1,\dots,T. \tag{EC.21}$$

It is not difficult to check that $\bar\alpha_j$ is a feasible solution to (EC.20a) under $(\bar\sigma_j,\bar z)$ (i.e., $\bar\alpha_j$ satisfies the two constraints in (EC.20a)). In addition, for any $j\in[J]$, we also have

$$\begin{aligned}b\cdot\sum_{t'=t}^{T}\sigma_{t'j}^{(1)}r_j(\alpha_{t'j}^{(1)})+(1-b)\cdot\sum_{t'=t}^{T}\sigma_{t'j}^{(2)}r_j(\alpha_{t'j}^{(2)})&=\sum_{t'=t}^{T}\bar\sigma_{t'j}\left(\frac{b\sigma_{t'j}^{(1)}}{\bar\sigma_{t'j}}r_j(\alpha_{t'j}^{(1)})+\frac{(1-b)\sigma_{t'j}^{(2)}}{\bar\sigma_{t'j}}r_j(\alpha_{t'j}^{(2)})\right)\\&\le\sum_{t'=t}^{T}\bar\sigma_{t'j}r_j(\bar\alpha_{t'j}),\end{aligned} \tag{EC.22}$$

where the inequality follows from the concavity of $r_j(\cdot)$ (by Assumption A3).

Now, we proceed to prove Lemma EC.1 by induction. For the base case ($j=J$), note that $b\cdot R_{tJ}^{\mathsf{D}}(\sigma_J^{(1)},z^{(1)})+(1-b)\cdot R_{tJ}^{\mathsf{D}}(\sigma_J^{(2)},z^{(2)})\le\sum_{t'=t}^{T}\bar\sigma_{t'J}r_J(\bar\alpha_{t'J})\le R_{tJ}^{\mathsf{D}}(\bar\sigma_J,\bar z)$, where the first inequality follows from (EC.22) and $R_{t,J+1}^{\mathsf{D}}(\cdot,\cdot)=0$, and the second inequality follows from the feasibility of $\bar\alpha_J$ under $(\bar\sigma_J,\bar z)$. This establishes the joint concavity of $R_{tJ}^{\mathsf{D}}(\cdot,\cdot)$. Let us now suppose that $R_{t,j+1}^{\mathsf{D}}(\cdot,\cdot)$ is jointly concave. We will next show that $R_{tj}^{\mathsf{D}}(\cdot,\cdot)$ is also jointly concave. For notational convenience, define $\sigma_{j+1}^{\dagger},\sigma_{j+1}^{\ddagger},z^{\dagger},z^{\ddagger}$ as follows:

$$(\sigma_{j+1}^{\dagger},z^{\dagger})=\left(\sigma_j^{(1)}\zeta_j-\zeta_j\sigma_j^{(1)}\circ\alpha_j^{(1)},\ z^{(1)}-\left(\sum_{t'=t}^{T}\sigma_{t'j}^{(1)}\alpha_{t'j}^{(1)}\right)e_j\right), \tag{EC.23}$$

$$(\sigma_{j+1}^{\ddagger},z^{\ddagger})=\left(\sigma_j^{(2)}\zeta_j-\zeta_j\sigma_j^{(2)}\circ\alpha_j^{(2)},\ z^{(2)}-\left(\sum_{t'=t}^{T}\sigma_{t'j}^{(2)}\alpha_{t'j}^{(2)}\right)e_j\right). \tag{EC.24}$$

From the induction hypothesis, we have:

$$\begin{aligned}b\cdot R_{t,j+1}^{\mathsf{D}}(\sigma_{j+1}^{\dagger},z^{\dagger})+(1-b)\cdot R_{t,j+1}^{\mathsf{D}}(\sigma_{j+1}^{\ddagger},z^{\ddagger})&\le R_{t,j+1}^{\mathsf{D}}\left(b\cdot\sigma_{j+1}^{\dagger}+(1-b)\cdot\sigma_{j+1}^{\ddagger},b\cdot z^{\dagger}+(1-b)\cdot z^{\ddagger}\right)\\&=R_{t,j+1}^{\mathsf{D}}\left(\bar\sigma_j\zeta_j-\zeta_j\bar\sigma_j\circ\bar\alpha_j,\ \bar z-\left(\sum_{t'=t}^{T}\bar\sigma_{t'j}\bar\alpha_{t'j}\right)e_j\right).\end{aligned} \tag{EC.25}$$

Combining (EC.22) and (EC.25) yields

$$
\begin{aligned}
&b\cdot R^{\mathsf{D}}_{tj}(\sigma_j^{(1)}, z^{(1)}) + (1-b)\cdot R^{\mathsf{D}}_{tj}(\sigma_j^{(2)}, z^{(2)})\\
&\qquad \le \sum_{t'=t}^{T} \bar{\sigma}_{t'j} r_j(\bar{\alpha}_{t'j}) + R^{\mathsf{D}}_{t,j+1}\left(\bar{\sigma}_j\zeta_j - \zeta_j\bar{\sigma}_j \circ \bar{\alpha}_j,\ \bar{z} - \left(\sum_{t'=t}^{T}\bar{\sigma}_{t'j}\bar{\alpha}_{t'j}\right) e_j\right) \le R^{\mathsf{D}}_{tj}(\bar{\sigma}_j, \bar{y}),
\end{aligned}
$$

where the last inequality follows from the feasibility of $\bar{\alpha}_j$ in (EC.20a) under $(\bar{\sigma}_j, \bar{z})$. This concludes the proof of Lemma EC.1. $\square$

EC.5. Proof of Theorem 1

Let $D_{tj}(\alpha_t)$ denote the realized demand for product j in period t under conditional purchase rates α_t. We prove Theorem 1 by induction, starting from $t = T$. Note that, by definition,

$$
\begin{aligned}
\rho^*_T(y_T) = \max_{(\alpha_{Tj})_{j=1}^J} \quad & \sum_{j=1}^{J} \sigma_j(\alpha_{T1}, \ldots, \alpha_{TJ}) r_j(\alpha_{Tj}) && \text{(EC.26a)}\\
\text{s.t.} \quad & D_{Tj}(\alpha_T) \le y_{Tj}, \quad \forall j \in [J] && \text{(EC.26b)}\\
& \alpha_{Tj} \in \Omega_\alpha, \quad \forall j \in [J] && \text{(EC.26c)}
\end{aligned}
$$

where the constraints must hold almost surely, and

$$
\begin{aligned}
\rho^{\mathsf{D}}_T(y_T) = \max_{(\alpha_{Tj})_{j=1}^J} \quad & \sum_{j=1}^{J} \sigma_j(\alpha_{T1}, \ldots, \alpha_{TJ}) r_j(\alpha_{Tj}) && \text{(EC.27a)}\\
\text{s.t.} \quad & \sigma_j(\alpha_{T1}, \ldots, \alpha_{TJ})\alpha_{Tj} \le y_j, \quad \forall j \in [J] && \text{(EC.27b)}\\
& \alpha_{Tj} \in \Omega_\alpha, \quad \forall j \in [J] && \text{(EC.27c)}
\end{aligned}
$$

Fix y_T. Let $\alpha^*_T = (\alpha^*_{Tj})$ be the optimal solution to (EC.26). Since constraint (EC.26b) holds almost surely, it must also hold in expectation. Hence, α^*_T is a feasible solution to (EC.27) and, therefore, $\rho_T(y) = \sum_{j=1}^{J} \sigma_j(\alpha^*_T)\alpha^*_{Tj} p_j(\alpha^*_{Tj}) \le \rho^{\mathsf{D}}_T(y)$, proving the base case.

Now, suppose that $\rho^*_{t+1}(y_{t+1}) \le \rho^{\mathsf{D}}_{t+1}(y_{t+1})$ for all y_{t+1}. We want to show that $\rho^*_t(y_t) \le \rho^{\mathsf{D}}_t(y_t)$ for all y_t. Again, fix y_t. Let $\alpha^*_t = (\alpha^*_{tj})$ be the optimal solution to (5). Since the conditional purchase rates α^*_t satisfy the constraints in $S(y_t)$ almost surely, they also satisfy these constraints in expectation, i.e., $\alpha^*_t \in S^D(y_t)$. Therefore,

$$
\begin{aligned}
\rho^*_t(y_t) &= \sum_{j=1}^{J} \sigma_j(\alpha^*_t) r_j(\alpha^*_{tj}) + \sum_{j=1}^{J} \sigma_j(\alpha^*_t)\alpha^*_{tj}\rho^*_{t+1}(y_t - e_j) + \left(1 - \sum_{j=1}^{J}\sigma_j(\alpha^*_t)\alpha^*_{tj}\right)\rho_{t+1}(y_t)\\
&\le \sum_{j=1}^{J} \sigma_j(\alpha^*_t) r_j(\alpha^*_{tj}) + \sum_{j=1}^{J} \sigma_j(\alpha^*_t)\alpha^*_{tj}\rho^{\mathsf{D}}_{t+1}(y_t - e_j) + \left(1 - \sum_{j=1}^{J}\sigma_j(\alpha^*_t)\alpha^*_{tj}\right)\rho^{\mathsf{D}}_{t+1}(y_t)\\
&\le \sum_{j=1}^{J} \sigma_j(\alpha^*_t) r_j(\alpha^*_{tj}) + \rho^{\mathsf{D}}_{t+1}\left(y_t - \sum_{j=1}^{J}\sigma_j(\alpha^*_t)\alpha^*_{tj} e_j\right) \le \rho^{\mathsf{D}}_t(y_t),
\end{aligned}
$$

where the first inequality follows from induction hypothesis, the second inequality follows from Jensen's inequality and the joint concavity of $\rho^{\mathsf{D}}_{t+1}(\cdot)$ (by Lemma 3), and the final inequality follows since $\alpha^*_t \in S^D(y_t)$. This concludes the proof of Theorem 1. $\square$

EC.6. Proof of Lemma 4

Fix $y = (y_j)$. For any $j \in [J]$ and $\sigma_j = (\sigma_{tj})_{t=1}^T$, define:

$$W_j(\sigma_j) := \max_{(\alpha_{tj})_{t=1}^T} \sum_{t=1}^T \sigma_{tj} r_j(\alpha_{tj}) + W_{j+1}(\sigma_j \zeta_j - \zeta_j \sigma_j \circ \alpha_j) \tag{EC.28a}$$

$$\text{s.t.} \sum_{t=1}^T \sigma_{tj} \alpha_{tj} \le y_j \tag{EC.28b}$$

$$\alpha_{tj} \in \Omega_\alpha, \quad \forall t \in [T], \tag{EC.28c}$$

where $W_{J+1}(\cdot) = 0$. Note that, by construction, $W_j(\sigma_j) = R_{1j}^{\mathsf{D}}(\sigma_j, y)$, where $R_{tj}^{\mathsf{D}}(\cdot,\cdot)$ is as defined in (EC.20a). Moreover, since $\rho_1^D(y) = R_{t1}^{\mathsf{D}}(\Lambda, y)$, where Λ is the vector whose elements are all equal to λ, we also have $V^{\mathsf{D}} = W_1(\Lambda)$.

Before proceeding with the proof, we state the following two claims.

Claim EC.1. *$W_j(\sigma_j)$ is jointly concave in σ_j for any $j \in [J]$.*

Claim EC.2. *For any $\sigma_j = (\sigma_{tj})_{t=1}^T$, let $\sigma_{j,[i]}$ denote the i^{th} smallest element of σ_j. If σ_j, σ_j' are two T-dimensional vectors where $\sigma_{j,[t]} = \sigma'_{j,[t]}$ for all $t \in [T]$, then $W_j(\sigma_j) = W_j(\sigma_j')$.*

The first claim follows directly from Lemma EC.1, i.e., since $W_j(\sigma_j) = R_{1j}^{\mathsf{D}}(\sigma_j, y)$. The second claim is obvious due to the stationarity of the deterministic problem, i.e., time periods are interchangeable with the same parameter ζ_j and functions $r_j(\cdot)$ for each j.

Fix j. We will prove Lemma 4 by showing that if $\sigma_{1j} = \cdots = \sigma_{Tj} = q$ for some $q \in [0,1]$, then there exists an optimal solution α_j^* such that $\alpha_{1j}^* = \cdots = \alpha_{Tj}^*$. (Since we start with constant arrival rates to product 1 (i.e, $\sigma_{t1} = \lambda$ for all $t \in [T]$), this proves the lemma because, by choosing intensity α_j^*, the vector of customer arrivals for the next product is also constant.)

We only show the proof for the case of $T = 2$. (We omit the proof for the general case because it uses similar arguments.) Let $\alpha_j^* = (\alpha_{1j}^*, \alpha_{2j}^*)$ be an optimal solution to (EC.28a) with $\alpha_{1j}^* \neq \alpha_{2j}^*$. We are going to show that there exists another solution $\hat{\alpha}$ whose elements are equal and whose objective is at least as good as that of α^*. Define α_j' to be a copy of α_j^* but with element 1 and 2 interchanged (i.e., $\alpha_{1j}' = \alpha_{2j}^*$ and $\alpha_{2j}' = \alpha_{1j}^*$). Note that the set $\{\alpha_j^*, \alpha_j'\}$ contains all possible permutations of the elements of α_j^*. Now define $\hat{\alpha}_j = \frac{1}{2}\alpha_j^* + \frac{1}{2}\alpha_j'$. By construction, the elements of $\hat{\alpha}_j$ are all equal. Moreover, $\hat{\alpha}_j$ also satisfies (EC.28b) since $\sigma_{tj} = q$ for all $t \in [T]$.

From the concavity of $r_j(\cdot)$, we know that $r_j(\hat{\alpha}_{1j}) + r_j(\hat{\alpha}_{2j}) = 2 \cdot r_j\left(\frac{1}{2}\alpha_{1j}^* + \frac{1}{2}\alpha_{2j}^*\right) \ge r_j(\alpha_{1j}^*) + r_j(\alpha_{2j}^*)$. Hence, $\sum_{t=1}^T q r_j(\hat{\alpha}_{tj}) \ge \sum_{t=1}^T q r_j(\alpha_{tj}^*)$. Moreover, we also know that

$$\begin{aligned} W_{j+1}(q\zeta_j - \zeta_j q \hat{\alpha}_j) &= W_{j+1}(q(1-\hat{\alpha}_j)\zeta_j) = W_{j+1}\left(\frac{1}{2} q\left[(1-\alpha_j^*)\zeta_j\right] + \frac{1}{2}\sigma_{tj}\left[(1-\alpha_j')\zeta_j\right]\right) \\ &\ge \frac{1}{2} W_{j+1}\left(q\left[(1-\alpha_j^*)\zeta_j\right]\right) + \frac{1}{2} W_{j+1}\left(q\left[(1-\alpha_j')\zeta_j\right]\right) = W_{j+1}\left(q\left[(1-\alpha_j^*)\zeta_j\right]\right), \end{aligned}$$

where the inequality follows from the joint concavity of $W_{j+1}(\cdot)$ (Claim EC.1) and the last equality follows from the fact that interchanging time indices results in the same revenue-to-go (Claim EC.2). Therefore, $\hat{\alpha}_j$ results in an objective (EC.28a) that is at least as large as that of α_j^*. This concludes the proof for $T=2$.

Note that to prove Lemma 4 for $T>2$, starting from an optimal solution α_j^* whose elements may not all be equal, we need to construct *all* possible permutations of elements of α_j^*. Next, we define $\hat{\alpha}_j$ to be a linear combination of these vectors, with equal weight. The remainder of the proof (i.e., showing that the constructed solution has an objective that is at least as large as that of α_j^*) proceeds in the same way. This completes the proof. □

EC.7. Proof of Lemma 5

We start by defining the following function $F_j(\alpha_j;\sigma_j) := \sigma_j r_j(\alpha_j) + Q_{j+1}(\sigma_j(1-\alpha_j)\zeta_j)$. Note that $Q_j(\sigma_j) = \max_{\alpha_j \in S_j^Q(\sigma_j)} F_j(\alpha_j;\sigma_j)$. The derivative of F_j with respect to α_j is given by

$$F_j'(\alpha_j;\sigma_j) = \sigma_j\left[r_j'(\alpha_j) - \zeta_j \cdot Q_{j+1}'(\sigma_j(1-\alpha_j)\zeta_j)\right]. \tag{EC.29}$$

In order to show that $a_j^D < u$ for all $j\in[J]$, we need to show that $F_j'(u;\sigma_j)<0$ whenever $\sigma_j>0$. It is sufficient that we show $Q_j'(\sigma_j)\geq 0$ for all $\sigma_j\geq 0$ and $j\in[J]$. To see why, note that together with Assumption A3, this implies $F_j'(u;\sigma_j) = \sigma_j\left[r_j'(u) - \zeta_j\cdot Q_{j+1}'(\sigma_j(1-u)\zeta_j)\right] < 0$. It turns out that $Q_j'(\sigma_j)\geq 0$ is also a consequence of Assumption A3 but the proof requires an inductive argument, see below.

We prove that $Q_j'(\sigma_j)\geq 0$ by induction. An important property that we will use in this proof is the fact that $Q_j'(\sigma_j)$ is non-increasing in σ_j (c.f. Claim EC.1 in the proof of Lemma 4). We start with the base case $j=J$. Note that

$$Q_J(\sigma_J) = \begin{cases} \sigma_J r_J(\alpha_J^{\max}), & \text{if } \sigma_J \leq \frac{y_J}{\alpha_J^{\max}T}, \\ \sigma_J r_J\left(\frac{y_J}{\sigma_J T}\right), & \text{if } \sigma_J > \frac{y_J}{\alpha_J^{\max}T}, \end{cases} \tag{EC.30}$$

so we have:

$$Q_J'(\sigma_J) = \begin{cases} r_J(\alpha_J^{\max}), & \text{if } \sigma_J \leq \frac{y_J}{\alpha_J^{\max}T}, \\ r_J\left(\frac{y_J}{\sigma_J T}\right) - \frac{y_J}{\sigma_J T}\cdot r_J'\left(\frac{y_J}{\sigma_J T}\right), & \text{if } \sigma_J > \frac{y_J}{\alpha_J^{\max}T}. \end{cases} \tag{EC.31}$$

From (EC.31), note that if $r_J(\alpha) - \alpha r_J'(\alpha)\geq 0$ for any $\alpha\in[0,u]$, we have $Q_J'(\sigma_J)\geq 0$ for any $\sigma_J\geq 0$. But this is true because of the concavity of r_J. (Recall that concavity of r_j implies that $r_j(0)\leq r_j(\alpha) + (0-\alpha)r_j'(\alpha)$ for any α.) We state this observation formally below.

Claim EC.3. *$r_j(\alpha) - \alpha r_j'(\alpha)\geq 0$ for any $\alpha\in[0,u]$.*

Now, assume that $Q'_{j+1}(\sigma_{j+1}) \geq 0$ for any $\sigma_{j+1} \geq 0$. We next show that this implies $Q'_j(\sigma_j) \geq 0$ for any $\sigma_j \geq 0$. Let $\alpha_j^{\max,F}(\sigma_j)$ denote the unconstrained maximizer of $F_j(\cdot;\sigma_j)$. (Note the distinction with $\alpha_j^{\max}$ which is the unconstrained maximizer of $r_j(\cdot)$.) We will also need the following result:

Claim EC.4. *$\alpha_j^{\max,F}(\sigma_j)$ is non-decreasing in σ_j.*

To show Claim EC.4, note from (EC.29) that $\alpha_j^{\max,F}(\sigma_j)$ is the intersection point of two curves: $r'_j(\alpha_j)$ and $h(\alpha_j;\sigma_j) := \zeta_j \cdot Q'_{j+1}(\sigma_j(1-\alpha_j)\zeta_j)$. Due to strict concavity of $r_j(\cdot)$, we know that $r'_j(\alpha_j)$ is a strictly decreasing function of α_j. On the other hand, $h(\alpha_j;\sigma_j)$ is an increasing function of α_j (again, due to concavity of Q_{j+1} from Claim EC.1). Therefore, they intersect at exactly one point $\alpha_j^{\max,F}(\sigma_j)$. Now consider two points σ_j, σ'_j such that $\sigma_j \leq \sigma'_j$. Since $Q'_{j+1}(\sigma_{j+1})$ is non-increasing in σ_{j+1}, this means that $h(\alpha_j;\sigma_j) \geq h(\alpha_j;\sigma'_j)$ for all α_j. Therefore, the point of intersection of $r'_j(\alpha_j)$ with $h(\alpha_j;\sigma'_j)$ is to the right of its intersection with $h(\alpha_j;\sigma_j)$. In other words, $\alpha_j^{\max,F}(\sigma_j) \leq \alpha_j^{\max,F}(\sigma'_j)$. This establishes our claim.

We continue with our proof. Define the following: $\sigma_j^\ell := \{\sigma_j \geq 0 : \alpha_j^{\max,F}(\sigma_j) = \frac{y_j}{T\sigma_j}\}$. Note that σ_j^ℓ is in fact a singleton since the RHS is decreasing in σ_j and approaches zero, while the LHS is non-decreasing in σ_j (Claim EC.4), so the two curves intersect at exactly one point. For any $\sigma_j \geq \sigma_j^\ell$, we have $\alpha_j^{\max,F}(\sigma_j) \geq \frac{y_j}{T\sigma_j}$. To see why, note that $\alpha_j^{\max,F}(\sigma_j) \geq \alpha_j^{\max,F}(\sigma_j^\ell) = \frac{y_j}{T\sigma_j^\ell} \geq \frac{y_j}{T\sigma_j}$, where the first inequality follows from Claim EC.4, the equality follows from the definition of σ_j^ℓ, and last inequality is because $\sigma_j \geq \sigma_j^\ell$.

Note that, whenever $\alpha_j^{\max,F}(\sigma_j) \geq \frac{y_j}{T\sigma_j}$, then it follows that $\alpha_j^{\max,F}(\sigma_j)$ is not a feasible solution to $Q_j(\sigma_j) = \max_{\alpha_j \in S_j^Q(\sigma_j)} F_j(\alpha_j;\sigma_j)$, where $S_j^Q(\sigma_j) = \{\alpha_j \in \Omega_\alpha : \sigma_j\alpha_j \leq \frac{y_j}{T}\}$. In this case, since $F_j(\cdot;\sigma_j)$ is concave in α_j, the optimal value occurs when $\alpha_j = \frac{y_j}{T\sigma_j}$. Therefore, we have $Q_j(\sigma_j) = \sigma_j r_j(\frac{y_j}{\sigma_j T}) + Q_{j+1}(\zeta_j\sigma_j - \zeta_j\frac{y_j}{T})$, for $\sigma_j > \sigma_j^\ell$. This identity implies that, for any $\sigma_j \geq \sigma_j^\ell$, we have $Q'_j(\sigma_j) = [r_j(\frac{y_j}{\sigma_j T}) - \frac{y_j}{\sigma_j T} r'_j(\frac{y_j}{\sigma_j T})] + \zeta_j Q'_{j+1}(\zeta_j\sigma_j - \zeta_j\frac{y_j}{T}) \geq 0$, where the inequality follows from Claim EC.3 and the induction hypothesis. Since $Q'_j(\sigma_j)$ is non-increasing in σ_j (Claim EC.1), we conclude that $Q'_j(\sigma_j) \geq 0$ for all $\sigma_j \geq 0$.

This completes the induction; therefore, $F'_j(u;\sigma_j) < 0$ for all $j \in [J]$ and $\sigma_j > 0$. $\square$

EC.8. Proof of Theorem 2

Let $p^D = (p_j^D)_{j=1}^J$ where $p_j^D = p_j(a_j^D)$ and let $\hat{p} = \max_j p_j^D$. Also, let $v_j^D = \sigma_j(a^D)a_j^D$ (by definition, $\mathbf{E}[D_{tj}(p^D)] = v_j^D$). Fix $k \geq 1$. Consider an alternative policy FP′ that always applies p^D in every periods regardless of out-of-stock. At the end of period T, FP′ pays a penalty $\hat{p}$ for each unit of oversold product. Obviously, $V^{k,FP'} \leq V^{k,FP}$. Moreover, by definition,

$$\begin{aligned} V^{k,FP'} &= V^{k,D} - \hat{p} \cdot \sum_{j=1}^J \mathbf{E}\left[\left(\sum_{t=1}^{T^k} D_{tj}(p^D) - y_j^k\right)^+\right] \geq V^{k,D} - \hat{p} \cdot \sum_{j=1}^J \mathbf{E}\left[\left(\sum_{t=1}^{T^k} D_{tj}(p^D) - \sum_{t=1}^{T^k} v_j^D\right)^+\right] \\ &= V^{k,D} - \hat{p} \cdot \sum_{j=1}^J \sqrt{T^k v_j^D(1-v_j^D)} \geq V^{k,D} - \frac{\hat{p}J}{2} \cdot \sqrt{T^k}, \end{aligned}$$

where the first inequality follows because $\sum_{t=1}^{T^k} v_j^D \le y_j^k$ (by inventory constraints), the second equality follows because $\sum_{t=1}^{T^k} D_{tj}(p^D)$ is a Binomial random variable with T^k trials and success probability v_j^D, and the last inequality follows since $v_j^D(1-v_j^D) \le \frac{1}{4}$. This completes the proof. $\square$

EC.9. Proof of Theorem 3

Let $Q = ||-A^{-1}M^{-1}||_\infty$ (by definition of δ_t^σ, we have $||\delta_t^\sigma||_\infty \le Q \cdot ||\delta_t^v||_\infty$) and define $\varphi > 0$ to be the largest number satisfying:

$$\varphi \le \min_j y_j, \tag{EC.32}$$

$$\varphi \le \min\left\{\min_j v_j^D, \min_j (1-v_j^D)\right\}, \tag{EC.33}$$

$$\varphi \le Q^{-1} \cdot \min\left\{\min_j \sigma_j^D, \min_j (1-\sigma_j^D)\right\}, \tag{EC.34}$$

$$\varphi \le (1+Q)^{-1} \cdot \min_j \sigma_j^D (1-a_j^D). \tag{EC.35}$$

Fix $k \ge 1$. (The constant φ defined above is independent of k.) Without loss of generality, we will assume throughout the proof that $T^k = k$ (i.e., $T = 1$) and $v_j^D = y_j$ (i.e., the inventory constraints in DET are all binding). To prove Theorem 3, we proceed in several steps. In Step 1, we show that, as long as $\delta_t^{k,v}$ is small, we have $\alpha_t^k = \hat{\alpha}_t^k$; in Step 2, we show that $\delta_t^{k,v}$ is indeed small for at least τ^k periods and τ^k is very close to k (in expectation); in Step 3, we finally show the expected revenue loss bound in Theorem 3 by bounding the revenue loss during the first $\tau^k - 1$ periods and simply throwing away all the revenues collected after period τ^k.

Step 1

Let $y_t^k = (y_{tj}^k)_{j=1}^J$ where y_{tj}^k denote the remaining inventory of product j at the beginning of period t. Also, let $\hat{y}_t^k = (\hat{y}_{tj}^k)_{j=1}^J$ where $\hat{y}_{tj}^k$ is defined as follows:

$$\hat{y}_{tj}^k = (k-t+1)\left[y_j - \sum_{s=1}^{t-1} \frac{\Delta_{sj}^k}{k-s}\right] = (k-t+1)[y_j - \delta_{tj}^{k,v}]. \tag{EC.36}$$

Let $\hat{\varphi} = \varphi/2$. We state a lemma.

Lemma EC.2. *Suppose that $\alpha_s^k = \hat{\alpha}_s^k$ and $y_s^k = \hat{y}_s^k$ for all $s \le t-1$. If $||\delta_t^{k,v}||_\infty < \hat{\varphi}$, then $\alpha_t^k = \hat{\alpha}_t^k$ and $y_t^k = \hat{y}_t^k$.*

Proof. The proof is by induction. At the beginning of period 1, we have $\alpha_1^k = \hat{\alpha}_1^k$ and $y_1^k = \hat{y}_1^k$. This is our base case. Now, suppose that it is true for all $s \le t-1$. By definition of CC policy, $\hat{\alpha}_{t1}^k = \frac{\hat{v}_{t1}^k}{\lambda} = \frac{v_1^D - \delta_{t1}^{k,v}}{\lambda}$ and $\hat{\alpha}_{tj}^k = \frac{\hat{v}_{tj}^k}{\hat{\sigma}_{tj}^k} = \frac{v_j^D - \delta_{tj}^{k,v}}{\sigma_j^D + \delta_{tj}^{k,\sigma}}$ $\forall j \ge 2$. It is not difficult to check that condition $||\delta_t^{k,v}||_\infty < \hat{\varphi}$ guarantees $\hat{\alpha}_{tj}^k \in (0,1)$ for all j (see (EC.33)-(EC.35)). So, $\hat{\alpha}_t^k$ is feasible and, therefore,

$\alpha_t^k = \hat{\alpha}_t^k$. As for the remaining inventory for product j at the beginning of period t, $||\delta_t^{k,v}||_\infty < \hat{\varphi}$ implies $\hat{y}_{tj}^k \geq 0$ (see (EC.32)). Moreover,

$$\begin{aligned} y_{tj}^k &= y_{t-1,j}^k - D_{t-1,j}^k(\alpha_{t-1,j}^k) = \hat{y}_{t-1,j}^k - D_{t-1,j}^k(\hat{\alpha}_{t-1,j}^k) = (k-t+2)\left[y_j - \sum_{s=1}^{t-2}\frac{\Delta_{sj}^k}{k-s}\right] - (\hat{v}_{t-1,j}^k + \Delta_{t-1,j}^k) \\ &= (k-t+2)\left[y_j - \sum_{s=1}^{t-2}\frac{\Delta_{sj}^k}{k-s}\right] - \left[v_j^D - \sum_{s=1}^{t-2}\frac{\Delta_{sj}^k}{k-s}\right] - \Delta_{t-1,j}^k = (k-t+1)\left[y_j - \sum_{s=1}^{t-1}\frac{\Delta_{sj}^k}{k-s}\right] \;=\; \hat{y}_{tj}^k, \end{aligned}$$

where the second equality follows from induction hypothesis. This completes the proof. $\square$

<u>Step 2</u>

Let τ^k denote the minimum of k and the first time t when the condition $||\delta_t^{k,v}||_\infty < \hat{\varphi}$ is violated (τ^k is a stopping time; it is defined as $\min\{t \leq k : ||\delta_t^{k,v}||_\infty \geq \hat{\varphi}\}$ if $||\delta_t^{k,v}||_\infty \geq \hat{\varphi}$ for some $t \leq k$, otherwise it is defined k). The following lemma tells us that τ^k is very close to k.

Lemma EC.3. *There exists $\Psi > 0$ independent of $k \geq 1$ such that $\mathbf{E}[k - \tau^k] \leq 1 + \Psi \log k$.*

Proof. The key is to note that the sequence $\{\delta_{tj}^{k,v}\}$ is a Martingale with respect to the filtration $\{\Im_t\}$, where $\Im_t$ is the observed history up to the beginning of period t. This implies that $\{|\delta_{tj}^{k,v}|\}$ is a sub-Martingale. So, we can use Doob's Maximal inequality (see David Williams, *Probability with Martingales*, Cambridge University Press (2010), pp. 137) to bound the following probability:

$$P(\tau^k \leq t) \;=\; P\left(\sup_{j,\, s\leq t} |\delta_{sj}^{k,v}| \geq \hat{\varphi}\right) \leq P\left(\sum_{j=1}^J \sup_{s\leq t} |\delta_{sj}^{k,v}| \geq \hat{\varphi}\right) \leq \sum_{j=1}^J P\left(\sup_{s\leq t} |\delta_{sj}^{k,v}| \geq \frac{\hat{\varphi}}{J}\right) \leq \sum_{j=1}^J \frac{\mathbf{E}[(\delta_{tj}^{k,v})^2]}{(\hat{\varphi}/J)^2}.$$

Since $\mathbf{E}[\tau^k] = \sum_{t=1}^k P(\tau^k \geq t) = k - \sum_{t=2}^k P(\tau^k < t) = k - 1 - \sum_{t=2}^{k-1} P(\tau^k < t)$ and

$$\sum_{t=2}^{k-1} \mathbf{E}[(\delta_{tj}^{k,v})^2] \;=\; \sum_{t=2}^{k-1}\left[\frac{\mathbf{E}[(\Delta_{t-1,j}^k)^2]}{(k-t+1)^2} + \frac{\mathbf{E}[(\Delta_{t-2,j}^k)^2]}{(k-t+2)^2} + \cdots + \frac{\mathbf{E}[(\Delta_{1,j}^k)^2]}{(k-1)^2}\right] \;=\; O(\log k),$$

(since we assume at most one arrival per period, $|\Delta_{tj}^k| \leq 1$) we conclude that there exists a constant $\Psi > 0$ independent of $\theta > 0$ such that $\mathbf{E}[k - \tau^k] \leq 1 + \Psi\, \log k$. This completes the proof. $\square$

<u>Step 3</u>

We will now bound the expected loss of CC policy. Let $R^{k,CC}(s,t)$ denote the total revenues earned under CC policy during periods $s, s+1, \ldots, t$. Observe that

$$\begin{aligned} V^{k,D} - V^{k,CC} &\leq \mathbf{E}\left[\sum_{t=1}^{\tau^k - 1} \tilde{V}^D - R^{k,CC}(1, \tau^k - 1)\right] + (k - \tau^k + 1)\tilde{V}^D \\ &\leq \mathbf{E}\left[\sum_{t=1}^{\tau^k - 1} \tilde{V}^D - R^{k,CC}(1, \tau^k - 1)\right] + (2 + \Psi \log k)\tilde{V}^D, \end{aligned}$$

where $\tilde{V}^D$ is as defined in Section 5 and the last inequality follows from Lemma EC.3. So, what remains is to bound the expected loss during the first $\tau^k - 1$ periods.

Let $\epsilon_{tj}^k = \hat{\alpha}_{tj}^k - a_j^D$. By definition of τ^k,

$$\begin{aligned}
\mathbf{E}\left[R^{k,CC}(1,\tau^k-1)\right] &= \mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}\sigma_j(\hat{\alpha}_{tj}^k)r_j(\hat{\alpha}_{tj}^k)\right] = \mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}(\sigma_j^D+\delta_{tj}^{k,\sigma})r_j(a_j^D+\epsilon_{tj}^k)\right] \\
&= \mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}\sigma_j^D r_j(a_j^D+\epsilon_{tj}^k)\right] + \mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}\delta_{tj}^{k,\sigma} r_j(a_j^D+\epsilon_{tj}^k)\right] := \Phi_1+\Phi_2.
\end{aligned}$$

We bound Φ_1 and Φ_2 separately. We start with Φ_1. By Taylor's expansion, $r_j(a_j^D+\epsilon_{tj}^k) \geq r_j(a_j^D) + r_j'(a_j^D)\epsilon_{tj}^k - \frac{\omega}{2}(\epsilon_{tj}^k)^2$. Also, for $t<\tau^k$, from (17), we have:

$$\epsilon_{tj}^k = -\frac{\delta_{tj}^{k,v}}{\sigma_j^D} - \frac{(v_j^D-\delta_{tj}^{k,v})\delta_{tj}^{k,\sigma}}{\sigma_j^D(\sigma_j^D+\delta_{tj}^{k,\sigma})} \tag{EC.37}$$

$$= -\frac{\delta_{tj}^{k,v}}{\sigma_j^D} - \frac{v_j^D\delta_{tj}^{k,\sigma}}{(\sigma_j^D)^2} + \frac{\delta_{tj}^{k,v}\delta_{tj}^{k,\sigma}}{(\sigma_j^D)^2} + \frac{(v_j^D-\delta_{tj}^{k,v})(\delta_{tj}^{k,\sigma})^2}{(\sigma_j^D)^2(\sigma_j^D+\delta_{tj}^{k,\sigma})}. \tag{EC.38}$$

We will use identity (EC.38) to compute a lower and upper bound for $\mathbf{E}[\sum_{t=1}^{\tau^k-1}\epsilon_{tj}^k]$ and identity (EC.37) to compute a lower bound for $\mathbf{E}[\sum_{t=1}^{\tau^k-1} -\frac{\omega}{2}(\epsilon_{tj}^k)^2]$. We make several observations:

1. Since τ^k-1 is a stopping time, by Stopping Time theorem, $\mathbf{E}[\sum_{t=1}^{\tau^k-1}\delta_{tj}^{k,v}] = \mathbf{E}[\sum_{t=1}^{\tau^k-1}\delta_{tj}^{k,\sigma}] = 0$.
2. By definition, $\delta_{t,2:J}^{k,\sigma} = -A^{-1}M^{-1}\delta_{t,1:J-1}^{k,v}$. This means that $\delta_{tj}^{k,\sigma} = \sum_{i=1}^{J}\theta_{ij}\delta_{ti}^{k,v}$ for some constants θ_{ij} and we can write:

$$\begin{aligned}
\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\delta_{tj}^{k,v}\delta_{tj}^{k,\sigma}\right] &= \mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{i=1}^{J}\theta_{ij}\delta_{tj}^{k,v}\delta_{ti}^{k,v}\right] = \mathbf{E}\left[\sum_{t=1}^{k}\sum_{i=1}^{J}\theta_{ij}\delta_{tj}^{k,v}\delta_{ti}^{k,v}\right] - \mathbf{E}\left[\sum_{t=\tau^k}^{k}\sum_{i=1}^{J}\theta_{ij}\delta_{tj}^{k,v}\delta_{ti}^{k,v}\right] \\
&= \mathbf{E}\left[\sum_{t=1}^{k}\theta_{jj}(\delta_{tj}^{k,v})^2\right] - \mathbf{E}\left[\sum_{t=\tau^k}^{k}\theta_{jj}(\delta_{tj}^{k,v})^2\right] \geq -2\cdot\sup_j|\theta_{jj}|\cdot\mathbf{E}\left[\sum_{t=1}^{k}(\delta_{tj}^{k,v})^2\right] = -O(\log k),
\end{aligned}$$

where the third equality follow since $\mathbf{E}[\delta_{tj}^{k,v}\delta_{ti}^{k,v}]=0$ for $i\neq j$ and the last equality follows by the same argument as in Step 2. Similarly, we also have

$$\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\delta_{tj}^{k,\sigma}\delta_{tj}^{k,v}\right] \leq 2\cdot\sup_j|\theta_{jj}|\cdot\mathbf{E}\left[\sum_{t=1}^{k}(\delta_{tj}^{k,v})^2\right] = O(\log k).$$

3. As for the term $\frac{(v_j^D-\delta_{tj}^{k,v})(\delta_{tj}^{k,\sigma})^2}{(\sigma_j^D)^2(\sigma_j^D+\delta_{tj}^{k,\sigma})}$, for $t<\tau^k$, we have $|\delta_{tj}^{k,v}|\leq\hat{\varphi}=\varphi/2\leq v_j^D/2$ and $|\delta_{tj}^{k,\sigma}|\leq Q\hat{\varphi} = Q\varphi/2\leq\sigma_j^D/2)$. So, we can bound:

$$\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\frac{(v_j^D-\delta_{tj}^{k,v})(\delta_{tj}^{k,\sigma})^2}{(\sigma_j^D)^2(\sigma_j^D+\delta_{tj}^{k,\sigma})}\right] \leq \frac{3v_j^D}{(\sigma_j^D)^3}\cdot\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}(\delta_{tj}^{k,\sigma})^2\right] \leq \frac{3v_j^D}{(\sigma_j^D)^3}\cdot\mathbf{E}\left[\sum_{t=1}^{k}(\delta_{tj}^{k,\sigma})^2\right] = O(\log k),$$

where the last equality follows since

$$\mathbf{E}\left[\sum_{t=1}^{k}\left(\delta_{tj}^{k,\sigma}\right)^2\right]=\mathbf{E}\left[\sum_{t=1}^{k}\left(\sum_{i=1}^{J}\theta_{ij}\delta_{ti}^{k,v}\right)^2\right]=\mathbf{E}\left[\sum_{t=1}^{k}\sum_{i=1}^{J}\theta_{ij}^2(\delta_{ti}^{k,v})^2\right]=O(\log k).$$

Similarly, we can also bound:

$$\begin{aligned}\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\frac{(v_j^D-\delta_{tj}^{k,v})(\delta_{tj}^{k,\sigma})^2}{(\sigma_j^D)^2(\sigma_j^D+\delta_{tj}^{k,\sigma})}\right] &\geq -\frac{3v_j^D}{(\sigma_j^D)^3}\cdot\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}(\delta_{tj}^{k,\sigma})^2\right]=-\frac{3v_j^D}{(\sigma_j^D)^3}\cdot\left\{\mathbf{E}\left[\sum_{t=1}^{k}(\delta_{tj}^{k,\sigma})^2\right]-\mathbf{E}\left[\sum_{t=\tau^k}^{k}(\delta_{tj}^{k,\sigma})^2\right]\right\}\\ &\geq -\frac{6v_j^D}{(\sigma_j^D)^3}\cdot\mathbf{E}\left[\sum_{t=1}^{k}(\delta_{tj}^{k,\sigma})^2\right] = -O(\log k).\end{aligned}$$

4. Put the first three observations together with (EC.38), we have: $\left|\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\epsilon_{tj}^k\right]\right|=\Theta(\log k)$.
5. We now compute a lower bound for $\mathbf{E}[\sum_{t=1}^{\tau^k-1}-\frac{\omega}{2}(\epsilon_{tj}^k)^2]$. Note that, using inequality $(a+b)^2\leq 2\cdot(a^2+b^2)$, we can bound:

$$(\epsilon_{tj}^k)^2\leq 2\cdot\left[\frac{(\delta_{tj}^{k,v})^2}{(\sigma_j^D)^2}+\frac{(v_j^D-\delta_{tj}^{v})^2(\delta_{tj}^{\sigma})^2}{(\sigma_j^D)^2(\sigma_j^D+\delta_{tj}^{\sigma})^2}\right]\leq 2\cdot\left[\frac{(\delta_{tj}^{k,v})^2}{(\sigma_j^D)^2}+\frac{(\frac{3}{2}v_j^D)^2(\delta_{tj}^{\sigma})^2}{(\sigma_j^D)^2(\frac{1}{2}\sigma_j^D)^2}\right],$$

   where the second inequality follows from the discussions in observation no. 3. By similar arguments as in observations no. 2 and 3, we have: $\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}-\frac{\omega}{2}(\epsilon_{tj}^k)^2\right]\geq -O(\log k)$.
6. Put the results in observations no. 4 and 5 together, we have:

$$\begin{aligned}\Phi_1&=\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}\sigma_j^D r_j(a_j^D+\epsilon_{tj}^k)\right]\\ &\geq\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}\sigma_j^D\left(r_j(a_j^D)+r_j'(a_j^D)\epsilon_{tj}^k-\frac{\omega}{2}(\epsilon_{tj}^k)^2\right)\right]\geq\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\tilde{V}^D\right]-O(\log k).\end{aligned}$$

We will now compute a lower bound for Φ_2. By Taylor's expansion,

$$\Phi_2=\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}\delta_{tj}^{k,\sigma}r_j(a_j^D+\epsilon_{tj}^k)\right]=\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\sum_{j=1}^{J}\delta_{tj}^{k,\sigma}\left(r_j(a_j^D)+r_j'(a_j^D)\epsilon_{tj}^k+\frac{r_j''(\xi_{tj})}{2}(\epsilon_{tj}^k)^2\right)\right]$$

for some $\{\xi_{tj}\}$. By Stopping Time theorem, $\mathbf{E}[\sum_{t=1}^{\tau^k-1}\delta_{tj}^{k,\sigma}]=0$. As for $\mathbf{E}[\sum_{t=1}^{\tau^k}\delta_{tj}^{k,\sigma}\epsilon_{tj}^k]$, using identity (EC.37), it is not difficult to see that $\mathbf{E}[\sum_{t=1}^{\tau^k}\delta_{tj}^{k,\sigma}\epsilon_{tj}^k]\geq -O(\log k)$ (by similar arguments used in observations no. 2 and 3 above). To bound $\mathbf{E}[\sum_{t=1}^{\tau^k-1}\delta^{k,\delta}r_j''(\xi_{tj})(\epsilon_{tj}^k)^2]$, simply notice that

$$\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\delta_{tj}^{k,\delta}r_j''(\xi_{tj})(\epsilon_{tj}^k)^2\right]\geq-\omega\cdot\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}|\delta_{tj}^{k,\delta}|(\epsilon_{tj}^k)^2\right]\geq-\omega Q\hat{\varphi}\cdot\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}(\epsilon_{tj}^k)^2\right]\geq -O(\log k),$$

where the second inequality follows by the definition of τ^k (see observation no. 3 above) and the last inequality follows by the result in observation no. 5. We conclude that $\Phi_2\geq -O(\log k)$.

Put all things together, $V^{k,D}-V^{k,CC}\leq\mathbf{E}\left[\sum_{t=1}^{\tau^k-1}\tilde{V}^D-R^{k,CC}(1,\tau^k-1)\right]+(2+\Psi\log k)\tilde{V}^D=O(\log k)$. This completes the proof. $\square$

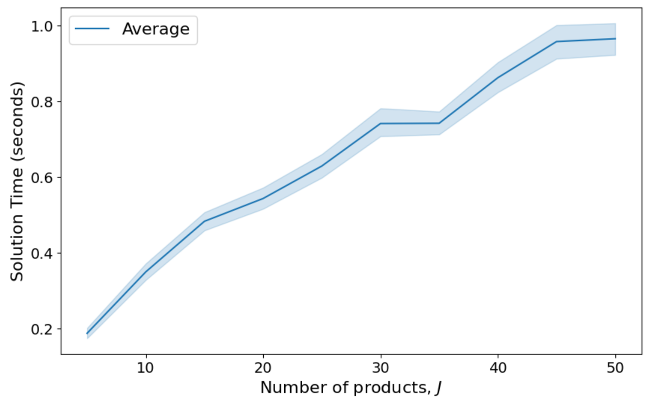

Figure EC.1 **Average and 95% confidence intervals of the solution times of DET (using the DP algorithm) plotted against the number of products**

EC.10. Proof of Lemma EC.5

In the deterministic model, recall that λ customers arrive to the first position. Consider the population of customers that arrive to product i (regardless of its position). Note that, out of those, a_i^D is the fraction that buy product i and $\Gamma_i = (1-a_i^D)(1-\zeta_i)$ is the fraction not buying and leaving immediately. So, out of customers who reach product i, $\mu_i = a_i^D + \Gamma_i = 1-(1-a_i^D)\zeta_i$ is the fraction not continuing with the search.

Consider an arbitrary ranking θ and an arbitrary position j. Suppose that products i and k are in positions j and $j+1$, respectively, i.e., $\theta_j = i$ and $\theta_{j+1} = k$. We examine whether swapping positions of products i and k will improve the total revenue. Let $L_j(\theta)$ be the total revenue generated from products in position 1 through $j-1$ out of the population that arrives in position 1. Let $R_{j+2}(\theta)$ be the total revenue generated from products in position $j+2$ through J out of the population that arrives at position $j+2$. The total revenue generated under ranking θ is equal to:

$$\lambda L_j(\theta) + \lambda\left(\prod_{\ell=1}^{j-1}(1-\mu_{\theta_\ell})\right)\left[a_i^D p_i(a_i^D) + (1-\mu_i)a_k^D p_k(a_k^D)\right] + \lambda\left(\prod_{\ell=1}^{j+1}(1-\mu_{\theta_\ell})\right)R_{j+2}(\theta).$$

Note that $L_j(\theta)$, $R_{j+2}(\theta)$, $\lambda\prod_{\ell=1}^{j-1}(1-\mu_{\theta_\ell})$, and $\lambda\prod_{\ell=1}^{j+1}(1-\mu_{\theta_\ell})$ remain the same even if we swap the position of i and k. Therefore, the total revenue with (i,k) is greater compared to (k,i) iff $a_i^D p_i(a_i^D) + (1-\mu_i)a_k^D p_k(a_k^D) \geq a_k^D p_k(a_k^D) + (1-\mu_k)a_i^D p_i(a_i^D)$. Rearranging terms and substituting the expression for μ_i and μ_k, this is equivalent to the condition $\frac{a_i^D p_i(a_i^D)}{1-(1-a_i^D)\zeta_i} \geq \frac{a_k^D p_k(a_k^D)}{1-(1-a_k^D)\zeta_k}$. Hence, given a^D, we know that the optimal ranking is in the order of decreasing index ν_i. $\square$

EC.11. Simulation Results

Figure EC.1 shows the computation time of DET., Tables EC.1 to EC.3 provide simulation results from Section 8.

Table EC.1 Linear conditional purchase probability

(a) $\zeta = 0.75$

J	LF	k	**FP Gap** (%)	**CC Gap** (%)	**MNL-p Gap** (%)
5	1	4	$\mathbf{9.9 \pm 1.2}$	20.7 ± 1.4	21.8 ± 2.4
5	1	8	$\mathbf{4.9 \pm 0.9}$	12.1 ± 0.9	18.0 ± 2.3
5	1	16	$\mathbf{2.8 \pm 0.5}$	7.6 ± 0.6	15.6 ± 2.2
5	1	32	$\mathbf{1.8 \pm 0.3}$	4.3 ± 0.5	14.6 ± 2.2
5	6	4	15.5 ± 1.2	$\mathbf{7.9 \pm 1.1}$	67.6 ± 4.5
5	6	8	7.9 ± 0.8	$\mathbf{3.5 \pm 0.7}$	65.2 ± 4.8
5	6	16	4.0 ± 0.6	$\mathbf{2.0 \pm 0.3}$	64.0 ± 4.9
5	6	32	2.1 ± 0.4	$\mathbf{1.3 \pm 0.2}$	63.5 ± 5.0
5	20	4	16.3 ± 1.2	$\mathbf{6.3 \pm 1.0}$	98.8 ± 1.4
5	20	8	8.4 ± 0.8	$\mathbf{2.8 \pm 0.5}$	98.8 ± 1.5
5	20	16	4.9 ± 0.6	$\mathbf{1.6 \pm 0.3}$	98.8 ± 1.5
5	20	32	2.8 ± 0.4	$\mathbf{1.1 \pm 0.1}$	98.8 ± 1.5
10	1	4	$\mathbf{11.3 \pm 1.0}$	17.2 ± 1.2	29.4 ± 2.8
10	1	8	$\mathbf{6.3 \pm 0.7}$	10.9 ± 0.7	26.1 ± 2.8
10	1	16	$\mathbf{4.1 \pm 0.5}$	7.4 ± 0.6	24.4 ± 2.9
10	1	32	$\mathbf{3.0 \pm 0.4}$	5.8 ± 0.5	23.4 ± 2.9
10	6	4	13.4 ± 0.7	$\mathbf{5.6 \pm 0.7}$	61.1 ± 3.0
10	6	8	6.0 ± 0.5	$\mathbf{2.2 \pm 0.4}$	58.7 ± 3.2
10	6	16	2.6 ± 0.3	$\mathbf{1.4 \pm 0.2}$	57.3 ± 3.3
10	6	32	1.6 ± 0.2	$\mathbf{0.9 \pm 0.2}$	56.7 ± 3.3
10	20	4	14.7 ± 0.7	$\mathbf{4.3 \pm 0.7}$	72.8 ± 3.1
10	20	8	7.2 ± 0.6	$\mathbf{1.6 \pm 0.3}$	71.1 ± 3.3
10	20	16	3.3 ± 0.4	$\mathbf{1.0 \pm 0.2}$	70.2 ± 3.4
10	20	32	1.7 ± 0.3	$\mathbf{0.6 \pm 0.1}$	69.7 ± 3.4

(b) $\zeta = 0.95$

J	LF	k	**FP Gap** (%)	**CC Gap** (%)	**MNL-p Gap** (%)
5	1	4	$\mathbf{19.2 \pm 1.7}$	21.8 ± 1.6	46.8 ± 3.4
5	1	8	$\mathbf{12.0 \pm 1.0}$	12.9 ± 0.8	29.3 ± 3.0
5	1	16	$\mathbf{2.7 \pm 0.4}$	5.4 ± 0.6	27.8 ± 3.1
5	1	32	$\mathbf{1.6 \pm 0.3}$	3.4 ± 0.4	27.1 ± 3.1
5	6	4	15.8 ± 1.2	$\mathbf{6.8 \pm 0.8}$	96.5 ± 2.9
5	6	8	7.8 ± 0.9	$\mathbf{3.0 \pm 0.5}$	87.8 ± 4.2
5	6	16	3.7 ± 0.6	$\mathbf{1.8 \pm 0.4}$	87.4 ± 4.3
5	6	32	2.2 ± 0.4	$\mathbf{0.9 \pm 0.2}$	87.1 ± 4.3
5	20	4	15.9 ± 1.1	$\mathbf{6.0 \pm 1.2}$	100 ± 0.0
5	20	8	8.4 ± 0.9	$\mathbf{2.2 \pm 0.6}$	100 ± 0.0
5	20	16	4.6 ± 0.7	$\mathbf{1.0 \pm 0.4}$	100 ± 0.0
5	20	32	2.4 ± 0.5	$\mathbf{0.6 \pm 0.2}$	100 ± 0.0
10	1	4	$\mathbf{10.2 \pm 1.2}$	12.9 ± 0.7	40.8 ± 4.2
10	1	8	$\mathbf{5.1 \pm 0.9}$	6.9 ± 0.6	36.9 ± 4.2
10	1	16	$\mathbf{2.5 \pm 0.3}$	4.5 ± 0.3	34.9 ± 4.4
10	1	32	$\mathbf{1.7 \pm 0.2}$	3.0 ± 0.2	34.0 ± 4.3
10	6	4	14.8 ± 0.7	$\mathbf{4.1 \pm 0.6}$	76.0 ± 3.0
10	6	8	6.9 ± 0.5	$\mathbf{1.5 \pm 0.3}$	74.4 ± 3.2
10	6	16	3.1 ± 0.4	$\mathbf{1.1 \pm 0.2}$	73.6 ± 3.3
10	6	32	1.7 ± 0.3	$\mathbf{0.5 \pm 0.1}$	73.2 ± 3.4
10	20	4	15.0 ± 0.8	$\mathbf{3.7 \pm 0.6}$	96.3 ± 2.0
10	20	8	7.1 ± 0.7	$\mathbf{1.5 \pm 0.2}$	96.1 ± 2.1
10	20	16	3.3 ± 0.4	$\mathbf{1.0 \pm 0.2}$	96.0 ± 2.2
10	20	32	1.6 ± 0.2	$\mathbf{0.7 \pm 0.1}$	95.9 ± 2.2

Table EC.2 Logistic conditional purchase probability

(a) $\zeta = 0.75$

J	LF	k	**FP Gap** (%)	**CC Gap** (%)	**MNL-p Gap** (%)
5	1	4	$\mathbf{11.0 \pm 1.5}$	20.2 ± 2.1	18.9 ± 2.3
5	1	8	$\mathbf{5.3 \pm 1.1}$	13.3 ± 1.6	13.5 ± 2.2
5	1	16	$\mathbf{3.1 \pm 0.6}$	8.2 ± 1.1	10.9 ± 2.3
5	1	32	$\mathbf{2.1 \pm 0.3}$	5.2 ± 0.8	9.4 ± 2.4
5	6	4	15.9 ± 1.3	$\mathbf{8.4 \pm 1.3}$	55.3 ± 4.3
5	6	8	8.1 ± 0.9	$\mathbf{3.9 \pm 0.7}$	52.1 ± 4.4
5	6	16	4.1 ± 0.6	$\mathbf{2.2 \pm 0.3}$	50.7 ± 4.7
5	6	32	2.2 ± 0.4	$\mathbf{1.4 \pm 0.2}$	50.2 ± 4.7
5	20	4	16.7 ± 1.3	$\mathbf{7.9 \pm 1.1}$	77.3 ± 3.3
5	20	8	8.6 ± 0.9	$\mathbf{3.7 \pm 0.6}$	76.0 ± 3.6
5	20	16	4.9 ± 0.6	$\mathbf{2.0 \pm 0.3}$	75.6 ± 3.6
5	20	32	2.8 ± 0.4	$\mathbf{1.3 \pm 0.2}$	75.5 ± 3.6
10	1	4	$\mathbf{12.1 \pm 1.1}$	17.7 ± 1.6	29.7 ± 2.8
10	1	8	$\mathbf{6.8 \pm 0.8}$	12.4 ± 1.2	26.1 ± 2.8
10	1	16	$\mathbf{4.6 \pm 0.6}$	9.0 ± 1.0	24.3 ± 2.9
10	1	32	$\mathbf{3.7 \pm 0.5}$	6.9 ± 0.8	23.5 ± 3.0
10	6	4	13.9 ± 0.8	$\mathbf{6.3 \pm 0.8}$	48.5 ± 2.9
10	6	8	6.3 ± 0.6	$\mathbf{2.6 \pm 0.4}$	44.5 ± 3.0
10	6	16	2.7 ± 0.3	$\mathbf{1.5 \pm 0.3}$	42.6 ± 3.2
10	6	32	1.6 ± 0.2	$\mathbf{0.9 \pm 0.2}$	41.6 ± 3.2
10	20	4	15.0 ± 0.7	$\mathbf{6.2 \pm 0.7}$	56.3 ± 3.0
10	20	8	7.3 ± 0.6	$\mathbf{2.3 \pm 0.5}$	53.4 ± 3.2
10	20	16	3.3 ± 0.4	$\mathbf{1.3 \pm 0.3}$	51.7 ± 3.4
10	20	32	1.7 ± 0.3	$\mathbf{0.8 \pm 0.1}$	51.0 ± 3.0

(b) $\zeta = 0.95$

J	LF	k	**FP Gap** (%)	**CC Gap** (%)	**MNL-p Gap** (%)
5	1	4	$\mathbf{19.0 \pm 1.1}$	20.0 ± 1.2	25.7 ± 3.4
5	1	8	$\mathbf{11.9 \pm 0.9}$	12.0 ± 0.6	21.1 ± 3.5
5	1	16	7.5 ± 0.5	$\mathbf{7.2 \pm 0.3}$	19.2 ± 3.5
5	1	32	5.0 ± 0.4	$\mathbf{4.4 \pm 0.2}$	18.2 ± 3.6
5	6	4	15.6 ± 1.2	$\mathbf{8.1 \pm 1.2}$	72.9 ± 3.3
5	6	8	7.4 ± 0.9	$\mathbf{4.1 \pm 0.7}$	71.6 ± 3.4
5	6	16	4.0 ± 0.6	$\mathbf{2.3 \pm 0.4}$	71.4 ± 3.5
5	6	32	2.5 ± 0.4	$\mathbf{1.5 \pm 0.2}$	71.1 ± 3.5
5	20	4	16.3 ± 1.2	$\mathbf{8.1 \pm 1.2}$	88.4 ± 3.0
5	20	8	8.6 ± 0.9	$\mathbf{3.5 \pm 0.6}$	87.6 ± 3.0
5	20	16	4.7 ± 0.6	$\mathbf{1.9 \pm 0.3}$	87.4 ± 3.0
5	20	32	2.7 ± 0.5	$\mathbf{1.2 \pm 0.2}$	87.4 ± 3.0
10	1	4	11.6 ± 0.7	$\mathbf{11.5 \pm 1.0}$	34.4 ± 3.1
10	1	8	$\mathbf{5.5 \pm 0.5}$	6.1 ± 0.7	30.0 ± 3.2
10	1	16	$\mathbf{2.5 \pm 0.4}$	3.7 ± 0.5	27.9 ± 3.4
10	1	32	$\mathbf{1.5 \pm 0.3}$	2.4 ± 0.3	26.8 ± 3.4
10	6	4	15.3 ± 0.7	$\mathbf{5.9 \pm 0.7}$	60.9 ± 3.1
10	6	8	7.2 ± 0.5	$\mathbf{2.3 \pm 0.3}$	57.9 ± 3.4
10	6	16	3.2 ± 0.4	$\mathbf{1.4 \pm 0.3}$	56.6 ± 3.5
10	6	32	1.8 ± 0.3	$\mathbf{0.7 \pm 0.1}$	56.0 ± 3.6
10	20	4	15.4 ± 0.8	$\mathbf{5.7 \pm 0.7}$	74.8 ± 2.7
10	20	8	7.3 ± 0.7	$\mathbf{2.2 \pm 0.3}$	73.3 ± 3.0
10	20	16	3.3 ± 0.4	$\mathbf{1.2 \pm 0.3}$	72.4 ± 3.1
10	20	32	1.6 ± 0.2	$\mathbf{0.7 \pm 0.2}$	72.0 ± 3.1

Table EC.3 Exponential conditional purchase probability

(a) $\zeta = 0.75$

J	**LF**	k	**FP Gap** (%)	**CC Gap** (%)	**MNL-p Gap** (%)
5	1	4	**13.6 ± 2.1**	23.0 ± 2.3	15.7 ± 3.9
5	1	8	**7.6 ± 1.8**	14.1 ± 1.5	10.1 ± 1.6
5	1	16	**4.1 ± 0.7**	7.5 ± 0.9	6.0 ± 1.0
5	1	32	**2.5 ± 0.5**	4.4 ± 0.6	4.4 ± 1.0
5	6	4	16.7 ± 1.6	**13.2 ± 1.6**	20.0 ± 2.0
5	6	8	8.8 ± 1.3	**6.6 ± 1.1**	11.7 ± 1.7
5	6	16	4.1 ± 0.8	**3.6 ± 0.5**	6.7 ± 1.7
5	6	32	2.4 ± 0.5	**2.1 ± 0.2**	4.2 ± 1.8
5	20	4	17.2 ± 1.5	**10.7 ± 1.4**	15.1 ± 2.0
5	20	8	8.9 ± 1.2	**5.6 ± 0.8**	7.4 ± 1.8
5	20	16	5.0 ± 0.7	**3.0 ± 0.5**	3.2 ± 1.8
5	20	32	2.8 ± 0.4	1.7 ± 0.3	**0.5 ± 1.9**
10	1	4	**14.4 ± 1.6**	21.0 ± 1.6	18.2 ± 2.1
10	1	8	**7.7 ± 1.2**	12.8 ± 1.1	12.4 ± 1.6
10	1	16	**4.5 ± 0.6**	8.2 ± 0.8	8.7 ± 1.2
10	1	32	**3.3 ± 0.5**	5.7 ± 0.7	6.8 ± 1.3
10	6	4	15.1 ± 1.1	**11.3 ± 1.1**	26.7 ± 2.0
10	6	8	7.1 ± 0.8	**5.2 ± 0.6**	19.4 ± 2.0
10	6	16	3.1 ± 0.4	**3.0 ± 0.4**	15.2 ± 2.0
10	6	32	**1.7 ± 0.3**	1.8 ± 0.3	13.1 ± 2.1
10	20	4	15.7 ± 1.0	**9.5 ± 1.0**	26.2 ± 2.2
10	20	8	7.6 ± 0.7	**4.1 ± 0.6**	19.2 ± 2.2
10	20	16	3.6 ± 0.5	**2.1 ± 0.4**	15.8 ± 2.2
10	20	32	1.9 ± 0.4	**1.3 ± 0.2**	13.6 ± 2.3

(b) $\zeta = 0.95$

J	**LF**	k	**FP Gap** (%)	**CC Gap** (%)	**MNL-p Gap** (%)
5	1	4	**13.4 ± 1.9**	20.7 ± 1.7	17.0 ± 2.0
5	1	8	**7.4 ± 1.4**	11.5 ± 1.2	12.2 ± 1.5
5	1	16	**3.7 ± 0.6**	6.5 ± 0.6	9.1 ± 1.2
5	1	32	**2.2 ± 0.5**	3.6 ± 0.5	7.3 ± 1.2
5	6	4	16.3 ± 1.2	**12.4 ± 1.3**	21.4 ± 2.1
5	6	8	7.8 ± 0.9	**6.3 ± 0.8**	13.7 ± 1.9
5	6	16	4.4 ± 0.7	**3.6 ± 0.5**	9.9 ± 2.0
5	6	32	2.7 ± 0.5	**2.2 ± 0.3**	7.3 ± 2.2
5	20	4	16.8 ± 1.5	**10.9 ± 1.5**	17.1 ± 2.1
5	20	8	9.1 ± 1.3	**5.2 ± 0.8**	9.2 ± 2.0
5	20	16	5.0 ± 0.7	**2.7 ± 0.5**	5.4 ± 2.1
5	20	32	2.8 ± 0.5	**1.7 ± 0.3**	3.2 ± 2.3
10	1	4	**14.0 ± 1.2**	17.4 ± 1.2	21.5 ± 2.0
10	1	8	**7.3 ± 0.7**	9.2 ± 0.8	15.7 ± 1.6
10	1	16	**3.8 ± 0.5**	5.4 ± 0.5	12.4 ± 1.6
10	1	32	**2.3 ± 0.3**	3.8 ± 0.3	11.0 ± 1.5
10	6	4	16.3 ± 0.9	**9.7 ± 1.0**	29.2 ± 2.1
10	6	8	7.8 ± 0.7	**4.2 ± 0.6**	22.7 ± 2.1
10	6	16	3.5 ± 0.5	**2.3 ± 0.4**	19.1 ± 2.1
10	6	32	2.0 ± 0.4	**1.2 ± 0.2**	17.4 ± 2.2
10	20	4	16.2 ± 1.0	**8.5 ± 1.0**	22.9 ± 2.0
10	20	8	7.7 ± 0.8	**3.6 ± 0.5**	16.1 ± 2.0
10	20	16	3.4 ± 0.5	**1.8 ± 0.4**	12.3 ± 2.1
10	20	32	1.8 ± 0.3	**0.9 ± 0.2**	10.3 ± 2.2

.