

# Supplements for “Product Service Outsourcing: Impact of Environment Uncertainty and Partial Observability”

---

## A1. Limited Resource Budget

In the following analysis, we consider a product service outsourcing setting with a budget constraint. In such a setting, because the service environment is uncertain, a brand can only maintain the required service level in a probabilistic sense. That is, the brand will maintain resource level  $\bar{k}$  when the corresponding cost  $c(\bar{k}, d_i)$  is less than the budget  $\bar{w}$  but maintains a reduced level  $\hat{k} < \bar{k}$  such that  $c(\hat{k}, d_i) = \bar{w}$ . With fixed budget  $\bar{w}$ , the benefit of outsourcing (besides product value gain  $v^o - v$ ) can be partitioned into three regions depending on the service environment  $d_i$ . The following proposition illustrates the drivers of outsourcing benefits under different levels of environment uncertainty.

### Proposition A1 (Value Drivers of TS)

Let  $H(x) = \frac{1}{\beta}(\bar{w}/x - 1)$ . Define three regions of service environment as

$$\Omega_1 = \{d_i : d_i \leq H(\bar{k})\}, \quad \Omega_2 = \{d_i : H(\bar{k}) < d_i \leq H(\alpha\bar{k})\}, \quad \Omega_3 = \{d_i : d_i > H(\alpha\bar{k})\}$$

(a) If  $d_i \in \Omega_1$ , then the outsourcing benefit is driven entirely by cost savings, which is linearly increasing in  $d_i$ . (b) If  $d_i \in \Omega_2$ , then the outsourcing benefit is driven by both higher service levels and cost savings. In this region, the outsourcing benefit is decreasing in  $d_i$  if  $\alpha > \pi e^{-1}/\bar{k}$ . (c) If  $d_i \in \Omega_3$ , then the outsourcing benefit is driven purely by improved service levels. In this region, the outsourcing benefit is unimodal in  $d_i$ .

Part (a) captures settings where the environment is favorable. In this region, outsourcing benefits the brand through cost savings (besides product value gain  $(v^o - v)$ ), and the benefit is increasing even as the environment becomes less favorable. However, as the environment becomes more challenging, part (b) suggests that the benefit is driven by higher service levels and lower costs. While this appears to increase the value of outsourcing, it turns out that the benefit can be decreasing in this region. Part of the intuition is that the higher service level may not offset the increased cost (due to a more challenging environment), resulting in a lower outsourcing benefit. As the environment becomes even more challenging, part (c) suggests that the benefit is unimodal and eventually approaches zero.

With a budget constraint, the brand can achieve its target service level (i.e., ensuring  $\bar{k}$  amount of resources) in case (a) only if the product service is carried out in-house, but can achieve in both cases (a) and (b) if outsourced. However, when the environment is highly challenging (i.e., case (c)), the target service level cannot be attained even with outsourcing. Thus, budget constraints

hurt the benefit of outsourcing under moderately to highly challenging environments. As a result, the expected cumulative outsourcing benefit is lower than that without a budget constraint (or with a larger budget). The following corollary formalizes the above observation.

**Corollary A1 (Impact of Fixed Budget)**

*Suppose that a minimum service level  $\bar{\theta}$  (equivalently  $\bar{k}$ ) is desired with a fixed budget  $\bar{w}$ . The benefit of outsourcing declines in the available budget. Consequently, the brand needs a longer planning horizon to benefit from outsourcing.*

While budget constraint lowers the potential benefit of outsourcing, it does not eliminate such potential benefit. If the product value gain from outsourcing is significant enough such that  $v^o - v \approx \tau$ , then the brand is always better off switching to outsourcing regardless of budget limitations.

**Proof of Proposition A1.**

*Part (a).* In the first region when the environment uncertainty is low, then the benefit comes from cost savings in maintaining resource level  $\bar{k}$ , which is given by  $B_1(d_i) = (1 - \alpha)C\bar{k}(1 + \beta d_i)$ . In this region, the benefit is linearly increasing in the service environment.

*Part (b).* In the second region when the environment uncertainty is medium, the benefit comes from a combination of higher service level and lower cost. In this region, the in-house operation maintains a lower service level with resource  $\hat{k}$  at a cost of  $\bar{w}$ , whereas the third-party provider still maintains  $\bar{k}$  at a cost lower than  $\bar{w}$ . The benefit is therefore given by (ignoring the product value gain  $v^o - v$ )  $B_2(d_i) = \pi(\theta(\bar{k}) - \theta(\hat{k})) + \bar{w} - \alpha c(\bar{k}, d_i)$ , where  $\hat{k} = \bar{w}/(C(1 + \beta d_i))$ . For notational ease, we at times use superscript ' and '' to denote first order and second derivatives. It follows that

$$B_2'(d_i) = -\pi\theta'(\hat{k})\frac{\partial\hat{k}}{\partial d_i} - \alpha\frac{\partial c(\bar{k}, d_i)}{\partial d_i}, \quad B_2''(d_i) = -\pi\theta''(\hat{k})\left(\frac{\partial\hat{k}}{\partial d_i}\right)^2 - \pi\theta'(\hat{k})\frac{\partial^2\hat{k}}{\partial d_i^2}.$$

Note that  $\theta'(\hat{k}) = \lambda e^{-\lambda\hat{k}}$ ,  $\theta''(\hat{k}) = -\lambda^2 e^{-\lambda\hat{k}}$ ,  $\frac{\partial\hat{k}}{\partial d_i} = \frac{-\bar{w}\beta}{C(1+\beta d_i)^2}$ , and  $\frac{\partial^2\hat{k}}{\partial d_i^2} = \frac{2\bar{w}\beta^2}{C(1+\beta d_i)^3}$ . Substituting these terms into  $B_2''(d_i)$ , we have

$$B_2''(d_i) = \pi\lambda\hat{k}e^{-\lambda\hat{k}}\left(\frac{\beta}{1+\beta d_i}\right)^2\left\{\frac{\lambda\bar{w}}{C(1+\beta d_i)} - 2\right\}$$

It follows that when  $\bar{w}$  is large (i.e.,  $\bar{w} \geq \frac{2}{\lambda}C(1 + \beta d_i)$ ), then the benefit is convex in  $d_i$ . Conversely, when  $\bar{w}$  is small, then the benefit is more likely to be concave in  $d_i$ . Notice that as the environment becomes more challenging ( $d_i$  increases), then the benefit is more likely to be concave. Furthermore, note that

$$B_2'(d_i) = \pi\lambda e^{-\lambda\hat{k}}\frac{\bar{w}\beta}{C(1+\beta d_i)^2} - \alpha C\beta\bar{k} = \pi\lambda\hat{k}e^{-\lambda\hat{k}}\frac{\beta}{1+\beta d_i} - \alpha C\beta\bar{k}.$$

Note that  $\lambda \widehat{k} e^{-\lambda \widehat{k}} \leq e^{-1}$ , and hence we have  $B'_2(d_i) < \pi e^{-1} \frac{\beta}{1+\beta d_i} - \alpha C \beta \bar{k}$ . It follows that if  $\alpha > \pi e^{-1} / (C \beta \bar{k})$ , then the benefit of outsourcing is strictly decreasing in  $d_i$  in this region. That is, unless the third-party provider is extremely efficient (very small  $\alpha$ ), the benefit of outsourcing in this region is likely to decrease as the service environment becomes more challenging.

*Part (c).* Finally, in the third region when the environment is highly challenging (i.e., high  $d_i$ ), the benefit is driven by differences in service level provided as both in-house operation and third-party provider incur the same cost  $\bar{w}$ . The benefit is therefore given by  $B_3(d_i) = \pi \left( \theta(\widehat{k}^o) - \theta(\widehat{k}) \right)$ , where  $\widehat{k}$  and  $\widehat{k}^o$  are bounded by the budget  $\bar{w}$ . Note that  $\widehat{k} = \alpha \widehat{k}^o$ , hence

$$B'_3(d_i) = \pi \left( -\alpha \lambda e^{-\alpha \lambda \widehat{k}^o} + \lambda e^{-\lambda \widehat{k}^o} \right) \frac{\partial \widehat{k}^o}{\partial d_i},$$

$$B''_3(d_i) = \pi \left( \alpha^2 \lambda^2 e^{-\alpha \lambda \widehat{k}^o} - \lambda^2 e^{-\lambda \widehat{k}^o} \right) \left( \frac{\partial \widehat{k}^o}{\partial d_i} \right)^2 + \pi \left( -\alpha \lambda e^{-\alpha \lambda \widehat{k}^o} + \lambda e^{-\lambda \widehat{k}^o} \right) \frac{\partial^2 \widehat{k}^o}{\partial d_i^2}.$$

When  $B'_3(d_i) = 0$ , we have  $\alpha \lambda e^{-\alpha \lambda \widehat{k}^o} = \lambda e^{-\lambda \widehat{k}^o}$ . Substituting this expression into  $B''_3(d_i)$ , we have

$$B''_3(d_i) = \pi \alpha \lambda^2 e^{-\alpha \lambda \widehat{k}^o} (\alpha - 1) \left( \frac{\partial \widehat{k}^o}{\partial d_i} \right)^2 < 0.$$

It follows that the benefit in the third region is unimodal in the service environment  $d_i$ .  $\square$

**Proof of Corollary A1.** Observe that the maximum gain (besides those in regions 1 and 2) is given by  $(1 - \alpha) \pi e^{-\lambda \alpha \widehat{k}^o}$ , where  $\widehat{k}^o = \frac{\bar{w}}{\alpha C (1 + \beta d_i)}$ . It follows that the maximum benefit is bounded by  $(1 - \alpha) \pi$ . In contrast, with no budget limit, the benefit is given by  $(1 - \alpha) C \bar{k} (1 + \beta d_i)$ , which grows in  $d_i$  without bound. It follows that if  $d_i > \frac{1}{\beta} \left( \frac{\pi}{C \bar{k}} - 1 \right)$ , then the benefit of no budget constraint exceeds that with budget constraint. By definition, in  $\Omega_3$  we have  $d_i > \frac{1}{\beta} \left( \frac{\bar{w}}{\alpha C \bar{k}} - 1 \right)$ . Thus, the condition  $d_i > \frac{1}{\beta} \left( \frac{\pi}{C \bar{k}} - 1 \right)$  is always satisfied in region 3. As such, all else being equal, the budget constraint reduces the value of outsourcing. It then follows that the brand needs a longer planning horizon  $T$  for the accrued benefit of outsourcing (if attractive) to outweigh the fixed switching cost  $\tau$ .  $\square$

## A2. Proofs

Note that for notational ease, we at times use superscript ' and '' to denote first order and second derivatives when there is no ambiguity. Otherwise, we use  $\partial$  to denote derivatives.

### Proof of Lemma 1.

*Part (a).* Note that  $G'_k(k|v, d) = \pi \theta'(k) - c'_k(k, d)$  and  $G''_k(k|v, d) = \pi \theta''(k) - c''_k(k, d)$ . Because  $\theta''(k) \leq 0$  (due to concavity of  $\theta(\cdot)$  function) and  $c''_k(k, d) = 0$  (due to the linearity of  $c(k, d)$  function), it follows that  $G''_k(k|v, d) \leq 0$ .

*Part (b).* Note that  $G''_{k,d}(k|v, d) \geq 0$  (due to supermodular property of  $c(k, d)$ ), it follows that  $G'_k(k|v, d)$  increases in  $d$ . Substituting this into  $\pi \theta'(k) = c'_k(k, d_i)$ , we have the RHS increasing in

$d_i$ . To maintain the equality, it is necessary to reduce  $k$ . Note that the above first-order condition does not depend on future periods' value function because resource maintenance cost is a variable cost incurred in every period.  $\square$

**Proof of Lemma 2.** The proof follows analogously to the proof of Lemma 1. Specifically, notice that  $G_k^{o'}(k|v, d) = \pi\theta'(k) - c_k^{S'}(k, d)$  and  $G_k^{o''}(k|v, d) = \pi\theta''(k) - c_k^{S''}(k, d)$ . Because  $\theta''(k) \leq 0$  (due to concavity of  $\theta(\cdot)$  function) and  $c_k^{S''}(k, d) \geq 0$  (due to convexity of  $c^S(k, d)$  function), it follows that  $G_k^{o''}(k|v, d) \leq 0$ . In addition, observe that  $G_{k,d}^{o''}(k|v, d) \geq 0$  (due to supermodular property of  $c^S(k, d)$ ), it follows that  $G_k^{o'}(k|v, d)$  increases in  $d$ . Substituting this into  $\pi\theta'(k) = c_k^{S'}(k, d_i^{b*})$ , we have the RHS increasing in  $d_i^{b*}$ . To maintain the equality, it is necessary to reduce  $k$ . The lemma statement then follows.  $\square$

**Proof of Proposition 1.**

*Part (a).* Note that  $s_i = 0$  is equivalent to the brand having full observability into the true market environment, and therefore the brand observes  $d_i$  even with the third-party service. Given that full observability is a relaxation of the original problem with partial observability, it follows that the value function under  $s_i = 0$  must dominate that under  $s_i > 0$ . Therefore, the value function under  $s_i = 0$  must be an upper bound on any achievable performance under  $s_i > 0$ .

*Part (b).* By Lemma 1, with direct service, the optimal resource level in an arbitrary period  $i$ ,  $k_i^*$ , satisfies  $\pi\theta'(k_i^*) = c'_{k_i}(k_i^*, d_i)$  (denote as E1). Similarly, by Lemma 2, with third-party service, the optimal resource level in period  $i$  satisfies  $\pi\theta'(k_i^o) = c_{k_i}^{S'}(k_i^o, d_i^{b*})$  (denote as E2). Since  $s_i = 0$ , we have  $d_i^{b*} = d_i$ . Recall that all else equal  $c_k^{S'}(k, d) \leq c'_k(k, d)$  for any arbitrary  $k$  and  $d$  (see §3.6.2). Additionally, because the left-hand side of E1 and E2 are identical (for the same resource level), it follows that  $k_i^o \geq k_i^*$ .

In addition, recall that under direct service the brand's value function is  $V_i^{DS}(d_i) = \max_{k_i} \{G(k_i, d_i) + \gamma \mathbf{E}_{e_{i+1}} V_{i+1}^{DS}(d_{i+1})\}$ ,  $i = 1, \dots, T$ , whereas under the third-party service (with  $s_i = 0$ ), the brand's value function is simplified to  $V_i^{TS}(d_i) = \max_{k_i} \{G^o(k_i, d_i) + \gamma \mathbf{E}_{e_{i+1}} V_{i+1}^{TS}(d_{i+1})\}$ ,  $i = 1, \dots, T$ . Now, suppose the brand endorses a resource level in each period that is optimal under the direct service and imposes such series of resource levels for the third-party service, then we must have  $G(k_i^*, d_i) \leq G^o(k_i^*, d_i)$  because  $c^S(k, d) \leq c(k, d)$  for any arbitrary  $k$  and  $d$ . It follows that  $V_i^{DS}(d_i) \leq V_i^{TS}(d_i)|_{k_i^o \doteq k_i^*}$ , that is, when the resources levels are set identically based on the direct service setting. However, the brand could optimally choose  $k_i^o$  based on the third-party service setting and this can only improve its value function; that is,  $V_i^{DS}(d_i)|_{k_i^o \doteq k_i^*} \leq V_i^{TS}(d_i)|_{k_i^o}$  when  $k_i^o$  is optimally chosen. The proposition statement then follows.  $\square$

**Proof of Proposition 2.**

*Part (a).* When the service environment is stable, then the expected value of switching to the third-party service provider is the sum of discounted benefits over the  $T$  time periods. Leveraging Lemmas 1 and 2, we have

$$\begin{aligned} G(k_i^*) &= v - \frac{C(1 + \beta d_i)}{\lambda} \left( 1 - \ln \left( \frac{C(1 + \beta d_i)}{\lambda \pi} \right) \right), \\ G(k_i^o) &= v^o - \frac{\alpha C(1 + \beta d_i)}{\lambda} \left( 1 - \ln \left( \frac{\alpha C(1 + \beta d_i)}{\lambda \pi} \right) \right). \end{aligned}$$

The benefit of switching to a third-party service provider is therefore given by

$$G(k_i^o) - G(k_i^*) = v^o - v + \frac{C(1 + \beta d_i)}{\lambda} \left( 1 - \ln \left( \frac{C(1 + \beta d_i)}{\lambda \pi} \right) \right) - \frac{\alpha C(1 + \beta d_i)}{\lambda} \left( 1 - \ln \left( \frac{\alpha C(1 + \beta d_i)}{\lambda \pi} \right) \right).$$

Using the definition of  $L(x) = x(1 - \ln(x))$  and  $A_i = C(1 + \beta d_i)/(\lambda \pi)$ , the above expression can be simplified to  $G(k_i^o) - G(k_i^*) = v^o - v + \pi(L(A_i) - L(\alpha A_i)) > 0$ , where the inequality follows from the fact that  $v^o > v$  and  $L(x)$  is an increasing function of  $x$  for any  $x \leq 1$ . Given  $d_i = d$  for all  $i$ , it follows that  $G(k_i^o) - G(k_i^*)$  is a constant in each period. The sum of discounted values of  $G(k_i^o) - G(k_i^*)$  over  $T$  periods is therefore given by  $B(T) = \frac{1 - \gamma^{T-1}}{1 - \gamma} (G(k_i^o) - G(k_i^*))$ . The brand, therefore, benefits from outsourcing product service as long as the  $B(T) \geq \tau$ . Therefore, the boundary planning horizon  $\bar{T}$  is obtained by considering the indifference point of  $B(\bar{T}) = \tau$ . The exact expression follows from rearranging the terms in  $B(\bar{T}) - \tau = 0$  and substituting  $G(k_i^o)$  and  $G(k_i^*)$  into  $B(\bar{T})$ .

*Part (b)(i).* Observe that  $\bar{T}$  satisfies  $B(\bar{T}) - \tau = 0$ , which is equivalent to

$$\left[ v^o - v + \pi(L(A_i) - L(\alpha A_i)) \right] \frac{1 - \gamma^{\bar{T}-1}}{1 - \gamma} - \tau = 0.$$

Treating the terms in the square bracket as a function of  $d_i$ , the above expression can be re-written as  $\mathcal{J}(d_i) \frac{1 - \gamma^{\bar{T}-1}}{1 - \gamma} - \tau = 0$ . Notice that if  $\mathcal{J}(d_i)$  increases, then  $\bar{T}$  must decrease to maintain the equality, and vice versa. In other words, the directional change in  $\bar{T}$  is the opposite of the directional change in  $\mathcal{J}(d_i)$ . Now, observe that  $\partial \mathcal{J}(d_i) / \partial d_i \sim \alpha \ln(\alpha d_i) - \ln(d_i)$ . Furthermore,  $\partial^2 \mathcal{J}(d_i) / \partial d_i^2 \sim (\alpha - 1) / d_i < 0$ . Therefore,  $\mathcal{J}(d_i)$  is concave in  $d_i$ , suggesting that  $\mathcal{J}(d_i)$  initially increases and eventually decreases in  $d_i$ . As such, this means that  $\bar{T}$  must be convex: initially decrease and then increase in  $d_i$ .

*Part (b)(ii).* First consider the case when the brand has full observability of the service environment. Because  $L(x)$  is concave in  $x$ , by Jensen's inequality, we have  $\mathbb{E}_x L(x) \leq L(\mathbb{E}_x x)$ . It follows that, starting from period  $i$ , the expected benefit in any subsequent period  $i < j \leq T$  must be such that  $\mathbb{E}_{d_j} L(A_j) \leq L(\mathbb{E}_{d_j} A_j)$ . Note that  $L(x) - L(\alpha x)$  is also concave in  $x$  because  $L''(x) - L''(\alpha x) = -(1 - \alpha) / x < 0$ . Therefore, we have  $\mathbb{E}_{d_j} (G(k_j^o) - G(k_j^*)) \leq G(\mathbb{E}_{d_j} k_j^o) - G(\mathbb{E}_{d_j} k_j^*) = G(k_i^o) - G(k_i^*) = v^o - v + \pi(L(A_i) - L(\alpha A_i))$ , where the first equality follows from the fact that  $\mathbb{E}[d_j | d_i] = d_i$  by the property of MMFE. It then follows that under uncertain  $d_i$ , the sum of discounted expected

benefit is strictly less than that attained under constant service environment  $d_i$ . This implies that  $\bar{T}$  must be increased to compensate for the reduced benefit of service outsourcing; that is,  $\bar{T}$  must be increased to render outsourcing attractive. Now, by Proposition 1(a), the brand's value function is upper bounded by the full observability case, and therefore for any  $s_i > 0$ , we have  $V_i^{TS}(d_i^b, s_i) \leq V_i^{TS}(d_i, 0)$  such that the benefit of outsourcing is strictly lower than that under the full observability case. Therefore,  $\bar{T}$  must be further increased beyond the full observability case to offset the switching cost  $\tau$ .  $\square$

**Proof of Proposition 3.** With a minimum service requirement, the benefit of outsourcing in the current period is given by  $B_i = R^o(\bar{k}) - c^S(\bar{k}, d_i) - (R(\bar{k}) - c(\bar{k}, d_i)) = v^o - v + (1 - \alpha)C\bar{k}(1 + \beta d_i)$ . At the beginning of a planning horizon of  $T$ , the expected sum of discounted benefit is given by  $B = \sum_{i=1}^T \gamma^{i-1} \mathbf{E}[B_i] = \sum_{i=1}^T \gamma^{i-1} \mathbf{E}[v^o - v + (1 - \alpha)C\bar{k}(1 + \beta d_i)] = \sum_{i=1}^T \gamma^{i-1} (v^o - v + (1 - \alpha)C\bar{k}(1 + \beta \mathbf{E}[d_i])) = \sum_{i=1}^T \gamma^{i-1} (v^o - v + (1 - \alpha)C\bar{k}(1 + \beta d_i))$ , where the last equality follows from the martingale property. It follows that  $B = (v^o - v + (1 - \alpha)C\bar{k}(1 + \beta d_i)) \frac{1 - \gamma^T}{1 - \gamma}$ . The brand benefits from outsourcing if  $B \geq \tau$ . The proposition statement follows by re-arranging the above inequality.  $\square$

**Proof of Lemma 3.** First note that without learning and monitoring effort, the brand's belief is simply  $d_i^b$  (as opposed to the partially learning updated belief  $d_i^{b*}$ ). As such, we only need to consider the brand's belief  $d_i^b$ . Part (a). By definition, we have  $\xi_i^b = \prod_{k=i^o}^{i-1} e_k / e_k^b = \prod_{k=i^o}^{i-1} \prod_{m \in |I_i \setminus I_i^b|} \xi_{k,m}$ , where the notation  $m \in |I_i \setminus I_i^b|$  refers to those random events that the brand does not observe due to outsourcing. Because  $\xi_{k,m}$ 's are independent across  $k$  and  $m$ , and lognormally distributed with mean equal to 1, it follows that  $\xi_i^b$ , as a multiplication of lognormal random variables, is also lognormally distributed with mean 1 and variance parameter  $s_i^2$ . The variance parameter captures the cumulative uncertainty captured by these random events. Therefore, we have  $\mathbf{E}[d_i] = \mathbf{E}[d_i^b \xi_i^b] = \mathbf{E}[d_i^b] \mathbf{E}[\xi_i^b] = \mathbf{E}[d_i^b]$ , where the second equality follows from the fact that  $d_i^b$  and  $\xi_i^b$  are independent. Part (b). Observe that  $P(d_i^b - d_i \geq X | s_i) = P(d_i^b - \xi_i^b d_i^b \geq X | s_i) = P(d_i^b(1 - \xi_i^b) \geq X | s_i) = P(1 - \xi_i^b \geq X/d_i^b | s_i) = P(\xi_i^b \leq 1 - X/d_i^b | s_i)$ . The lemma statement follows from the fact that  $\xi_i^b$  is lognormally distributed with mean equal to 1 and arbitrary variance parameter  $s_i^2$ . Finally, let  $\Gamma(s) = (A + s^2/2)/(\sqrt{2}s)$ , we have  $\Gamma'(s) = \frac{1}{\sqrt{2}}(\frac{1}{2} - A/s^2)$ . Substituting  $A = \ln(1 - X/d_i^b) \leq 0$  into  $\Gamma'(s)$ , we have  $\Gamma'(s_i) \geq 0$ .  $\square$

**Proof of Proposition 4.**

*Part (a).* Consider any period  $i > 0$ . Suppose that partial observability has resulted in the brand holding a belief  $d_i^b$  with the variance parameter of the information gap as  $s_i^2$ . The brand sets a target resource level  $\bar{k}$  with payment  $w_i = c^S(\bar{k}, d_i^b)$  to satisfy the provider's participation constraint. If  $d_i^b \geq d_i$ , then the provider's participation constraint is satisfied (but not binding). In this case, the value of outsourcing is reduced by  $c^S(\bar{k}, d_i^b) - c^S(\bar{k}, d_i) \geq 0$ , where the inequality follows from

the fact that  $d_i^b \geq d_i$ . On the other hand, if  $d_i^b < d_i$ , then the provider's participation constraint is violated (because  $c^S(\bar{k}, d_i) > c^S(\bar{k}, d_i^b)$ ). This forces the brand to adjust its payment to  $c^S(\bar{k}, d_i)$  based on actual environment  $d_i$ . The gain from outsourcing is therefore exactly the same as if the brand observes the service environment  $d_i$ . In the above payment structure, the brand acts on the point estimate of  $d_i^b$ , although the brand is aware that  $d_i^b$  is not perfect. Nevertheless, the payment structure is valid even if the brand incorporates the uncertainties surrounding  $d_i^b$ . To see this, recall that the functional form of  $c^S(\bar{k}, d_i^b)$  is given by  $c^S(\bar{k}, d_i^b) = \alpha C \bar{k} (1 + \beta d_i^b)$ . Suppose that the brand recognizes that the true environment is random which we denote as  $\tilde{d}_i = d_i^b \xi_i^b$  with some estimated distribution function of  $\xi_i^b$  as  $F_i(\xi_i^b)$ . Then for any given  $\bar{k}$  the expected service provision cost is given by  $\mathbb{E}[c^S(\bar{k}, \tilde{d}_i)] = \int_{\xi_i^b} c^S(\bar{k}, \tilde{d}_i) dF_i(\xi_i^b) = \int_{\xi_i^b} \alpha C \bar{k} (1 + \beta \tilde{d}_i) dF_i(\xi_i^b) = \alpha C \bar{k} (1 + \beta \int_{\xi_i^b} \tilde{d}_i dF_i(\xi_i^b)) = \alpha C \bar{k} (1 + \beta d_i^b \mathbb{E}[\xi_i^b]) = \alpha C \bar{k} (1 + \beta d_i^b) = c^S(\bar{k}, d_i^b)$ , where the second last equality follows from the MMFE property (as  $d_i^b$  is an unbiased estimate of  $\tilde{d}_i$ ). Such a contract is similar to the specified capacity and quality (SCQ) contract studied in Feng et al. (2019). Note that the diminishing marginal effect is captured in the customer's satisfaction function  $\theta(\bar{k})$  so the functional form of  $c^S(\bar{k}, d_i^b)$  still allows the non-linear effect of resource level and customer satisfaction.

*Part (b).* The proposition statement follows from part (c) of Lemma 3.

*Part (c).* Note that the expected value loss from the belief drifting above the actual environment in an arbitrary period after outsourcing is given by

$$\mathbb{E} [c^S(\bar{k}, d_i^b) - c^S(\bar{k}, d_i) | d_i \leq d_i^b] = \int_{\xi_i^b \leq 1} (c^S(\bar{k}, d_i^b) - c^S(\bar{k}, d_i^b \xi_i^b)) dF_i(\xi_i^b),$$

where  $\xi_i^b$  is the cumulative loss of visibility by period  $i$  and  $F_i(\cdot)$  denote the lognormal distribution of  $\xi_i^b$  with mean 1 and variance  $s_i^2$ . Substituting the lognormal distribution into the above expression and recognizing that  $c^S(k, d) = \alpha C k (1 + \beta d)$ , we obtain

$$\mathbb{E} [c^S(\bar{k}, d_i^b) - c^S(\bar{k}, d_i) | d_i \leq d_i^b] = \alpha C \beta \bar{k} d_i^b \frac{1}{F_i(1)} \int_{\xi_i^b \leq 1} (1 - \xi_i^b) dF_i(\xi_i^b) = \alpha C \beta \bar{k} d_i^b \left( 1 - \frac{1 - \operatorname{erf}\left(\frac{s_i}{2\sqrt{2}}\right)}{1 - \operatorname{erf}\left(\frac{-s_i}{2\sqrt{2}}\right)} \right).$$

As  $s_i$  increases (visibility worsens),  $\lim_{s_i \rightarrow \infty} \mathbb{E} [c^S(\bar{k}, d_i^b) - c^S(\bar{k}, d_i) | d_i \leq d_i^b] \rightarrow \alpha C \beta \bar{k} d_i^b$ . Therefore, there exists a threshold level of visibility  $\bar{s}_i$  such that beyond which the loss in value due to partial observability satisfies  $\mathbb{E} [c^S(\bar{k}, d_i^b) - c^S(\bar{k}, d_i) | d_i \leq d_i^b] \geq \bar{\Delta}_i$ , where  $\bar{\Delta}_i$  represents the expected value gain necessary in period  $i$  as implied by  $V_0^{DS}(d_0) + \tau \leq V_0^{TS}(d_0, s_0)$  at the start of the planning horizon. Furthermore, note that the partial observability variance is cumulative over the planning horizon, so that  $s_i^2$  is never decreasing. Therefore, if  $s_i > \bar{s}_i$  at some period  $i$ , then the condition holds for all of the remaining periods. It then follows that there exists a  $\bar{s}_i$  such that if  $s_i > \bar{s}_i$  occurs within the planning horizon, then the value of outsourcing is strictly negative.  $\square$

### Proof of Proposition 5.

*Part (a).* We first show that the optimal PBE reporting strategy must be of a threshold type. Observe that the service provider's profit in period  $i$  is given by  $u = w_i - c^S(\bar{k}, d_i) - c_R$ , where  $w_i$  satisfies  $w_i = c^S(\bar{k}, d_i^b)$ . All else being equal, the provider's payoff  $w_i$  is therefore increasing in the brand's belief  $d_i^b$ . It follows that any incentive compatible reporting strategy  $R(d_i)$  must satisfy the conditions that (a)  $R(d_i) = d_i$  if  $c^S(\bar{k}, d_i) \geq \mathbb{E}_{\xi_i^b}[c^S(\bar{k}, d_i^b \xi_i^b) | \emptyset] + c^R \Rightarrow d_i \geq d_i^b \mathbb{E}(\xi_i^b | \emptyset) + c^R / (\alpha C \beta \bar{k})$  and (b)  $R(d_i) = \emptyset$  if  $c^S(\bar{k}, d_i) < \mathbb{E}_{\xi_i^b}[c^S(\bar{k}, d_i^b \xi_i^b) | \emptyset] + c^R \Rightarrow d_i < d_i^b \mathbb{E}(\xi_i^b | \emptyset) + c^R / (\alpha C \beta \bar{k})$ . The two conditions (a) and (b) imply that any PBE optimal strategy must be of a threshold type and hence (11) must hold.

It remains to be shown that the optimal threshold satisfies condition (12). Notice that if the provider discloses true environment information  $d_i$ , then its expected payoff is  $w_i - c^R = \mathbb{E}[c^S(\bar{k}, d_i^b) | d_i] - c^R = c^S(\bar{k}, d_i) - c^R$ . Because any optimal disclosure strategy is of a threshold type, there must exist a threshold  $d_i^R$  such that the provider does not disclose the true environment when  $d_i < d_i^R$  but reports the true environment otherwise. For any given  $d_i^R$ , if the provider does not disclose, then the brand's (adjusted) belief about the service environment is given by

$$\mathbb{E}[d_i^b \xi_i^b | \emptyset] = \mathbb{E}[d_i^b \xi_i^b | d_i^b \xi_i^b < d_i^R] = d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R / d_i^b]. \quad (\text{A-1})$$

At the boundary point of  $d_i^R$ , the provider is indifferent between disclosure and non-disclosure. Thus, we have

$$c^S(\bar{k}, d_i^R) - c^R = \mathbb{E}[c^S(\bar{k}, d_i^b \xi_i^b) | d_i^b \xi_i^b < d_i^R] = c^S(\bar{k}, d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R / d_i^b]). \quad (\text{A-2})$$

Recall that  $c^S(k, d) = \alpha C \cdot k(1 + \beta d)$ , therefore (A-2) can be expressed as

$$\begin{aligned} \alpha C \cdot \bar{k}(1 + \beta d_i^R) - c^R &= \alpha C \cdot \bar{k}(1 + \beta d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R / d_i^b]) \\ &\Leftrightarrow 1 + \beta d_i^R - \frac{c^R}{\alpha C \cdot \bar{k}} = 1 + \beta d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R / d_i^b] \\ &\Leftrightarrow \beta d_i^R = \beta d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R / d_i^b] + \frac{c^R}{\alpha C \cdot \bar{k}} \\ &\Leftrightarrow d_i^R = d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R / d_i^b] + \frac{c^R}{\alpha C \beta \bar{k}}. \end{aligned} \quad (\text{A-3})$$

Let  $F_i(\cdot)$  denote the distribution of  $\xi_i^b$ . We have

$$\mathbb{E}[\xi_i^b | \xi_i^b \leq \tau] = F_i(\tau)^{-1} \int_{x \leq \tau} x dF_i(x). \quad (\text{A-4})$$

Letting  $\tau = d_i^R / d_i^b$  and substituting (A-4) into (A-3) yield the expression (12).

To show the uniqueness of  $d_i^R$ , it is sufficient to show that  $c^S(\bar{k}, d_i^R) - c^S(\bar{k}, d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R / d_i^b])$  is monotone in  $d_i^R$ . Note that  $c^S(\bar{k}, d_i^R)$  is linearly increasing in  $d_i^R$ . In addition, we can evaluate

the directional impact of  $d_i^R$  by examining how  $d_i^R$  influences  $\mathbb{E}[\xi_i^b | \xi_i^b < d_i^R/d_i^b]$ . For notational ease define  $G(k) = \mathbb{E}[\xi_i^b | \xi_i^b < k]$ . It follows that  $\lim_{k \rightarrow \infty} G(k) = 1$  and  $\lim_{k \rightarrow 0} G(k) = 0$ . Furthermore, it is clear that  $G(k)$  monotonically increasing in  $k$ . By (A-2), since  $c^S(\bar{k}, \infty) - c^R > c^S(\bar{k}, d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < (\infty/d_i^b)]) = c^S(\bar{k}, d_i^b)$  and  $c^S(\bar{k}, 0) - c^R < c^S(\bar{k}, d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < (0/d_i^b)]) = c^S(\bar{k}, 0)$ , it follows that there must exist a unique  $d_i^R$  that satisfies (A-2).

*Part (b).* The proposition statement follows from the fact that (i) if the provider reports the true environment then the brand's belief is updated to the true environment (given the assumption that the provider does not distort information) and (ii) the threshold reporting level  $d_i^R$  is exactly the indifference point where the provider chooses non-disclosure. Recall that by (A-1), the brand's updated belief  $d_i^{b*}$  is given by  $d_i^{b*} = \mathbb{E}[d_i^b \xi_i^b | \emptyset] = d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R/d_i^b]$ . By (A-3), we have

$$d_i^R = d_i^b \mathbb{E}[\xi_i^b | \xi_i^b < d_i^R/d_i^b] + \frac{c^R}{\alpha C \beta \bar{k}} \Leftrightarrow d_i^R = \mathbb{E}[d_i^b \xi_i^b | \emptyset] + \frac{c^R}{\alpha C \beta \bar{k}} \Leftrightarrow \mathbb{E}[d_i^b \xi_i^b | \emptyset] = d_i^R - \frac{c^R}{\alpha C \beta \bar{k}}.$$

It follows that if the provider does not report, then the brand's updated belief is given by  $d_i^{b*} = \mathbb{E}[d_i^b \xi_i^b | \emptyset] = d_i^R - \frac{c^R}{\alpha C \beta \bar{k}}$ ; that is, the brand's belief about the service environment under non-disclosure is exactly the equilibrium threshold reporting point with a downward adjustment of the scaled reporting cost  $c^R/(\alpha C \beta \bar{k})$ .  $\square$

**Proof of Corollary 1.** Define

$$\mathcal{L}(d_i^R) = d_i^R - d_i^b F_i^{-1}(d_i^R/d_i^b) \int_{\xi \leq d_i^R/d_i^b} \xi dF_i(\xi) - \frac{c^R}{\alpha C \beta \bar{k}}.$$

*Part (a).* We have  $\partial d_i^R / \partial c^R = -(\partial \mathcal{L} / \partial c^R) / (\partial \mathcal{L} / \partial d_i^R)$ . By the proof of Proposition 5, we have  $\partial \mathcal{L} / \partial d_i^R \geq 0$ . Furthermore, note that  $\partial \mathcal{L} / \partial c^R = -1/(\alpha C \beta \bar{k}) < 0$ . It then follows that  $\partial d_i^R / \partial c^R \geq 0$ .

*Part (b).* We have  $\partial d_i^R / \partial s_i = -(\partial \mathcal{L} / \partial s_i) / (\partial \mathcal{L} / \partial d_i^R)$ . Analogous to part (a), we have  $\partial \mathcal{L} / \partial d_i^R \geq 0$ . Furthermore, under the lognormal approximation we have (recognizing that  $\mathbb{E}[\xi_i^b] = 1$ )

$$\mathbb{E}[\xi_i^b | \xi_i^b \leq \tau] = \Phi\left(\frac{\ln(\tau) - s_i^2/2}{s_i}\right) / \Phi\left(\frac{\ln(\tau) + s_i^2/2}{s_i}\right), \quad (\text{A-5})$$

where  $\tau = d_i^R/d_i^b$  and  $\Phi(\cdot)$  denote the standard normal distribution. Notice that  $\mathcal{L}(d_i^R)$  can be equivalently written as  $\mathcal{L}(d_i^R) = d_i^R - d_i^b \mathbb{E}[\xi_i^b | \xi_i^b \leq \tau] - \frac{c^R}{\alpha C \beta \bar{k}}$ . We have

$$\frac{\partial \mathcal{L}}{\partial s_i} = -d_i^b \partial_{s_i} \left( \frac{\Phi(\ln(d_i^R/d_i^b)/s_i - s_i/2)}{\Phi(\ln(d_i^R/d_i^b)/s_i + s_i/2)} \right).$$

Observe that

$$\lim_{s \rightarrow \infty} \frac{\Phi(\ln(d_i^R/d_i^b)/s_i - s_i/2)}{\Phi(\ln(d_i^R/d_i^b)/s_i + s_i/2)} \rightarrow 0 \quad \text{and} \quad \lim_{s \rightarrow 0} \frac{\Phi(\ln(d_i^R/d_i^b)/s_i - s_i/2)}{\Phi(\ln(d_i^R/d_i^b)/s_i + s_i/2)} \rightarrow 1.$$

Furthermore, let  $\Delta = \ln(d_i^R/d_i^b)$ , then the following derivatives are equivalent:

$$\partial_{s_i} \left( \frac{\Phi(\ln(d_i^R/d_i^b)/s_i - s_i/2)}{\Phi(\ln(d_i^R/d_i^b)/s_i + s_i/2)} \right) \Leftrightarrow \partial_{s_i} \left( \frac{\int_{-\infty}^{\frac{\Delta}{s_i} - \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt}{\int_{-\infty}^{\frac{\Delta}{s_i} + \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt} \right)$$

Taking the derivative on the right-hand side of the above expression, we have

$$\partial_{s_i} \left( \frac{\int_{-\infty}^{\frac{\Delta}{s_i} - \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt}{\int_{-\infty}^{\frac{\Delta}{s_i} + \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt} \right) = \frac{e^{-\frac{1}{2}(\frac{\Delta}{s_i} - \frac{s_i}{2})^2} \left( -\frac{\Delta}{s_i^2} - \frac{1}{2} \right)}{\int_{-\infty}^{\frac{\Delta}{s_i} + \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt} - \frac{e^{-\frac{1}{2}(\frac{\Delta}{s_i} + \frac{s_i}{2})^2} \left( -\frac{\Delta}{s_i^2} + \frac{1}{2} \right)}{\left( \int_{-\infty}^{\frac{\Delta}{s_i} - \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt \right)^2}$$

The sign of the above expression can be determined by evaluating the sign of

$$D = e^{-\frac{1}{2}(\frac{\Delta}{s_i} - \frac{s_i}{2})^2} \left( -\frac{\Delta}{s_i^2} - \frac{1}{2} \right) \left( \int_{-\infty}^{\frac{\Delta}{s_i} + \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt \right) - e^{-\frac{1}{2}(\frac{\Delta}{s_i} + \frac{s_i}{2})^2} \left( -\frac{\Delta}{s_i^2} + \frac{1}{2} \right) \left( \int_{-\infty}^{\frac{\Delta}{s_i} - \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt \right).$$

Notice that (a)  $\left( \int_{-\infty}^{\frac{\Delta}{s_i} + \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt \right) > \left( \int_{-\infty}^{\frac{\Delta}{s_i} - \frac{s_i}{2}} e^{-\frac{t^2}{2}} dt \right)$ , (b)  $\frac{\Delta}{s_i^2} + \frac{1}{2} > \frac{\Delta}{s_i^2} - \frac{1}{2}$ , and (c)  $e^{-\frac{1}{2}(\frac{\Delta}{s_i} - \frac{s_i}{2})^2} > e^{-\frac{1}{2}(\frac{\Delta}{s_i} + \frac{s_i}{2})^2}$ . These three inequalities together imply that in the above expression  $D < 0$ . Recognizing the  $-d_i^b$  factor in  $\partial \mathcal{L} / \partial s_i$ , we can conclude that  $\partial d_i^R / \partial s_i \leq 0$ .  $\square$

### Proof of Proposition 6.

*Part (a).* Consider any period where the brand's partial observability is captured by  $\xi_i^b$ . Note that  $\xi_i^b$  is unbiased and  $\mathbb{E}[\xi_i^b] = 1$ ; that is, the service environment remains the same in expectation (by the MMFE property). Had the brand maintained product service in-house, its cost would be given by  $c(\bar{k}, d_i)$ , where  $d_i$  is the true environment since the brand provides product service directly. In contrast, if the brand outsources the product service to a third-party provider, its payment is given by  $c^S(\bar{k}, d_i^b)$ , where  $d_i^b$  is the brand's belief about the service environment. The expected cost savings due to outsourcing in period  $i$  can be expressed as

$$\begin{aligned} \mathbb{E} [c(\bar{k}, d_i) - c^S(\bar{k}, d_i^b)] &= \mathbb{E} [c(\bar{k}, d_i^b \xi_i^b) - c^S(\bar{k}, d_i^b)] \\ &= \int_{\xi_i^b < 1} (c(\bar{k}, d_i^b \xi_i^b) - c^S(\bar{k}, d_i^b)) dF_i(\xi_i^b) + \int_{\xi_i^b \geq 1} (c(\bar{k}, d_i^b \xi_i^b) - c^S(\bar{k}, d_i^b)) dF_i(\xi_i^b). \end{aligned}$$

Substituting  $c(k, d) = Ck(1 + \beta d)$  and  $c^S(k, d) = \alpha c(k, d)$  into the above expression, we have

$$\begin{aligned} \mathbb{E} [C(\bar{k}, d_i) - c^S(\bar{k}, d_i^b)] &= C\bar{k} \int_{\xi_i^b < 1} ((1 + \beta d_i^b \xi_i^b) - \alpha(1 + \beta d_i^b)) dF_i(\xi_i^b) \\ &\quad + C\bar{k} \int_{\xi_i^b \geq 1} ((1 + \beta d_i^b \xi_i^b) - \alpha(1 + \beta d_i^b \xi_i^b)) dF_i(\xi_i^b) \\ &= C\bar{k}\beta d_i^b \left( 1 - \alpha F_i(1) - \alpha \int_{\xi_i^b \geq 1} \xi_i^b dF_i(\xi_i^b) \right) \leq C\bar{k}\beta d_i^b \left( 1 - \alpha \int_{\xi_i^b \geq 1} \xi_i^b dF_i(\xi_i^b) \right). \end{aligned}$$

The proposition follows by setting  $1 - \alpha \int_{\xi_i^b \geq 1} \xi_i^b dF_i(\xi_i^b) \leq 0$ .

*Part (b).* By Proposition 5, in each period when the service provider does not report, the brand's belief is adjusted downwards from  $d_i^b$  to  $d_i^R / (\alpha C \beta k_i^o)$ . By the definition of PBE threshold  $d_i^R$ , we have  $d_i^b > d_i^R \geq d_i$ . Let  $\xi_i^b$  and  $\xi_i^{b'}$  denote the cumulative uncertainty that has not been learned by the brand in period  $i$  just before and after the belief updating respectively (assuming the service

provider does not disclose true environment condition in period  $i$ ). By the martingale property, we have  $\mathbb{E}[d_i^b \xi_i^b] = \mathbb{E}[d_i^R \xi_i^{b'}] = d_i$ . It follows that  $\mathbb{E}[\xi_i^{b'}] = d_i^b/d_i^R \mathbb{E}[\xi_i^b] > \mathbb{E}[\xi_i^b]$ , implying that a subset of random events in  $\xi_i^b$  is learned by belief updating and that the learned events are biased downwards (i.e., with an expectation less than 1). Therefore, the remaining environment uncertainty in the brand's belief is reduced by the belief updating in each period (as long as the service provider continues to choose non-disclosure). Since in each period the uncertainty in the brand's belief is strictly decreasing when  $d_i^R$  approaches  $d_i$ , we have  $\lim_{i \rightarrow \infty} \int_{\xi_i^b \geq 1} \xi_i^b dF_i(\xi_i^b) \rightarrow 0$ . As such, there must exist a finite threshold  $\bar{i}$  beyond which  $\int_{\xi_i^b \geq 1} \xi_i^b dF_i(\xi_i^b) \leq 1/\alpha$  for any  $i \geq \bar{i}$ , assuming that the service provider continues to choose non-disclosure.  $\square$

**Proof of Proposition 7.**

*Part (a).* The statement follows from the assumption that if the provider reports the true environment, the report is truthful. It follows that the brand knows the true environment whenever the provider reports, and thus we must have  $y_i^* = 0$ . If on the other hand the provider does not report, then the brand can infer that  $d_i^b > d_i$ , where monitoring creates potential value by calibrating its belief with the true service environment.

*Part (b).* We prove the proposition statement through backward induction. First consider the last period  $T$ . If the brand does not exert monitoring effort and the provider does not report the true environment, then its belief is updated downwards from  $d_T^b$  to  $d_T^{b*} = d_T^R - c^R/(\alpha C \beta \bar{k})$ . Without monitoring, the brand's payment to the provider is given by  $c^S(\bar{k}, d_T^{b*})$ . In contrast, with monitoring, the brand's payment depends on actual service environment  $d_T$  and it incurs a lower payment whenever  $d_T \leq d_T^{b*}$  (and incurs the same payment  $d_T^{b*}$  when  $d_T^{b*} < d_T \leq d_T^R$  since the provider absorbs the reporting cost  $c^R$ ). Therefore, the value difference between monitoring and not monitoring can be derived using (3), which can be simplified as

$$\begin{aligned} MV_T(y_T = 1) &= c^S(\bar{k}, d_T^{b*}) - \int_{d_T \leq d_T^{b*}} c^S(\bar{k}, d_T) d\widehat{F}_T(d_T) - M \\ &= c^S(\bar{k}, d_T^{b*}) - \int_{\xi_T^{b'} \leq 1} c^S(\bar{k}, d_T^{b*} \xi_T^{b'}) dF_T(\xi_T^{b'}) - M, \end{aligned}$$

where we use  $\widehat{F}_T(d_T)$  to denote the distribution of  $d_T$ ,  $F_T(d_T)$  to denote the distribution of  $\xi_T^{b'}$ , where  $\xi_T^{b'}$  to denote the cumulative environment uncertainty not learned by the brand (after adjusting its belief downward due to the service provider's non-disclosure). Recall that  $c^S(k, d) = \alpha C k (1 + \beta d)$ . Substituting into the above expression, we have

$$MV_T(y_T = 1) = \alpha C \bar{k} \beta d_T^{b*} \left( 1 - \int_{\xi_T^{b'} \leq 1} \xi_T^{b'} dF_T(\xi_T^{b'}) \right) - M.$$

Let  $s_T$  denote the variance parameter of  $\xi_T^{b'}$  and set  $H(s_T) = 1 - \int_{\xi_T^{b'} \leq 1} \xi_T^{b'} dF_T(\xi_T^{b'})$ . It follows that  $y_T^* = 1$  if and only if

$$MV_T(y_T = 1) \geq 0 \Leftrightarrow d_T^{b*} \geq \frac{M}{\alpha C \bar{k} \beta H(s_T)} := \bar{d}_T^b(s_T)$$

In addition, under the lognormal matching,  $H(s)$  is a monotone increasing function of  $s$  because  $H(s)$  can be simplified to  $H(s) = \frac{1}{2}(1 + \operatorname{erf}(\frac{s}{2\sqrt{2}}))$ , where  $\operatorname{erf}(\cdot)$  is the standard error function and we have taken the fact that  $\mathbb{E}[\xi_T^{b'}] = 1$  (i.e., the brand's belief remains unbiased after updating). It follows immediately that  $H'(s) \geq 0$  since  $\operatorname{erf}(\cdot)$  is a monotone increasing function. Thus, we have  $\partial \bar{d}_T^b(s_T) / \partial s_T \leq 0$ . This completes the proof for the last period  $T$ .

Suppose the threshold policy holds for period  $i + 1$ . Consider the monitoring decision in period  $i$ . The value of monitoring is given by

$$MV_i(y_i = 1) = c^S(\bar{k}, d_i^{b*}) - \int_{\xi_i^{b'} \leq 1} c^S(\bar{k}, d_i^{b*} \xi_i^{b'}) dF_i(\xi_i^{b'}) - M + \gamma (\mathbb{E} V_{i+1}(d_{i+1}^b | d_i) - \mathbb{E} V_{i+1}(d_{i+1}^b | d_i^{b*})). \quad (\text{A-6})$$

Observe that the first three terms in the above expression have a similar structure to that for the last period, and thus  $c^S(\bar{k}, d_i^{b*}) - \int_{\xi_i^{b'} \leq 1} c^S(\bar{k}, d_i^{b*} \xi_i^{b'}) dF_i(\xi_i^{b'}) - M$  must be monotonically increasing in  $d_i^{b*}$ . It then remains to be shown that the last term is also monotonically increasing in  $d_i^{b*}$ . Note that for any random realizations of  $\xi_{i+1}^b$ , we have  $V_{i+1}(d_{i+1}^b | d_i^{b*}) \leq V_{i+1}(d_{i+1}^b | \hat{d}_i^{b*})$  for any  $d_i^{b*} \geq \hat{d}_i^{b*}$  because  $c^S(\bar{k}, d_i^{b*}) \geq c^S(\bar{k}, \hat{d}_i^{b*})$  (as the expected payment to the provider is higher under belief  $d_i^{b*} \xi_i^{b'} \xi_{i+1}^b$  than that under  $\hat{d}_i^{b*} \xi_i^{b'} \xi_{i+1}^b$ ). Because in period  $i + 1$  a threshold policy is optimal, let  $\bar{d}_{i+1}^b$  denote the optimal threshold. There are three cases to consider.

*Case 1.*  $d_i^{b*} \xi_i^{b'} \xi_{i+1}^b > d_i \xi_{i+1}^b > \bar{d}_{i+1}^b$ . In this case, the belief is above the threshold in both cases and hence it is optimal to monitor. As such, we have  $\mathbb{E} V_{i+1}(d_{i+1}^b | d_i) - \mathbb{E} V_{i+1}(d_{i+1}^b | d_i^{b*}) = 0$ ; that is, the marginal benefit of monitoring in period  $i$  does not carry over to the next period. In this case, the threshold policy must be optimal since the monitoring decision in period  $i$  is decoupled from future periods.

*Case 2.*  $d_i^{b*} \xi_i^{b'} \xi_{i+1}^b > \bar{d}_{i+1}^b > d_i \xi_{i+1}^b$ . It is then optimal to monitor in period  $i + 1$  if monitoring was not conducted in period  $i$  and vice versa. The carry-over marginal benefit of monitoring in period  $i$  is thus the cost savings of not having to monitor in period  $i + 1$ . Because it is optimal to monitor only if the cost of monitoring is less than the benefit obtained, it follows that  $\mathbb{E} V_{i+1}(d_{i+1}^b | d_i^{b*}) - \mathbb{E} V_{i+1}(d_{i+1}^b | d_i) \leq 0$ . Further, such carry-over benefit is increasing in  $d_i^{b*}$ . To see this, note that the second term does not depend on  $d_i^{b*}$  (because  $y_{i+1}^* = 1$ ) while monitoring benefit is directly associated with  $d_{i+1}^b = d_i^{b*} \xi_i^{b'} \xi_{i+1}^b$ , which is increasing in  $d_i^{b*}$  (i.e., all else being equal, monitoring reduces the expected payment to a larger extent as  $d_i^{b*}$  increases). It is worth pointing out that  $V_{i+1}(d_{i+1}^b | d_i^{b*}) = V_{i+1}(d_i^{b*} \xi_i^{b'} \xi_{i+1}^b)$  decreases in  $d_i^{b*}$ , but the benefit of monitoring is increasing.

*Case 3.*  $\bar{d}_{i+1}^b > d_i^{b*} \xi_i^{b'} \xi_{i+1}^b > d_i \xi_{i+1}^{b'}$ . In this case, it is not optimal to monitor in period  $i+1$  in both cases. It then follows that  $\mathbb{E} V_{i+1}(d_i^{b*} \xi_i^{b'} \xi_{i+1}^b) - \mathbb{E} V_{i+1}(d_i \xi_{i+1}^{b'})$  monotonically increases in  $d_i^{b*}$  because both  $\mathbb{E} V_{i+1}(d_i^{b*} \xi_i^{b'} \xi_{i+1}^b)$  and  $\mathbb{E} V_{i+1}(d_i \xi_{i+1}^{b'})$  represent the sum of discounted value from period  $i+1$  on-wards, and their difference is captured by the sum of discounted cost differences that take the same structure as that in the current period  $\mathbb{E}[c^S(\bar{k}, d_i^{b*} \xi_i^{b'}) - c^S(\bar{k}, d_i)]$ , which is monotonically increasing in  $d_i^{b*}$ .

Thus, in all three cases, the marginal value of monitoring in the current period is monotonically increasing in  $d_i^{b*}$ , implying that a threshold policy is also optimal for period  $i$ . This completes the proposition statement.  $\square$

**Proof of Corollary 2.** By the proof of Proposition 7, the value of monitoring is captured by current cost savings and the marginal benefit of reduced cost for future periods. The threshold for the last period  $T$  is given by  $\bar{d}_T^b = M/\alpha C \bar{k} \beta H(s_T)$ . The corollary statements then follow by taking derivatives of  $\bar{d}_T^b$  with respect to  $\alpha$ ,  $\bar{k}$ , and  $M$ , respectively. Suppose these directional results hold for period  $i+1$ , and consider period  $i$ . By Proposition 7, the optimal monitoring policy is again a threshold type that satisfies (A-6). Define

$$\mathcal{L}(\bar{d}_i^b) = c^S(\bar{k}, \bar{d}_i^b) - \int_{\xi_i^{b'}} c^S(\bar{k}, \bar{d}_i^b \xi_i^{b'}) dF_i(\xi_i^{b'}) - M + \gamma \left( \mathbb{E} V_{i+1}(d_{i+1}^b | d_i) - \mathbb{E} V_{i+1}(d_{i+1}^b | \bar{d}_i^b) \right).$$

The corollary statements then follow by applying the implicit function theorem on  $\mathcal{L}(\bar{d}_i^b)$  and recognizing that  $\partial \mathcal{L}(\bar{d}_i^b) / \partial \bar{d}_i^b \geq 0$  (from the proof of Proposition 7). Since these directional results also hold for any future periods by the supposition, the corollary statements therefore must hold for any arbitrary period.  $\square$

**Proof of Corollary 3.** The corollary statement follows from Proposition 7 by recognizing that the brand's belief following any monitoring effort must satisfy  $d_i \leq d_i^{b*}$ , where the inequality follows from the fact that monitoring is optimal only if the provider does not report true environment condition (implying  $d_i^{b*} \geq d_i$ ). All else being equal,  $P(d_{i+1} | d_i > x) \geq P(d_{i+1} | d_i^{b*} > x)$  for any given  $x$  because  $d_{i+1} | d_i = d_i \xi_{i+1}^b$  and  $d_{i+1} | d_i^{b*} = d_i^{b*} \xi_i^{b'} \xi_{i+1}^b$ . It follows then  $P(d_{i+1} | d_i > \bar{d}_{i+1}^b) \geq P(d_{i+1} | d_i^{b*} > \bar{d}_{i+1}^b)$ . Thus for any realized belief path, the probability that the true environment condition is more challenging than the brand's belief is higher after the brand's monitoring effort, implying that the provider is more likely to report the true environment. Conversely, whenever the provider reports true environment, we have  $d_i \geq d_i^{b*}$ . Therefore, we have  $P(d_{i+1}^b | d_i > \bar{d}_{i+1}^b) \geq P(d_{i+1}^b | d_i^{b*} > \bar{d}_{i+1}^b)$ , implying that the brand is more likely to conduct monitoring efforts following provider's reporting effort.  $\square$

**Proof of Proposition 8.** The key impact of partial observability is when the brand overpays the service provider based on its own belief of the service environment,  $d_i^b$ , as opposed to the true

environment condition  $d_i$ . Under the passive monitoring strategy, the service provider updates the brand if and only if the true service environment exceeds the threshold  $d_i^R$  (i.e., only when  $d_i \geq d_i^R$ ). Therefore, the brand suffers only when  $d_i < d_i^R$ . Consider the cost implications under passive monitoring when  $d_i < d_i^R$ . The brand will update its belief to  $d_i^{b*} = d_i^R - c^R/(\alpha C \beta \bar{k})$  (per Proposition 5). For any given target resource level  $\bar{k}$ , the cost incurred for the brand is given by  $c^S(\bar{k}, d_i^{b*}) = \alpha c(\bar{k}, d_i^{b*})$ . If the brand were to provide the service directly, on the other hand, the cost incurred is  $c(\bar{k}, d_i)$ . The expected cost over-run with outsourcing is therefore given by

$$\begin{aligned} \Delta_c &= c^S(\bar{k}, d_i^{b*}) - \mathbb{E}[c(\bar{k}, d_i)] = \alpha c(\bar{k}, d_i^{b*}) - c(\bar{k}, d_i^{b*}) \mathbb{E}[\xi_i^{b'} | \xi_i^{b'} \leq 1] P(\xi_i^{b'} \leq 1) \\ &= \alpha c(\bar{k}, d_i^{b*}) - c\left(\bar{k}, d_i^{b*} \Phi\left(-\frac{1}{2} s_i^2\right)\right), \end{aligned}$$

where the second equality follows from the matching approximation where  $\xi_i^{b'}$  is lognormally distributed with  $\mathbb{E}[\xi_i^{b'}] = 1$ ,  $\Phi$  is the standard normal cumulative distribution, and  $s_i^2$  captures the cumulative variance parameter of the environment uncertainty that has not been learned by the brand. Notice that in the worst-case scenario we have  $\lim_{s_i^2 \rightarrow \infty} \Phi\left(-\frac{1}{2} s_i^2\right) \rightarrow 0$ , we have  $\Delta_c \leq \alpha c(\bar{k}, d_i^{b*}) - c(\bar{k}, d_i^{b*} \cdot 0) = \alpha c(\bar{k}, d_i^{b*}) - c(\bar{k}, 0)$ . Since  $\alpha c(\bar{k}, 0) - c(\bar{k}, 0) < 0$  and  $c(\bar{k}, d_i^{b*})$  is monotonically increasing in  $d_i^{b*}$ , there must exist a unique  $\bar{d}_i^b$  such that  $\alpha c(\bar{k}, \bar{d}_i^b) - c(\bar{k}, 0) = 0$ . It follows that for all  $d_i^{b*} \leq \bar{d}_i^b$ , we have  $\alpha c(\bar{k}, d_i^{b*}) - c(\bar{k}, 0) \leq 0$ , implying that the brand is better off with outsourcing than direct service provision. Given the fact that  $d_i^{b*} = d_i^R - c^R/(\alpha C \beta \bar{k})$ , there is a unique  $\bar{d}_i^R$  such that the brand is better off whenever  $d_i^R \leq \bar{d}_i^R$ .

Furthermore, by Proposition 5, there is a finite number of periods where  $d_i \leq \bar{d}_i^R$ . Let  $N$  denote this finite number of periods. Define  $\bar{d}^R = \max_{i \in N} \{\bar{d}_i^R\} = \bar{d}_1^R$ , where the last equality follows from the fact that  $\bar{d}_i^R$  is non-increasing in  $i$  (the number of periods the service provider has not reported). Since for all  $i \in N$ , we have  $\bar{d}_i^R \leq \bar{d}^R$ , we must have  $\alpha c(\bar{k}, \bar{d}_i^R) - c(\bar{k}, 0) \leq 0$  for all  $i \in N$ . Since any sequence of the provider's non-disclosure periods can be viewed as a renewal process (as the service environment evolution is martingale), the above argument holds for any sequence of non-disclosure periods. For the sensitivity statement, it follows from the fact that  $\alpha c(\bar{k}, \hat{d}_i^{b*})$  is decreasing in  $\alpha$  and therefore a decrease in  $\alpha$  (improved efficiency) implies a larger  $\bar{d}_i^R$  and hence  $\bar{d}^R$ . This completes the proof.  $\square$

**Proof of Proposition 9.** For any given myopic policy, the optimal monitoring policy can be derived from the optimal policy characterized in Proposition 7. In particular, it is optimal to set  $y^* = 0$  if the provider reports true environment condition in period  $i$ . In contrast, if the provider does not report, then the marginal benefit of monitoring for the current period is given by (see proof of Proposition 7)

$$\mathbb{E}[c^S(\bar{k}, d_i^{b*}) - c^S(\bar{k}, d_i^{b*} \xi_i^{b'} | \xi_i^{b'} \leq 1)] - M = \alpha C \bar{k} \beta d_i^{b*} \mathbb{E}[(1 - \xi_i^{b'}) | \xi_i^{b'} \leq 1] - M$$

$$= 2\alpha C\bar{k}\beta d_i^b \left( \frac{\operatorname{erf}\left(\frac{s_i}{2\sqrt{2}}\right)}{1 + \operatorname{erf}\left(\frac{s_i}{2\sqrt{2}}\right)} \right) - M,$$

where the last equality follows from the matching lognormal approximation. Observe that the benefit of monitoring is increasing in  $s_i$ . Let  $s_i \rightarrow \infty$ , we have  $\mathbb{E}[c^S(\bar{k}, d_i^b) - c^S(\bar{k}, d_i^b \xi_i^b)] - M \rightarrow \alpha C\bar{k}\beta d_i^b - M$ . Setting  $\alpha C\bar{k}\beta d_i^b - M \geq 0$  yields the threshold  $\widehat{d}_i^b = M(\alpha C\bar{k}\beta)^{-1}$ . Next, for any myopic policy that considers only the current period, the benefit of monitoring cannot exceed  $2\alpha C\bar{k}\beta d_i^b H(s_i) - M$ , where  $H(s_i)$  is defined in the proof of Proposition 7. Note that  $2\alpha C\bar{k}\beta d_i^b H(s_i) \leq 2\alpha C\bar{k}\beta d_i^b H(1) = \alpha C\bar{k}\beta d_i^b$ . When future marginal benefits are considered, then the global optimal threshold  $\bar{d}_i^b$  must satisfy  $\bar{d}_i^b \leq M(2\alpha C\bar{k}\beta d_i^b H(s_i))^{-1} \leq \widehat{d}_i^b$ . Finally, observe that the marginal benefit of monitoring in the current period on future periods cannot exceed the monitoring cost  $M$  discounted one period back. To see this, note that any marginal benefit carried over to the next period is embedded in the potential lower  $d_{i+1}^b$  for future periods. However, one can always recover the lowered  $d_{i+1}^b$  in the next period by monitoring in period  $i+1$  with a monitoring cost  $M$ . As such, the maximum marginal benefit for monitoring in period  $i$  cannot exceed  $2\alpha C\bar{k}\beta d_i^b H(s_i) + M(1+r)^{-1}$ , and hence the global optimal threshold cannot be lower than  $2\alpha C\bar{k}\beta d_i^b H(s_i) + M(1+r)^{-1} - M \geq 0 \rightarrow \bar{d}_i^b \geq r/(1+r)M(2\alpha C\bar{k}\beta H(s_i))^{-1}$ . The proposition statement then follows by setting  $s_i \rightarrow \infty$ .  $\square$