Electronic Companion for *Shall We Only Store Popular Products? Warehouse Assortment Selection for E-Companies*

EC.1. Proofs of Statements

Proof of Proposition 1 Proof of part (i): Consider any $\mathcal{S},\mathcal{S}'\subseteq\mathcal{N}$ with $\mathcal{S}\subseteq\mathcal{S}'$, we have

$$C(\mathcal{T}|\mathcal{S})=\begin{cases}0 & \text{if } \mathcal{T}\subseteq\mathcal{S}\\ G(\mathcal{T}) & \text{if } \mathcal{T}\nsubseteq\mathcal{S}\end{cases}=\begin{cases}0 & \text{if } \mathcal{T}\subseteq\mathcal{S}\\ G(\mathcal{T}) & \text{if } \mathcal{T}\nsubseteq\mathcal{S} \text{ and } \mathcal{T}\subseteq\mathcal{S}'\\ G(\mathcal{T}) & \text{if } \mathcal{T}\nsubseteq\mathcal{S}'\end{cases},$$

and

$$C(\mathcal{T}|\mathcal{S}')=\begin{cases}0 & \text{if } \mathcal{T}\subseteq\mathcal{S}\\ 0 & \text{if } \mathcal{T}\nsubseteq\mathcal{S} \text{ and } \mathcal{T}\subseteq\mathcal{S}'\\ G(\mathcal{T}) & \text{if } \mathcal{T}\nsubseteq\mathcal{S}'\end{cases}.$$

So, given $G(\mathcal{T})\geq 0\ \forall\mathcal{T}\subseteq\mathcal{N}$, we have $C(\mathcal{T}|\mathcal{S})\geq C(\mathcal{T}|\mathcal{S}')\ \forall\mathcal{T}\subseteq\mathcal{N}$.

Thus, given $\pi(\mathcal{T})\geq 0\ \forall\mathcal{T}\subseteq\mathcal{N}$, we have $f(\mathcal{S})=\sum_{\mathcal{T}\subseteq\mathcal{N}}\pi(\mathcal{T})C(\mathcal{T}|\mathcal{S})\geq\sum_{\mathcal{T}\subseteq\mathcal{N}}\pi(\mathcal{T})C(\mathcal{T}|\mathcal{S}')=f(\mathcal{S}')$. That is, the larger $\mathcal{S}$ is, the lower the total fulfillment costs. Since we want to minimize the cost, (CP') is equivalent to (CP).

Proof of part (ii): Consider any $\mathcal{S},\mathcal{S}'\subseteq\mathcal{N}$ with $\mathcal{S}\subseteq\mathcal{S}'$, we have

$$C(\mathcal{T}|\mathcal{S})=\begin{cases}0 & \text{if } \mathcal{T}\subseteq\mathcal{S}\\ G(\mathcal{T}\backslash\mathcal{S}) & \text{if } \mathcal{T}\nsubseteq\mathcal{S}\end{cases}=\begin{cases}0 & \text{if } \mathcal{T}\subseteq\mathcal{S}\\ G(\mathcal{T}\backslash\mathcal{S}) & \text{if } \mathcal{T}\nsubseteq\mathcal{S} \text{ and } \mathcal{T}\subseteq\mathcal{S}'\\ G(\mathcal{T}\backslash\mathcal{S}) & \text{if } \mathcal{T}\nsubseteq\mathcal{S}'\end{cases},$$

and

$$C(\mathcal{T}|\mathcal{S}')=\begin{cases}0 & \text{if } \mathcal{T}\subseteq\mathcal{S}\\ 0 & \text{if } \mathcal{T}\nsubseteq\mathcal{S} \text{ and } \mathcal{T}\subseteq\mathcal{S}'\\ G(\mathcal{T}\backslash\mathcal{S}') & \text{if } \mathcal{T}\nsubseteq\mathcal{S}'\end{cases}.$$

Since $(\mathcal{T}\backslash\mathcal{S}')\subseteq(\mathcal{T}\backslash\mathcal{S})\ \forall\mathcal{T}\subseteq\mathcal{N}$ and $G(X)\leq G(Y)\ \forall X\subseteq Y\subseteq\mathcal{N}$ (the type-II cost function is increasing), then we have $G(\mathcal{T}\backslash\mathcal{S})\geq G(\mathcal{T}\backslash\mathcal{S}')$. So, $C(\mathcal{T}|\mathcal{S})\geq C(\mathcal{T}|\mathcal{S}')\ \forall\mathcal{T}\subseteq\mathcal{N}$.

Thus, given $\pi(\mathcal{T})\geq 0\ \forall\mathcal{T}\subseteq\mathcal{N}$, we have $f(\mathcal{S})=\sum_{\mathcal{T}\subseteq\mathcal{N}}\pi(\mathcal{T})C(\mathcal{T}|\mathcal{S})\geq\sum_{\mathcal{T}\subseteq\mathcal{N}}\pi(\mathcal{T})C(\mathcal{T}|\mathcal{S}')=f(\mathcal{S}')$. That is, the larger $\mathcal{S}$ is, the lower the total fulfillment costs. Since we want to minimize the cost, problem (CP') is equivalent to problem (CP). □

Proof of Theorem 1 Proof of part (i): For type-I cost functions and any $\mathcal{S},\mathcal{S}'\subseteq\mathcal{N}$, take any set $\mathcal{T}\subseteq\mathcal{N}$. There are five cases: (a) $\mathcal{T}\subseteq\mathcal{S}'\cap\mathcal{S}$; (b) $\mathcal{T}\subseteq\mathcal{S}'$ but $\mathcal{T}\nsubseteq\mathcal{S}$; (c) $\mathcal{T}\nsubseteq\mathcal{S}'$ but $\mathcal{T}\subseteq\mathcal{S}$; (d) $\mathcal{T}\subseteq\mathcal{S}'\cup\mathcal{S}$ but $\mathcal{T}\nsubseteq\mathcal{S}'$ and $\mathcal{T}\nsubseteq\mathcal{S}$; and (e) $\mathcal{T}\nsubseteq\mathcal{S}'\cup\mathcal{S}$.

Under case (a) $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})=C(\mathcal{T}|\mathcal{S}')=C(\mathcal{T}|\mathcal{S})=C(\mathcal{T}|\mathcal{S}'\cup\mathcal{S})=0$; under case (b) $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})=C(\mathcal{T}|\mathcal{S})=G(\mathcal{T})$ and $C(\mathcal{T}|\mathcal{S}')=C(\mathcal{T}|\mathcal{S}'\cup\mathcal{S})=0$; under case (c) $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})=C(\mathcal{T}|\mathcal{S}')=G(\mathcal{T})$ and $C(\mathcal{T}|\mathcal{S})=C(\mathcal{T}|\mathcal{S}'\cup\mathcal{S})=0$; under case (d) $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})=C(\mathcal{T}|\mathcal{S}')=C(\mathcal{T}|\mathcal{S})=G(\mathcal{T})$ and $C(\mathcal{T}|\mathcal{S}'\cup\mathcal{S})=0$; and under case (e) $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})=C(\mathcal{T}|\mathcal{S}')=C(\mathcal{T}|\mathcal{S})=C(\mathcal{T}|\mathcal{S}'\cup\mathcal{S})=G(\mathcal{T})$.

For cases (a)-(c) and (e), $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})+C(\mathcal{T}|\mathcal{S}'\cup\mathcal{S})=C(\mathcal{T}|\mathcal{S}')+C(\mathcal{T}|\mathcal{S})$. For case (d) $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})+C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})=G(\mathcal{T})\leq 2G(\mathcal{T})=C(\mathcal{T}|\mathcal{S}')+C(\mathcal{T}|\mathcal{S})$. Thus, for any given $\mathcal{T}\subseteq\mathcal{N}$, $C(\mathcal{T}|\mathcal{S})$ is submodular in $\mathcal{S}$. Taking the weighted sum over all $\mathcal{T}$, we have $f(\mathcal{S})$ is submodular in $\mathcal{S}$.

Proof of part (*ii*): For type-II cost functions with $G(\cdot)$ being submodular and any $\mathcal{S},\mathcal{S}'\subseteq\mathcal{N}$, we have

$$\begin{aligned}
C(\mathcal{T}|\mathcal{S})+C(\mathcal{T}|\mathcal{S}') &= G(\mathcal{T}\backslash\mathcal{S})+G(\mathcal{T}\backslash\mathcal{S}')\\
&\geq G\left((\mathcal{T}\backslash\mathcal{S})\cap(\mathcal{T}\backslash\mathcal{S}')\right)+G\left((\mathcal{T}\backslash\mathcal{S})\cup(\mathcal{T}\backslash\mathcal{S}')\right)\\
&= G\left((\mathcal{T}\backslash(\mathcal{S}\cap\mathcal{S}')\right)+G\left((\mathcal{T}\backslash(\mathcal{S}\cup\mathcal{S}')\right)\\
&= C(\mathcal{T}|\mathcal{S}\cap\mathcal{S}')+C(\mathcal{T}|\mathcal{S}\cup\mathcal{S}').
\end{aligned}$$

Thus, $C(\mathcal{T}|\mathcal{S})$ is submodular in $\mathcal{S}$. Taking the weighted sum over all $\mathcal{T}$, we have $f(\mathcal{S})$ is submodular in $\mathcal{S}$.

If $G(\mathcal{S})$ is size-based, i.e., $G(\mathcal{S})=c(|\mathcal{S}|)\ \forall\mathcal{S}\subseteq\mathcal{N}$ for some function $c(\cdot)$, then $G(\mathcal{S}')+G(\mathcal{S})\geq G(\mathcal{S}'\cap\mathcal{S})+G(\mathcal{S}'\cup\mathcal{S})$ is equivalent to $c(|\mathcal{S}'|)+c(|\mathcal{S}|)\geq c(|\mathcal{S}'\cap\mathcal{S}|)+c(|\mathcal{S}'\cup\mathcal{S}|)$. Note that $|\mathcal{S}'|+|\mathcal{S}|=|\mathcal{S}'\cap\mathcal{S}|+|\mathcal{S}'\cup\mathcal{S}|$ and $|\mathcal{S}'\cap\mathcal{S}|\leq|\mathcal{S}'|,|\mathcal{S}|\leq|\mathcal{S}'\cup\mathcal{S}|$. Hence, $c(|\mathcal{S}'|)+c(|\mathcal{S}|)\geq c(|\mathcal{S}'\cap\mathcal{S}|)+c(|\mathcal{S}'\cup\mathcal{S}|)$ is equivalent to $c(\cdot)$ is concave. Thus, $G(\mathcal{S})$ is submodular is equivalent to $c(\cdot)$ is concave. ☐

Proof of Corollary 1 In terms of (OFRM-C), let $f(\mathcal{S})=\sum_{m=1}^{M}\pi_m\mathbb{I}(\mathcal{T}_m\nsubseteq\mathcal{S})$. It is obvious that the objective function in (OFRM-C) is decreasing and submodular. In terms of (OFRM), let $\mathrm{OFR}(\mathcal{S})=\sum_{m=1}^{M}\pi_m\mathbb{I}(\mathcal{T}_m\subseteq\mathcal{S})=1-f(\mathcal{S})$. On the one hand, for any $\mathcal{S},\mathcal{S}'\subseteq\mathcal{N}$ with $\mathcal{S}\subseteq\mathcal{S}'$, we have

$$C(\mathcal{T}|\mathcal{S})=\begin{cases}0 \text{ if } \mathcal{T}\subseteq\mathcal{S}\\ 1 \text{ if } \mathcal{T}\nsubseteq\mathcal{S}\end{cases}=\begin{cases}0 \text{ if } \mathcal{T}\subseteq\mathcal{S}\\ 1 \text{ if } \mathcal{T}\nsubseteq\mathcal{S} \text{ and } \mathcal{T}\subseteq\mathcal{S}'\\ 1 \text{ if } \mathcal{T}\nsubseteq\mathcal{S}'\end{cases},$$

and

$$C(\mathcal{T}|\mathcal{S}')=\begin{cases}0 \text{ if } \mathcal{T}\subseteq\mathcal{S}\\ 0 \text{ if } \mathcal{T}\nsubseteq\mathcal{S} \text{ and } \mathcal{T}\subseteq\mathcal{S}'\\ 1 \text{ if } \mathcal{T}\nsubseteq\mathcal{S}'\end{cases}.$$

Hence, we have $C(\mathcal{T}|\mathcal{S})\geq C(\mathcal{T}|\mathcal{S}')\ \forall\mathcal{T}\subseteq\mathcal{N}$. So, given $\pi(\mathcal{T})\geq 0\ \forall\mathcal{T}\subseteq\mathcal{N}$, we have $\mathrm{OFR}(\mathcal{S})=1-f(\mathcal{S})=1-\sum_{\mathcal{T}\subseteq\mathcal{N}}\pi(\mathcal{T})C(\mathcal{T}|\mathcal{S})\leq 1-\sum_{\mathcal{T}\subseteq\mathcal{N}}\pi(\mathcal{T})C(\mathcal{T}|\mathcal{S}')=1-f(\mathcal{S}')=\mathrm{OFR}(\mathcal{S}')$. That is, $\mathrm{OFR}(\cdot)$ is increasing.

On the other hand, for any $\mathcal{S},\mathcal{S}'\subseteq\mathcal{N}$, take any set $\mathcal{T}\subseteq\mathcal{N}$. There are five cases: (a) $\mathcal{T}\subseteq\mathcal{S}'\cap\mathcal{S}$; (b) $\mathcal{T}\subseteq\mathcal{S}'$ but $\mathcal{T}\nsubseteq\mathcal{S}$; (c) $\mathcal{T}\nsubseteq\mathcal{S}'$ but $\mathcal{T}\subseteq\mathcal{S}$; (d) $\mathcal{T}\subseteq\mathcal{S}'\cup\mathcal{S}$ but $\mathcal{T}\nsubseteq\mathcal{S}'$ and $\mathcal{T}\nsubseteq\mathcal{S}$; and (e) $\mathcal{T}\nsubseteq\mathcal{S}'\cup\mathcal{S}$. Under case (a) $C(\mathcal{T}|\mathcal{S}'\cap\mathcal{S})=$

$C(\mathcal{T}|\mathcal{S}') = C(\mathcal{T}|\mathcal{S}) = C(\mathcal{T}|\mathcal{S}' \cup \mathcal{S}) = 0$; under case (b) $C(\mathcal{T}|\mathcal{S}' \cap \mathcal{S}) = C(\mathcal{T}|\mathcal{S}) = 1$ and $C(\mathcal{T}|\mathcal{S}') = C(\mathcal{T}|\mathcal{S}' \cup \mathcal{S}) = 0$; under case (c) $C(\mathcal{T}|\mathcal{S}' \cap \mathcal{S}) = C(\mathcal{T}|\mathcal{S}') = 1$ and $C(\mathcal{T}|\mathcal{S}) = C(\mathcal{T}|\mathcal{S}' \cup \mathcal{S}) = 0$; under case (d) $C(\mathcal{T}|\mathcal{S}' \cap \mathcal{S}) = C(\mathcal{T}|\mathcal{S}') = C(\mathcal{T}|\mathcal{S}) = 1$ and $C(\mathcal{T}|\mathcal{S}' \cup \mathcal{S}) = 0$; and under case (e) $C(\mathcal{T}|\mathcal{S}' \cap \mathcal{S}) = C(\mathcal{T}|\mathcal{S}') = C(\mathcal{T}|\mathcal{S}) = C(\mathcal{T}|\mathcal{S}' \cup \mathcal{S}) = 1$. For cases (a)-(c) and (e), $C(\mathcal{T}|\mathcal{S}' \cap \mathcal{S}) + C(\mathcal{T}|\mathcal{S}' \cup \mathcal{S}) = C(\mathcal{T}|\mathcal{S}') + C(\mathcal{T}|\mathcal{S})$. For case (d) $C(\mathcal{T}|\mathcal{S}' \cap \mathcal{S}) + C(\mathcal{T}|\mathcal{S}' \cap \mathcal{S}) = 1 \le 2 = C(\mathcal{T}|\mathcal{S}') + C(\mathcal{T}|\mathcal{S})$. Hence, for any given $\mathcal{T} \subseteq \mathcal{N}$, $C(\mathcal{T}|\mathcal{S})$ is submodular in $\mathcal{S}$. So, $1 - C(\mathcal{T}|\mathcal{S})$ is supermodular in $\mathcal{S}$. Taking the weighted sum over all $\mathcal{T} \subseteq \mathcal{N}$, we have OFR($\mathcal{S}$) is supermodular in $\mathcal{S}$. □

Proof of Theorem 2 We prove by a reduction from the Densest k-Subgraph (DkS) Problem, which is known to be NP-hard (Corneil and Perl 1984).

We start by presenting the DkS problem. Consider a undirected simple graph $G(V,E)$ with $V = \{1, 2, \dots, N\}$, $E \subseteq \{\{i,j\} | i,j \in V, i \neq j\}$ (loops are permitted), and $|E| = m$. Given a parameter k, the goal of the DkS problem is to find a subgraph of G induced on k vertices that contains the largest number of edges.

Let A be the adjacency matrix of G, then the DkS problem can be formulated as follows

$$\begin{aligned} \max_{\boldsymbol{x}} \ & \boldsymbol{x}^\top A \boldsymbol{x} \\ \text{s.t.} \ & \textstyle\sum_{i=1}^N x_i = k \\ & x_i = \{0,1\} \quad i \in \{1, 2, \dots, N\}, \end{aligned} \tag{EC.1}$$

where x_i a binary decision variable which equals to 1 if vertex i is in the densest k-subgraph and 0 otherwise.

Now, we construct an instance of the (OFRM) problem with $L = 2$ and cardinality constraint k. Assume that there are N different items and let $\mathcal{N} = \{1, 2, \dots, N\}$. For any $i, j \in \mathcal{N}$, subset $\{i,j\}$ has a probability of $\frac{1}{m}$ to be chosen if $\{i,j\} \in E$, and the rest subsets are never been chosen. Let $B = \{b_{i,j}\}_{N \times N}$ with $b_{i,j} = \frac{1}{2m}$ $\forall \{i,j\} \in E$, and 0 otherwise, then the (OFRM) problem with $L = 2$ for this instance can be formulated as follows.

$$\begin{aligned} \max_{\boldsymbol{x}} \ & \boldsymbol{x}^\top B \boldsymbol{x} \\ \text{s.t.} \ & \textstyle\sum_{i=1}^N x_i \le K \\ & x_i = \{0,1\} \quad i \in \mathcal{N}, \end{aligned} \tag{EC.2}$$

where x_i a binary decision variable which equals to 1 if product i is selected in assortment $\mathcal{S}$ and 0 otherwise. Moreover, according to Proposition 1, Problem (EC.2) is equivalent to

$$\begin{aligned} \max_{\boldsymbol{x}} \ & \tfrac{1}{2m} \cdot \boldsymbol{x}^\top A \boldsymbol{x} \\ \text{s.t.} \ & \textstyle\sum_{i=1}^N x_i = K \\ & x_i = \{0,1\} \quad i \in \mathcal{N}. \end{aligned} \tag{EC.3}$$

Since when $K = k$, Problem (EC.3) is equivalent to Problem (EC.1). Thus, we conclude that the (OFRM) problem with $L = 2$ is NP-hard.

Furthermore, by the equivalence of the (OFRM) problem and the (OFRM-C) problem, we also conclude that the (OFRM-C) problem with $L = 2$ is NP-hard. □

Proof of Proposition 2 We will show that Problem (5) can be reduced to a knapsack problem, which can be solved using dynamic programming.

We start by presenting the common $0-1$ knapsack problem. Assume that there are n items with $v_1,\dots,v_n$ being the associated item values and $w_1,\dots,w_n$ being the associated item sizes, where item sizes are assumed to be positive integers. The goal of the knapsack problem is to find a subset of items such that the total value (of the subset of items) is maximized and the total size (of the subset of items) does not exceed the knapsack size $W \in \mathbb{N}^+$. We assume that the items are indexed such that $v_1 \le v_2 \le \cdots \le v_n$. Let $\boldsymbol{x} = [x_1,\dots,x_n]^{\mathcal{T}}$ be the decision variable with $x_i = 1$ if item i is included in the interested subset and 0 otherwise. Then, the knapsack problem is formulated as follows.

$$\begin{aligned} \max_{\boldsymbol{x}\in\{0,1\}^n} \ & \sum_{i=1}^{n} v_i x_i \\ \text{s.t.} \ & \sum_{i=1}^{n} w_i x_i \le W. \end{aligned} \tag{EC.4}$$

Now, we construct an instance of the knapsack problem with $n = \tilde{M}$ items. We let $v_i = \pi_i$ and $w_i = s_i \ \forall i \in [\tilde{M}]$ be the item value and item size, respectively. Additionally, we let the knapsack size $W = K$. So, compare both problem formulations, solving Problem (5) is equivalent to solve the constructed knapsack Problem (EC.4).

Generally, the $0-1$ knapsack problem is (binary) NP-complete (Williamson and Shmoys 2011) and can be solved via a dynamic programming algorithm (Algorithm 1).

Algorithm 1: Dynamic Programming Algorithm for $0-1$ Knapsack Problem

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Input  : $v_1,\dots,v_n,\ w_1,\dots,w_n,\ W$
Output: $m_{n,W}$
$m_{i,0} \leftarrow 0 \ \forall i \in [n]$ ;
$m_{0,j} \leftarrow 0 \ \forall j \in [W]$ ;
for $i \leftarrow 1$ to $n$ do do
    for $j \leftarrow 1$ to $W$ do do
        if $w_i \le j$ then
            $m_{i,j} = \max\{m_{i-1,j-w_i} + v_i,\ m_{i-1,j}\}$ ;
        else
            $m_{i,j} = m_{i-1,j}$ ;
        end
    end
end
return $m_{n,W}$
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The computational complexity of Algorithm 1 is $\mathcal{O}(nW)$, which means the general $0-1$ knapsack problem can be solved in pseudo-polynomial time (given that W can be arbitrary large). However, in our special case, $n=\tilde{M}\le N$ and $W=K\le N$ is capped by the total number of products, the constructed $0-1$ knapsack problem can be solved in run time $\mathcal{O}(NK)$. Thus, Problem (5) can be solved in polynomial time using dynamic programming Algorithm 1. $\square$

Proof of Proposition 3 Without loss of generality, we assume $c(n)=n$. Then, problem (CP) is equivalent to the following integer programming

$$\begin{aligned} \max_{\boldsymbol{\xi}} \ & \textstyle\sum_{n=1}^{N}\left(\sum_{m:\mathcal{T}_m\ni n}\pi_m\right)\xi_n \\ \text{s.t.} \ & \sum_{n=1}^{N}\xi_n\le K \\ & \xi_n\in\{0,1\} \qquad\qquad \forall n\in[N], \end{aligned} \tag{EC.5}$$

where $\boldsymbol{\xi}\in\{0,1\}^N$ denotes the binary decision variable with $\xi_n=1$ if product n is included in the assortment $\mathcal{S}$ and 0 otherwise. Problem (EC.5) is a knapsack problem with the same weight, as a result, the optimal policy is to select K ξ_ns with the largest $\sum_{m:\mathcal{T}_m\ni n}\pi_m$ among the N products, setting them to be 1. Note that $\sum_{m:\mathcal{T}_m\ni n}\pi_m$ is the marginal choice probability of product n. Thus, the MCI policy is optimal. $\square$

Proof of Proposition 4 Considering type-II cost function with $G(\mathcal{T})=\sum_{n\in\mathcal{T}}\kappa_n$, (CP) can be reformulated as the following integer programming

$$\begin{aligned} \max_{\boldsymbol{\xi}} \ & \textstyle\sum_{n=1}^{N}\left(\sum_{m:\mathcal{T}_m\ni n}\pi_m\right)\kappa_n\xi_n \\ \text{s.t.} \ & \sum_{n=1}^{N}\xi_n\le K \\ & \xi_n\in\{0,1\} \qquad\qquad \forall n\in[N], \end{aligned} \tag{EC.6}$$

where $\boldsymbol{\xi}\in\{0,1\}^N$ denotes the binary decision variable with $\xi_n=1$ if product n is included in the assortment $\mathcal{S}$ and 0 otherwise. Problem (EC.6) is a knapsack problem with the same weight, as a result, the optimal policy is to select K ξ_ns with the largest $\left(\sum_{m:\mathcal{T}_m\ni n}\pi_m\right)\kappa_n$ among the N products, setting them to be 1. Note that $\left(\sum_{m:\mathcal{T}_m\ni n}\pi_m\right)\kappa_n=\omega_n\kappa_n$ is the choice-weighted fulfillment cost of product n. Thus, the modified MCI policy described in Proposition 4 is optimal. $\square$

Proof of Theorem 3 Without loss of generality, we assume the elements in $\mathcal{N}$ are already indexed according to $\mathcal{I}$. For the first half, we need to show that $\omega_i\ge\omega_j$ for all $i,j\in\mathcal{N}$ with $i<j$. Consider arbitrary $i,j\in\mathcal{N}$ with $i<j$. Since the elements in $\mathcal{N}$ are indexed according to $\mathcal{I}$, which is a dominant indexing rule w.r.t. the choice model π, then for any $\mathcal{T}\subseteq\mathcal{N}$, we have $\pi(\mathcal{T}\cup\{i\})\ge\pi(\mathcal{T}\cup\{j\})$. So,

$$\begin{aligned} \omega_i &= \sum_{\mathcal{T}\subseteq\mathcal{N}}\pi(\mathcal{T})\cdot\mathbb{I}(i\in\mathcal{T}) = \sum_{\mathcal{T}\subseteq\mathcal{N}\setminus\{i\}}\pi(\mathcal{T}\cup\{i\}) = \sum_{\mathcal{T}\subseteq\mathcal{N}\setminus\{i,j\}}\pi(\mathcal{T}\cup\{i\})+\pi(\mathcal{T}\cup\{i,j\}) \\ &\ge \sum_{\mathcal{T}\subseteq\mathcal{N}\setminus\{i,j\}}\pi(\mathcal{T}\cup\{j\})+\pi(\mathcal{T}\cup\{i,j\}) = \sum_{\mathcal{T}\subseteq\mathcal{N}\setminus\{j\}}\pi(\mathcal{T}\cup\{j\})=\omega_j. \end{aligned}$$

Thus, if a dominant indexing rule exists, then it is an MCI.

For the second half, we need to show that $\mathcal{S}^*=\{1,2,\ldots,K\}$ is the optimal solution.

(i) According to part 1 of Proposition 1, problem (CP') is equivalent to problem (CP) under type-I size-based cost functions for any choice model. So, we only have to prove $\mathcal{S}^*$ is an optimal solution for problem (CP').

We prove the proposition by constructing a contradiction. Assume that $\mathcal{S}^0 = \{1,2,\dots,l-1,m,l+1,\dots,K\}$ such that $f(\mathcal{S}^0) < f(\mathcal{S}^*)$ for some $l \in \{1,\dots,K\}$ and $m \in \{K+1,\dots,N\}$.

Let $\tilde{\mathcal{S}} = \{1,\dots,l-1,l+1,\dots,K\}$. Then, we have $\mathcal{S}^* = \tilde{\mathcal{S}} \cup \{l\}$ and $\mathcal{S}^0 = \tilde{\mathcal{S}} \cup \{m\}$. So,

$$\begin{aligned} f(\mathcal{S}^*) &= \sum_{\mathcal{T}\subseteq\mathcal{N}} \pi(\mathcal{T})C(\mathcal{T}|\mathcal{S}^*) = \sum_{\mathcal{T}\subseteq\mathcal{N},\mathcal{T}\not\subseteq\mathcal{S}^*} \pi(\mathcal{T})c(|\mathcal{T}|) = \sum_{n=1}^{N}\left(\sum_{\mathcal{T}\subseteq\mathcal{N},\mathcal{T}\not\subseteq\mathcal{S}^*,|\mathcal{T}|=n} \pi(\mathcal{T})c(n)\right) \\ &= \sum_{n=1}^{N} c(n)\left(1-\sum_{\mathcal{T}\subseteq\mathcal{S}^*,|\mathcal{T}|=n}\pi(\mathcal{T})\right) = \sum_{n=1}^{N} c(n)\left(1-\sum_{\mathcal{T}\subseteq\tilde{\mathcal{S}},|\mathcal{T}|=n}\pi(\mathcal{T}) - \sum_{\mathcal{T}\subseteq\tilde{\mathcal{S}},|\mathcal{T}|=n-1}\pi(\mathcal{T}\cup\{l\})\right), \end{aligned}$$

and

$$\begin{aligned} f(\mathcal{S}^0) &= \sum_{\mathcal{T}\subseteq\mathcal{N}} \pi(\mathcal{T})C(\mathcal{T}|\mathcal{S}^0) = \sum_{\mathcal{T}\subseteq\mathcal{N},\mathcal{T}\not\subseteq\mathcal{S}^0} \pi(\mathcal{T})c(|\mathcal{T}|) = \sum_{n=1}^{N} c(n)\left(1-\sum_{\mathcal{T}\subseteq\mathcal{S}^0,|\mathcal{T}|=n}\pi(\mathcal{T})\right) \\ &= \sum_{n=1}^{N} c(n)\left(1-\sum_{\mathcal{T}\subseteq\tilde{\mathcal{S}},|\mathcal{T}|=n}\pi(\mathcal{T}) - \sum_{\mathcal{T}\subseteq\tilde{\mathcal{S}},|\mathcal{T}|=n-1}\pi(\mathcal{T}\cup\{m\})\right). \end{aligned}$$

Since $\mathcal{I}$ is a dominant indexing rule w.r.t. the choice model π, for any $\mathcal{T} \subseteq \tilde{\mathcal{S}}$, we have $\pi(\mathcal{T}\cup\{l\}) \geq \pi(\mathcal{T}\cup\{m\})$. So, $f(\mathcal{S}^0) \geq f(\mathcal{S}^*)$. It contradicts the assumption. In a similar sense, we can iteratively replace elements in $\{1,\dots,K\}$ with elements in $\{K+1,\dots,N\}$ and show that selecting $\mathcal{S}^* = \{1,\dots,K\}$ has the lower cost compared to that of any other $\mathcal{S}^0 \subseteq \mathcal{N}$ with $|\mathcal{S}^0| = K$.

Thus, $f(\mathcal{S}^*) \leq f(\mathcal{S})$ $\forall \mathcal{S} \subseteq \mathcal{N}$ with $|\mathcal{S}| = K$. This proves that $\mathcal{S}^*$ is optimal.

(ii) According to part 2 of Proposition 1, problem (CP') is equivalent to problem (CP) under increasing type-II size-based cost functions for any choice model. So, we only have to prove $\mathcal{S}^*$ is an optimal solution for problem (CP').

We prove this by constructing a contradiction. Assume that $\mathcal{S}^0 = \{1,2,\dots,l-1,m,l+1,\dots,K\}$ such that $f(\mathcal{S}^0) < f(\mathcal{S}^*)$ for some $l \in \{1,\dots,K\}$ and $m \in \{K+1,\dots,N\}$.

Let $\tilde{\mathcal{S}} = \{1, \ldots, l-1, l+1, \ldots, K\}$, $\bar{\mathcal{S}} = \{K+1, \ldots, m-1, m+1, \ldots, N\}$. Then, we have $\mathcal{S}^* = \tilde{\mathcal{S}} \cup \{l\}$ and $\mathcal{S}^0 = \tilde{\mathcal{S}} \cup \{m\}$. Additionally, we let $\bar{\mathcal{S}^*} = \bar{\mathcal{S}} \cup \{m\}$ and $\bar{\mathcal{S}^0} = \bar{\mathcal{S}} \cup \{l\}$. So,

$$\begin{aligned}
f(\mathcal{S}^*) &= \sum_{\mathcal{T} \subseteq \mathcal{N}} \pi(\mathcal{T}) C(\mathcal{T}|\mathcal{S}^*) = \sum_{\mathcal{T} \subseteq \mathcal{N}, \mathcal{T} \not\subseteq \mathcal{S}^*} \pi(\mathcal{T}) c(|\mathcal{T} \backslash \mathcal{S}^*|) \\
&= \sum_{n=1}^{N-K} \left(\sum_{\mathcal{T} \subseteq \mathcal{N}, \mathcal{T} \not\subseteq \mathcal{S}^*, |\mathcal{T} \backslash \mathcal{S}^*| = n} \pi(\mathcal{T}) c(n) \right) = \sum_{n=1}^{N-K} c(n) \left(\sum_{X \subseteq \mathcal{S}^*} \sum_{Y \subseteq \bar{\mathcal{S}^*}, |Y| = n} \pi(X \cup Y) \right) \\
&= \sum_{n=1}^{N-K} c(n) \left(\sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}^*}, |Y| = n} \pi(X \cup Y) + \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}^*}, |Y| = n} \pi(X \cup Y \cup \{l\}) \right) \\
&= \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y) + \sum_{n=1}^{N-K} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n-1} \pi(X \cup Y \cup \{m\}) \\
&\quad + \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y \cup \{l\}) + \sum_{n=1}^{N-K} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n-1} \pi(X \cup Y \cup \{l\} \cup \{m\}) \\
&= \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y) + \sum_{n=1}^{N-K} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n-1} \pi(X \cup Y \cup \{l\} \cup \{m\}) \\
&\quad + \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y \cup \{l\}) + \sum_{n=0}^{N-K-1} c(n+1) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y \cup \{m\}),
\end{aligned}$$

and

$$\begin{aligned}
f(\mathcal{S}^0) &= \sum_{\mathcal{T} \subseteq \mathcal{N}} \pi(\mathcal{T}) C(\mathcal{T}|\mathcal{S}^0) = \sum_{\mathcal{T} \subseteq \mathcal{N}, \mathcal{T} \not\subseteq \mathcal{S}^0} \pi(\mathcal{T}) c(|\mathcal{T} \backslash \mathcal{S}^0|) \\
&= \sum_{n=1}^{N-K} \left(\sum_{\mathcal{T} \subseteq \mathcal{N}, \mathcal{T} \not\subseteq \mathcal{S}^0, |\mathcal{T} \backslash \mathcal{S}^0| = n} \pi(\mathcal{T}) c(n) \right) = \sum_{n=1}^{N-K} c(n) \left(\sum_{X \subseteq \mathcal{S}^0} \sum_{Y \subseteq \bar{\mathcal{S}^0}, |Y| = n} \pi(X \cup Y) \right) \\
&= \sum_{n=1}^{N-K} c(n) \left(\sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}^0}, |Y| = n} \pi(X \cup Y) + \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}^0}, |Y| = n} \pi(X \cup Y \cup \{m\}) \right) \\
&= \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y) + \sum_{n=1}^{N-K} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n-1} \pi(X \cup Y \cup \{l\}) \\
&\quad + \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y \cup \{m\}) + \sum_{n=1}^{N-K} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n-1} \pi(X \cup Y \cup \{m\} \cup \{l\}) \\
&= \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y) + \sum_{n=1}^{N-K} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n-1} \pi(X \cup Y \cup \{m\} \cup \{l\}) \\
&\quad + \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y \cup \{m\}) + \sum_{n=0}^{N-K-1} c(n+1) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y| = n} \pi(X \cup Y \cup \{l\}).
\end{aligned}$$

Since $\mathcal{I}$ is dominant w.r.t. π, then for any $X \subseteq \tilde{\mathcal{S}}$ and $Y \subseteq \bar{\mathcal{S}}$, we have $\pi(X \cup Y \cup \{l\}) \geq \pi(X \cup Y \cup \{m\})$. So,

$$\begin{aligned}
&f(\mathcal{S}^*) - f(\mathcal{S}^0) \\
= &\sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{l\}) + \sum_{n=0}^{N-K-1} c(n+1) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{m\}) \\
&- \sum_{n=1}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{m\}) - \sum_{n=0}^{N-K-1} c(n+1) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{l\}) \\
= &\sum_{n=0}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{l\}) + \sum_{n=0}^{N-K-1} c(n+1) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{m\}) \\
&- \sum_{n=0}^{N-K-1} c(n) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{m\}) - \sum_{n=0}^{N-K-1} c(n+1) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} \pi(X \cup Y \cup \{l\}) \\
&- c(0) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=0} \pi(X \cup Y \cup \{l\}) + c(0) \sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=0} \pi(X \cup Y \cup \{m\}) \\
= &\sum_{n=0}^{N-K-1} [c(n) - c(n+1)] \left[\sum_{X \subseteq \tilde{\mathcal{S}}} \sum_{Y \subseteq \bar{\mathcal{S}}, |Y|=n} (\pi(X \cup Y \cup \{l\}) - \pi(X \cup Y \cup \{m\})) \right] \\
\leq &\ 0.
\end{aligned}$$

It contradicts the assumption. In a similar sense, we can iteratively replace elements in $\{1, \ldots, K\}$ with elements in $\{K+1, \ldots, N\}$ and show that selecting $\mathcal{S}^* = \{1, \ldots, K\}$ has the lower cost compared to that of any other $\mathcal{S}^0 \subseteq \mathcal{N}$ with $|\mathcal{S}^0| = K$.

Thus, $f(\mathcal{S}^*) \leq f(\mathcal{S})$ $\forall \mathcal{S} \subseteq \mathcal{N}$ with $|\mathcal{S}| = K$. This proves that $\mathcal{S}^*$ is optimal. □

Proof of Proposition 6 For the first half, we claim that indexing products such that $p_1 \geq p_2 \geq \cdots \geq p_N$ is dominant w.r.t. the ICM.

Since $\omega_n = \sum_{\mathcal{T} \subseteq \mathcal{N}} \pi_{ICM}(\mathcal{T}) \cdot \mathbb{I}(n \in \mathcal{T}) = p_n$ $\forall n \in \mathcal{N}$, then such indexing rule is exactly an MCI. For any subset $\mathcal{T} \subseteq \mathcal{N}$ and two distinct elements $l, m \in \mathcal{N} \backslash \mathcal{T}$ with $l < m$. Let $\mathcal{T}' = \mathcal{T} \cup \{l\}$ and $\mathcal{T}'' = \mathcal{T} \cup \{m\}$. Then, we have $p_l \geq p_m$, $1 - p_m \geq 1 - p_l$, and

$$\begin{aligned}
\pi_{ICM}(\mathcal{T}') &= \pi_{ICM}(\mathcal{T} \cup \{l\}) \\
&= p_l \textstyle\prod_{i \in \mathcal{T}} p_i \prod_{j \in \mathcal{N} \backslash \mathcal{T}, j \neq l} (1 - p_j) \\
&= p_l (1 - p_m) \textstyle\prod_{i \in \mathcal{T}} p_i \prod_{j \in \mathcal{N} \backslash \mathcal{T}, j \neq l, j \neq m} (1 - p_j) \\
&\geq p_m (1 - p_l) \textstyle\prod_{i \in \mathcal{T}} p_i \prod_{j \in \mathcal{N} \backslash \mathcal{T}, j \neq l, j \neq m} (1 - p_j) \\
&= \pi_{ICM}(\mathcal{T} \cup \{m\}) = \pi_{ICM}(\mathcal{T}'').
\end{aligned}$$

In a similar sense, we can iteratively pick a pair of elements (i_l, i_m) with $i_l, i_m \in \mathcal{N} \backslash \mathcal{T}$ and $i_l < i_m$, and verify that $\pi_{ICM}(\mathcal{T}' \cup \{i_l\}) \geq \pi_{ICM}(\mathcal{T}'' \cup \{i_m\})$. Thus, we conclude this indexing rule is dominant w.r.t the ICM.

For the second half, it holds directly from Theorem 3 as we find the MCI is dominant w.r.t. the ICM. □

Proof of Proposition 7 For the first half, we claim that indexing products such that $U_1 \succeq_1 U_2 \succeq_1 \cdots \succeq_1 U_N$ is dominant w.r.t. the MC-RIUM.

The choice probability of a set $\mathcal{T} \subseteq \mathcal{N}$ under MC-RIUM can be represented as

$$\begin{aligned}
\pi_{MC-RIUM}(\mathcal{T}) &= \mathbb{P}\left(Q = |\mathcal{T}|, \min_{i\in\mathcal{T}} U_i > \max_{j\in\mathcal{N}^+\setminus\mathcal{T}} U_j\right) + \mathbb{P}\left(Q > |\mathcal{T}|, \min_{i\in\mathcal{T}} U_i > U_0, U_0 > \max_{k\in\mathcal{N}\setminus\mathcal{T}} U_k\right) \\
&= \mathbb{P}\left(Q = |\mathcal{T}|\right) \cdot \mathbb{P}\left(\min_{i\in\mathcal{T}} U_i > \max_{j\in\mathcal{N}^+\setminus\mathcal{T}} U_j\right) + \mathbb{P}\left(Q > |\mathcal{T}|\right) \cdot \mathbb{P}\left(\min_{i\in\mathcal{T}} U_i > U_0, U_0 > \max_{k\in\mathcal{N}\setminus\mathcal{T}} U_k\right) \\
&= \mathbb{P}(Q = |\mathcal{T}|) \cdot \mathbb{P}\left(\mathcal{T} = \operatorname*{arg\,max}_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|} \sum_{i\in\mathcal{S}} U_i\right) \\
&\quad + \mathbb{P}\left(Q > |\mathcal{T}|\right) \cdot \mathbb{P}\left(\mathcal{T} = \operatorname*{arg\,max}_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|} \sum_{i\in\mathcal{S}} U_i, \mathcal{T}\cup\{0\} = \operatorname*{arg\,max}_{\mathcal{S}'\subseteq\mathcal{N}^+, |\mathcal{S}'|=|\mathcal{T}|+1} \sum_{i\in\mathcal{S}'} U_i\right) \\
&= \mathbb{P}\left(Q = |\mathcal{T}|\right) \cdot \mathbb{P}\left(\sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|} \sum_{i\in\mathcal{S}} U_i\right) \\
&\quad + \mathbb{P}\left(Q > |\mathcal{T}|\right) \cdot \mathbb{P}\left(\sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|} \sum_{i\in\mathcal{S}} U_i, \sum_{t\in\mathcal{T}} U_t + U_0 = \max_{\mathcal{S}'\subseteq\mathcal{N}^+, |\mathcal{S}'|=|\mathcal{T}|+1} \sum_{i\in\mathcal{S}'} U_i\right).
\end{aligned}$$

Given that $U_1 \succeq_1 U_2 \succeq_1 \cdots \succeq_1 U_N$, if $U_l \succeq_1 U_m$, we have

$$\begin{aligned}
\omega_l &= \sum_{\mathcal{T}\subseteq\mathcal{N}} \pi_{MC-RIUM}(\mathcal{T}) \cdot \mathbb{I}(l \in \mathcal{T}) \\
&= \sum_{\mathcal{T}\subseteq\mathcal{N}, l\in\mathcal{T}} \Bigg[\mathbb{P}\left(Q = |\mathcal{T}|\right) \cdot \mathbb{P}\left(\sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|} \sum_{i\in\mathcal{S}} U_i\right) \\
&\quad + \mathbb{P}\left(Q > |\mathcal{T}|\right) \cdot \mathbb{P}\left(\sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|} \sum_{i\in\mathcal{S}} U_i, \sum_{t\in\mathcal{T}} U_t + U_0 = \max_{\mathcal{S}'\subseteq\mathcal{N}^+, |\mathcal{S}'|=|\mathcal{T}|+1} \sum_{i\in\mathcal{S}'} U_i\right)\Bigg] \\
&= \sum_{\mathcal{T}\subseteq\mathcal{N}\setminus\{l,m\}} \Bigg[\mathbb{P}(Q = |\mathcal{T}|+1) \cdot \mathbb{P}\left(U_l + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+1} \sum_{i\in\mathcal{S}} U_i\right) \\
&\quad + \mathbb{P}(Q = |\mathcal{T}|+2) \cdot \mathbb{P}\left(U_l + U_m + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+2} \sum_{i\in\mathcal{S}} U_i\right) \\
&\quad + \mathbb{P}(Q > |\mathcal{T}|+1) \cdot \mathbb{P}\left(\begin{array}{l} U_l + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+1} \sum_{i\in\mathcal{S}} U_i, \\ U_l + \sum_{t\in\mathcal{T}} U_t + U_0 = \max_{\mathcal{S}'\subseteq\mathcal{N}^+, |\mathcal{S}'|=|\mathcal{T}|+2} \sum_{i\in\mathcal{S}'} U_i \end{array}\right) \\
&\quad + \mathbb{P}(Q > |\mathcal{T}|+2) \cdot \mathbb{P}\left(\begin{array}{l} U_l + U_m + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+2} \sum_{i\in\mathcal{S}} U_i, \\ U_l + U_m + \sum_{t\in\mathcal{T}} U_t + U_0 = \max_{\mathcal{S}'\subseteq\mathcal{N}^+, |\mathcal{S}'|=|\mathcal{T}|+3} \sum_{i\in\mathcal{S}'} U_i) \end{array}\right)\Bigg] \\
&\geq \sum_{\mathcal{T}\subseteq\mathcal{N}\setminus\{l,m\}} \Bigg[\mathbb{P}(Q = |\mathcal{T}|+1) \cdot \mathbb{P}\left(U_m + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+1} \sum_{i\in\mathcal{S}} U_i\right) \\
&\quad + \mathbb{P}(Q = |\mathcal{T}|+2) \cdot \mathbb{P}\left(U_l + U_m + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+2} \sum_{i\in\mathcal{S}} U_i\right) \\
&\quad + \mathbb{P}(Q > |\mathcal{T}|+1) \cdot \mathbb{P}\left(\begin{array}{l} U_m + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+1} \sum_{i\in\mathcal{S}} U_i, \\ U_m + \sum_{t\in\mathcal{T}} U_t + U_0 = \max_{\mathcal{S}'\subseteq\mathcal{N}^+, |\mathcal{S}'|=|\mathcal{T}|+2} \sum_{i\in\mathcal{S}'} U_i \end{array}\right) \\
&\quad + \mathbb{P}(Q > |\mathcal{T}|+2) \cdot \mathbb{P}\left(\begin{array}{l} U_l + U_m + \sum_{t\in\mathcal{T}} U_t = \max_{\mathcal{S}\subseteq\mathcal{N}^+, |\mathcal{S}|=|\mathcal{T}|+2} \sum_{i\in\mathcal{S}} U_i, \\ U_l + U_m + \sum_{t\in\mathcal{T}} U_t + U_0 = \max_{\mathcal{S}'\subseteq\mathcal{N}^+, |\mathcal{S}'|=|\mathcal{T}|+3} \sum_{i\in\mathcal{S}'} U_i) \end{array}\right)\Bigg] \\
&= \omega_m.
\end{aligned}$$

Hence, $U_l \succeq_1 U_m$ implies $\omega_l \geq \omega_m$ $\forall l, m \in \mathcal{N}$ and $l \neq m$. So, indexing products such that $U_1 \succeq_1 U_2 \succeq_1 \cdots \succeq_1 U_N$ is an MCI. For any subset $\mathcal{T} \subseteq \mathcal{N}$ with $|\mathcal{T}| = k$ for some non-negative integer k and two distinct elements $l, m \in \mathcal{N}\setminus\mathcal{T}$ with

$l < m$. Let $\mathcal{T}' = \mathcal{T} \cup \{l\}$ and $\mathcal{T}'' = \mathcal{T} \cup \{m\}$. Then, we have $U_l \succeq_1 U_m$ and

$$\begin{aligned}
\pi_{MC-RIUM}(\mathcal{T}') &= \pi_{MC-RIUM}(\mathcal{T} \cup \{l\}) \\
&= \mathbb{P}(Q = k+1) \cdot \mathbb{P}\left(\sum_{t \in \mathcal{T} \cup \{l\}} U_t = \max_{\mathcal{S} \subseteq \mathcal{N}^+, |\mathcal{S}| = k+1} \sum_{i \in \mathcal{S}} U_i \right) \\
&\quad + \mathbb{P}(Q > k+1) \cdot \mathbb{P}\left(\begin{array}{l} \displaystyle\sum_{t \in \mathcal{T} \cup \{l\}} U_t = \max_{\mathcal{S} \subseteq \mathcal{N}^+, |\mathcal{S}| = k+1} \sum_{i \in \mathcal{S}} U_i, \\ \displaystyle\sum_{t \in \mathcal{T} \cup \{l\}} U_t + U_0 = \max_{\mathcal{S}' \subseteq \mathcal{N}^+, |\mathcal{S}'| = k+2} \sum_{i \in \mathcal{S}'} U_i \end{array} \right) \\
&= \mathbb{P}(Q = k+1) \cdot \mathbb{P}\left(U_l + \sum_{t \in \mathcal{T}} U_t = \max_{\mathcal{S} \subseteq \mathcal{N}^+, |\mathcal{S}| = k+1} \sum_{i \in \mathcal{S}} U_i \right) \\
&\quad + \mathbb{P}(Q > k+1) \cdot \mathbb{P}\left(\begin{array}{l} \displaystyle U_l + \sum_{t \in \mathcal{T}} U_t = \max_{\mathcal{S} \subseteq \mathcal{N}^+, |\mathcal{S}| = k+1} \sum_{i \in \mathcal{S}} U_i, \\ \displaystyle U_l + \sum_{t \in \mathcal{T}} U_t + U_0 = \max_{\mathcal{S}' \subseteq \mathcal{N}^+, |\mathcal{S}'| = k+2} \sum_{i \in \mathcal{S}'} U_i \end{array} \right) \\
&\geq \mathbb{P}(Q = k+1) \cdot \mathbb{P}\left(U_m + \sum_{t \in \mathcal{T}} U_t = \max_{\mathcal{S} \subseteq \mathcal{N}^+, |\mathcal{S}| = k+1} \sum_{i \in \mathcal{S}} U_i \right) \\
&\quad + \mathbb{P}(Q > k+1) \cdot \mathbb{P}\left(\begin{array}{l} \displaystyle U_m + \sum_{t \in \mathcal{T}} (V_t + \epsilon_t) = \max_{\mathcal{S} \subseteq \mathcal{N}^+, |\mathcal{S}| = k+1} \sum_{i \in \mathcal{S}} U_i, \\ \displaystyle U_m + \sum_{t \in \mathcal{T}} V_t + \epsilon_t + U_0 = \max_{\mathcal{S}' \subseteq \mathcal{N}^+, |\mathcal{S}'| = k+2} \sum_{i \in \mathcal{S}'} U_i \end{array} \right) \\
&= \pi_{MC-RIUM}(\mathcal{T} \cup \{m\}) = \pi_{MC-RIUM}(\mathcal{T}'').
\end{aligned}$$

In a similar sense, we can repeatedly pick a pair of elements (i_l, i_m) with $i_l, i_m \in \mathcal{N} \backslash \mathcal{T}$ and $i_l < i_m$, and verify that $\pi_{MC-RIUM}(\mathcal{T}' \cup \{i_l\}) \geq \pi_{MC-RIUM}(\mathcal{T}'' \cup \{i_m\})$. Thus, we conclude this indexing rule is dominant w.r.t the MC-RIUM.

For the second half, it holds directly from Theorem 3 as we find the MCI is dominant w.r.t. the MC-RIUM. $\square$

Proof of Corollary 2 Assume that random utilities have the form $U_n = V_n + \epsilon_n \ \forall n \in \mathcal{N}^+$ and $\epsilon_1, \ldots, \epsilon_N$ are independent and identically distributed. For any $l, m \in \mathcal{N}$, if $V_l \geq V_m$, then we have

$$\mathbb{P}(U_l \leq u) = \mathbb{P}(V_l + \epsilon_l \leq u) \leq \mathbb{P}(V_m + \epsilon_l \leq u) = \mathbb{P}(U_m \leq u).$$

By the definition of first-order stochastic dominance, the decreasing order of the utility in the FSD sense is equivalent to the decreasing order of the deterministic utility.

Thus, according to Proposition 7, indexing products such that $V_1 \geq V_2 \geq \ldots V_N$ is an MCI and is dominant w.r.t. the MC-RIUM. $\square$

Proof of Proposition 8 For the first half, if $\beta_{il} \geq \beta_{im} \ \forall i \in \mathcal{N} \backslash \{l, m\}$ for any $l, m \in \mathcal{N}$ with $V_l \geq V_m$, we claim that indexing products such that $V_1 \geq V_2 \geq \cdots \geq V_N$ is dominant w.r.t. the BundleMVL-L model.

The choice probability of a set $\mathcal{T} \subseteq \mathcal{N}$ under BundleMVL-L model can be represented as $\pi_{BundleMVL-L}(\mathcal{T}) = \frac{V_{\mathcal{T}}}{1 + \sum_{\mathcal{T}' \subseteq \mathcal{N}, |\mathcal{T}'| \leq L} V_{\mathcal{T}'}}$, where $V_{\mathcal{T}} = \exp\left(\sum_{i \in \mathcal{T}} V_i + \sum_{i,j \in \mathcal{T}, i<j} \beta_{ij} \right)$. Given $\beta_{il} \geq \beta_{im} \ \forall i \in \mathcal{N} \backslash \{l, m\}$ for any $l, m \in \mathcal{N}$

with $V_l \geq V_m$, if $V_l \geq V_m$, we have

$$\begin{aligned}
\omega_l &= \sum_{\mathcal{T}\subseteq\mathcal{N},|\mathcal{T}|\leq L} \pi_{BundleMVL-L}(\mathcal{T})\cdot\mathbb{I}(l\in\mathcal{T})\\
&= \sum_{\mathcal{T}\subseteq\mathcal{N},|\mathcal{T}|\leq L,l\in\mathcal{T}} \frac{\exp\left(\sum_{i\in\mathcal{T}}V_i+\sum_{i,j\in\mathcal{T},i<j}\beta_{ij}\right)}{1+\sum_{\mathcal{T}'\subseteq\mathcal{N},|\mathcal{T}'|\leq L}\exp\left(\sum_{i\in\mathcal{T}'}V_i+\sum_{i,j\in\mathcal{T}',i<j}\beta_{ij}\right)}\\
&= \sum_{\mathcal{T}\subseteq\mathcal{N}\backslash\{l,m\},|\mathcal{T}|\leq L-1} \frac{\exp\left((V_l+\sum_{j\in\mathcal{T},j\neq l}\beta_{jl})+\sum_{i\in\mathcal{T}}V_i+\sum_{i,j\in\mathcal{T},i<j}\beta_{ij}\right)}{1+\sum_{\mathcal{T}'\subseteq\mathcal{N},|\mathcal{T}'|\leq L}\exp\left(\sum_{i\in\mathcal{T}'}V_i+\sum_{i,j\in\mathcal{T}',i<j}\beta_{ij}\right)}\\
&\quad+\sum_{\mathcal{T}\subseteq\mathcal{N}\backslash\{l,m\},|\mathcal{T}|\leq L-2} \frac{\exp\left((V_l+\sum_{j\in\mathcal{T},j\neq l}\beta_{jl}+V_m+\sum_{j\in\mathcal{T},j\neq m}\beta_{jm})+\sum_{i\in\mathcal{T}}V_i+\sum_{i,j\in\mathcal{T},i<j}\beta_{ij}\right)}{1+\sum_{\mathcal{T}'\subseteq\mathcal{N},|\mathcal{T}'|\leq L}\exp\left(\sum_{i\in\mathcal{T}'}V_i+\sum_{i,j\in\mathcal{T}',i<j}\beta_{ij}\right)}\\
&\geq \sum_{\mathcal{T}\subseteq\mathcal{N}\backslash\{l,m\},|\mathcal{T}|\leq L-1} \frac{\exp\left((V_m+\sum_{j\in\mathcal{T},j\neq m}\beta_{jm})+\sum_{i\in\mathcal{T}}V_i+\sum_{i,j\in\mathcal{T},i<j}\beta_{ij}\right)}{1+\sum_{\mathcal{T}'\subseteq\mathcal{N},|\mathcal{T}'|\leq L}\exp\left(\sum_{i\in\mathcal{T}'}V_i+\sum_{i,j\in\mathcal{T}',i<j}\beta_{ij}\right)}\\
&\quad+\sum_{\mathcal{T}\subseteq\mathcal{N}\backslash\{l,m\},|\mathcal{T}|\leq L-2} \frac{\exp\left((V_l+\sum_{j\in\mathcal{T},j\neq l}\beta_{jl}+V_m+\sum_{j\in\mathcal{T},j\neq m}\beta_{jm})+\sum_{i\in\mathcal{T}}V_i+\sum_{i,j\in\mathcal{T},i<j}\beta_{ij}\right)}{1+\sum_{\mathcal{T}'\subseteq\mathcal{N},|\mathcal{T}'|\leq L}\exp\left(\sum_{i\in\mathcal{T}'}V_i+\sum_{i,j\in\mathcal{T}',i<j}\beta_{ij}\right)}\\
&= \omega_m.
\end{aligned}$$

Hence, $V_l \geq V_m$ implies $\omega_l \geq \omega_m$ $\forall l,m\in\mathcal{N}$ and $l\neq m$. So, indexing products such that $V_1\geq V_2\geq\cdots\geq V_N$ is an MCI. For any subset $\mathcal{T}\subseteq\mathcal{N}$ with $|\mathcal{T}|=k\leq L-1$ for some non-negative integer k and two distinct elements $l,m\in\mathcal{N}\backslash\mathcal{T}$ with $l<m$. Let $\mathcal{T}'=\mathcal{T}\cup\{l\}$ and $\mathcal{T}''=\mathcal{T}\cup\{m\}$. Then, we have $V_l\geq V_m$ and

$$\begin{aligned}
\pi_{BundleMVL-L}(\mathcal{T}') = \pi_{BundleMVL-L}(\mathcal{T}\cup\{l\}) &= \frac{\exp\left((V_l+\sum_{j\in\mathcal{T},j\neq l}\beta_{jl})+\sum_{i\in\mathcal{T}}V_i+\sum_{i,j\in\mathcal{T},i<j}\beta_{ij}\right)}{1+\sum_{\mathcal{R}\subseteq\mathcal{N},|\mathcal{R}|\leq L}\exp\left(\sum_{i\in\mathcal{R}}V_i+\sum_{i,j\in\mathcal{R},i<j}\beta_{ij}\right)}\\
&\geq \frac{\exp\left((V_m+\sum_{j\in\mathcal{T},j\neq m}\beta_{jm})+\sum_{i\in\mathcal{T}}V_i+\sum_{i,j\in\mathcal{T},i<j}\beta_{ij}\right)}{1+\sum_{\mathcal{R}\subseteq\mathcal{N},|\mathcal{R}|\leq L}\exp\left(\sum_{i\in\mathcal{R}}V_i+\sum_{i,j\in\mathcal{R},i<j}\beta_{ij}\right)}\\
&= \pi_{BundleMVL-L}(\mathcal{T}\cup\{m\}) = \pi_{BundleMVL-L}(\mathcal{T}'').
\end{aligned}$$

In a similar sense, we can repeatedly pick a pair of elements (i_l,i_m) with $i_l,i_m\in\mathcal{N}\backslash\mathcal{T}$ and $i_l<i_m$, and verify that $\pi_{BundleMVL-L}(\mathcal{T}'\cup\{i_l\})\geq\pi_{BundleMVL-L}(\mathcal{T}''\cup\{i_m\})$. Thus, we conclude this indexing rule is dominant w.r.t the BundleMVL-L model provided $\beta_{il}\geq\beta_{im}$ $\forall i\in\mathcal{N}\backslash\{l,m\}$ for any $l,m\in\mathcal{N}$ with $V_l\geq V_m$.

For the second half, it holds from Theorem 3 as we find the MCI is dominant w.r.t. the BundleMVL-L model provided $\beta_{il}\geq\beta_{im}$ $\forall i\in\mathcal{N}\backslash\{l,m\}$ for any $l,m\in\mathcal{N}$ with $V_l\geq V_m$. $\square$

EC.2. A Method Utilizing Benders Decomposition for Solving (CP) with Type-I Cost Functions

First, we reformulate Problem (3) as follows

$$\begin{aligned}
\max_{\boldsymbol{\xi},\boldsymbol{\lambda}}\ & \sum_{m=1}^{M}\pi_m g_m\lambda_m,\\
\text{s.t. } & \lambda_m\leq h_m(\boldsymbol{\xi}), \quad \forall m\in[M],\\
& \sum_{n=1}^{N}\xi_n=K,\\
& \xi_n\in\{0,1\}, \qquad \forall n\in[N],
\end{aligned} \tag{EC.7}$$

where subproblem $h_m(\boldsymbol{\xi})$ is defined as follows

$$\begin{aligned} h_m(\boldsymbol{\xi}) = \max_{\zeta_m} \ & \zeta_m, \\ \text{s.t. } & \zeta_m \le \xi_n, \ \forall n \in \mathcal{T}_m, \\ & \zeta_m \ge 0. \end{aligned} \tag{EC.8}$$

Similar to the analysis of the MILP (3), when $\boldsymbol{\xi}$ is binary, the constraints in Problem (EC.8) automatically forces $\boldsymbol{\zeta}$ to be the correct binary value. Let $A^{(m)}(\boldsymbol{\xi}) = \{\zeta | \zeta \le \xi_n \ \forall n \in \mathcal{T}_m, \ \zeta \ge 0\}$ be the feasible set of ζ_m for the subProblem (EC.8), then we can rewrite the main Problem (EC.7) as

$$\begin{aligned} \max_{\boldsymbol{\xi},\boldsymbol{\lambda}} \ & \sum_{m=1}^{M} \pi_m g_m \lambda_m, \\ \text{s.t. } & \lambda_m \le \zeta_m, \qquad \forall m \in [M], \ \zeta_m \in A^m(\boldsymbol{\xi}), \\ & \sum_{n=1}^{N} \xi_n = K, \\ & \xi_n \in \{0,1\}, \qquad \forall n \in [N]. \end{aligned} \tag{EC.9}$$

According to the straightforward structure of subProblem (EC.8), we can easily derive an optimal solution. We present this formally in the following proposition.

PROPOSITION EC.1. *Given $\boldsymbol{\xi} \in \{0,1\}^N$, the optimal solution of Problem (EC.8) for order $m \in [M]$ with the assortment represented by $\boldsymbol{\xi}$ is*

$$\zeta_m = \begin{cases} 0 \ \textit{if} \ \min_{j \in \mathcal{T}_m} \xi_j < 1, \\ 1 \ \textit{otherwise}. \end{cases}$$

Proposition EC.1 provides us with a simple approach to test whether an integer solution $(\boldsymbol{\xi}, \boldsymbol{\lambda})$ to Problem (EC.8) violates any constraint or not, i.e., we generate ζ_m according to Proposition EC.1 and compare it with λ_m for each $m \in [M]$. If λ_m is less or equal to ζ_m, then the mth set of constraints remains unviolated; otherwise, if λ_m is strictly larger than ζ_m, then we have identified a violated constraint and add it to the formulation. This procedure is inspired by Bertsimas and Mišić (2019), which proposes a Benders decomposition method to solve the product line design problem under the classic single-purchase rank list model. Note that the structure of Problem (3) is simpler than that presented in Bertsimas and Mišić (2019), so applying Benders decomposition to our problem is expected be more efficient.

EC.3. Supplementary Materials for Numerical Experiments

EC.3.1. Supplementary Materials for Section 6.1

In-sample and out-of-sample comparisons of order fill rates under different methods for RDCs are shown in Table EC.1 and Table EC.2. In-sample and out-of-sample comparisons of order fill rates under different methods for CDCs are shown in Table EC.3 and Table EC.4.

Table EC.1 In-Sample OFR Comparison Results for RDCs

DC Code	# SKUs	SKU Cap	# Diff Orders	Current OFR (%)	OPT OFR (%)	MCIP OFR (%)	Avg RP OFR (%)	OPT IMP Current (%)	MCIP IMP Current (%)	$\frac{ORF_{MCIP}}{ORF_{OPT}}$ (%)	MILP Time (s)
RRSZX001	3985	3601	3901	98.84	99.7	99.6	89.22	0.87	0.76	99.9	2.18
RRSZX002	2915	1932	2730	88.04	97.14	96.82	63.05	10.34	9.98	99.67	1.01
RRSZX003	3323	2441	3205	93.24	98.59	98.48	70.49	5.74	5.62	99.89	0.82
RRSZX004	3463	2525	3410	94.39	98.76	98.58	71.18	4.63	4.44	99.82	0.89
RRSZX020	3256	2580	3204	96.61	99.21	99.16	78.53	2.68	2.63	99.95	0.85
RRSZX022	47	31	26	66.1	86.44	84.75	46.02	30.77	28.21	98.04	0.14
RRSZX023	922	658	852	96.39	98.57	98.44	70.41	2.26	2.13	99.87	0.29
RRSZX033	2992	1905	2940	92.82	97.91	97.76	62.17	5.49	5.32	99.84	0.87
RRSZX038	3084	1911	3054	83.94	96.64	96.52	59.62	15.13	14.98	99.88	0.88
RRSZX039	1436	1041	1313	94.57	99.34	99.17	70.92	5.04	4.86	99.83	0.42
RRSZX040	1148	1049	1044	99.16	99.8	99.71	90.72	0.64	0.55	99.9	0.33
RRSZX041	4279	3054	4335	94.85	99.0	98.94	67.94	4.38	4.31	99.93	1.12
RRSZX043	4128	3102	4140	96.59	99.35	99.27	72.13	2.86	2.77	99.92	1.67
RRSZX044	3630	2720	3571	95.22	98.78	98.71	71.91	3.74	3.66	99.92	1.19
RRSZX045	3748	2792	3772	95.21	98.87	98.69	71.98	3.84	3.66	99.82	1.22
RRSZX058	4363	3230	4515	97.11	99.44	99.38	72.34	2.4	2.34	99.94	1.88
RRSZX059	4444	3410	4568	97.03	99.52	99.46	74.33	2.57	2.51	99.94	1.72
RRSZX060	4061	3018	4170	97.29	99.38	99.29	72.05	2.15	2.06	99.9	1.39
RRSZX073	3002	1986	3026	91.06	97.54	97.23	64.51	7.12	6.78	99.68	1.47
RRSZX074	3055	2070	3138	93.35	98.36	98.23	66.78	5.37	5.23	99.86	1.16
RRSZX082	3475	2298	3535	93.91	98.43	98.3	65.0	4.81	4.68	99.87	1.31
RRSZX083	3655	2568	3645	95.26	98.9	98.78	68.85	3.82	3.69	99.88	1.25
RRSZX084	3988	2960	3937	95.79	99.21	99.09	72.82	3.56	3.44	99.88	1.57
RRSZX094	1597	957	1569	84.46	94.86	94.5	58.49	12.32	11.9	99.62	0.6
RRSZX095	2879	1884	2817	92.21	97.67	97.39	64.51	5.93	5.63	99.72	0.92
RRSZX110	4108	3189	3911	95.72	98.93	98.88	74.96	3.36	3.31	99.95	1.04

Table EC.2 Out-of-Sample OFR Comparison Results for RDCs

DC Code	# SKUs	SKU Cap	# Diff Orders	HS OPT OFR (%)	Test OPT OFR (%)	Test MCIP OFR (%)	Test MCIP IMP Over Test OPT (%)
RRSZX001	3985	3601	3505	100.0	98.63	98.85	0.22
RRSZX002	2915	1932	2323	99.63	95.44	95.49	0.05
RRSZX003	3323	2441	2764	99.89	96.53	95.9	-0.65
RRSZX004	3463	2525	2868	99.91	96.83	97.21	0.4
RRSZX020	3256	2580	2685	100.0	97.95	98.0	0.04
RRSZX022	47	31	13	100.0	48.91	42.39	-13.33
RRSZX023	922	658	686	99.82	97.48	97.72	0.25
RRSZX033	2992	1905	2465	99.66	95.16	95.89	0.77
RRSZX038	3084	1911	2658	99.33	94.64	94.98	0.35
RRSZX039	1436	1041	1058	99.95	96.81	96.92	0.11
RRSZX040	1148	1049	904	100.0	98.33	98.66	0.34
RRSZX041	4279	3054	3986	99.88	97.78	97.52	-0.28
RRSZX043	4128	3102	3753	99.95	98.23	97.81	-0.44
RRSZX044	3630	2720	3287	99.92	96.58	96.53	-0.05
RRSZX045	3748	2792	3534	99.92	96.68	95.98	-0.73
RRSZX058	4363	3230	3846	99.98	98.42	98.26	-0.15
RRSZX059	4444	3410	3868	100.0	98.16	98.05	-0.11
RRSZX060	4061	3018	3595	99.98	98.03	97.97	-0.06
RRSZX073	3002	1986	2676	99.56	93.35	94.5	1.23
RRSZX074	3055	2070	2760	99.75	96.45	96.64	0.2
RRSZX082	3475	2298	3114	99.69	95.99	96.12	0.13
RRSZX083	3655	2568	3251	99.86	96.15	96.15	-0.0
RRSZX084	3988	2960	3422	99.95	97.72	97.38	-0.35
RRSZX094	1597	957	1264	99.22	93.17	93.4	0.25
RRSZX095	2879	1884	2415	99.7	96.56	96.29	-0.28
RRSZX110	4108	3189	3320	100.0	97.64	97.65	0.01

Table EC.3 In-Sample OFR Comparison Results for CDCs

DC Code	# SKUs	SKU Cap	# Diff Orders	Current OFR (%)	OPT OFR (%)	MCIP OFR (%)	Avg RP OFR (%)	OPT IMP Current (%)	MCIP IMP Current (%)	$\frac{ORF_{MCIP}}{ORF_{OPT}}$ (%)	MILP Time (s)
RRSZX031	4410	3973	4631	99.22	99.8	99.74	89.43	0.58	0.52	99.94	1.55
RRSZX042	3981	3680	4021	98.68	99.82	99.72	91.33	1.16	1.05	99.9	1.58
RRSZX056	4084	3722	4136	99.22	99.85	99.79	90.35	0.64	0.58	99.94	1.67
RRSZX072	3838	3420	3888	97.4	99.64	99.55	87.42	2.3	2.2	99.91	1.61
RRSZX081	4087	3870	4042	99.54	99.91	99.83	93.96	0.38	0.29	99.92	1.55
RRSZX093	4030	3659	4011	98.65	99.74	99.67	89.82	1.11	1.04	99.93	1.06
RRSZX104	3648	2754	3494	95.75	99.03	98.92	72.31	3.43	3.32	99.9	0.84

Table EC.4 Out-of-Sample OFR Comparison Results for CDCs

DC Code	# SKUs	SKU Cap	# Diff Orders	HS OPT OFR (%)	Test OPT OFR (%)	Test MCIP OFR (%)	Test MCIP IMP Over Test OPT (%)
RRSZX031	4410	3973	4144	100.0	98.25	98.1	-0.15
RRSZX042	3981	3680	3651	100.0	99.06	99.26	0.2
RRSZX056	4084	3722	3634	100.0	99.66	99.49	-0.17
RRSZX072	3838	3420	3577	100.0	98.17	98.07	-0.1
RRSZX081	4087	3870	3480	100.0	99.68	99.84	0.16
RRSZX093	4030	3659	3635	100.0	99.25	98.53	-0.73
RRSZX104	3648	2754	3103	99.94	96.35	96.8	0.47

EC.3.2. Supplementary Materials for Section 6.2

Detail of the chosen cities is listed in Table EC.5.

Table EC.5 Details of Chosen Cities

	City Code	# SKU	# Distinct Orders	Avg MILP Solving Time
City 1	b3bbd6bcd84bc84f8bb874d96cee51e6	5143	5554	1.54
City 2	e7e6252a02709c4f1bfab796ebd3efe2	5140	5663	1.60
City 3	2942fa707f340db57611c88fca53a211	4952	5264	1.47
City 4	08641489dbf16de3f0fbdb8095d8721d	4984	5268	1.47
City 5	b0cadecbe35d998f1758ff45f23dffbb	4841	4910	1.35

Figures of OFR comparisons, PFR comparisons, and the ratios comparisons for City 2 to 5 are shown in Figure EC.1 to Figure EC.12.

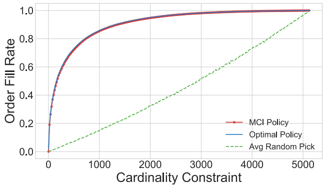

Figure EC.1 City 2 OFRs

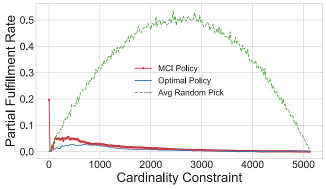

Figure EC.2 City 2 PFRs

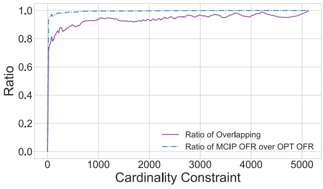

Figure EC.3 City 2 Ratios

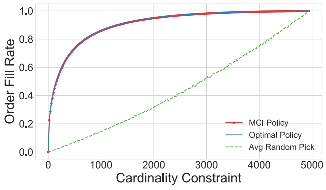

Figure EC.4 City 3 OFRs

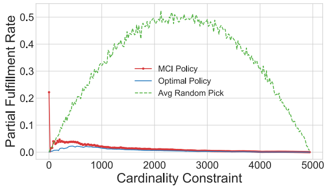

Figure EC.5 City 3 PFRs

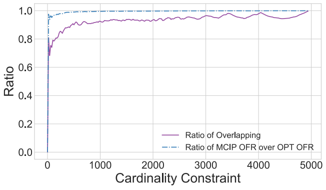

Figure EC.6 City 3 Ratios

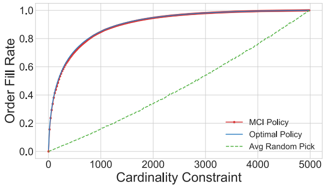

Figure EC.7 City 4 OFRs

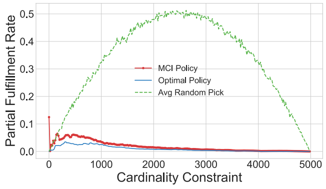

Figure EC.8 City 4 PFRs

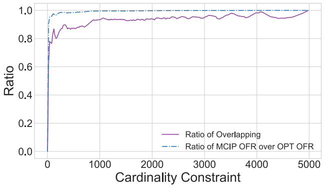

Figure EC.9 City 4 Ratios

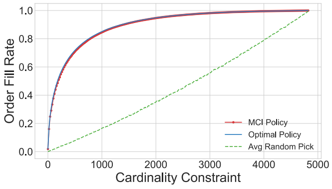

Figure EC.10 City 5 OFRs

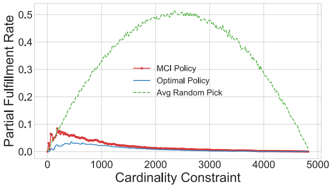

Figure EC.11 City 5 PFRs

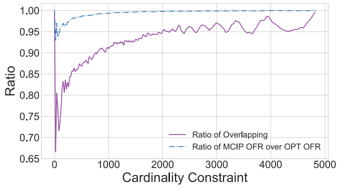

Figure EC.12 City 5 Ratios

EC.3.3. Experiments on General Cost Functions

We further test the MCI policy on some general cost functions. The experiment settings are the same as that in Section 6.1.

EC.3.3.1. Size-Based Cost Functions We fist consider both the type-I and type-II size-based cost functions with $c(n) = n^{\alpha}\ \forall n \in \{1, 2, \dots\}$, where $\alpha = 1/3, 1/2, 1, 2$. Table EC.6 reports the average cost reduction ratio of applying the MCI policy and the MILP optimal solution compared to the current policy at different distribution centers. It demonstrates the high potential for additional fulfillment cost reduction by carefully selecting the warehouse assortment for all (either concave, linear, or convex) cost functions. Specifically, the MCI policy results in a significant average cost reduction ranging from 33.76% to 80.44% under both types of cost functions. Indeed, as is shown in Proposition 3, when the cost function is type-II size-based with $\alpha = 1$, the MCI policy is optimal. Besides, we find that for most cases, as α becomes smaller (from 2 to 1/3), the average cost reduction by applying the MCI policy becomes larger under type-I size-based cost functions; while the reverse is true under type-II size-based cost functions. Note that the more different cost functions are from linear functions, the larger gap the MCI policy performs in comparison to the MILP optimal solution. However, although formulation (3) can be solved within seconds considering type-I cost functions, it takes much longer (more than half an hour) for solving (CP) with type-II cost functions through MILP formulation (4) when N and M are larger than 3000. In this regard, the MCI policy with near-optimal performance solved in milliseconds is quite acceptable.

Table EC.6 In-Sample Average Cost Reduction Comparison Results

DC Type	Cost Type	α	MCIP AVG Cost Reduction Over Current	OPT AVG Cost Reduction Over Current	DC Type	Cost Type	α	MCIP AVG Cost Reduction Over Current	OPT AVG Cost Reduction Over Current	DC Type	Cost Type	α	MCIP AVG Cost Reduction Over Current	OPT AVG Cost Reduction Over Current
LTC	Type I	0.33	38.04%	43.62%	RDC	Type I	0.33	72.02%	74.49%	CDC	Type I	0.33	72.81%	79.24%
		0.5	37.98%	43.26%			0.5	71.74%	74.01%			0.5	72.42%	78.37%
		1	37.43%	42.12%			1	70.68%	72.97%			1	71.05%	76.59%
		2	33.76%	48.35%			2	66.86%	76.66%			2	66.73%	83.59%
	Type II	0.33	39.18%	44.03%		Type II	0.33	72.55%	75.20%		Type II	0.33	74.15%	80.63%
		0.5	39.73%	43.41%			0.5	72.59%	74.55%			0.5	74.56%	79.50%
		1	41.23%	41.23%			1	72.71%	72.71%			1	76.09%	76.09%
		2	43.63%	49.24%			2	72.98%	77.46%			2	80.44%	82.69%

Moreover, since RDCs usually have more SKU capacities than LTCs and CDCs usually have more SKU capacities than RDCs, Table EC.6 implies that as the SKU capacities become larger, the average cost reduction of using the MCI policy becomes more evident. This finding complements the result in Section 6.1 where the MCI policy results in larger OFR improvement for facilities with smaller SKU capacities.

EC.3.3.2. Type-II Cost Functions with $G(\mathcal{T}) = \sum_{n \in \mathcal{T}} \kappa_n$

In this subsection, we specifically consider the case where the cost function is type-II with $G(\mathcal{T}) = \sum_{n \in \mathcal{T}} \kappa_n$, which is discussed in the beginning of Section 5.1. Specifically, we designate the product-specific additional fulfillment cost κ_n as the volume or weight of the product in our experiments. According to Proposition 4, a modified MCI policy (MMCIP), which selects the K products with the largest choice-weighted fulfillment costs to store, achieves optimality. In these experiments, we engage in a comparative analysis of the cost reduction achieved by applying both MMCIP and the standard MCI policy (MCIP). The results of the comparison are summarized in Table EC.7. We observe that although applying the standard MCI policy leads to reduced additional fulfillment costs in most instances compared to the current practice, the modified MCI policy consistently delivers more substantial cost reductions across all test cases. These findings suggest that when the additional fulfillment cost exhibits a type-II structure and is intricately related to product-specific features, it is important to incorporate these features into the design of the assortment selection policy. Indeed, a minor adjustment to the standard MCI policy can significantly enhance its performance.

Table EC.7 In-Sample Average Type-II Cost with $G(\mathcal{T}) = \sum_{n \in \mathcal{T}} \kappa_n$ Reduction Comparison Results

DC Type	Product-Specific Cost (κ_n) Based on	MCIP AVG Cost Reduction Over Current	MMCIP AVG Cost Reduction Over Current	MMCIP AVG IMP Over MCIP
LTC	volume	-5.38%	65.79%	48.12%
	weight	35.00%	66.25%	45.32%
RDC	volume	56.27%	81.99%	53.17%
	weight	67.27%	82.61%	44.87%
CDC	volume	48.47%	91.47%	74.77%
	weight	76.65%	90.95%	61.89%

Additionally, recall from Table EC.6 that it is verified the MMCIP reduces to the MCIP and achieves optimality when the cost function is type-II with $G(\mathcal{T}) = \sum_{n \in \mathcal{T}} \kappa_n$ and the product-specific additional fulfillment cost is uniform across all products (type-II sized-based with $\alpha = 1$).

Owing to space constraints, all detailed results of the experiments conducted on different cost functions within this subsection are available in `https://docs.google.com/spreadsheets/d/1OUI8j4YfKRGoZqS_LHstto6SItuu0e8QMwClqgBUEZs/edit?usp=sharing`.