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Online Supplement: Optimal consolidation of polling locations

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1. Derivation of constraint set (8)

We employ the technique introduced by Marianov and Serra (2002) to formulate a linear integer chance constraint that reflects our wait time goal for each polling location. By considering each polling location separately, we ensure that the resource allocation decisions are equitable. Assuming the queueing system is in steady state, the probability that the wait time w_j is greater than τ for a random voter entering the queue at polling location $j \in J$ with $m \in \{1, 2, \ldots, c_j\}$ server resources each with a service rate of μ_j is described by (Chan and Lin 2003):

$$P(w_j > \tau) = \Big(\frac{m\rho_m}{m - \lambda'_j/\mu_j}\Big) e^{-(m\mu_j - \lambda'_j)\tau}$$

where

$$\rho_0 = \Big[\sum_{k=0}^{m-1} \frac{(\lambda'_j/\mu_j)^k}{k!} + \frac{(\lambda'_j/\mu_j)^m}{(m-1)!(m - \lambda'_j/\mu_j)}\Big]^{-1}$$

and

$$\rho_k = \Big(\frac{(\lambda'_j/\mu_j)^k}{k!}\Big)\rho_0 \quad \text{for } 1 \le k \le m.$$

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The wait time goal can then be restated as:

$$\Big(\frac{m\rho_m}{m-\lambda'_j/\mu_j}\Big)e^{-(m\mu_j-\lambda'_j)\tau}\leq 1-\alpha.$$

Root-finding methods can identify a parameter $\lambda_{\alpha,m}$ such that the previous inequality holds at equality if m resources are allocated to $j\in J$. Thus, the wait time goal can be restated using the constraint $\sum_{i\in I}\lambda x_{i,j}\leq\lambda_{\alpha,m}$ for all $i\in I$ (Marianov and Serra 2002). Since the number of resources distributed to polling location $j\in J$ is endogenous to the PLCP, the constraint can be formulated as

$$\sum_{i\in I}\lambda_i x_{i,j}\leq y_{j,1}\lambda_{\alpha,1}+\sum_{m=2}^{c_j}y_{jm}(\lambda_{\alpha,m}-\lambda_{\alpha,(m-1)})\quad\forall\, j\in J.$$

2. Proofs for the main text

Proof to Proposition 1. Suppose we have a solution $x\in\{0,1\}^{|I|\times|J|}$, $y\in\{0,1\}^{|J|\times c_j}$ that satisfies constraints (2)-(11). The solution must be valid for the server-resource inequalities. Suppose it is not. Then, for some $i\in I$ and $j\in J$,

$$y_{j,m_j^*}<x_{i,j}$$

Since it is not possible that $y_{j,m_j^*}<x_{i,j}$ when $x_{i,j}=0$, we can assume that $x_{i,j}=1$ and $y_{j,m_j^*}=0$.

By constraint set (6), it must be true that $x_{kj}=1$ for each $k\in I$ such that $j(k)=j$. By constraint set (10), we know that $y_{jm}=0$ for every $m\geq m_j^*$. Thus,

$$\begin{aligned}\sum_{k\in I:j(k)=j}\lambda_k &\leq \sum_{k\in I}\lambda_k x_{k,j}\\ &\leq y_{j,1}\lambda_{\alpha,1}+\sum_{m=2}^{c_j}y_{j,m}(\lambda_{\alpha,m}-\lambda_{\alpha,(m-1)})\\ &\leq \lambda_{\alpha,1}+\sum_{m=2}^{m_j^*-1}(\lambda_{\alpha,m}-\lambda_{\alpha,(m-1)})\\ &\leq \lambda_{\alpha,(m_j^*-1)}\end{aligned}\tag{1}$$

This is a contradiction on the definition of m_j^*. □

Proof to Proposition 2 The objective function (OFV) is non-negative for any feasible solution, since $p_i \geq 0$, $x_{i,j} \geq 0$, and $(d_{i,j} - d_{i,j(i)})^+ \geq 0$ for each $i \in I$ and $j \in J$. If there exists a feasible assignment with an OFV of zero, it must be optimal.

The OFV of the solution defined by x^* is

$$\sum_{i \in I} \sum_{j \in J} p_i (d_{i,j} - d_{i,j(i)})^+ x^*_{i,j} = \sum_{i \in I} p_i (d_{i,j} - d_{i,j(i)})^+ x^*_{i,j(i)} = 0$$

since $(d_{i,j} - d_{i,j(i)})^+ = \max\{0, 0\} = 0$. Thus, it must be optimal. □

Proof to Remark 1 We provide an example in which polling locations are consolidated in all feasible solutions. Suppose there are two population districts, $I = \{A, B\}$, each originally assigned to unique polling locations, $J = \{0, 1\}$. Let $j(A) = 0$ and $j(B) = 1$. Suppose $s_{\max} = 10$, $c_0 = c_1 = 10$, $n = 2$, and $R = \emptyset$. Suppose $\tau = 30$ minutes, $\mu_j = 1.005$ voters/minute, $\alpha = 0.95$, and $\lambda_A = \lambda_B = 5$.

We first show that the assignment $x_{A0} = 1$ and $x_{B1} = 1$ (and zero otherwise) does not satisfy the wait time constraints (8). Note that the value of $P(w > 30)$ is decreasing with an increasing number of server resources m distributed to a polling location. Thus, with $s_{\max} = 10$ we simply need to check whether the wait time constraints (8) are satisfied when 5 server resources are allocated to each polling location, otherwise the wait time constraint is violated for at least one polling location. The following is true:

$$P(w_0 > 30) = P(w_1 > 30) = \Big(\frac{5\rho_5}{5 - 5/1.005}\Big) e^{-30(5 \times 1.005 - 5)} \approx 0.094 > 0.05 = 1 - \alpha$$

Thus, the assignment of polling locations to their standard polling locations is not feasible when $s_{\max} = 10$.

However, suppose both population districts are assigned to the same polling location. Without loss of generality, let this be polling location 0. We then can distribute all 10 server resources to polling location 0 (since $10 \leq c_0$). The probability of waiting longer than 30 minutes is then:

$$P(w_0 > 30) = \Big(\frac{10\rho_{10}}{10 - 10/1.005}\Big) e^{-30(10 \times 1.005 - 10)} \approx 0.022 < 0.05 = 1 - \alpha$$

Thus, this consolidated assignment is feasible for the wait time constraint. □

Proof to Theorem 5 We provide a reduction of the bin packing problem to the PLCP. The bin packing problem is to identify whether a set of f objects of volume v_i can be assigned into q or fewer bins, each having a volume V.

Given an instance of the bin packing problem, we construct an instance to the PLCP. Let $I = \{1,\dots,f\}$, $n = q$, $\lambda_i = v_i$, $\lambda_{\alpha,1} = V$. Also let $J = \{0,1,2,\dots,q\}$ with $c_j = 1$ for each $j \in J$, $\hat{J} = \{0\}$, and $j(i) = 0$ for each $i \in I$. To instantiate the remainder of the parameters of the PLCP, we let $R = \emptyset$, $s_{\max} = \infty$, and $N_i = I \setminus \{i\}$ for each $i \in I$. With these parameter values, the feasible region of the PLCP reduces to

$$\sum_{i \in I} v_i x_{i,j} \leq V y_{j,1} \qquad \forall\ j \in J \setminus \hat{J} \tag{2}$$

$$\sum_{j \in J \setminus \hat{J}} y_{j,1} \leq q \tag{3}$$

$$\sum_{j \in J \setminus \hat{J}} x_{i,j} = 1 \qquad \forall\ i \in I \tag{4}$$

$$x_{i,j} \in \{0,1\} \qquad \forall\ i \in I, j \in J \setminus \hat{J} \tag{5}$$

$$y_{j,1} \in \{0,1\} \qquad \forall\ j \in J \setminus \hat{J} \tag{6}$$

The first constraint set (2) is a result of the wait time constraints (8) of the PLCP while the second constraint (3) is a result of constraint (3) of the PLCP.

Finding a solution that satisfies these constraints is equivalent to finding a solution to the bin packing problem. Thus, if there exists a polynomial time algorithm to determine if the feasible region of the PLCP is non-empty, the same algorithm can be used to solve the bin packing problem. Since the bin packing problem is NP-Complete (Garey 1979), it is NP-Complete to determine if the PLCP has a feasible solution. □

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