Online Appendix for "Truncated Balancing Policy for Perishable Inventory Management: Combating High Shortage Penalties"

Appendix. A. Marginal-Cost Accounting Scheme

As discussed in the main text, the marginal-cost accounting scheme assigns each period the costs that are "caused" by the decision made at this period (Levi et al. 2007). Following this logic, we next present the detailed derivations of the marginal-cost accounting scheme for the perishable inventory setting, and we remark that our definitions of the marginal costs are in line with those presented in Chao et al. (2015).

Marginal Shortage Penalty. Since we consider zero lead time, the marginal shortage penalty at each period is simply defined as the shortage penalty that occurs at this period. For $t = 1, ..., T$, given $\mathbf{x}_t, f_t$ and q_t, let $P_t(\mathbf{x}_t, f_t, q_t)$ denote the expected marginal shortage penalty at period t. Then, we have:

$$P_t(\mathbf{x}_t, f_t, q_t) := \beta^{t-1} p \mathbb{E}[(D_t - y_t)^+ | f_t].$$

Marginal Holding Cost. For $t = 1, ..., T$, given $\mathbf{x}_t$, f_t and q_t, let $H_t(\mathbf{x}_t, f_t, q_t)$ denote the expected marginal holding cost at period t, which is defined as the sum of all expected holding costs charged for units ordered at period t. In the perishable inventory setting, the future holding costs charged for q_t depend on the entire inventory vector $\mathbf{x}_t$. Similar to Nahmias (1975), we let $A_{0,t} = 0$, and for $k = 1, ..., K-1$, let $A_{k,t}$ be the total demand over periods $t, ..., t+k-1$ that cannot be satisfied by the inventory of $(x_{K-k,t}, ..., x_{K-1,t})$, i.e., the inventory that would have been outdated by the end of period $t+k-1$. Then:

$$A_{k,t} = (A_{k-1,t} + D_{t+k-1} - x_{K-k,t})^+, k = 1, ..., K-1.$$

Thus, for $k = 0, ..., K-1$, $\left(A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t}\right)^+$ denotes the total demand over periods $t, ..., t+k$ that cannot be satisfied by the inventory of $\mathbf{x}_t$, and $\left(q_t - \left(A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t}\right)^+\right)^+$ denotes the amount of q_t left in inventory at the end of period $t+k$. Then, the expected marginal holding cost is defined as follows:

$$H_t(\mathbf{x}_t, f_t, q_t) := \sum_{k=0}^{K-1} \beta^{t+k-1} h \mathbb{E}\left[\left(q_t - \left(A_{k,t} + D_{t+k} - \sum_{m=1}^{K-k-1} x_{m,t}\right)^+\right)^+ \middle| f_t\right],$$

where the sum over k is defined up to $T-t$ when $t+K-1 \geq T$.

Marginal Outdating Cost. For $t = 1, ..., T$, given $\mathbf{x}_t$, f_t and q_t, let $W_t(\mathbf{x}_t, f_t, q_t)$ denote the expected marginal outdating cost at period t, which is defined as the sum of all expected outdating costs charged for units ordered at period t, i.e., the expected outdating costs that occur at period $t+K-1$. Since units ordered in periods $T-K+2, ..., T$ will not outdate within the planning horizon, we have $W_t(\mathbf{x}_t, f_t, q_t) = 0$ for $t \geq T-K+2$. For $t \leq T-K+1$, $(q_t - A_{K-1,t} - D_{t+K-1})^+$ represents the amount of q_t that will be outdated at the end of period $t+K-1$. Then, the expected marginal outdating cost is defined as follows:

$$W_t(\mathbf{x}_t, f_t, q_t) := \beta^{t+K-1} w \mathbb{E}[(q_t - A_{K-1,t} - D_{t+K-1})^+ | f_t].$$

Appendix. B. Proofs of Analytical Results

Proof of Lemma 1. We prove this lemma in the following two steps:

Step 1: We first prove part (i) of the lemma by using induction. Part (i) of the lemma clearly holds for the first period in $\mathscr{T}_H$, because by construction, all products will be moved to age zero under policy IM at the first period in $\mathscr{T}_H$. Consider any period $t \in \mathscr{T}_H$. Assume that part (i) of the lemma holds for all periods in $[1,t) \cap \mathscr{T}_H$. We next show that it also holds for period t.

Recall that for any $t \in \mathscr{T}_H$, we let $\{\tau_1, ..., \tau_i\} = [1,t) \cap (\mathscr{T}_{HM} \cup \mathscr{T}_{HL})$, where $\tau_1 < ... < \tau_i$. Suppose $\{1, ..., K-1\} \cap \{t-\tau_j, j = 1, ..., i\}$ is an empty set. Then, part (i) of the lemma trivially holds for period t because by construction, all products under policy IM will be moved to age zero at period t. Without loss of generality, assume $\{1, ..., K-1\} \cap \{t-\tau_j, j = 1, ..., i\}$ is non-empty. Moreover, assume that $t - \tau_1 \leq K - 1$ (this is also without loss of generality because our analysis can be extended to consider $t - \tau_1 > K - 1$ by simply replacing $t - \tau_1$ with the largest $t - \tau_j$ that is less than or equal to $K - 1$). Then, Figure 3 below provides an illustration of different ages at period t.

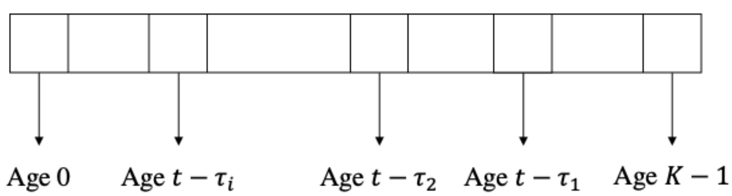

Figure 3 **Illustration of different ages at period t.**

By construction of policy IM, after the movements of units at period $t \in \mathscr{T}_H$, there could only be a positive inventory of age $0, t-\tau_i, t-\tau_{i-1}, ..., t-\tau_1$ under policy IM. Therefore, to prove that Inequality 3 and Equation 4 hold at t, it is sufficient to show that these results hold at t for $k = t-\tau_i, ..., t-\tau_1$.

By induction assumption, both Inequality 3 and Equation 4 hold for all periods in $[1,t) \cap \mathscr{T}_H$. Due to Inequality 3, the inventory under policy IM must be consumed no faster than that under policy TB. Therefore, before the movements of units at period t, we have $\sum_{m=k}^{K-1} x_{m,t}^{IM} \geq x_{k,t}^{B} + \sum_{m=k+1}^{K-1} x_{m,t}^{TB}$ for all $k = t-\tau_i, ..., t-\tau_1$. Then, for $k = t-\tau_1$, we can easily move some units of age k under policy IM to age $k-1$ such that $\sum_{m=k}^{K-1} \check{x}_{m,t}^{IM} = x_{k,t}^{B} + \sum_{m=k+1}^{K-1} x_{m,t}^{TB}$ is achieved before $\check{x}_{k,t}^{IM} = 0$ (because by construction, we have $\check{x}_{m,t}^{IM} = 0$ for all $m \geq t-\tau_1+1$). Therefore, we have that Equation 4 holds for $k = t-\tau_1$ at period t. Moreover, since we have $q_{\tau_1}^{B} \leq q_{\tau_1}^{TB}$ for $\tau_1 \in \mathscr{T}_{HM} \cup \mathscr{T}_{HL}$, we must have $x_{k,t}^{B} \leq x_{k,t}^{TB}$ for $k = t-\tau_1$. Therefore, we have $\sum_{m=k}^{K-1} \check{x}_{m,t}^{IM} \leq \sum_{m=k}^{K-1} x_{m,t}^{TB}$. That is, Inequality 3 holds for $k = t-\tau_1$ at period t. By following the same arguments, it can also be shown that both Inequality 3 and Equation 4 hold for $k = t-\tau_2, ..., t-\tau_i$ at period t.

Step 2: We next prove that $\forall t \in \mathscr{T}_{HU}$, we have $\hat{y}_t^{IM} = y_t^{TB} < y_t^{B}$ after the disposals of units. For any $t \in \mathscr{T}_{HU}$, we have $y_t^{TB} < y_t^{B}$ by definition of $\mathscr{T}_{HU}$. Therefore, it is sufficient to show $\hat{y}_t^{IM} = y_t^{TB}$. Recall that for $t \in \mathscr{T}_{HU}$, we have $y_t^{TB} < y_t^{IM}$ by construction, and we dispose $\min\{\check{x}_{0,t}^{IM}, y_t^{IM} - y_t^{TB}\}$ units of age zero under policy IM after the movements of units. Therefore, to prove $\hat{y}_t^{IM} = y_t^{TB}$, it is sufficient to prove

$\check{x}_{0,t}^{IM} \geq y_t^{IM} - y_t^{TB}$ so that exactly $y_t^{IM} - y_t^{TB}$ units of products are disposed of under policy IM. As shown in Step 1, we must have $\sum_{k=1}^{K-1} \check{x}_{k,t}^{IM} \leq \sum_{k=1}^{K-1} x_{k,t}^{TB}$ after the movements of units. Therefore, we have

$$\check{x}_{0,t}^{IM} = y_t^{IM} - \sum_{k=1}^{K-1} \check{x}_{k,t}^{IM} \geq y_t^{IM} - \sum_{k=1}^{K-1} x_{k,t}^{TB} \geq y_t^{IM} - y_t^{TB}.$$

Then, exactly $y_t^{IM} - y_t^{TB}$ units of products are disposed of under policy IM, we hence $\hat{y}_t^{IM} = y_t^{TB}$. □

Proof of Lemma 2. In order to prove this lemma, we first prove a structural property for the optimal cost-to-go function. At each period t, given $\mathbf{x}_t$ and f_t, let $C_t(\mathbf{x}_t, f_t)$ denote the optimal cost-to-go function. Further, let $C_{T+1}(\mathbf{x}_{T+1}, f_{T+1}) = 0$. Then, the optimality equation is defined as follows:

$$C_t(\mathbf{x}_t, f_t) = \min_{q_t \geq 0} \left\{ p\mathbb{E}[(D_t - y_t)^+ | f_t] + h\mathbb{E}[(y_t - D_t)^+ | f_t] + w\mathbb{E}[(x_{K-1,t} - D_t)^+ | f_t] + \beta\mathbb{E}[C_{t+1}(\mathbf{X}_{t+1}, F_{t+1})] \right\}.$$

For $k = 1, ..., K-1$, for the continuous case, let $C_t^{(k)}(\mathbf{x}_t, f_t)$ denote $\partial C_t(\mathbf{x}_t, f_t)/\partial x_{k,t}$ (the differentiability can be established by following Fries (1975)); for the discrete case, let $C_t^{(k)}(\mathbf{x}_t, f_t)$ denote the increase of $C_t(\mathbf{x}_t, f_t)$ caused by a unit increase of $x_{k,t}$. Then, we have the following result:

Lemma A1 *Under Assumption 1, for $t = 1, ..., T$, we have (i) $C_{t+1}^{(k)}(\boldsymbol{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K-1$, $\forall \boldsymbol{x}_{t+1}, f_{t+1}$; (ii) $C_{t+1}^{(i)}(\boldsymbol{x}_{t+1}, f_{t+1}) \leq C_{t+1}^{(j)}(\boldsymbol{x}_{t+1}, f_{t+1}), 1 \leq i < j \leq K-1$, $\forall \boldsymbol{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$; and (iii) $C_{t+1}^{(k)}(\boldsymbol{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \boldsymbol{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$.*

Based on part (i) of Lemma A1, if the initial inventory level at period t is less than or equal to $\bar{y}_t$, then the total inventory level after ordering under an optimal ordering decision rule must also be less than or equal to $\bar{y}_t$ (because ordering up to more than $\bar{y}_t$ increases both the expected cost at period t and the future optimal cost-to-go). Therefore, given that we start from zero inventory and an optimal ordering decision rule is followed at each period under policy IM, we must have $\sum_{k=1}^{K-1} x_{k,t+1}^{IM} \leq \bar{y}_t, \forall t = 1, ..., T$.

Recall that for each given sample path, units can be moved or disposed of during periods $t \in \mathscr{T}_{HM} \cup \mathscr{T}_{HL} \cup \mathscr{T}_{HU} = \{s_1, ..., s_n\}$. Consider two variations of policy IM, denoted by IM_1 and IM_2. In particular, under IM_1, we only allow movements of units (no disposal) at period s_1, after which an optimal ordering decision rule is followed and no movement or disposal of units is performed. Under IM_2, on top of the movements of units at s_1, we also allow disposal of units at s_1 (if $s_1 \in \mathscr{T}_{HU}$), but again, no movement or disposal of units is performed in future periods. Then, to show $\mathbb{E}[\mathscr{C}(IM)] \leq \mathbb{E}[\mathscr{C}(OPT)]$, it is sufficient to show $\mathbb{E}[\mathscr{C}(IM_1)] \leq \mathbb{E}[\mathscr{C}(OPT)]$ and $\mathbb{E}[\mathscr{C}(IM_2)] \leq \mathbb{E}[\mathscr{C}(IM_1)]$, because if these are true, then following a similar argument, movements and disposals of units at future periods can only further decrease the expected total cost.

We first prove $\mathbb{E}[\mathscr{C}(IM_1)] \leq \mathbb{E}[\mathscr{C}(OPT)]$. Consider the following two cases:

First, suppose the amount of outdates at s_1 under policies IM_1 and OPT are the same. Then, the total cost at s_1 under the two policies are the same (moving some units to age zero does not affect the total inventory level). The total inventory level at $s_1 + 1$ under the two policies must also be the same but the inventory vector under policy IM_1 is "younger", i.e., $\sum_{k=1}^{K-1} x_{k,s_1+1}^{IM_1} = \sum_{k=1}^{K-1} x_{k,s_1+1}^{OPT}$, $\sum_{k=m}^{K-1} x_{k,s_1+1}^{IM_1} \le \sum_{k=m}^{K-1} x_{k,s_1+1}^{OPT}, m=2,...,K-1$. By part (ii) of Lemma A1, policy IM_1 must have no more expected total cost than policy OPT.

Second, suppose there are $\xi \le \epsilon$ more units of outdates at s_1 under policy OPT than under policy IM_1 (this is only possible when some units of age $K-1$ are moved to younger positions under policy IM_1). Then, policy IM_1 has ξw less outdating cost than policy OPT at period s_1. Since moving units does not affect the total inventory level, the shortage penalty and holding cost of the two systems at s_1 are the same. Moreover, at the beginning of period s_1+1, we have $\sum_{k=1}^{K-1} x_{k,s_1+1}^{IM_1} = \sum_{k=1}^{K-1} x_{k,s_1+1}^{OPT} + \xi$, $\sum_{k=m}^{K-1} x_{k,s_1+1}^{IM_1} \le \sum_{k=m}^{K-1} x_{k,s_1+1}^{OPT} + \xi, m=2,...,K-1$. By parts (ii)-(iii) of Lemma A1, the expected cost-to-go at period s_1+1 under policy IM_1 is at most $\xi w/\beta$ more than that under policy OPT. Therefore, policy IM_1 has no more expected total cost than policy OPT.

Combining the above two cases, we have $\mathbb{E}[\mathscr{C}(IM_1)] \le \mathbb{E}[\mathscr{C}(OPT)]$.

We next prove $\mathbb{E}[\mathscr{C}(IM_2)] \le \mathbb{E}[\mathscr{C}(IM_1)]$. By construction, units are only disposed of under policy IM_2 at s_1 if $s_1 \in \mathscr{T}_{HU}$. If $s_1 \notin \mathscr{T}_{HU}$, then there is nothing to prove. If $s_1 \in \mathscr{T}_{HU}$, we have $y_{s_1}^{TB} < y_{s_1}^{B}$. Then, by definition of policy TB, we have $y_{s_1}^{TB} = y_{s_1}^{U}$, which provides an upper bound on the optimal total inventory level after ordering under given $\mathbf{x}_{s_1}^{TB}$ and f_{s_1}, that is, $y_{s_1}^{TB} \ge y_{s_1}^{OPT}(\mathbf{x}_{s_1}^{TB}, f_{s_1})$. Also, by construction of policy IM, the inventory vector under policy IM is "younger" than that under policy TB after the movements of units (i.e., Inequality 3). We next show that a "younger" inventory vector implies a smaller optimal total inventory level, implying that $y_{s_1}^{TB}$ also provides an upper bound on the optimal total inventory level for given $\check{\mathbf{x}}_{s_1}^{IM}$ and f_{s_1}.

Lemma A2 *Under Assumption 1, at any period t, suppose $\boldsymbol{x}_t, \boldsymbol{x}_t'$ satisfy $\sum_{m=k}^{K-1} x_{m,t} \le \sum_{m=k}^{K-1} x'_{m,t}, k=1,...,K-1$. Then, $y_t^{OPT}(\boldsymbol{x}_t, f_t) \le y_t^{OPT}(\boldsymbol{x}_t', f_t)$.*

The proof for Lemma A2 is built upon the $L^\natural$-convexity property. Given this result, we have $y_{s_1}^{OPT}(\check{\mathbf{x}}_{s_1}^{IM}, f_{s_1}) \le y_{s_1}^{OPT}(\mathbf{x}_{s_1}^{TB}, f_{s_1}) \le y_{s_1}^{TB}$. Due to convexity, disposing of some inventory under policy IM at s_1 so that the total inventory level after the disposal is reduced to $\hat{y}_{s_1}^{IM} = y_{s_1}^{TB}$ will not increase the expected total cost. That is, policy IM_2 has no more expected total cost than policy IM_1. □

Proof of Proposition 1. We start by proving a structural property on the optimal cost-to-go function under the marginal-cost accounting scheme. At each period t, given $\mathbf{x}_t$ and f_t, let $G_t(\mathbf{x}_t, f_t)$ denote the optimal cost-to-go function at period t under the marginal-cost accounting scheme, and let $\Gamma_t(\mathbf{x}_t, f_t, q_t) = P_t(\mathbf{x}_t, f_t, q_t) + H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t)$. Further, let $G_{T+1}(\mathbf{x}_{T+1}, f_{T+1}) = 0, \forall \mathbf{x}_{T+1}, f_{T+1}$. Then, the optimality equation under the marginal-cost accounting scheme is defined as follows:

$$G_t(\mathbf{x}_t, f_t) = \min_{q_t \ge 0} \left\{ \Gamma_t(\mathbf{x}_t, f_t, q_t) + \mathbb{E}[G_{t+1}(\mathbf{X}_{t+1}, F_{t+1}) | f_t] \right\}.$$

For $k=1,...,K-1$, for the continuous case, let $G_t^{(k)}(\mathbf{x}_t, f_t)$ denote $\partial G_t(\mathbf{x}_t, f_t)/\partial x_{k,t}$; for the discrete case, let $G_t^{(k)}(\mathbf{x}_t, f_t)$ denote the increase of $G_t(\mathbf{x}_t, f_t)$ caused by a unit increase of $x_{k,t}$. Then, we have the following result:

Lemma A3 *Under Assumption 1, for $t=1,...,T$, we have $G^{(k)}_{t+1}(\boldsymbol{x}_{t+1}, f_{t+1}) \leq 0, k=1,...,K-1$, $\forall \boldsymbol{x}_{t+1}, f_{t+1}$.*

The intuition why $G_{t+1}(\mathbf{x}_{t+1}, f_{t+1})$ is decreasing in the inventory levels is that, under the marginal-cost accounting scheme, the holding and outdating costs charged to the units in $\mathbf{x}_{t+1}$ are not included in $G_{t+1}(\mathbf{x}_{t+1}, f_{t+1})$ (since these units were ordered prior to period $t+1$). As a result, a higher inventory level could lead to smaller ordering quantities in future periods, which reduces the cost-to-go function.

With this result, we next prove the proposition by contradiction. Suppose for some period t, given $\mathbf{x}_t$ and f_t, we have $q^L_t > q^{OPT}_t$. Consider a policy L, under which q^L_t units are ordered at period t and an optimal ordering policy is followed in future periods. Then, the total expected cost of policy L is $\Gamma_t(\mathbf{x}_t, f_t, q^L_t) + \mathbb{E}[G_{t+1}(\mathbf{X}^L_{t+1}, F_{t+1})|f_t]$, where $\mathbf{X}^L_{t+1}$ denotes the inventory vector at period $t+1$ under policy L. On the other hand, the total expected cost of policy OPT is $\Gamma_t(\mathbf{x}_t, f_t, q^{OPT}_t) + \mathbb{E}[G_{t+1}(\mathbf{X}^{OPT}_{t+1}, F_{t+1})|f_t]$. By definition of q^L_t, we have $\Gamma_t(\mathbf{x}_t, f_t, q^L_t) < \Gamma_t(\mathbf{x}_t, f_t, q^{OPT}_t)$. Further, since $q^L_t > q^{OPT}_t$, we have $X^L_{k,t+1} \geq X^{OPT}_{k,t+1}, k=1,...,K-1$ with probability one. Therefore, by Lemma A3, we have $G_{t+1}(\mathbf{X}^L_{t+1}, F_{t+1}) \leq G_{t+1}(\mathbf{X}^{OPT}_{t+1}, F_{t+1})$ with probability one. Then:

$$\Gamma_t(\mathbf{x}_t, f_t, q^L_t) + \mathbb{E}[G_{t+1}(\mathbf{X}^L_{t+1}, F_{t+1})|f_t] < \Gamma_t(\mathbf{x}_t, f_t, q^{OPT}_t) + \mathbb{E}[G_{t+1}(\mathbf{X}^{OPT}_{t+1}, F_{t+1})|f_t].$$

That is, policy OPT is not optimal for periods $t,...,T$, which is a contradiction. □

Proof of Lemma 3. The proof of Lemma 3 is built on our construction of policy IM (§5.1) and the dynamic unit matching scheme (§5.2). We next prove the three statements in Lemma 3 one by one:

(i) Under any given sample path, if $t \in \mathscr{T}_P$, we have $y^B_t \geq y^{IM}_t$, and hence $P^B_t \leq P^{IM}_t$. If $t \in \mathscr{T}_{HU}$, we have $\hat{y}^{IM}_t = y^{TB}_t < y^B_t$ after the movements and disposals of units under policy IM. Therefore, we also have $P^B_t \leq P^{IM}_t$ in this case. Combining these two cases, we have $\sum_{t\in\mathscr{T}_P\cup\mathscr{T}_{HU}} P^B_t \leq \sum_{t=1}^{T} P^{IM}_t$ with probability one.

(ii) Recall that for each $t \in \mathscr{T}_{HM} \cup \mathscr{T}_{HL}$, the first q^B_t ($\leq q^{TB}_t$) units ordered under policy TB are permanently matched with units under policy IM. Therefore, $\sum_{t\in\mathscr{T}_{HM}\cup\mathscr{T}_{HL}} H^B_t$ is exactly the total holding cost charged to all matched units under policy TB. Further, by construction, there is no overlap in permanent matchings (i.e., each matched unit under policy TB is matched with a different unit under policy IM). Then, to prove $\sum_{t\in\mathscr{T}_{HM}\cup\mathscr{T}_{HL}} H^B_t \leq \sum_{t=1}^{T} H^{IM}_t$, it suffices to show that all matched units under policy TB stay in inventory no longer than their permanently matched units. To do so, we first observe that due to Equations 3-4, each pair of temporarily matched units will have the same age. Next, consider the following three cases:

Case 1: If a matched unit under policy TB is used to satisfy demand, call this unit u_1. By construction, u_1's permanently matched unit under policy IM is used to satisfy the same unit of demand. By Inequality 3, the inventory under policy IM must be consumed no faster than that under policy TB. Therefore, u_1's last temporarily matched unit under policy IM, which has the same age as u_1, must be consumed no earlier than u_1. Then, under the FIFO issuing policy, u_1's permanently matched unit, which is used to satisfy the same unit of demand as u_1, must be no younger than u_1. Further considering possible movements under policy IM, u_1 must have stayed in inventory no longer than its permanently matched unit.

Case 2: If a matched unit under policy TB is outdated, call this unit u_2. By construction, u_2's permanently matched unit under policy IM is outdated in the same period. Further considering possible movements under policy IM, u_2 must have stayed in inventory no longer than its permanently matched unit.

Case 3: If a matched unit under policy TB is still in inventory at the end of the planning horizon, call it u_3. Then, u_3's permanently matched unit under policy IM is defined as its last temporarily matched unit. Since each temporarily matched pair of units have the same age, and further considering possible movements under policy IM, u_3 must have stayed in inventory no longer than its permanently matched unit.

Since the permanent matchings are defined on a one-to-one basis and the above arguments are true for any given sample path, we must have $\sum_{t\in\mathscr{T}_{HM}\cup\mathscr{T}_{HL}} H_t^B \le \sum_{t=1}^T H_t^{IM}$ with probability one.

(iii) By definition of the dynamic unit-matching scheme, for a matched unit under policy TB that is outdated, the permanently matched unit under policy IM must be outdated at the same period. Since the permanent matchings are defined on a one-to-one correspondence and the above argument is true for any given sample path, we must have $\sum_{t\in\mathscr{T}_{HM}\cup\mathscr{T}_{HL}} W_t^B \le \sum_{t=1}^T W_t^{IM}$ with probability one. □

Proof of Lemma 4. Let $\Gamma_t(\mathbf{x}_t,f_t,q_t)=P_t(\mathbf{x}_t,f_t,q_t)+H_t(\mathbf{x}_t,f_t,q_t)+W_t(\mathbf{x}_t,f_t,q_t)$. Consider the following three cases. First, suppose $q_t^{TB}=q_t^B$. Then clearly, $\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^{TB})=\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^B)$. Second, suppose $q_t^{TB}>q_t^B$. Then, we have $q_t^{TB}=q_t^L$. Since $q_t^{TB}=q_t^L$ minimizes $\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t)$, we must have $\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^{TB})\le\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^B)$. Finally, suppose $q_t^{TB}<q_t^B$. Then, we have $q_t^{TB}=q_t^U$. Given $\mathbf{x}_t$ and f_t, it is straightforward to check that $\Gamma_t(\mathbf{x}_t,f_t,q_t)$ is convex in q_t. Further, since q_t^L minimizes $\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t)$, and $q_t^B>q_t^{TB}=q_t^U\ge q_t^L$, we must have $\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^{TB})\le\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^B)$. By definition, for any given f_t, we have $\mathbb{E}[P_t^{TB}+H_t^{TB}+W_t^{TB}|f_t]=\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^{TB})$, $\mathbb{E}[P_t^B+H_t^B+W_t^B|f_t]=\Gamma_t(\mathbf{x}_t^{TB},f_t,q_t^B)$. Therefore, we have:

$$\mathbb{E}[P_t^{TB}+H_t^{TB}+W_t^{TB}|f_t]\le\mathbb{E}[P_t^B+H_t^B+W_t^B|f_t].$$

Let $\mathbb{1}(t\in\mathscr{T}_P)$, $\mathbb{1}(t\in\mathscr{T}_{HM})$, $\mathbb{1}(t\in\mathscr{T}_{HL})$ and $\mathbb{1}(t\in\mathscr{T}_{HU})$ be indicator functions. Then, $\mathbb{1}(t\in\mathscr{T}_P)+\mathbb{1}(t\in\mathscr{T}_{HM})+\mathbb{1}(t\in\mathscr{T}_{HL})+\mathbb{1}(t\in\mathscr{T}_{HU})=1$ with probability one. Further, for any given f_t, $\mathbb{1}(t\in\mathscr{T}_P)$, $\mathbb{1}(t\in\mathscr{T}_{HM})$, $\mathbb{1}(t\in\mathscr{T}_{HL})$ and $\mathbb{1}(t\in\mathscr{T}_{HU})$ are all deterministic. Therefore, we have the following result:

$$\begin{aligned}
\mathbb{E}[\mathscr{C}(TB)]&=\sum_{t=1}^T\mathbb{E}[\mathbb{E}[P_t^{TB}+H_t^{TB}+W_t^{TB}|F_t]]\\
&\le\sum_{t=1}^T\mathbb{E}[\mathbb{E}[P_t^B+H_t^B+W_t^B|F_t]]\\
&=\sum_{t=1}^T\mathbb{E}[\mathbb{E}[(P_t^B+H_t^B+W_t^B)(\mathbb{1}(t\in\mathscr{T}_P)+\mathbb{1}(t\in\mathscr{T}_{HM})+\mathbb{1}(t\in\mathscr{T}_{HL})+\mathbb{1}(t\in\mathscr{T}_{HU}))|F_t]]\\
&=\sum_{t=1}^T\mathbb{E}[\mathbb{E}[2P_t^B\mathbb{1}(t\in\mathscr{T}_P\cup\mathscr{T}_{HU})+2(W_t^B+H_t^B)\mathbb{1}(t\in\mathscr{T}_{HM}\cup\mathscr{T}_{HL})|F_t]]\\
&=\mathbb{E}\left[\sum_{t\in\mathscr{T}_P\cup\mathscr{T}_{HU}}2P_t^B+\sum_{t\in\mathscr{T}_{HM}\cup\mathscr{T}_{HL}}2(H_t^B+W_t^B)\right]\\
&\le\mathbb{E}\left[2\sum_{t=1}^T P_t^{IM}+2\sum_{t=1}^T(H_t^{IM}+W_t^{IM})\right]\\
&=2\mathbb{E}[\mathscr{C}(IM)],
\end{aligned}$$

where the second inequality follows from Lemma 3, which completes the proof. □

Proof of Theorem 1. Since we have $\mathbb{E}[\mathscr{C}(IM)] \leq \mathbb{E}[\mathscr{C}(OPT)]$ from Lemma 2 and $\mathbb{E}[\mathscr{C}(TB)] \leq 2\mathbb{E}[\mathscr{C}(IM)]$ from Lemma 4, we have $\mathbb{E}[\mathscr{C}(TB)] \leq 2\mathbb{E}[\mathscr{C}(IM)] \leq 2\mathbb{E}[\mathscr{C}(OPT)]$, which completes the proof. □

Proof of Theorem 2. We first prove $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(OPT)]} = 1$. In order to prove this result, we next construct a policy A and prove that $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$ and $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(OPT)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$, respectively. For ease of exposition and in line with our motivating example where platelets are transfused in integer units, we prove these results by focusing on integer demands. However, we later briefly discuss how our analysis and results can be extended to consider continuous demands as well. Recall that there exists $M > 0$ such that $\mathbb{P}(D_t \leq M) = 1$ for all periods $t = 1, ..., T$. At any period t, given f_t, let $M_t \leq M$ denote the smallest number such that $\mathbb{P}(D_t > M_t | F_t = f_t) = 0$ (for notational simplicity, we omit the dependence of M_t on f_t). Equivalently, given f_t, M_t is the largest number on which D_t has a positive probability mass. At any period t, for any given $\mathbf{x}_t$ and f_t, let policy A be such that if $\sum_{k=1}^{K-1} x_{k,t} < M_t$, we order up to M_t; if $\sum_{k=1}^{K-1} x_{k,t} \geq M_t$, we order nothing. That is, the ordering quantity under policy A is given by $q_t^A = \max\{M_t - \sum_{k=1}^{K-1} x_{k,t}, 0\}$.

In order to prove $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$, it is sufficient to prove that for any given $\mathbf{x}_t$ and f_t at period t, if p is higher than a threshold, we have $q_t^{TB} = q_t^A$. By definitions of q_t^L and q_t^{TB}, we have $q_t^L \leq q_t^{TB} \leq q_t^A$ (because ordering q_t^A already results in zero shortage penalty). Then, it is sufficient to show that if p is higher than a threshold, we have $q_t^L = q_t^A$. Recall that q_t^L is the minimizer of $P_t(\mathbf{x}_t, f_t, q_t) + H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t)$. We next prove that for any $q_t < q_t^A$, we have $P_t(\mathbf{x}_t, f_t, q_t) + H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t) > P_t(\mathbf{x}_t, f_t, q_t^A) + H_t(\mathbf{x}_t, f_t, q_t^A) + W_t(\mathbf{x}_t, f_t, q_t^A)$ when p is higher than a threshold. For any $q_t < q_t^A$, we clearly have $P_t(\mathbf{x}_t, f_t, q_t) > P_t(\mathbf{x}_t, f_t, q_t^A)$ and $H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t) \leq H_t(\mathbf{x}_t, f_t, q_t^A) + W_t(\mathbf{x}_t, f_t, q_t^A)$. Moreover, we have $P_t(\mathbf{x}_t, f_t, q_t) - P_t(\mathbf{x}_t, f_t, q_t^A) \geq p \times \mathbb{P}(D_t = M_t | F_t = f_t) \times (q_t^A - q_t)$, while $H_t(\mathbf{x}_t, f_t, q_t^A) + W_t(\mathbf{x}_t, f_t, q_t^A) - H_t(\mathbf{x}_t, f_t, q_t) - W_t(\mathbf{x}_t, f_t, q_t) \leq (hK + w) \times (q_t^A - q_t)$. Clearly, as long as $p > \frac{hK+w}{\mathbb{P}(D_t = M_t | F_t = f_t)}$, we have $P_t(\mathbf{x}_t, f_t, q_t) + H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t) > P_t(\mathbf{x}_t, f_t, q_t^A) + H_t(\mathbf{x}_t, f_t, q_t^A) + W_t(\mathbf{x}_t, f_t, q_t^A)$. This implies that $q_t^L = q_t^A$ when p is high.[11] Therefore, when p is sufficiently high, we have $q_t^{TB} = q_t^A$, and hence $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$.

In order to prove $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(OPT)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$, it is sufficient to prove that for any given $\mathbf{x}_t$ and f_t at period t, if p is higher than a threshold, we have $q_t^{OPT} = q_t^A$. For any p, we must have $q_t^{OPT} \leq q_t^A$ (because by part (i) of Lemma A1, further increasing the ordering quantity increases both the expected cost at period t and the expected cost-to-go of the next period). In the previous step, we have shown that when p is higher than a threshold, we have $q_t^L = q_t^A$, which implies $q_t^{OPT} \geq q_t^L = q_t^A$. Therefore, we have $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(OPT)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$.

Given both $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$ and $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(OPT)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$, we must have $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(OPT)]} = 1$.

We next prove $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(OPT)]} = 2$. From Theorem 1, we already know that $\frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(OPT)]} \leq 2$ for any p. Therefore, it is sufficient to prove $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(OPT)]} \geq 2$. Since $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(OPT)]}{\mathbb{E}[\mathscr{C}(A)]} = 1$, it is sufficient to show $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(A)]} \geq 2$. For any given $\mathbf{x}_t$ and f_t at period t, by construction of policies A and B, we must have $q_t^A \geq q_t^B$. We next construct a hypothetical policy A' and prove that $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(A')]}{\mathbb{E}[\mathscr{C}(A)]} \geq 1$ and $\lim_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(A')]} = 2$, respectively.

[11] For continuous demand, we may not have exactly $q_t^L = q_t^A$; however, by following similar analysis, it can be shown that (1) q_t^L will converge to q_t^A when p goes to infinity; and (2) the expected shortage penalty from ordering q_t^L will converge to zero when p goes to infinity. These results are sufficient for the rest of our analysis to hold.

At period t, given $\mathbf{x}_t$ and f_t, we let $q_t^{A'} = q_t^A$. That is, let $q_t^{A'} = \max\{M_t - \sum_{k=1}^{K-1} x_{k,t}, 0\}$. Then, we have $q_t^{A'} \geq q_t^B$. At the beginning of period $t+1$, we dispose of some inventory from system A' such that the inventory levels in Systems A' and B are exactly the same. Then, we repeat this procedure to define the ordering quantities and disposals of units under policy A' for future periods. We define the total cost in System A', i.e., $\mathscr{C}(A')$, by using the marginal cost accounting scheme, that is, given $\mathbf{x}_t$, f_t, and the ordering quantity $q_t^{A'}$, the expected cost at period t is given by $P_t(\mathbf{x}_t, f_t, q_t^{A'}) + H_t(\mathbf{x}_t, f_t, q_t^{A'}) + W_t(\mathbf{x}_t, f_t, q_t^{A'})$, where the marginal cost functions are defined in the same way as those presented in Appendix A as if no units will be disposed of. As shown above, when p is higher than a threshold, policy A is exactly the same as policy OPT. Further, as shown in Proposition 1, the optimal cost-to-go function under the marginal cost accounting scheme is decreasing in the inventory levels. Therefore, under the marginal cost accounting scheme, disposing of some units from inventory will lead to a higher optimal cost-to-go for future periods. Then, we have $\lim\limits_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(A')]}{\mathbb{E}[\mathscr{C}(A)]} \geq 1$.

By construction, systems A' and B will have the same inventory levels at the beginning of each period (after the unit disposals in System A'). Therefore, to prove $\lim\limits_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(A')]} = 2$, it is sufficient to prove

$$\lim_{p\to\infty} \left(P_t(\mathbf{x}_t, f_t, q_t^B) + H_t(\mathbf{x}_t, f_t, q_t^B) + W_t(\mathbf{x}_t, f_t, q_t^B)\right) = 2 \lim_{p\to\infty} \left(P_t(\mathbf{x}_t, f_t, q_t^{A'}) + H_t(\mathbf{x}_t, f_t, q_t^{A'}) + W_t(\mathbf{x}_t, f_t, q_t^{A'})\right).$$

By definitions of policies A' and B, we have $P_t(\mathbf{x}_t, f_t, q_t^{A'}) = 0$ and $P_t(\mathbf{x}_t, f_t, q_t^B) = H_t(\mathbf{x}_t, f_t, q_t^B) + W_t(\mathbf{x}_t, f_t, q_t^B)$. For any q_t such that $q_t < q_t^{A'}$, we can always find a sufficiently high shortage penalty p under which we have $P_t(\mathbf{x}_t, f_t, q_t) > H_t(\mathbf{x}_t, f_t, q_t) + W_t(\mathbf{x}_t, f_t, q_t)$. Therefore, we must have $\lim\limits_{p\to\infty} q_t^B = q_t^{A'}$, which implies

$$\lim_{p\to\infty} \left(H_t(\mathbf{x}_t, f_t, q_t^B) + W_t(\mathbf{x}_t, f_t, q_t^B)\right) = \lim_{p\to\infty} \left(H_t(\mathbf{x}_t, f_t, q_t^{A'}) + W_t(\mathbf{x}_t, f_t, q_t^{A'})\right).$$

Then, $\lim_{p\to\infty} \left(P_t(\mathbf{x}_t, f_t, q_t^B) + H_t(\mathbf{x}_t, f_t, q_t^B) + W_t(\mathbf{x}_t, f_t, q_t^B)\right) = 2\lim_{p\to\infty} \left(H_t(\mathbf{x}_t, f_t, q_t^{A'}) + W_t(\mathbf{x}_t, f_t, q_t^{A'})\right)$. This result implies that $\lim\limits_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(A')]} = 2$. Therefore, we must have $\lim\limits_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(OPT)]} = 2$.

Finally, the proof for $\lim\limits_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(PB)]}{\mathbb{E}[\mathscr{C}(OPT)]} \geq 1 + \frac{Kh+w}{2(K-1)h+w}$ follows similar steps as for $\lim\limits_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(B)]}{\mathbb{E}[\mathscr{C}(OPT)]} = 2$. The only difference is that under policy PB, we have $P_t(\mathbf{x}_t, f_t, q_t^{PB}) = \frac{Kh+w}{2(K-1)h+w}\left(H_t(\mathbf{x}_t, f_t, q_t^{PB}) + W_t(\mathbf{x}_t, f_t, q_t^{PB})\right)$. As a result, we have $\lim_{p\to\infty} \left(P_t(\mathbf{x}_t, f_t, q_t^{PB}) + H_t(\mathbf{x}_t, f_t, q_t^{PB}) + W_t(\mathbf{x}_t, f_t, q_t^{PB})\right) = \left(1 + \frac{Kh+w}{2(K-1)h+w}\right) \lim_{p\to\infty} \left(H_t(\mathbf{x}_t, f_t, q_t^{A'}) + W_t(\mathbf{x}_t, f_t, q_t^{A'})\right)$. Therefore, $\lim\limits_{p\to\infty} \frac{\mathbb{E}[\mathscr{C}(PB)]}{\mathbb{E}[\mathscr{C}(OPT)]} \geq 1 + \frac{Kh+w}{2(K-1)h+w}$. $\square$

Proof of Proposition 2. We prove this proposition in the following two steps:

Step 1: We first prove that if the condition $h \leq \frac{1-\gamma}{\gamma} p + \frac{1-\beta\gamma}{\beta\gamma} w$ holds, then we have that for any period $t = 1, ..., T$, $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}$ such that $\sum\limits_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$.

The claim is clearly true for T because we have $C_{T+1}(\mathbf{x}_{T+1}, f_{T+1}) = 0, \forall \mathbf{x}_{T+1}, f_{T+1}$. For any t such that $2 \leq t \leq T$, assume that the claim is true for $t, ..., T$. We next show that it is also true for $t-1$, i.e., $C_t^{(k)}(\mathbf{x}_t, f_t) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_t$ such that $\sum\limits_{k=1}^{K-1} x_{k,t} < \bar{y}_{t-1}$, $\forall f_t$. At period t, for any given $\mathbf{x}_t$ and f_t such that $\sum\limits_{k=1}^{K-1} x_{k,t} < \bar{y}_{t-1}$, let $y_t^{OPT}(\mathbf{x}_t, f_t)$ denote the total inventory level after ordering at period t by following an optimal ordering policy. By induction assumption and Step 2 of the proof for Lemma A1, we know that $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K-1, \forall \mathbf{x}_{t+1}, f_{t+1}$. Hence, we have $y_t^{OPT}(\mathbf{x}_t, f_t) \leq \bar{y}_t$. Consider two cases:

Case 1: $y_t^{OPT}(\mathbf{x}_t, f_t) < \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon, x'_{m,t} = x_{m,t}, \forall m \neq k$ (i.e., System 2 starts with ϵ more units of age k), and ϵ is positive but sufficiently small such that $y_t^{OPT}(\mathbf{x}_t, f_t) + \epsilon \leq \bar{y}_t$. Let System 1 follow an optimal ordering policy, and let System 2 order the same quantity as System 1 at period t and follow an optimal ordering policy afterward. Since $y_t^{OPT}(\mathbf{x}_t, f_t) + \epsilon \leq \bar{y}_t$, the total inventory level after ordering is no more than $\bar{y}_t$ in both Systems 1 and 2. Since $\mathbb{P}(D_t \leq \bar{y}_t | f_t) = \Phi_t(\bar{y}_t) \leq \gamma$, there will be at most $\gamma h \epsilon$ more expected holding cost and at least $(1-\gamma)p\epsilon$ less expected shortage penalty in System 2 than in System 1 at period t. Suppose $d_t > \bar{y}_t$. Then the two systems will be exactly the same from period $t+1$. Suppose $d_t \leq \bar{y}_t$. Let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ under Systems 1 and 2, respectively. Then, by construction, we have $x_{k,t+1} \leq x'_{k,t+1}, k = 1, ..., K-1$. Assume that there are $\xi \leq \epsilon$ more units of outdates in System 2 than in System 1 at period t. Then $\sum_{k=1}^{K-1} x'_{k,t+1} - \sum_{k=1}^{K-1} x_{k,t+1} = \epsilon - \xi$. By induction assumption, we have $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$. Therefore, the expected total cost in System 2 is at most $\gamma h \epsilon - (1-\gamma)p\epsilon + \gamma w \epsilon = \gamma(h+w)\epsilon - (1-\gamma)p\epsilon \leq w\epsilon/\beta$ more than that in System 1, where the inequality holds because we have $h \leq \frac{1-\gamma}{\gamma}p + \frac{1-\beta\gamma}{\beta\gamma}w$.

Case 2: $y_t^{OPT}(\mathbf{x}_t, f_t) = \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon, x'_{m,t} = x_{m,t}, \forall m \neq k$ (i.e., System 2 starts with ϵ more units of age k), and ϵ is positive but sufficiently small such that $\sum_{k=1}^{K-1} x_{k,t} + \epsilon \leq \bar{y}_t$. Let System 1 follow an optimal ordering policy, and let System 2 order up to $\bar{y}_t$ at period t and follow an optimal ordering policy afterward. Then, by construction, the total inventory level after ordering is equal to $\bar{y}_t$ in both Systems 1 and 2. For any demand realization d_t, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ under Systems 1 and 2, respectively. Then, by construction, we have $x_{1,t+1} \geq x'_{1,t+1}, x_{k,t+1} \leq x'_{k,t+1}, k = 2, ..., K-1$. Assume that there are $\xi \leq \epsilon$ more units of outdates in System 2 than in System 1 at period t. Then $\sum_{k=2}^{K-1} x'_{k,t+1} - \sum_{k=2}^{K-1} x_{k,t+1} \leq \epsilon - \xi$. By induction assumption and Step 2 for the proof of Lemma A1, we have $0 \leq C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$. Therefore, the expected total cost in System 2 is at most $w\epsilon \leq w\epsilon/\beta$ more than that in System 1.

Therefore, we have $C_t^{(k)}(\mathbf{x}_t, f_t) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_t$ such that $\sum_{k=1}^{K-1} x_{k,t} < \bar{y}_{t-1}$, $\forall f_t$.

Step 2: We next prove that if for $t = 1, ..., T$, $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$, then we have that Assumption 1 holds. First, given that for $t = 1, ..., T$, $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$, we know from Step 2 of the proof for Lemma A1 that for $t = 1, ..., T$, $C_{t+1}^{(k)}(\mathbf{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K-1$, $\forall \mathbf{x}_{t+1}, \forall f_{t+1}$. Given this, we know that at any period t, if the inventory level after ordering at period t satisfies $y_t \leq \bar{y}_t$, then we must have $y_s \leq \bar{y}_s$ for all $s > t$ if an optimal ordering policy defined in §3 is followed at each period.

Next, we prove that Assumption 1 holds by using induction. Since $C_{T+1}^{(k)}(\mathbf{x}_{T+1}, f_{T+1}) = 0, k = 1, ..., K-1$, issuing products of age $K-1$ at T clearly results in less cost than issuing younger products. Also, how we issue products of age less than $K-1$ at T does not affect the total cost. Thus, Assumption 1 holds for T.

Assume that Assumption 1 holds for $t+1, .., T$. We next show that it also holds for t. That is, we would like to show that if an optimal ordering policy is implemented at each period and if $y_t \leq \bar{y}_t$, then FIFO is an optimal issuing decision rule for period t. As discussed above, given $y_t \le \bar{y}_t$ and an optimal ordering policy is implemented at each period, we must have $y_s \le \bar{y}_s, \forall s > t$. Then, by induction assumption, FIFO is an optimal issuing decision rule starting from period $t+1$. Then, we make the following observation: Although now the problem is to find an optimal issuing policy under a fixed ordering policy (instead of finding an optimal ordering policy under a fixed issuing policy), the optimal cost-to-go at period $t+1$ is exactly the same as before, i.e., $C_{t+1}(\mathbf{x}_{t+1}, f_{t+1})$, because starting from period $t+1$, the optimal system is the one that follows an optimal ordering policy and the FIFO issuing policy (which is exactly the same as before).

Then, at period t, issuing products of age $K-1$ results in a lower expected total cost than issuing younger products because we have $C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \le w/\beta, k = 1, ..., K-1$. Thus, an optimal issuing policy will issue as many oldest products as possible at period t. Let A be such an issuing policy (but may not necessarily issue older products before younger ones among products with age $\le K-2$). Then, the costs that occur at period t by following FIFO and A are exactly the same. Further, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ by following FIFO an A, respectively. Then, we have $\sum_{k=1}^{K-1} x_{k,t+1} = \sum_{k=1}^{K-1} x'_{k,t+1}$ and $\sum_{k=m}^{K-1} x_{k,t+1} \le \sum_{k=m}^{K-1} x'_{k,t+1}, m = 2, ..., K-1$. From Step 2 of the proof for Lemma A1, we know that if for $t = 1, ..., T,\ C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \le w/\beta, k = 1, ..., K-1,\ \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t,\ \forall f_{t+1}$, then we have for $t = 1, ..., T,\ C^{(i)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \le C^{(j)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}), 1 \le i < j \le K-1,\ \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t,\ \forall f_{t+1}$. Therefore, the expected cost-to-go under FIFO is no more than that under A. That is, FIFO is optimal at period t. $\square$

Proof of Lemma A1. We prove this lemma in the following two steps:

Step 1: We first prove that under Assumption 1, statement (iii) holds for all $t = 1, ..., T$, i.e., for all $t = 1, ..., T,\ C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \le w/\beta, k = 1, ..., K-1,\ \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t,\ \forall f_{t+1}$. Suppose this is not true, i.e., at some period $t+1$, there exist some $k \in \{1, ..., K-1\}$ and some $\mathbf{x}_{t+1}, f_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$ and $C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) > w/\beta$. We next show that Assumption 1 is violated.

Given $\mathbf{x}_{t+1}, f_{t+1}$ defined above. Let ϵ be a small positive number that satisfies $\sum_{k=1}^{K-1} x_{k,t+1} + 2\epsilon \le \bar{y}_t$. Suppose the inventory levels after ordering at period t are given by $x_{k-1,t} = x_{k,t+1} + \epsilon, x_{K-1,t} = \epsilon$, and $x_{m-1,t} = x_{m,t+1}, m = 1, ..., k-1, k+1, ..., K-1$ (clearly, we have $y_t \le \bar{y}_t$), and suppose demand for future periods is defined by f_{t+1}. We next show that issuing younger products from inventory can result in a lower expected total cost than issuing older products from inventory. That is, there exist $\boldsymbol{u}_t$ and $\boldsymbol{u}'_t$ that satisfy the conditions stated in Assumption 1, but $\boldsymbol{u}_t$, which satisfies the FIFO rule, results in a high expected cost-to-go than $\boldsymbol{u}'_t$. In particular, consider two issuing decisions $\boldsymbol{u}_t, \boldsymbol{u}'_t$ such that $u_{t,K-1} = \epsilon, u_{t,m} = 0, \forall m \ne K-1$, and $u'_{k-1,t} = \epsilon, u'_{t,m} = 0, \forall m \ne k-1$ (clearly, these two issuing decisions satisfy the conditions stated in Assumption 1). Then, $\boldsymbol{u}'_t$ will result in ϵ more units of outdate at period t than $\boldsymbol{u}_t$, while $\boldsymbol{u}_t$ will result in ϵ more units of age k at the beginning of period $t+1$ than $\boldsymbol{u}'_t$. Further, under $\boldsymbol{u}'_t$, the inventory vector at the beginning of period $t+1$ is exactly $\mathbf{x}_{t+1}$. Since $C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) > w/\beta$, $\boldsymbol{u}_t$ results in a higher expected cost-to-go than $\boldsymbol{u}'_t$, which is a contradiction.

Step 2: We next prove that if statement (iii) holds for all $t=1,...,T$, then statements (i) and (ii) both holds for all $t=1,...,T$. Statements (i) and (ii) clearly both hold for T since $C_{T+1}(\mathbf{x}_{T+1},f_{T+1})=0,\forall\mathbf{x}_{T+1},f_{T+1}$. Assume that Statements (i) and (ii) hold for some $t\geq 2$. We next show that they also hold for $t-1$.

We first show $C_t^{(k)}(\mathbf{x}_t,f_t)\geq 0, k=1,...,K-1,\forall\mathbf{x}_t,f_t$. Consider the following two cases.

Case 1: Suppose $\sum_{k=1}^{K-1} x_{k,t}<\bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t}=x_{k,t}+\epsilon$ and $x'_{m,t}=x_{m,t}, m=1,...,k-1,k+1,...,K-1$ (i.e., System 2 has ϵ more units of age k), and ϵ satisfies $\epsilon>0$ and $\sum_{k=1}^{K-1} x_{k,t}+\epsilon\leq\bar{y}_t$. Let System 2 follow an optimal ordering policy at each period, and let System 1 order ϵ more units than System 2 at t and follow an optimal ordering policy afterward. Then, it is sufficient to show that System 1 has no more expected total cost than System 2. Clearly, the shortage penalty and holding cost at period t in the two systems are the same. Also, the total inventory level after ordering in both systems must be no more than $\bar{y}_t$ (because by induction assumption, we have $C_{t+1}^{(k)}(\mathbf{x}_{t+1},f_{t+1})\geq 0, k=1,...,K-1, \forall\mathbf{x}_{t+1},f_{t+1}$; hence, ordering up to more than $\bar{y}_t$ increases both the expected cost at period t and the future optimal cost-to-go). For any realization of d_t, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ for Systems 1 and 2, respectively. Assume that there are $\xi\leq\epsilon$ more units of outdates in System 2 than in System 1 at period t. Then, System 2 has $w\xi$ more outdating cost than System 1, and we have $x_{1,t+1}\geq x'_{1,t+1}, x_{k,t+1}\leq x'_{k,t+1},\forall k=2,...,K-1$, and $\sum_{k=1}^{K-1} x_{k,t+1}=\sum_{k=1}^{K-1} x'_{k,t+1}+\xi\leq\bar{y}_t$. By statement (iii) of the lemma as well as the induction assumption, we have $C_{t+1}^{(i)}(\mathbf{x}_{t+1},f_{t+1})\leq C_{t+1}^{(j)}(\mathbf{x}_{t+1},f_{t+1})\leq w/\beta, 1\leq i<j\leq K-1, \forall\mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1}<\bar{y}_t$, $\forall f_{t+1}$. Then, we have $C_{t+1}(\mathbf{x}_{t+1},f_{t+1})-C_{t+1}(\mathbf{x}'_{t+1},f_{t+1})\leq w\xi/\beta$. Therefore, System 1 must have no more expected total cost than System 2.

Case 2: Suppose $\sum_{k=1}^{K-1} x_{k,t}\geq\bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t}=x_{k,t}+\epsilon$ and $x'_{m,t}=x_{m,t}, m=1,...,k-1,k+1,...,K-1$ (i.e., System 2 has ϵ more units of age k), and ϵ is any positive number. Let System 2 follow an optimal ordering policy at each period, and let System 1 order nothing at period t and follow an optimal ordering policy afterward. Then, it is sufficient to show that System 1 has no more expected total cost than System 2. Let y_t and y'_t be the total inventory levels after ordering at period t in Systems 1 and 2, respectively. Then, $\bar{y}_t\leq y_t\leq y'_t$. Thus the expected cost at period t in System 1 is no more than that in System 2 (because a total inventory level of $\Phi_t^{-1}\left(\frac{p}{p+h}\right)$, which is no more than $\bar{y}_t$, minimizes the expected sum of shortage penalty and holding cost at period t, and the costs are convex in the ordering quantity). For any demand realization d_t, let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t+1$ for Systems 1 and 2, respectively. Then we have $x_{k,t+1}\leq x'_{k,t+1}, k=1,...,K-1$. By induction assumption, we have $C_{t+1}^{(k)}(\mathbf{x}_{t+1},f_{t+1})\geq 0, k=1,...,K-1,\forall\mathbf{x}_{t+1},f_{t+1}$. Therefore, System 1 has no more expected total cost than System 2.

Combining both Cases 1 and 2, we have $C_t^{(k)}(\mathbf{x}_t,f_t)\geq 0, k=1,...,K-1,\forall\mathbf{x}_t,f_t$.

We next show $C_t^{(i)}(\mathbf{x}_t,f_t)\leq C_t^{(j)}(\mathbf{x}_t,f_t), 1\leq i<j\leq K-1$, $\forall\mathbf{x}_t$ such that $\sum_{k=1}^{K-1} x_{k,t}<\bar{y}_{t-1}$, $\forall f_t$. Given $\mathbf{x}_t,f_t$ such that $\sum_{k=1}^{K-1} x_{k,t}<\bar{y}_{t-1}$, consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}'_t$ and System 2 starts from $\mathbf{x}''_t$, where $x'_{i,t}=x_{i,t}+\epsilon, x'_{k,t}=x_{k,t},\forall k\neq i$, and $x''_{j,t}=x_{j,t}+\epsilon, x''_{k,t}=$

$x_{k,t}, \forall k \neq j$ (i.e., System 1 starts with ϵ more units of age i and System 2 with ϵ more units of age j), and ϵ is positive but sufficiently small such that $\sum_{k=1}^{K-1} x_{k,t} + \epsilon \leq \bar{y}_t$. Let System 2 follow an optimal ordering policy at each period, and let System 1 order the same amount as System 2 at t and follow an optimal policy afterward. Then, it is sufficient to show that System 1 has no more expected total cost than System 2. For any demand realization d_t, let $\mathbf{x}'_{t+1}$ and $\mathbf{x}''_{t+1}$ be the inventory vectors at period $t+1$ in Systems 1 and 2, respectively. Then, we have $x'_{k,t+1} = x''_{k,t+1}, k = 1, ..., i, x'_{i+1,t+1} \geq x''_{i+1,t+1}$, and $x'_{k,t+1} \leq x''_{k,t+1}, \forall k = i+2, ..., K-1$. Assume that there are $\xi \leq \epsilon$ more units of outdates in System 2 than in System 1 at period t. Then, we have $\sum_{k=1}^{K-1} x'_{k,t+1} = \sum_{k=1}^{K-1} x''_{k,t+1} + \xi \leq \bar{y}_t$. By statement (iii) of the lemma as well as the induction assumption, we have $C_{t+1}^{(i)}(\mathbf{x}_{t+1}, f_{t+1}) \leq C_{t+1}^{(j)}(\mathbf{x}_{t+1}, f_{t+1}) \leq w/\beta, 1 \leq i < j \leq K-1, \forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$. Therefore, System 1 has no more expected total cost than System 2, which completes the proof. □

Proof of Lemma A2. To prove this lemma, we first show that the optimal cost-to-go function after a state transformation satisfies a property called $L^\natural$-convex (which is formally defined below). We start with the state transformation. Let $s_{k,t} = \sum_{m=1}^{k} x_{m,t}$ denote the total inventory level of age less than or equal to k at the beginning of period t, $k = 1, ..., K-1$. Then, the transformed state variables satisfy $0 \leq s_{1,t} \leq ... \leq s_{K-1,t}$. Also, the single-period expected cost under the transformed state variables is given by:

$$\Lambda_t(\mathbf{s}_t, f_t, q_t) = p\mathbb{E}[(D_t - q_t - s_{K-1,t})^+ | f_t] + h\mathbb{E}[(q_t + s_{K-1,t} - D_t)^+ | f_t] + w\mathbb{E}[(s_{K-1,t} - s_{K-2,t} - D_t)^+ | f_t],$$

For $t = 1, ..., T$, let $J_t(\mathbf{s}_t, f_t)$ be the optimal cost-to-go function under the transformed state variables. Let $J_{T+1} = 0$. Then, the optimality equation under the transformed state variables is:

$$J_t(\mathbf{s}_t, f_t) = \min_{q_t \geq 0} \{\Lambda_t(\mathbf{s}_t, f_t, q_t) + \beta\mathbb{E}[J_{t+1}(\mathbf{S}_{t+1}, F_{t+1})]\},$$

At any period t, given $\mathbf{s}_t$ and f_t, let $R_t(\mathbf{s}_t, f_t, \xi_t)$ denote the expected cost-to-go function at period t if $q_t = -\xi_t$ units are ordered at t and an optimal ordering policy is applied afterwards. Then:

$$R_t(\mathbf{s}_t, f_t, \xi_t) = \Lambda_t(\mathbf{s}_t, f_t, -\xi_t) + \beta\mathbb{E}[J_{t+1}(\mathbf{S}_{t+1}, F_{t+1})],$$

$$J_t(\mathbf{s}_t, f_t) = \min_{\xi_t \leq 0} R_t(\mathbf{s}_t, f_t, \xi_t).$$

Next, we introduce the definitions of submodularity and $L^\natural$-convexity on $\mathbb{F}^n$ ($\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{Z}$) (Zipkin 2008, Chen et al. 2014). Let $\vee$ and $\wedge$ be the componentwise maximum and minimum respectively, let $\mathbb{F}_+$ and $\mathbb{F}_-$ be the set of nonnegative and nonpositive elements in $\mathbb{F}$ respectively, and let $\mathbf{e}$ be the vector of 1's.

Definition 1 (Submodularity). A real-valued function $g(\mathbf{x})$ defined on a lattice $X \subset \mathbb{F}^n$ (i.e., $\forall \mathbf{x}, \mathbf{x}' \in X, \mathbf{x} \vee \mathbf{x}' \in X$ and $\mathbf{x} \wedge \mathbf{x}' \in X$) is submodular if $\forall \mathbf{x}, \mathbf{x}' \in X$:

$$g(\mathbf{x}) + g(\mathbf{x}') \geq g(\mathbf{x} \vee \mathbf{x}') + g(\mathbf{x} \wedge \mathbf{x}').$$

Definition 2 ($L^\natural$-convexity). A real-valued function $g(\mathbf{x})$ defined on an $L^\natural$-convex set $X \subset \mathbb{F}^n$ (i.e., $\forall \mathbf{x}, \mathbf{x}' \in X, \forall \alpha \in \mathbb{F}_+, \mathbf{x} \vee (\mathbf{x}' - \alpha\mathbf{e}) \in X$ and $(\mathbf{x} + \alpha\mathbf{e}) \wedge \mathbf{x}' \in X$) is $L^\natural$-convex if function $\phi(\mathbf{x}, \xi) = g(\mathbf{x} - \xi\mathbf{e})$, $\xi \leq 0$, is submodular on $X \times \mathbb{F}_-$.

It is straightforward to check that the set of all feasible $(\mathbf{s}_t, \xi_t)$ is an $L^\natural$-convex set because each associated constraint involves only one variable or two variables with opposite signs. Assume for any f_{t+1}, $J_{t+1}(\mathbf{s}_{t+1}, f_{t+1})$ is $L^\natural$-convex with respect to $\mathbf{s}_{t+1}$ (this clearly holds for $t = T$). We next show that for any given f_t, $R_t(\mathbf{s}_t, f_t, \xi_t)$ is $L^\natural$-convex with respect to $\mathbf{s}_t$ and ξ_t, and $J_t(\mathbf{s}_t, f_t)$ is $L^\natural$-convex with respect to $\mathbf{s}_t$.

To do so, we reformulate the problem at period t into a two-stage problem. Besides the ordering decision before demand realization, we assume that we also decide how to issue the inventory of each age to satisfy demand after demand realization. Given demand realization d_t, let $\eta_{0,t}, \eta_{1,t}, ..., \eta_{K-1,t}$ denote the amount of issued inventory among $-\xi_t, -\xi_t + s_{1,t}, ..., -\xi_t + s_{K-1,t}$ respectively. Let $s_{0,t} = 0$. Then, we have

$$\eta_{k,t} \geq 0, k = 0, 1, ..., K-1; \tag{5}$$

$$\eta_{K-1,t} \leq d_t; \tag{6}$$

$$\eta_{0,t} \leq -\xi_t, \eta_{k,t} - \eta_{k-1,t} \leq s_{k,t} - s_{k-1,t}, k = 1, ..., K-1, \tag{7}$$

where Inequality 5 ensures that the decision variables $\eta_k, t, k = 0, ..., K$ are nonnegative; Inequality 6 ensures that the total amount of issuing quantity $\eta_{K-1,t}$ is no more than the demand d_t; and Inequality 7 ensures that the issuing quantity of each age is no more than the inventory level of that age. Note that we do not need the constraints $\eta_{k-1,t} \leq \eta_{k,t}, k = 1, ..., K-1$ because a solution with $\eta_{k-1,t} > \eta_{k,t}$ implies that some units of age less than or equal to $k-1$ are moved to the position of age k, which is clearly not optimal under Assumption 1. Also, we have $s_{k,t+1} = -\xi_t + s_{k-1,t} - \eta_{k-1,t}, k = 1, ..., K-1$, and the single-period cost at period t is given by $p(d_t - \eta_{K-1,t}) + h(-\xi_t + s_{K-1,t} - \eta_{K-1,t}) + w(s_{K-1,t} - s_{K-2,t} - \eta_{K-1,t} + \eta_{K-2,t})$. This two-stage reformulation is equivalent to our original problem because under our model assumptions, demand should be met as much as possible and older products should be issued first, i.e., the optimal solution of $\eta_{k,t}$ is simply:

$$\eta^*_{k,t} = \min\{-\xi_t + s_{k,t}, (d_t - (s_{K-1,t} - s_{k,t}))^+\}, k = 1, ..., K-1.$$

Let $v_{k,t} = s_{k,t} - \eta_{k,t}, k = 0, ..., K-1$. Then, the constraints can be re-written as $s_{k,t} - v_{k,t} \geq 0, k = 0, ..., K-1; s_{K-1,t} - v_{K-1,t} \leq d_t; -v_{0,t} \leq -\xi_t; v_{k-1,t} \leq v_{k,t}, k = 1, ..., K-1$. Let $\mathbf{v}_t = (v_{0,t}, v_{1,t}, ..., v_{K-1,t})$. Then, the set of all possible $(\mathbf{s}_t, \xi_t, \mathbf{v}_t)$ is $L^\natural$-convex because each associated constraint involves either one variable or two variables with opposite signs. For given realization of f_{t+1}, the cost-to-go at period t is given by

$$\Pi_t(\mathbf{s}_t, f_{t+1}, \xi_t, \mathbf{v}_t) = p(d_t - s_{K-1,t} + v_{K-1,t}) + h(-\xi_t + v_{K-1,t}) + w(v_{K-1,t} - v_{K-2,t}) + \beta J_{t+1}(\mathbf{s}_{t+1}, f_{t+1}),$$

where $s_{k,t+1} = -\xi_t + v_{k-1,t}, k = 1, ..., K-1$. Then, we have $R_t(\mathbf{s}_t, f_t, \xi_t) = \mathbb{E}[\min_{\mathbf{V}_t} \Pi_t(\mathbf{s}_t, F_{t+1}, \xi_t, \mathbf{V}_t)|f_t]$. Since $L^\natural$-convexity is preserved under minimization and expectation (Zipkin 2008), to show $R_t(\mathbf{s}_t, f_t, \xi_t)$ is $L^\natural$-convex with respect to $\mathbf{s}_t$ and ξ_t, it is sufficient to show that for any realization of f_{t+1}, $\Pi_t(\mathbf{s}_t, f_{t+1}, \xi_t, \mathbf{v}_t)$ is $L^\natural$-convex with respect to $\mathbf{s}_t, \xi_t$ and $\mathbf{v}_t$. Further, since it is straightforward to verify that linear functions are $L^\natural$-convex, it remains to show that for any realization of f_{t+1}, $J_{t+1}(\mathbf{s}_{t+1}, f_{t+1})$ is $L^\natural$-convex with respect to $\mathbf{s}_t, \xi_t$ and $\mathbf{v}_t$.

Let $\mathbf{v}'_t = (v_{0,t}, ..., v_{K-2,t})$. Then, we have $\mathbf{s}_{t+1} = \mathbf{v}'_t - \xi_t \mathbf{e}$, and $J_{t+1}(\mathbf{s}_{t+1}, f_{t+1}) = J_{t+1}(\mathbf{v}'_t - \xi_t \mathbf{e}, f_{t+1})$. By induction assumption, for any f_{t+1}, $J_{t+1}(\mathbf{s}_{t+1}, f_{t+1})$ is $L^\natural$-convex with respect to $\mathbf{s}_{t+1}$. Then, by Lemma 1 in

Zipkin (2008), $J_{t+1}(\mathbf{v}'_t - \xi_t \mathbf{e}, f_{t+1})$ is $L^\natural$-convex with respect to $\mathbf{v}'_t$ and ξ_t. Since $\mathbf{v}'_t$ is simply the first $K-1$ dimensions of $\mathbf{v}_t$, we have $J_{t+1}(\mathbf{v}'_t - \xi_t \mathbf{e}, f_{t+1})$ is $L^\natural$-convex with respect to $\mathbf{s}_t, \xi_t$ and $\mathbf{v}_t$.

Therefore, $R_t(\mathbf{s}_t, f_t, \xi_t)$ is $L^\natural$-convex with respect to $\mathbf{s}_t$ and ξ_t, and hence $J_t(\mathbf{s}_t, f_t)$ is $L^\natural$-convex with respect to $\mathbf{s}_t$. Then, the optimal solution ξ^*_t is non-decreasing in $\mathbf{s}_t$ and the sensitivity is bounded by one (Zipkin 2008, Lemma 3), i.e., $0 \le \frac{\partial \xi^*_t}{s_{k,t}} \le 1$. Hence, we have $-1 \le \frac{\partial q^*_t}{s_{k,t}} \le 0$, which completes the proof. $\square$

Proof of Lemma A3. We prove this lemma by induction. The conclusion in Lemma A3 is clearly true for T since $G_{T+1}(\mathbf{x}_{T+1}, f_{T+1}) = 0, \forall \mathbf{x}_{T+1}, f_{T+1}$. Assume that the conclusion is true for some $t \ge 2$. We next show that it is also true for $t-1$, i.e., $G^{(k)}_t(\mathbf{x}_t, f_t) \le 0, k = 1, ..., K-1, \forall \mathbf{x}_t, f_t$. Consider the following two cases:

Case 1: Suppose $\sum_{k=1}^{K-1} x_{k,t} < \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon$ and $x'_{m,t} = x_{m,t}, m = 1, ..., k-1, k+1, ..., K-1$ (i.e., System 2 has ϵ more units of age k), and ϵ is positive but sufficiently small such that $\sum_{k=1}^{K-1} x_{k,t} + \epsilon \le \bar{y}_t$. Let System 1 follow an optimal ordering policy at each period. To define the ordering policy in System 2, consider a given sample path. Let $t_0 \in (t, t+K-1]$ be such that at the beginning of each period $t, ..., t_0 - 1$, there are still some products that were ordered in periods $< t$ in System 2, while at the beginning of period t_0, all of these products are gone (either used to satisfy demand or outdated) and all products in inventory were ordered in periods $\ge t$. Then, for each period $t, ..., t_0 - 1$, let System 2 order up to the same level as System 1 (order nothing if the inventory level in System 2 before ordering is higher than the inventory level in System 1 after ordering), and let System 2 follow an optimal ordering policy in periods $\ge t_0$.

Then, to prove the lemma, it is sufficient to show that the expected cost in periods $\ge t$ under the marginal-cost accounting scheme in System 2 is no more than that in System 1. By definition of t_0, no units ordered in periods $\ge t$ will be outdated by the beginning of period t_0. Then, the total cost under the marginal-cost accounting scheme in each system is comprised of: i) the shortage penalties that occur in periods $t, ..., t_0 - 1$, ii) the holding costs that occur in periods $t, ..., t_0 - 1$ charged for units ordered in periods $\ge t$, and iii) the total costs (including shortage penalties, holding and outdating costs) that occur in periods $\ge t_0$. (Note that by construction, all products in inventory at the beginning of period t_0 were ordered in periods $\ge t$ in both systems.)

i) Consider the shortage penalties that occur in periods $t, ..., t_0 - 1$. By construction, there is at least the same amount of inventory in System 2 as that in System 1 at each period $t, ..., t_0 - 1$ after ordering. Therefore, the shortage penalty at each period $t, ..., t_0 - 1$ in System 2 is no more than that in System 1 with probability one.

ii) Consider the holding costs that occur in periods $t, ..., t_0 - 1$ charged for units ordered in periods $\ge t$. If the ordering quantity in System 2 is zero for all $t, ..., t_0 - 1$, then there is nothing to prove. Otherwise, let $s_0 \in [t, t_0)$ be the first period among $[t, t_0)$ such that the ordering quantity in System 2 is strictly positive. Let q_s and q'_s be the ordering quantities in Systems 1 and 2 at period s, respectively, and y_s and y'_s be the total inventory level after ordering in Systems 1 and 2 at period s, respectively. Then, we have $q'_s = 0, y'_s \ge$

$y_s, \forall s = t, ..., s_0 - 1$ and $y'_{s_0} = y_{s_0}$. Since System 2 started with more inventory than System 1 at period t, the amount of outdates in System 2 is at least as much as that in System 1 at each period $t, ..., t_0 - 1$. Therefore, we have $q'_s \geq q_s$, and $y'_s = y_s, \forall s = s_0 + 1, ..., t_0 - 1$, and $\sum_{k=1}^{K-1} x'_{k,s} \leq \sum_{k=1}^{K-1} x_{k,s}, \forall s = s_0 + 1, ..., t_0$.

Consider $s \in [s_0 + 1, ..., t_0]$. Then, it is sufficient to show that the holding cost that occurs at each period $s - 1$ charged for units ordered in periods $\geq t$ in System 2 is no more than that in System 1 with probability one (since no units were ordered in System 2 in periods $t, ..., s_0 - 1$). Units ordered at period t will be of age $s - t$ at period s. Since System 2 started with more inventory than System 1 at period t, we must have $\sum_{k=s-t+1}^{K-1} x'_{k,s} \geq \sum_{k=s-t+1}^{K-1} x_{k,s}$. Since we also have $\sum_{k=1}^{K-1} x'_{k,s} \leq \sum_{k=1}^{K-1} x_{k,s}$, we must have $\sum_{k=1}^{s-t} x'_{k,s} \leq \sum_{k=1}^{s-t} x_{k,s}$. By definition of t_0, none of the units ordered in periods $\geq t$ will be outdated by the beginning of period s. Therefore, the holding cost at period $s - 1$ charged for units ordered in periods $\geq t$ is exactly determined by the inventory level of these units at period s. Given $\sum_{k=1}^{s-t} x'_{k,s} \leq \sum_{k=1}^{s-t} x_{k,s}$, the holding cost that occurs at period $s - 1$ charged for units ordered in periods $\geq t$ in System 2 is no more than that in System 1 with probability one.

iii) Consider the total costs that occur in periods $\geq t_0$. If the ordering quantity in System 2 is zero for all $t, ..., t_0 - 1$, then System 2 will be empty at the beginning of period t_0, and the expected total cost that occurs in periods $\geq t_0$ in System 2 is no more than that in System 1 because $C^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \geq 0, k = 1, ..., K - 1$, $\forall \mathbf{x}_{t+1}, f_{t+1}$ (Lemma A1). Otherwise, s_0 is well-defined, and we have $\sum_{k=1}^{K-1} x'_{k,t_0} \leq \sum_{k=1}^{K-1} x_{k,t_0}; q'_s = 0, \forall s = t, ..., s_0 - 1$; and $q'_s \geq q_s, \forall s = s_0 + 1, ..., t_0 - 2$. Then, we have $\sum_{k=m}^{K-1} x'_{k,t_0} \leq \sum_{k=m}^{K-1} x_{k,t_0}, m = 1, ..., K - 1$. By Lemma A1, we have $0 \leq C^{(i)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq C^{(j)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}), 1 \leq i < j \leq K - 1$, $\forall \mathbf{x}_{t+1}$ such that $\sum_{k=1}^{K-1} x_{k,t+1} < \bar{y}_t$, $\forall f_{t+1}$. Therefore, the expected total cost that occurs in periods $\geq t_0$ in System 2 is no more than that in System 1.

Case 2: Suppose $\sum_{k=1}^{K-1} x_{k,t} \geq \bar{y}_t$. Consider the following two systems (both following FIFO issuing policy): System 1 starts from $\mathbf{x}_t$ and System 2 starts from $\mathbf{x}'_t$, where $x'_{k,t} = x_{k,t} + \epsilon$ and $x'_{m,t} = x_{m,t}, m = 1, ..., k - 1, k + 1, ..., K - 1$ (i.e., System 2 has ϵ more units of age k), and ϵ is any positive number. Let both Systems 1 and 2 follow an optimal ordering policy at each period. Since $\sum_{k=1}^{K-1} x_{k,t} \geq \bar{y}_t$, the ordering quantities in both systems are zero at period t. Let y_t and y'_t be the total inventory levels after ordering in Systems 1 and 2, respectively. Then, $\bar{y}_t \leq y_t \leq y'_t$. Hence, the expected marginal shortage penalty at period t in System 2 is no more than that in System 1, and there is no marginal holding or outdating cost at period t in either system (since nothing was ordered at period t). Let $\mathbf{x}_{t+1}$ and $\mathbf{x}'_{t+1}$ be the inventory vectors at period $t + 1$ for Systems 1 and 2, respectively. Then we have $x_{k,t+1} \leq x'_{k,t+1}, k = 1, ..., K - 1$. By induction assumption, we have $G^{(k)}_{t+1}(\mathbf{x}_{t+1}, f_{t+1}) \leq 0, k = 1, ..., K - 1, \forall \mathbf{x}_{t+1}, f_{t+1}$. Therefore, under the marginal-cost accounting scheme, System 2 has no more expected total cost than System 1, which completes the proof. □

Appendix. C. A Tight Example for Policy TB

In this section, we present an example to show that the performance bound of two for policy TB presented in Theorem 1 is tight. Consider an instance with product lifetime $K=1$, planning horizon $T=1$, and cost parameters $p=2$, $h=1$ and $w \gg 1$. Assume demand for $t=1$ satisfies $\mathbb{P}(D_1=1)=\frac{1}{2}, \mathbb{P}(D_1=0)=\frac{1}{2}$. Since $T=1$, it is straightforward to check that Assumption 1 holds because after demand realization at period $t=1$, issuing the oldest products first (i.e., FIFO) can help avoid outdate while without increasing future cost.

Suppose there is no initial inventory in the system at the beginning of $t=1$. Then, the balancing ordering quantity at $t=1$ (i.e., q_1^B) is determined by the unique solution to the following equation:

$$\frac{1}{2}p(1-q_1^B)=\frac{1}{2}(h+w)q_1^B.$$

Hence, $q_1^B=\frac{p}{p+h+w}=\frac{2}{3+w}$. Further, q_1^L (i.e., the lower bound on the optimal ordering quantity defined in §4) is given by the minimizer of $\frac{1}{2}p(1-q_1)+\frac{1}{2}(h+w)q_1$. Then, we have $q_1^L=0$ for any $w>1$. Let $q_1^U=\Phi_1^{-1}\left(\frac{p}{p+h}\right)$. Then, we have $q_1^U=\Phi_1^{-1}\left(\frac{2}{3}\right)=1$. Clearly, we have $q_1^L<q_1^B<q_1^U$. Therefore, we have $q_1^{TB}=q_1^B=\frac{2}{3+w}$. Accordingly, the expected total cost of policy TB is given by $\frac{1}{2}p(1-q_1^B)+\frac{1}{2}(h+w)q_1^B=p(1-q_1^B)$.

Consider an alternative policy A, which orders $q_1^A=0$. The expected total cost under policy A is $\frac{1}{2}p$. Under any cost parameters, we must have $\mathbb{E}[\mathscr{C}(OPT)]\leq\mathbb{E}[\mathscr{C}(A)]$. Then, when $w\to\infty$, we have:

$$\frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(OPT)]}\geq\frac{\mathbb{E}[\mathscr{C}(TB)]}{\mathbb{E}[\mathscr{C}(A)]}\to 2,$$

which implies that the performance bound of two for policy TB presented in Theorem 1 is tight. This result implies that under small shortage penalties but large outdating costs, the TB policy may not perform very well. However, we believe that this need not be a major concern in practice because in many practical settings, such as in blood inventory management, the shortage penalty is typically very high.

Moreover, we note that by using a similar construction, it can be shown that the expected cost of the existing policies presented in the literature (e.g., policies PB, DB) can also achieve twice (or even higher than twice) that of an optimal policy when the unit shortage penalty is substantially lower than the unit outdating cost. We next present an example to show that $\frac{\mathbb{E}[\mathscr{C}(PB)]}{\mathbb{E}[\mathscr{C}(OPT)]}$ can be strictly higher than two. Consider an instance with $K=T$. Let $p=2, h=1, w\gg 1, \beta=\frac{w}{1+w}$ (by Proposition 2, FIFO is clearly optimal for this example). Assume demand for $t=1$ satisfies $\mathbb{P}(D_1=1)=\frac{1}{2}, \mathbb{P}(D_1=0)=\frac{1}{2}$, and assume $D_t=0, \forall t\geq 2$ with probability one. Given that $h=1$ and $\beta=\frac{w}{1+w}$, we have $h+\beta(h+w)=h+w$. Therefore, if a unit is not consumed at period 1 and eventually outdated at the end of period T, the total discounted holding and outdating is given by $h+w$. Then, q_1^{PB} can be obtained by solving $\frac{1}{2}p(1-q_1^B)=\frac{Kh+w}{2(K-1)h+w}\times\frac{1}{2}(h+w)q_1^B$. Then, we have $q_1^{PB}=\frac{p}{p+(h+w)\frac{Kh+w}{2(K-1)h+w}}$. Accordingly, the expected total cost of policy TB is given by $\frac{1}{2}p(1-q_1^{PB})+\frac{1}{2}(h+w)q_1^{PB}=\frac{1}{2}p(1-q_1^{PB})\times\left(2+\frac{(K-2)h}{Kh+w}\right)$. Consider an alternative policy A, which orders $q_1^A=0$. The expected total cost under policy A is $\frac{1}{2}p$. Then, when $w=T=K\to\infty$, we have:

$$\frac{\mathbb{E}[\mathscr{C}(PB)]}{\mathbb{E}[\mathscr{C}(OPT)]}\geq\frac{\mathbb{E}[\mathscr{C}(PB)]}{\mathbb{E}[\mathscr{C}(A)]}\to 2.5.$$

Moreover, as shown above, under the current set of model parameters, we have $q_1^L=0, q_1^U=1$, which implies $q_1^L<q_1^{PB}<q_t^U$; therefore, the above ratio will stay the same even if we use q_t^L, q_t^U to truncate q_t^{PB}.