Online Supplement to
"Employees versus Contractors: An Operational Perspective"

Appendix Roadmap

This appendix is composed of two sections. Appendix A compares the performances of the staffing models through simulation. Appendix B presents the primary proofs of the analytical results of this paper. Due to the M&SOM 16-page limit for online appendices, the remaining proofs are available in the electronic companion, available for access on the authors' websites.

A Simulation Results

In this section, we conduct numerical simulations to further compare the staffing models. Throughout this section, we consider a world of two states with equal probabilities, i.e., $n = 2$ and $\alpha_1 = \alpha_2 = 1/2$. We first use the setting of Section 5.3 with $r = c$, which allows for a fair comparison of employees and contractors, and then we consider asymmetric labor costs to see how the performances of different models are sensitive to the cost parameters.

In particular, we look at uniform distributions, exponential distributions, and triangular distributions. We parameterize a triangular distribution as Tri(a,c,b) with $a \leq c \leq b$, with a being the minimum, b the maximum and c the mode. From Proposition 8, we know that the profit bound Π^* can be attained with a hybrid strategy in this setting. Therefore, we will use Π^* as a benchmark to evaluate the profit of the employees-only and contractors-only models.

Table 1: Parameters: $\mathbb{E}[D_1] = 1$, $\mathbb{E}[D_2] = 10$, $p = 10$, $r = c = 5$

D_1	D_2	Π^{emp}/Π^*	Π^{cont}/Π^*
U[0,2]	U[9,11]	0.2002	0.9612
U[0,2]	U[8,12]	0.2108	0.9686
U[0,2]	U[7,13]	0.2225	0.9751
U[0,2]	U[6,14]	0.2356	0.9809
U[0,2]	U[5,15]	0.2503	0.9860
U[0,2]	Exp(1/10)	0.4717	0.9984
U[0,2]	Tri(0,10,20)	0.2758	0.9929
U[0,2]	Tri(5,10,15)	0.2267	0.9767
U[0,2]	Tri(9,10,11)	0.1970	0.9582

D_1	D_2	Π^{emp}/Π^*	Π^{cont}/Π^*
Exp(1)	Tri(0,10,20)	0.2600	0.9924
Exp(1)	Tri(5,10,15)	0.2306	0.9822
Exp(1)	Tri(9,10,11)	0.2007	0.9728
Exp(1)	U[9,11]	0.2042	0.9741
Exp(1)	U[8,12]	0.2151	0.9778
Exp(1)	U[7,13]	0.2272	0.9812
Exp(1)	U[6,14]	0.2404	0.9845
Exp(1)	U[5,15]	0.2548	0.9877

Our first set of results, presented in Table 1, look at situations with a "high demand" and a "low demand" scenarios, specifically $\mathbb{E}[D_2] = 10 \times \mathbb{E}[D_1]$. Recall that, in this setting, the profit bound

Π^* is also the profit of hybrid staffing. It is clear from the table that using contractors consistently and significantly dominates using employees, and that hybrid staffing just adds a few percentage points of profit compared to contractors only. This finding verifies and strengthens our claim in Section 4.3 that contractors offer a valuable kind of flexibility for a rapidly expanding platform, and confirms that hybrid provides a limited advantage in this setting.

Table 2: Parameters: $\mathbb{E}[D_1] = \mathbb{E}[D_2] = 10$, $p = 10$

Table 2(a): $r = c = 5$

D_1	D_2	Π^{emp}/Π^*	Π^{cont}/Π^*
Exp(1/10)	U[0,20]	0.9716	0.9969
Exp(1/10)	U[1,19]	0.9719	0.9903
Exp(1/10)	U[2,18]	0.9721	0.9809
Exp(1/10)	U[3,17]	0.9721	0.9694
Exp(1/10)	U[4,16]	0.9720	0.9558
Exp(1/10)	U[5,15]	0.9717	0.9404
Exp(1/10)	U[6,14]	0.9712	0.9227
Exp(1/10)	U[7,13]	0.9706	0.9021
Exp(1/10)	U[8,12]	0.9697	0.8773
Exp(1/10)	U[9,11]	0.9687	0.8452

Table 2(b): $r = c = 6$

D_1	D_2	Π^{emp}/Π^*	Π^{cont}/Π^*
Exp(1/10)	U[0,20]	0.9560	0.9983
Exp(1/10)	U[1,19]	0.9516	0.9913
Exp(1/10)	U[2,18]	0.9466	0.9809
Exp(1/10)	U[3,17]	0.9411	0.9684
Exp(1/10)	U[4,16]	0.9349	0.9545
Exp(1/10)	U[5,15]	0.9279	0.9393
Exp(1/10)	U[6,14]	0.9199	0.9229
Exp(1/10)	U[7,13]	0.9110	0.9048
Exp(1/10)	U[8,12]	0.9007	0.8841
Exp(1/10)	U[9,11]	0.8890	0.8581

In Table 2, we look at a world with $\mathbb{E}[D_1] = \mathbb{E}[D_2]$. Table 2 directly shows that there is no clear dominance relation between using employees and using contractors. This further confirms our insights from Section 4.3: even in case that are not favorable to them, contractors cannot be arbitrarily bad compared to employees. We also note that the advantage of hybrid is again limited, at most a 12% increase in profit across all our simulations. This is quite far from the tight bound of a 100% increase in profit proved in Proposition 9. This discrepancy illustrates that the distributions that are particularly favorable to hybrid settings, and therefore close to the 100% bound, are not particularly realistic. Comparing Tables 2(a) and 2(b), we can also see that the performances of different models are sensitive to the cost parameter r. For example, when $D_1 \sim \text{Exp}(1/10)$ and $D_2 \sim \text{U}[4, 16]$, using employees is better than contractors if $r = c = 5$ but worse if $r = c = 6$.

While many of our results use $r = c$ to focus on the operational advantages of employees and contractors rather than simple direct cost advantages, Table 3 studies asymmetric costs: we fix the contractor's reservation wage r and vary the employee cost c. As before, we consider uniform, exponential, and triangular distributions in two different worlds: one where $\mathbb{E}[D_2] = 10 \times \mathbb{E}[D_1]$ (Table 3(a)) and the other where $\mathbb{E}[D_1] = E[D_2]$ (Table 3(b)). We aim to understand what value of c can make the two staffing options equivalent, e.g., $\Pi^{\text{emp}}/\Pi^{\text{cont}} = 1$. When the two distributions have different magnitudes, in Table 3(a), we need to significantly lower the employee costs to make them competitive (c from 2.5 to 3, which is 40-50% cheaper than the contractor $r = 5$), regardless of the choice of distributions. However, Table 3(b) shows that when the magnitude of the two distributions are the same (which favors employees), only a slight increase in the employee cost to $c = 5.25$ (e.g., a 5% increase) is enough to make contractors competitive.

Table 3: Parameters: $p = 10$, $r = 5$

Table 3(a): $\mathbb{E}[D_1] = 1$, $\mathbb{E}[D_2] = 10$

D_1	D_2	c	$\Pi^{\text{emp}}/\Pi^{\text{cont}}$
		2.5	1.1984
U[0,2]	Tri(5,10,15)	2.75	1.0840
		3	0.9726
		2.5	1.1661
Tri(0,1,2)	U[5,15]	2.75	1.0465
		3	0.7245
		2.5	1.3870
Tri(0,1,2)	Exp(1/10)	2.75	1.2108
		3	1.0594
		2.5	1.2325
Exp(1)	U[5,15]	2.75	1.1061
		3	0.9331
		2.5	1.2182
Exp(1)	Tri(5,10,15)	2.75	1.1019
		3	0.9886

Table 3(b): $\mathbb{E}[D_1] = \mathbb{E}[D_2] = 10$

D_1	D_2	c	$\Pi^{\text{emp}}/\Pi^{\text{cont}}$
		4.5	1.1342
		4.75	1.0691
Tri(5,10,15)	U[5,15]	5	1.0051
		5.25	0.9422
		5.5	0.8803
		4.5	1.2575
		4.75	1.1630
Tri(5,10,15)	Exp(1/10)	5	1.0707
		5.25	0.9806
		5.5	0.8929
		4.5	1.2229
		4.75	1.1265
U[5,15]	Exp(1/10)	5	1.0336
		5.25	0.9444
		5.5	0.8587

B Proofs

Proof of Proposition 1. We first argue that the "sup" term in Eq. (9) is attained and can be replaced by a max, when $z \in [r/p, 1]$. The set $A(z) = \{s_i \geq 0 \mid \mathbb{E}[\min\{D_i, s_i\}] = zs_i\}$ is closed, as it is the pre-image of $\{0\}$ of the continuous function $s_i \mapsto \mathbb{E}[\min\{D_i, s_i\}] - s_i$ (as D_i has a continuous distribution). $A(z)$ is also bounded as the continuity of the distribution of D_i implies that $\lim_{s_i \to +\infty} \mathbb{E}[\min\{D_i, s_i\}] - zs_i = -\infty$ when $z > 0$, which means that $\sup(A(z)) \neq +\infty$, and $A(z) = \{0\}$ when $z = 0$. Therefore, $A(z)$ is compact and the "sup" term is attained.

We now prove the equivalency between Eqs. (4) and (10). Given any feasible solution $z \in [r/p, 1]$ to Eq. (10), $(w = r/z, (s_i(z))_{\forall i})$ is also a feasible solution to Eq. (4), with the same objective value. This follows from the fact that, for all i, we either have $s_i(z) = 0$ or $z_i(s_i(z)) = z$ (for the feasibility) and that $w\mathbb{E}[\min\{D_i, s_i(z)\}] = rs_i(z)$ (for the objective value).

Conversely, we will prove that there exists an optimal solution $(w, \mathbf{s})$ to Eq. (4) verifying the following conditions. Its utilization $z = w/p$ verifies $r/p \leq z \leq 1$ and its contractor hours verify $s_i = \max\{s \geq 0 \mid \mathbb{E}[\min\{D_i, s\}] = zs\}$. If this is true, then z is an optimal solution to Eq. (10) with the same objective value, and the proof of Proposition 1 follows.

As discussed in the paper, we can choose an optimal solution $(w, \mathbf{s})$ to Eq. (4) such that $r \leq w \leq p$, which implies $r/p \leq z \leq 1$. Consider such an optimal solution. For all i, we have $s_i = 0$ or $z_i(s_i) = z$, which implies that $\mathbb{E}[\min\{D_i, s_i\}] = zs_i$. Suppose there exist an i such that $s_i < \max\{s \geq 0 \mid \mathbb{E}[\min\{D_i, s\}] = zs\}$. Changing the value of s_i to $s_i(z) = \max\{s \geq 0 \mid \mathbb{E}[\min\{D_i, s\}] = zs\}$ would lead to a feasible solution to Eq. (4) with higher or equal objective value. $\square$

Proof of Proposition 2. In Eq. (1), the state i profit can be rewritten as

$$p\mathbb{E}[\min\{D_i, x\}] - cx = p\left(\frac{1}{x}\int_0^x \bar{F}_i(v)\mathrm{d}v - \frac{c}{p}\right)x = p\left(z_i(x) - \frac{c}{p}\right)x.$$

Similarly, in Eq. (10), the state i profit can be rewritten as

$$p\mathbb{E}[\min\{D_i, s_i(z)\}] - rs_i(z) = p\left(zs_i(z) - \frac{r}{p}s_i(z)\right) = p\left(z - \frac{r}{p}\right)s_i(z),$$

where the first equality follows from $\mathbb{E}[\min\{D_i, s_i(z)\}] = zs_i(z)$. Consequently, we can rewrite the two objective functions in Eqs. (1) and (10) as follows:

$$p\sum_{i=1}^n \alpha_i\left(z_i(x) - \frac{c}{p}\right)x = p\left(\sum_{i=1}^n \alpha_i z_i(x) - \frac{c}{p}\right)x = p\left(\widehat{z}(x) - \frac{c}{p}\right)x,$$
$$p\sum_{i=1}^n \alpha_i\left(z - \frac{r}{p}\right)s_i(z) = p\left(z - \frac{r}{p}\right)\sum_{i=1}^n \alpha_i s_i(z) = p\left(z - \frac{r}{p}\right)\widehat{s}(z).$$

$\square$

Proof of Proposition 3. We prove this proposition by constructing an instance with two states of the world that does not have a corresponding demand D. Let $D_1 \sim U[0,1]$ and D_2 be 1 with probability 1/2 and 0 otherwise, and let $\alpha_1 = \alpha_2 = 1/2$. We plot their corresponding utilization curves $z_1(s)$ and $z_2(s)$ respectively and average them horizontally to get the curve $z_h(s)$, as illustrated in Figure 13. When $s_1 \leq 2$, we have:

$$z_1(s) = \frac{1}{s}\int_0^s \bar{F}_1(v)\mathrm{d}v = \frac{1}{s}\int_0^s 1 - \frac{v}{2}\mathrm{d}v = 1 - \frac{s}{4}.$$

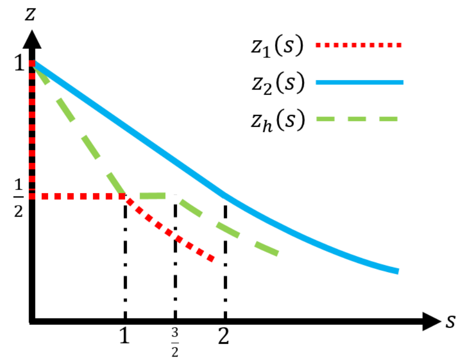

Figure 13: The horizontally-averaged utilization curve $z_h(x)$.

Similarly, when $s_2 \leq 1$, we have:

$$z_2(s) = \frac{1}{s}\int_0^s \bar{F}_2(v)\mathrm{d}v = \frac{1}{s}\int_0^s \frac{1}{2}\mathrm{d}v = \frac{1}{2}.$$

Consequently, over the interval $s \in [0, 3/2]$, we have

$$z_h(s) = \begin{cases} 1 - s/2, & \text{if } s \in [0,1] \\ 1/2, & \text{if } s \in [1, 3/2]. \end{cases}$$

This is horizontally-averaged utilization is plotted in Figure 13.

Now, suppose that there exists a demand distribution F_h that matches the utilization function $s \to z_h(s)$. That is, F_h satisfies $z_h(s) = \frac{1}{s}\int_0^s \bar{F}_h(v)dv$. Multiplying both sides by s and then differentiating, the condition becomes $\bar{F}_h(s) = z_h(s) + sz_h'(s)$. For $s < 1$, we have $\bar{F}_h(s) = 1 - s$. For $s \in (1, 2/3)$, we have $\bar{F}_h(s) = 1/2$. This implies a jump upward at 1, which contradicts F_h being a probability distribution.

Note that, for simplicity and clarity, the above counterexample used a discrete demand distribution $\bar{F}_2$, while our paper only considers continuous distributions. However, we can replace this discrete counterexample with a continuous approximation. For instance, we can define a set of random variables $\{D_2^\epsilon\}_{\epsilon>0}$, each with a continuous CDF

$$\bar{F}_2^\epsilon(v) = \begin{cases} 1 - \frac{v}{2\epsilon} & \text{if } v \in [0, \epsilon] \\ \frac{1}{2} & \text{if } v \in (\epsilon, 1) \\ \frac{1}{2} - \frac{v-1}{2\epsilon} & \text{if } v \in [1, 1+\epsilon] \\ 0 & \text{if } v \in (1+\epsilon, \infty) \end{cases},$$

and a corresponding utilization function $z_2^\epsilon(s)$. Since $\lim_{\epsilon\downarrow 0}\bar{F}_2^\epsilon(v)=\bar{F}_2(v),\ \forall v\geq 0$, it can be easily verified that $\lim_{\epsilon\downarrow 0} z_2^\epsilon(s)=z_2(s),\ \forall s\geq 0$, and that the resulting average utilization curve $z_h^\epsilon(s)$ (averaging $z_1(s)$ and $z_2^\epsilon(s)$ horizontally) also satisfies $\lim_{\epsilon\downarrow 0} z_h^\epsilon(s)=z_h(s),\ \forall s\geq 0$. Therefore, a small-enough choice ϵ gives the same contradiction as in the discrete counterexample. □

Proof of Proposition 4. Since we have assumed that $c\leq r$, Π^* is given by

$$\Pi^*=\sum_{i=1}^n \alpha_i\left[p\int_0^{q_i^*}\bar{F}_i(v)\mathrm{d}v-cq_i^*\right]$$

where $q_i^*=\bar{F}_i^{-1}(c/p)$. Since we have also assumed that $\bar{F}_1^{-1}(c/p)=\cdots=\bar{F}_n^{-1}(c/p)$, then, choosing $x=\bar{F}_1^{-1}(c/p)$, we have:

$$\Pi^*=\sum_{i=1}^n \alpha_i\left[p\int_0^{x}\bar{F}_i(v)\mathrm{d}v-cx\right].$$

The above expression also corresponds to the expected profit for employees only with x employees hour, which implies that $\Pi^{\text{emp}}\geq\Pi^*$. As $\Pi^*\geq\Pi^{\text{emp}}$ by definition, we have $\Pi^*=\Pi^{\text{emp}}$. □

Proof of Proposition 5. Since we have assumed that $r\leq c$, Π^* is then given by

$$\Pi^*=\sum_{i=1}^n \alpha_i\left[p\int_0^{q_i^*}\bar{F}_i(v)\mathrm{d}v-rq_i^*\right]$$

where $q_i^*=\bar{F}_i^{-1}(r/p)$. As we assume that $s_i(z)=\bar{F}_i^{-1}(r/p)$ for all i, then the equilibrium condition means we either have $\bar{F}_i^{-1}(r/p)=0$ or $z_i\left(\bar{F}_1^{-1}(r/p)\right)=z$. The first case would imply $r=p$ as F_i is continuous, but we assumed $r<p$ in this paper. Therefore, we have that $z_1\left(\bar{F}_1^{-1}(r/p)\right)=\cdots=z_n\left(\bar{F}_n^{-1}(r/p)\right)=z$. If the platform operates contractors only with utilization z, it will have $q_i^*=\bar{F}_i^{-1}(r/p)$ contractor hours in state $i=1,...,n$ and thus earn exactly Π^*:

$$\Pi^{\text{cont}}=\sum_{i=1}^n \alpha_i\left[p\int_0^{q_i^*}\bar{F}_i(v)\mathrm{d}v-rq_i^*\right]=\Pi^*.$$

□

Proof of Theorem 1. Let $\Pi_D^{\text{cont}}(z)=\mathbb{E}[p\min\{D,s_D(z)\}]-rs_D(z)$ be the contractors-only profit in a scenario with demand D and utilization z. Recall $s_D(z)=\max\{s\geq 0|\mathbb{E}[\min\{D,s\}]=zs\}$ is the number of contractor hours given utilization z (we use the subscript D in s_D for clarity). Let the corresponding optimal contractor profit be $\Pi_D^*=\max_{z\geq 0}\Pi_D^{\text{cont}}(z)$. The proof of the theorem relies on two (natural) technical results that we will prove later: result (A) is that for all z,

$\Pi_D^{\text{cont}}(z) \leq \Pi_{D'}^{\text{cont}}(z)$ if D is first-order stochastically dominated by D' (e.g., $\bar{F}_D \leq \bar{F}_{D'}$). And result (B) is that, for $\lambda > 0$, $\Pi_{\lambda D}^{\text{cont}}(z) = \lambda \Pi_D^{\text{cont}}(z)$.

Consider a family of ϵ-scaled distributions $D_1, \cdots, D_n$, as defined in Section 4.3 with corresponding optimal contractor utilization z_i^* for each scenario i. For scenario i, we have:

$$\begin{aligned}\Pi_{D_i}^* = \Pi_{D_i}^{\text{cont}}(z_i^*) &\leq \Pi_{(1+\epsilon)\lambda_i D_1}^{\text{cont}}(z_i^*) = (1+\epsilon)\lambda_i \Pi_{D_1}^{\text{cont}}(z_i^*) \\ &\leq (1+\epsilon)\lambda_i \Pi_{D_1}^{\text{cont}}(z_1^*) = (1+\epsilon)\Pi_{\lambda_i D_1}^{\text{cont}}(z_1^*) \\ &\leq (1+\epsilon)\Pi_{D_i}^{\text{cont}}(z_1^*),\end{aligned}$$

where we use (A) for the first and third inequalities, and (B) for the two equalities. The second inequality is simply the optimality of z_1^* in scenario 1. Therefore, if $r \leq c$, we have:

$$\Pi^* = \sum_i \alpha_i \Pi_{D_i}^* \leq (1+\epsilon) \sum_i \alpha_i \Pi_{D_i}^{\text{cont}}(z_1^*) = (1+\epsilon)\Pi^{\text{cont}}(z_1^*).$$

Therefore, choosing $w = pz_1^*$ proves the theorem. We now prove result (A). From Proposition 2, we have that $\Pi_D^{\text{cont}}(z) = p(z - r/p)s_D(z)$. Therefore, proving (A) is equivalent to proving that $s_D(z) \leq s_{D'}(z)$ when $\bar{F}_D \leq \bar{F}_{D'}$. We have $\mathbb{E}[\min\{D, s\}] = \int_0^s \bar{F}_D(u)\mathrm{d}u \leq \int_0^s \bar{F}_{D'}(u)\mathrm{d}u = \mathbb{E}[\min\{D', s\}]$. Therefore, $\mathbb{E}[\min\{D', s\}] - zs < 0 \implies \mathbb{E}[\min\{D', s\}] - zs < 0$. Furthermore, when $z > 0$, the function $s \mapsto \mathbb{E}[\min\{D, s\}] - zs$ is continuous with limit $-\infty$ for $s \to +\infty$. Therefore, $s_D(z)$ is also equal to $\inf\{s \geq 0, \mathbb{E}[\min\{D, s\}] - zs < 0\}$. Combined with the previous point, we get that $s_D(z) \leq s_{D'}(z)$, when $z > 0$, and $s_D(0) = 0 = s_{D'}(0)$ which concludes the proof of (A).

Similarly, proving (B) is equivalent to proving $s_{\lambda D}(z) = \lambda s_D(z)$ for any $\lambda > 0$, continuous distribution D and utilization z. This follows immediately from the equivalences below:

$$\begin{aligned}s_{\lambda D}(z) &= \max\{s \geq 0 | \mathbb{E}[\min\{\lambda D, s\}] = zs\} = \max\{s \geq 0 | \mathbb{E}[\min\{D, s/\lambda\}] = z(s/\lambda)\} \\ &= \max\{\lambda s' \geq 0 | \mathbb{E}[\min\{D, s'\}] = zs'\} = \lambda s_D(z),\end{aligned}$$

where the second line is just a change of variables $s' = s/\lambda$. □

Proof of Theorem 2. We construct the following instance: $\alpha_1 = 1 - c/p$, $\alpha_2 = c/p$, $D_1 \sim U[a, b]$ and $D_2 \sim U[\lambda a, \lambda b]$ for some $\lambda > b/a$. This instance is illustrated in Figure 6. As D_1 and D_2 are scaled versions of each other, the proof of Theorem 1 shows that the optimal contractor utilization is the same in both scenarios (we are not invoking the theorem directly since we did not assume $r \leq c$). Therefore, the number of contractors in each scenario is

$$s_1^* = \bar{F}_1^{-1}\left(\frac{r}{p}\right) = \frac{r}{p} \cdot a + \left(1 - \frac{r}{p}\right) b \quad \text{and} \quad s_2^* = \lambda s_1^*,$$

and the optimal contractor profit is

$$\Pi^{\text{cont}} = \alpha_1 \int_0^{s_1^*} \bar{F}_1(v)\mathrm{d}v + \alpha_2 \int_0^{s_2^*} \bar{F}_2(v)\mathrm{d}v = (\alpha_1 + \lambda\alpha_2) \int_0^{s_1^*} \bar{F}_1(v)\mathrm{d}v = (\alpha_1 + \lambda\alpha_2) \frac{(a + s_1^*)(1 - \frac{r}{p})}{2}.$$

If we let λ grow to infinity, we have $\lim_{\lambda\to\infty} \Pi^{\text{cont}} = \infty$.

Recall that if the platform wants to use employees only, it solves the average newsvendor problem and obtains $\widetilde{x} = \widetilde{F}^{-1}(c/p)$. In this instance, any point located in $[b, \lambda a]$ solves the average newsvendor, and without loss of optimality, we choose $\widetilde{x} = b$. Then,

$$\Pi^{\text{emp}} = \int_0^b \widetilde{F}(v)\mathrm{d}v = \alpha_1 \cdot \frac{(2a + (1 - \frac{c}{p})(b - a))(1 - \frac{c}{p}) - (b - a)(\frac{c}{p})^2}{2} + \alpha_2 \cdot b \left(1 - \frac{c}{p}\right)$$

which is independent of λ. Therefore, for any $\epsilon > 0$, we can always find a sufficiently large λ such that the ratio $\Pi^{\text{emp}}/\Pi^{\text{cont}} \leq \epsilon$. □

Before proving Theorems 3 and 4, we prove the following lemma, which provides a general tight lower bound for Π^{cont}/Π^* for any n and M as determined by the solution of an optimization problem. It is difficult to write $L(n, M)$ in closed form, but Theorems 3 and 4 are both consequences of this lemma. The limit $\lim_{M\to\infty} L(n, M)$, denoted by $L(n, \infty)$, equals $1/n$, corresponding to the result in Theorem 3. Meanwhile, $L(n, 1) = 1/(\sum_{i=1}^n 1/i)$ and Theorem 4 follows from proving that $L(n, M) \geq L(n, 1)/M$.

Lemma 1. *Let Π^{cont} and Π^* be as defined in Eqs.* (10) *and* (15)*. Assume $r = c$ and that there exists a number $M \geq 1$ such that, for all i and j in $\{1, ..., n\}$, we have $\alpha_i \bar{F}_i^{-1}(c/p)/(\alpha_j \bar{F}_j^{-1}(c/p)) \leq M$. Define the function $L(n, M)$ to be:*

$$L(n, M) = \inf_{\substack{A_1, \cdots, A_n > 0 \\ A_i/A_j \leq M\, \forall i,j}} \frac{1}{\sum_{i=1}^n \frac{A_i}{\sum_{j=1}^i A_j}} \tag{23}$$

Then, for all instances, we have $\Pi^{\text{cont}}/\Pi^* \geq L(n, M)$. This bound is tight in the following sense: for all n, $r = c < p$, and $\epsilon > 0$ there exists a set of demand distributions $(D_i^\epsilon)_{i\leq n}$ satisfying the above conditions, such that their optimal profit Π_ϵ^* and contractor profit $\Pi_\epsilon^{\text{cont}}$ satisfy:

$$\frac{\Pi_\epsilon^{\text{cont}}}{\Pi_\epsilon^*} \leq L(n, M) + \epsilon.$$

Proof of Lemma 1. For all $i = 1, ..., n$, let $\gamma_i = \bar{F}_i^{-1}(c/p)$ and $z_i^* = \int_0^{\gamma_i} \bar{F}_i(v)\mathrm{d}v/\gamma_i$, which are the optimal newsvendor quantity and the optimal contractor utilization in each state i respectively.

Let $u_i^* = z_i^* - c/p$. W.l.o.g., assume $u_1^* \geq ... \geq u_n^*$. Since we have assumed that $r = c$, we have

$$\Pi^* = \sum_{i=1}^{n} \alpha_i \left[p \int_0^{\gamma_i} \bar{F}_i(v) \mathrm{d}v - c\gamma_i \right] = p \sum_{i=1}^{n} \alpha_i u_i^* \gamma_i. \tag{24}$$

Let $\widetilde{s}_i(u) = s_i(u + c/p)$, $u \in [0, 1 - c/p]$, which represents the contractor supply in state i given that the platform chooses the contractor utilization $u + c/p$. Directly from the definition, we have $\widetilde{s}_i(u_i^*) = s_i(z_i^*) = \gamma_i$. With this change of variable, in the contractors-only model with utilization $u + c/p$ (e.g., with the choice of wage $w = r/(u + c/p)$), the expected profit in state i is $\widetilde{\Pi}_i^{\text{cont}}(u) = pu\widetilde{s}_i(u)$ and the average profit is

$$\widetilde{\Pi}^{\text{cont}}(u) = \sum_{i=1}^{n} \alpha_i \widetilde{\Pi}_i^{\text{cont}}(u) = pu \sum_{i=1}^{n} \alpha_i \widetilde{s}_i(u).$$

Let $K \in \text{argmax}_{i=1,...,n}\, u_i^* \sum_{j=1}^{i} \alpha_j \gamma_j$, which implies

$$u_K^* \sum_{j=1}^{K} \alpha_j \gamma_j \geq u_i^* \sum_{j=1}^{i} \alpha_j \gamma_j \iff u_i^* \leq u_K^* \frac{\sum_{j=1}^{K} \alpha_j \gamma_j}{\sum_{j=1}^{i} \alpha_j \gamma_j}, \quad \text{for all } i = 1, ..., n. \tag{25}$$

Therefore, we have:

$$\frac{\Pi^{\text{cont}}}{\Pi^*} \geq \frac{\widetilde{\Pi}^{\text{cont}}(u_K^*)}{p \sum_{i=1}^{n} \alpha_i u_i^* \gamma_i} = \frac{u_K^* \sum_{i=1}^{n} \alpha_i \widetilde{s}_i(u_K^*)}{\sum_{i=1}^{n} \alpha_i u_i^* \gamma_i} \geq \frac{u_K^* \sum_{i=1}^{n} \alpha_i \widetilde{s}_i(u_K^*)}{\sum_{i=1}^{n} \alpha_i \gamma_i u_K^* \frac{\sum_{j=1}^{K} \alpha_j \gamma_j}{\sum_{j=1}^{i} \alpha_j \gamma_j}} \tag{26}$$

where the first inequality uses Eq. (24) and the fact that u_K^* is not necessarily the optimal contractor utilization, and the second inequality follows from Eq. (25).

From the definition of $s_i(z)$ in Eq. (9), it is easy to verify that $s_i(z)$ is weakly decreasing in z, and thus $\widetilde{s}_i(u)$ is weakly decreasing in u. Therefore, then we have $\widetilde{s}_i(u_K^*) \geq \widetilde{s}_i(u_i^*) = \gamma_i$, for $i = 1, ..., K$ (remember that we assumed $u_1^* \geq \cdots \geq u_n^*$), and $\widetilde{s}_i(u_K^*) \leq \widetilde{s}_i(u_i^*) = \gamma_i$, for $i = K + 1, ..., n$. Then, we have:

$$\sum_{i=1}^{n} \alpha_i \widetilde{s}_i(u_K^*) = \sum_{i=1}^{K} \alpha_i \underbrace{\widetilde{s}_i(u)}_{\geq \gamma_i} + \sum_{i=K+1}^{n} \alpha_i \underbrace{\widetilde{s}_i(u)}_{\geq 0} \geq \sum_{i=1}^{K} \alpha_i \gamma_i. \tag{27}$$

We can use this inequality to simplify the numerator of the RHS of Eq. (26):

$$\frac{\Pi^{\text{cont}}}{\Pi^*} \geq \frac{u_K^* \sum_{i=1}^{K} \alpha_i \gamma_i}{\sum_{i=1}^{n} \alpha_i \gamma_i u_K^* \frac{\sum_{j=1}^{K} \alpha_j \gamma_j}{\sum_{j=1}^{i} \alpha_j \gamma_j}} = \frac{1}{\sum_{i=1}^{n} \frac{\alpha_i \gamma_i}{\sum_{j=1}^{i} \alpha_j \gamma_j}} \tag{28}$$

Let $A_i = \alpha_i\gamma_i$, which we can assume is positive without loss of generality. The assumption that $\alpha_j\gamma_j/(\alpha_i\gamma_i) \geq M$ is equivalent to $A_i/A_j \leq M$, for all i and j in $\{1, ..., n\}$. Therefore, by the definition of $L(n, M)$, we have $\Pi^{\text{cont}}/\Pi^* \geq L(n, M)$. Next, we construct an instance to show that this bound is tight. Going back to Eq. (23) from Lemma 1, notice that since we have assumed M to be finite, the infimum over $A_1, \cdots, A_n$ will be attained, because the objective function is continuous and we can transform the feasible set to be compact by constructing new variables $a_{i,j} = A_i/A_j$. Therefore, there exists $\{A_i^*\}_{i=1}^n > 0$ verifying $A_i^*/A_j^* \leq M$, $\forall i, j$ such that $L(n, M) = \frac{1}{\sum_{i=1}^n \frac{A_i^*}{\sum_{j=1}^i A_j^*}}$.

Similar to what we did in the proof of Proposition 3, for clarity, we will use discrete demand distributions, but we could use continuous distributions by replacing the discrete distributions with continuous approximations. For all $i = 1, \cdots, n$, let $\alpha_i = A_i^*/\gamma$, with $\gamma = \sum_i A_i^*$ so that we have $\sum_{i=1}^n \alpha_i = 1$. Now, we define the following discrete distributions with value in $\{0, \gamma\}$:

$$\bar{F}_i(v) = \begin{cases} u_i + c/p, & v \in [0, \gamma) \\ 0, & v \geq \gamma \end{cases}, \quad i = 1, ..., n,$$

where $u_1 = 1 - c/p$ and $u_i = u_1 A_1^* / \sum_{j=1}^i A_j^*$ for $i = 2, ..., n$. For ease of exposition, let $z_i = u_i + c/p$, for all $i = 1, ..., n$.

Since $\bar{F}_i^{-1}(c/p) = \gamma$ for all i, Π^* is attained when the platform uses employees only with the quantity of employee hours being γ:

$$\Pi^* = \sum_{i=1}^n \alpha_i \gamma u_i = \sum_{i=1}^n A_i^* u_i.$$

We will now argue that if the platform uses contractors only, then the optimal contractor utilization must be one of the values z_i. To see this, we compare the platform's profit under choosing utilization $z \in (z_{i+1}, z_i)$ and its profit under choosing z_i. If the platform chooses $z \in (z_{i+1}, z_i)$, it will earn $pz_j\gamma - cs_j(z)$ for all $j = 1, ..., i$ and 0 for the rest of the states; if the platform chooses z_i, it will earn $pz_j\gamma - cs_j(z_i)$ for all $j = 1, ..., i$ and 0 for the rest of the states. It is easy to verify that $s_i(z)$ is decreasing in z. Therefore, $s_j(z) > s_j(z_i)$ for all $j = 1, ..., i$, and consequently, the profit under choosing $z \in (z_{i+1}, z_i)$ is strictly dominated by the profit under choosing z_i.

If the platform chooses z_i for some i, then the contractor supply in each state and the resulting profit in each state (which we denote by $\widetilde{\Pi}_j^{\text{cont}}(u_i)$ to indicate its dependence on $u_i = z_i - c/p$) will respectively be

$$s_j(z_i) = \begin{cases} \dfrac{z_j}{z_i}\gamma, & \text{if } 1 \le j \le i \\ 0, & \text{if } i+1 \le j \le n, \end{cases} \quad \text{and} \quad \widetilde{\Pi}_j^{\text{cont}}(u_i) = \begin{cases} u_i s_j(z_i), & \text{if } 1 \le j \le i \\ 0, & \text{if } i+1 \le j \le n. \end{cases}$$

Consequently, the platform earns on average

$$\widetilde{\Pi}^{\text{cont}}(u_i) = \sum_{j=1}^{n} \alpha_j \widetilde{\Pi}_j^{\text{cont}}(u_i) = \sum_{j=1}^{i} \alpha_j \gamma \frac{z_j}{z_i} u_i = u_i \sum_{j=1}^{i} A_j^* \frac{u_j + c/p}{u_i + c/p}.$$

Therefore,

$$\frac{\widetilde{\Pi}^{\text{cont}}(u_i)}{\Pi^*} = \frac{u_i \sum_{j=1}^{i} A_j^* \frac{u_j+c/p}{u_i+c/p}}{\sum_{i=1}^{n} A_i^* u_i} = \frac{\frac{U_1 A_1^*}{\sum_{j=1}^{i} A_j^*} \sum_{j=1}^{i} A_j^* \frac{u_j+c/p}{u_i+c/p}}{\sum_{i=1}^{n} A_i^* \frac{U_1 A_1^*}{\sum_{j=1}^{i} A_j^*}} = \frac{\frac{1}{\sum_{j=1}^{i} A_j^*} \sum_{j=1}^{i} A_j^* \frac{u_j+c/p}{u_i+c/p}}{\sum_{i=1}^{n} \frac{A_i^*}{\sum_{j=1}^{i} A_j^*}},$$

where we replaced $u_i = u_1 A_1^* / \sum_{j=1}^{i} A_j^*$ in both the numerator and the denominator. In the limit $c \to p$, we have that all u_i's go to 0 (by definition of u_i), and thus the numerator of the previous equation goes to 1:

$$\lim_{c \to p} \frac{\widetilde{\Pi}^{\text{cont}}(u_i)}{\Pi^*} = \lim_{c \to p} \frac{\frac{1}{\sum_{j=1}^{i} A_j^*} \sum_{j=1}^{i} A_j^* \frac{u_j+c/p}{u_i+c/p}}{\sum_{i=1}^{n} \frac{A_i^*}{\sum_{j=1}^{i} A_j^*}} = \frac{1}{\sum_{i=1}^{n} \frac{A_i^*}{\sum_{j=1}^{i} A_j^*}} = L(n, M).$$

Note that the above expression does not depend on i. This implies that

$$\lim_{c \to p} \frac{\Pi^{\text{cont}}}{\Pi^*} = \lim_{c \to p} \frac{\max_{i=1,\ldots,n} \widetilde{\Pi}^{\text{cont}}(u_i)}{\Pi^*} = L(n, M),$$

completing the proof. □

Proof of Theorem 3. For all i, let $\gamma_i = \bar{F}_i^{-1}(c/p)$. Per Lemma 1, we have

$$\frac{\Pi^{\text{cont}}}{\Pi^*} \ge L(n, M) \ge L(n, \infty) = \inf_{A_1, \ldots, A_n > 0} \frac{1}{\sum_{i=1}^{n} \frac{A_i}{\sum_{j=1}^{i} A_j}},$$

where the second inequality holds because the program of $L(n, \infty)$ has less constraints than that of $L(n, M)$. Since $\frac{A_i}{\sum_{j=1}^i A_j} \leq 1$, we have

$$\frac{1}{\sum_{i=1}^n \frac{A_i}{\sum_{j=1}^i A_j}} \geq \frac{1}{\sum_{i=1}^n 1} = \frac{1}{n},$$

and consequently, $\Pi^{\text{cont}}/\Pi^* \geq L(n, \infty) \geq 1/n$. Indeed, we next show that $L(n, \infty) = 1/n$. Consider $A_i = \epsilon^{-i}$, $\forall i$, where $\epsilon > 0$. Then, as $\epsilon \to 0$,

$$\frac{1}{\sum_{i=1}^n \frac{A_i}{\sum_{j=1}^i A_j}} = \frac{1}{\sum_{i=1}^n \frac{\epsilon^{-i}}{\sum_{j=1}^i \epsilon^{-j}}} = \frac{1}{\sum_{i=1}^n \frac{\epsilon^{-i}}{\frac{\epsilon^{-i}-1}{1-\epsilon}}} = \frac{1}{(1-\epsilon)\sum_{i=1}^n \frac{1}{1-\epsilon^i}} \to \frac{1}{n},$$

which implies $L(n, \infty) = 1/n$.

Per Lemma 1, this bound $1/n$ is tight, because using the above sequence $\{A_i\}$ and following the construction procedure in Lemma 1, we will eventually obtain

$$\lim_{\substack{c \to p \\ \epsilon \to 0}} \frac{\Pi^{\text{cont}}}{\Pi^*} = L(n, \infty) = \frac{1}{n}.$$

□

Proof of Theorem 4. For all i, let $\gamma_i = \bar{F}_i^{-1}(c/p)$. From Eq. (28) in the proof of Lemma 1, we have

$$\frac{\Pi^{\text{cont}}}{\Pi^*} \geq \frac{1}{\sum_{i=1}^n \frac{\alpha_i \gamma_i}{\sum_{j=1}^i \alpha_j \gamma_j}} = \frac{1}{\sum_{i=1}^n \frac{1}{\sum_{j=1}^i \frac{\alpha_j \gamma_j}{\alpha_i \gamma_i}}} \geq \frac{1}{\sum_{i=1}^n \frac{1}{\sum_{j=1}^i \frac{1}{M}}} = \frac{1}{M \sum_{i=1}^n 1/i},$$

where the second inequality comes from the assumption that $\alpha_j \gamma_j / (\alpha_i \gamma_i) \geq 1/M$, for all i and j.

When $M = 1$, from $\alpha_j \gamma_j / (\alpha_i \gamma_i) \geq 1/M$ and $\alpha_j \gamma_j / (\alpha_i \gamma_i) \leq M$, we know all $\alpha_i \gamma_i$'s are equal. Therefore, $L(n, 1) = 1/\sum_{i=1}^n 1/i$. Per Lemma 1, this bound is tight. □

Proof of Proposition 6. In the EP case, the proof follows exactly the same steps as the proof of Proposition 1, whereas the UD case requires more care.

First, note that the "sup" term in Eqs. (16) and (17) can be replaced by a "max", given $x \geq 0$ and $z \in [r/p, 1]$. Indeed, the sets $\{s \geq 0 \mid \mathbb{E}[\min\{D_i - x, s\}^+ = zs]\}$ and $\{s \mid \mathbb{E}[\min\{D_i, x+s\} = z(x+s)]\}$ are compact because they are closed (reverse image of a closed set by a continuous function) and bounded (0 below, and a distribution-dependent bound above).

In the case of EP, the objective function of (7) is non-decreasing in s_i for any i, which can be seen when re-writing the objective as:

$$\mathbb{E}[p \min\{D_i, x + s_i\} - cx - w z_i^{\text{EP}}(x, s_i) s_i] = \mathbb{E}[p \min\{D_i, x\} - cx + (p - w) \min\{D_i - x, s_i\}^+].$$

With this fact, we can follow the exact same steps as the proof of Proposition 1 to prove Proposition 6 in the case of EP (omitted here to avoid the redundancy).

Under UD, the objective function of (7) is not necessarily increasing in s_i, which prevents us from using the same approach. First, given any feasible solution (x, z) to (19), define $w = r/z$. By definition of s_i^{UD}, we have $s_i^{\mathrm{UD}}(x, z) = 0$ or $\mathbb{E}[\min\{D_i, x + s_i^{\mathrm{UD}}(x, z)\}] = z(x + s_i^{\mathrm{UD}}(x, z))$, and the latter implies $z_i^{\mathrm{UD}}(x, s_i^{\mathrm{UD}}(x, z)) = z = r/w$. Therefore, $(x, w, (s_i^{\mathrm{UD}}(x, z))_{1 \le i \le n})$ is a feasible solution to (7) with the same objective function value, which implies that the optimum profit of Eq. (7) is at least as high as the one of Eq. (19), and that we can convert solutions of Eq. (19) into solutions of Eq. (7).

Therefore, to conclude the proof, we will prove that we can modify at least one *optimal* solution to Eq. (7) to construct a solution to Eq. (19) with the same profit (we restrict the solution to be optimal because the statement is not true otherwise). This is the most challenging part of the proof. To start, let $(x, w, \mathbf{s})$ be an *optimal* solution to (7), such that $w \in [r, p]$ (as discussed in Section 4.1, such an optimal solution exists). Let $z = r/w \in [r/p, 1]$.

We can restrict our attention to $z \in [r/p, 1)$ as $z = 1$ is always suboptimal for continuous demand distributions. If $s_i > 0$, then the constraints of (19) imply that $z_i(z, s_i) = z$, which implies $\mathbb{E}[\min\{D_i, x + s_i\}] = z(x + s_i)$. Consider the function $f : t \mapsto \mathbb{E}[\min\{D_i, t\}] - zt = \int_0^t \bar{F}_i(v)\mathrm{d}v - zt$. We want to prove that if $z < 1$, then $f(t) = 0$ has a unique solution verifying $t > 0$. We have $f'(t) = \bar{F}_i(t) - z$. We have $f(0) = 0$ and $f'(0) = 1 - z > 0$. As f' is weakly decreasing and $\lim_{t\to\infty} f(t) = -\infty$, $f(t) = 0$ has exactly one solution $\bar{t} > 0$. Thus, as $s_i > 0$ is a solution to the equation $\mathbb{E}[\min\{D_i, x + s\}] = z(x + s)$, we must have $x + s_i = \bar{t}$. As this solution is unique, we have $s_i = \sup\{s \mid \mathbb{E}[\min\{D_i, x + s\}] = z(x + s)\} = \widehat{s}_i^{\mathrm{UD}}(x, z)$.

Therefore, if $s_i > 0$, $s_i = \widehat{s}_i^{\mathrm{UD}}(x, z)$ and the expected profit in state i is $\int_0^{x+\widehat{s}_i^{\mathrm{UD}}(x,z)}[\bar{F}_i(v) - r/p]\mathrm{d}v$. If $s_i = 0$, the profit is $\int_0^x [\bar{F}_i(v) - r/p]\mathrm{d}v$. Because we assumed that $(x, w, \mathbf{s})$ is an optimal solution to (7), then when $s_i > 0$, $s_i = \widehat{s}_i^{\mathrm{UD}}(x, z)$ and the profit must be higher or equal to the profit with $s_i = 0$, which gives us

$$\int_0^{x+\widehat{s}_i^{\mathrm{UD}}(x,z)} [\bar{F}_i(v) - r/p]\mathrm{d}v \ge \int_0^x [\bar{F}_i(v) - r/p]\mathrm{d}v \iff \int_x^{x+\widehat{s}_i^{\mathrm{UD}}(x,z)} [\bar{F}_i(v) - r/p]\mathrm{d}v \ge 0.$$

This was the condition in the definition of s_i^{UD} in Eq.(18), and therefore, $s_i > 0$ implies $s_i = s_i^{\mathrm{UD}}(x, z)$. Similarly, if $s_i = 0$, the profit must be lower or equal to choosing $s_i = \widehat{s}_i^{\mathrm{UD}}(x, z)$ (as s_i is optimal), and therefore $\int_x^{x+\widehat{s}_i^{\mathrm{UD}}(x,z)}[\bar{F}_i(v) - r/p]\mathrm{d}v \le 0$ and $s_i^{\mathrm{UD}}(x, z) = 0$. Therefore, we have shown that we always have $s_i^{\mathrm{UD}}(x, z) = s_i$. Thus, (x, z) is a feasible solution to problem (19) with the same objective value. This concludes the proof. $\square$

Proof of Proposition 7. We first prove the result for the case $r < \widetilde{m}$. Assume without loss of generality that $\widetilde{m} = p\bar{F}_1(\widetilde{x})$, i.e., the first demand distribution has the highest marginal revenue at $\widetilde{x}$. Consider the original definition of the hybrid staffing problem in Eq. (7). The best employees-only solution sets $x = \widetilde{x}$ and $s_i = 0$, $\forall i \in \{1, \cdots, n\}$. We will show that another feasible solution, with $x = \widetilde{x}$, $s_i = 0$, $\forall i \in \{2, \cdots, n\}$, and $s_1 > 0$ has higher expected profit than the employees-only solution. Per Eq. (7), the problem of choosing the optimal s_1 in this setting reduces to:

$$\begin{aligned} \max_{s_1, w} \quad & \mathbb{E}[p \min\{D_1, \widetilde{x} + s_1\} - w z_1(\widetilde{x}, s_1) s_1] \\ \text{s.t.} \quad & w \cdot z_1(\widetilde{x}, s_1) = r \text{ or } s_1 = 0 \end{aligned}$$

As we can always set $w = r/z_1(\widetilde{x}, s_1)$ when $s_1 > 0$, where z_1 could be either z_1^{EP} or z_1^{UD} depending on the version of the problem under consideration, the constraint can always be satisfied regardless of the value of s_1. Additionally, the last term of the objective is always equal to rs_1. Therefore, the optimal s_1 solves:

$$\max_{s_1} \quad \mathbb{E}[p \min\{D_1, \widetilde{x} + s_1\} - rs_1] = p \int_{\widetilde{x}}^{\widetilde{x}+s_1} \bar{F}_1(v) \mathrm{d}v - rs_1.$$

It turns out that the optimal s_1 is independent of the dispatch policy. The derivative of the objective function in $s_1 = 0$ is $p\bar{F}_1(\widetilde{x}) - r = \widetilde{m} - r$, which is strictly positive by the assumption that $r < \widetilde{m}$. Therefore, $s_1 > 0$ is optimal.

It remains to be shown that if $r \geq \widetilde{m}$, using employees only with $\widetilde{x}$ hours is the optimal strategy. Consider the relaxation of Problem (7) where we allow w to depend on the scenario:

$$\begin{aligned} \max_{\mathbf{s} \geq 0, \mathbf{w} \geq 0} \quad & \sum_{i=1}^{n} \alpha_i \mathbb{E}[(p - w_i) \min\{D_i, s_i\}] \\ \text{s.t.} \quad & w_i \cdot z_i(\widetilde{x}, s_i) = r \text{ or } s_i = 0 \qquad \forall i. \end{aligned} \tag{29}$$

This is a relaxation of (7), because we only need to add the constraint $w_1 = \cdots = w_n = w$ for the two problems to become equivalent. This relaxation is the "Dynamic Wage" setting introduced in Section 6.2, and studied in detail in the proof of Theorem 6. Specifically, Theorem 6 proves that using employees-only with $\widetilde{x}$ hours is an optimal strategy when $r \geq \widetilde{m}$. However, as we do not use contractors, this solution to (29) is also a feasible solution to the original problem (7). As (29) is a valid relaxation of (7), the employees-only solution $x = \widetilde{x}, \mathbf{s} = 0$ must also be optimal in (7), which concludes the proof. □

Proof of Theorem 5. Note that most of the proof will not be influenced by choice of EP or UD. We first construct an instance and prove that its unique optimal solution is a contractors-only one. In this instance, we have $c > r$ and two states of the world, each with probability 1/2. The two distributions are characterized by a set of parameters $\{p, r, d_1, d_2, k_1, k_2\}$:

$$\bar{F}_1(v) = \begin{cases} -\frac{v}{d_1} + 1, & \text{if } v \in [0, P_1] \\ -k_1(v - P_1) + \frac{r}{p}, & \text{if } v \in (P_1, P_1 + \frac{r}{k_1 p}] \end{cases}$$

$$\bar{F}_2(v) = \begin{cases} -\frac{v}{d_2} + 1, & \text{if } v \in [0, P_2] \\ -k_2(v - P_2) + \frac{r}{p}, & \text{if } v \in (P_2, P_2 + \frac{r}{k_2 p}] \end{cases}$$

where $P_1 = (1 - r/p)d_1$, $P_2 = (1 - r/p)d_2$. Both distributions are piecewise linear, with respective kinks at $(P_1, r/p)$ and $(P_2, r/p)$. Moreover, k_1 is chosen to be a large number and k_2 to be small (the exact numbers will be specified later). The two distributions are illustrated in Figure 8.

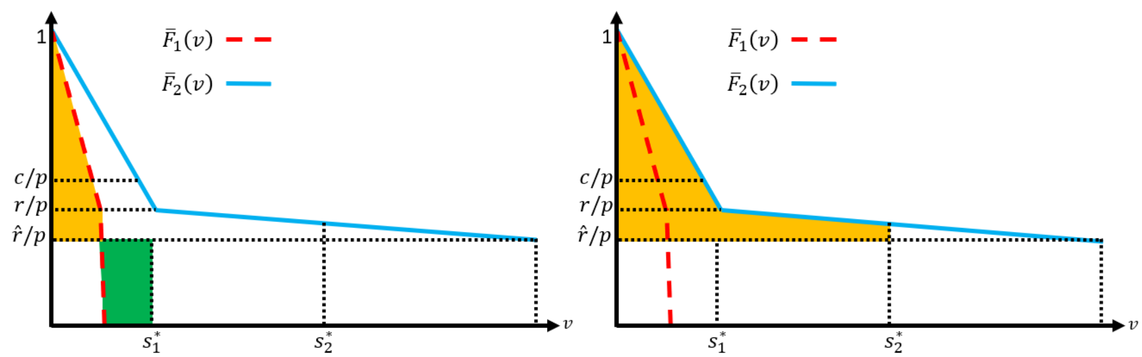

Figure 14: We illustrate the optimal contractors-only solution when the reservation wage is $\hat{r}$. The optimal contractor hours are (s_1^*, s_2^*). On the left, we show the profit in scenario 1: it is the sum of the yellow areas minus the green area (divided by p); green corresponds to losses due to overstaffing. On the right, the yellow area divided by p is the profit of scenario 2.

When the reservation wage is r, the solution to the relaxation of Problem (7) introduced in Eq. (29) of the previous proof is $x = 0, s_1^* = \bar{F}_1^{-1}(r/p), s_2^* = \bar{F}_1^{-1}(r/p)$. The proof of Theorem 6 shows this is the unique optimal solution when $r < c$. Note that the utilization of this solution in the two scenarios of the relaxation is the same: $z^* = (1 - r/p)/2$. Therefore, this solution is also feasible and the unique optimal solution of the original problem (7), as we can set $w^* = r/z^*$ to achieve s_1^* and s_2^*. As this is a contractors-only solution, there is no difference between EP and UD. Intuitively, this is a variant on Theorem 1: the two demand distributions are scaled versions of each other above the horizontal level r/p and $r < c$.

Now, consider the modified instance where r is replaced by $\hat{r} < r$. The two complementary distributions then look very different if we consider the curves above $\hat{r}/p$ instead of considering the curves above r/p. We specifically choose the following set of parameters: $\{p = 10,\ c = 3,\ r = 2.5,\ \hat{r} = 2,\ d_1 = 4,\ d_2 = 5,\ k_1 = 10^3,\ k_2 = 10^{-3}\}$. If the platform uses no employees ($x = 0$), then, solving the optimization problem (4) gives $s_1^* = 7.2271$ and $s_2^* = 47.5111$, yielding an expected profit of 16.2695. The profit from this solution in each state of the world is represented in Figure 14, where we can see that the rigid wage causes overstaffing in state 1 and understaffing in state 2.

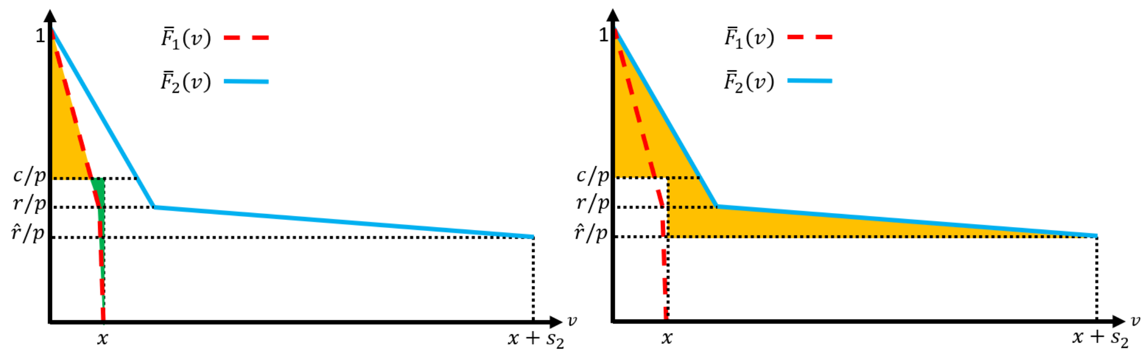

Figure 15: A hybrid solution yields more profit than an employees-only solution. The profit of the two scenarios is shown with the colored areas, as described in the caption of Fig. 14.

Next, we will show that adding employee hours can help reduce the mismatch cost, so that the platform may end up having a higher total profit, despite the fact that employees are assumed to be more costly. This will imply that when the reservation earning is $\hat{r}$, using only contractors is suboptimal, so the unique optimal solution is a hybrid one. Consider the following hybrid staffing strategy with $x = P_1 + r/(k_1 p) = 3.0003$, $s_1 = 0$, and $s_2 = P_2 + (r - \hat{r})/(k_2 p) - x = 50.7497$, as shown in Figure 15. The wage w is chosen such that $w\mathbb{E}[\min\{D_2 - x, s_2\}^+] = rs_2$ under EP or is chosen such that $w\mathbb{E}[\min\{D_2, x + s_2\}] = r(x + s_2)$ under UD. We first argue that the equilibrium constraint in each state is satisfied. In state 1, $s_1 = 0$ satisfies the equilibrium constraint. In state 2, the equilibrium constraint is automatically satisfied because of how the wage is chosen. The resulting total profit under this hybrid staffing strategy is 17.5934, which is strictly greater than 16.2695, the profit the platform will gain if it uses contractors only. Therefore, we have shown that when the reservation earning decreases from r to $\hat{r}$, using contractors only is no longer optimal, and the optimal solution switches to a hybrid one. □