

Online Appendix to Çakıcı and Karaesmen: *When Yield is Not the Only Supply Uncertainty: Newsvendor Model of a Trade Agent.*

EC.1. Preliminary results

The following property of the yield distributions is used in the proof of Theorem 1.

LEMMA EC.1. *Let s be a non-negative scalar and \mathbf{U} be a continuous random variable with support $[0, 1]$. If $\mathbb{E}[\mathbf{U}] < 1$, then $\frac{\mathbb{E}[\min(\mathbf{U}, s)]}{\mathbb{E}[\mathbf{U}]} > s$ is satisfied for all $s \in (0, 1)$.*

Proof of Lemma EC.1 Let $\tau : [0, 1] \rightarrow \mathbb{R}$ and $v : [0, 1] \rightarrow \mathbb{R}$ be two separate functions of s . We define $\tau(s) = s$ and $v(s) = \frac{\mathbb{E}[\min(\mathbf{U}, s)]}{\mathbb{E}[\mathbf{U}]}$. Note that $\tau(0) = v(0) = 0$ and $\tau(1) = v(1) = 1$. τ is a linear increasing function of s and $\frac{d\tau(s)}{ds} = 1$ for all $s \in [0, 1]$. v is a concave and nondecreasing function of s and $\frac{dv(0)}{ds} = \frac{1}{\mathbb{E}[\mathbf{U}]} > 1 = \frac{d\tau(0)}{ds}$ because $\mathbb{E}[\mathbf{U}] < 1$. Note that the functions $\tau(s)$ and $v(s)$ intersect at two points $s = 0$ and $s = 1$. Suppose there exists a $\tilde{s} \in (0, 1)$ such that $\tau(\tilde{s}) = v(\tilde{s})$ and $\tau(s) \neq v(s)$ for $s \in (0, 1) \setminus \{\tilde{s}\}$. We know

$$\tau(\tilde{s}) = \tau(0) + \int_0^{\tilde{s}} \frac{d\tau(s)}{ds} ds = v(0) + \int_0^{\tilde{s}} \frac{dv(s)}{ds} ds = v(\tilde{s}). \quad (\text{EC.1})$$

and $\frac{dv(s)}{ds} \leq \frac{dv(0)}{ds}$ for $s > 0$ because v is concave. If $\frac{dv(s)}{ds} > \frac{d\tau(s)}{ds}$ for all $s \in [0, \tilde{s}]$, then $v(\tilde{s}) > \tau(\tilde{s})$ must hold, which leads to a contradiction to (EC.1). Therefore, there must exist a unique $s^o \in (0, \tilde{s})$ such that $\frac{dv(s)}{ds} < \frac{d\tau(s)}{ds} = 1$ for all $s > s^o$. However, this would lead us to $v(1) < \tau(1)$ because

$$\tau(1) = \tau(\tilde{s}) + \int_{\tilde{s}}^1 \frac{d\tau(s)}{ds} ds \quad (\text{EC.2})$$

$$v(1) = v(\tilde{s}) + \int_{\tilde{s}}^1 \frac{dv(s)}{ds} ds = \tau(\tilde{s}) + \int_{\tilde{s}}^1 \frac{dv(s)}{ds} ds. \quad (\text{EC.3})$$

This leads to a contradiction as $v(1) = \tau(1)$ by definition. Therefore, $\tau(s)$ and $v(s)$ cannot intersect at any $\tilde{s} \in (0, 1)$. Furthermore $v(s) > \tau(s)$ for all $s \in (0, 1)$ because $\frac{dv(0)}{ds} > \frac{d\tau(0)}{ds}$. \square

EC.2. Proofs of analytical results

Proof of Theorem 1 By definition, $\Omega = \{b : w'(b) > 0\}$ as $\Omega = [0, \infty)$ is the support of \mathbf{Z} .

We let $R = \rho \mathbb{E}[\min(q\mathbf{U}, D)] - hq\mathbb{E}[\mathbf{U}]$ to simplify the notation. Then, the expected profit function becomes

$$\mathbb{E}[V^y(b)] = w(b)[R - bq] - \pi D, \quad (\text{EC.4})$$

$$\mathbb{E}[V^y(0)] = -\pi D. \quad (\text{EC.5})$$

Note that $\mathbb{E}[V^y(b)]$ is a differentiable function in Ω as w is assumed differentiable. We prove each result in turn:

(i): $R \leq 0 \Leftrightarrow \frac{\mathbb{E}[\min(\mathbf{U}, \frac{D}{q})]}{\mathbb{E}[\mathbf{U}]} \leq \frac{h}{\rho}$. In this case, $\mathbb{E}[V^y(b)]$ is decreasing in $b \in \Omega$ because $w(b) > 0$ for $b > 0$ and $\mathbb{E}[V^y(0)] > \mathbb{E}[V^y(b)]$ for all $b > 0$. Therefore, the optimal bid is $b^* = 0$ and it is the unique maximizer. Note that the conditions in Assumption 1 are not necessary to identify the optimal solution when $R \leq 0$.

(ii) and (iii): $R > 0 \Leftrightarrow \frac{\mathbb{E}[\min(\mathbf{U}, \frac{D}{q})]}{\mathbb{E}[\mathbf{U}]} > \frac{h}{\rho}$. The derivative of the expected profit function is

$$\frac{d\mathbb{E}[V^y(b)]}{db} = w'(b)R - w'(b)bq - w(b)q = w'(b)q \left[\frac{R}{q} - b - \frac{w(b)}{w'(b)} \right] \quad \text{for } b \in \Omega. \quad (\text{EC.6})$$

Given that $w'(b) > 0$ for $b \in \Omega$

$$\frac{d\mathbb{E}[V^y(b)]}{db} = 0 \Leftrightarrow \frac{R}{q} = b + \frac{w(b)}{w'(b)}. \quad (\text{EC.7})$$

$\frac{d\mathbb{E}[V^y(b)]}{db} = 0$ has a unique solution in Ω because $R > 0$ and $\left(b + \frac{w(b)}{w'(b)}\right)$ is strictly increasing by Assumption

1. Let b° be the unique solution that satisfies

$$b^\circ + \frac{w(b^\circ)}{w'(b^\circ)} = \frac{R}{q}. \quad (\text{EC.8})$$

b° could be an inflection point, a global minima or maxima of $\mathbb{E}[V^y(b)]$. We have

$$\frac{d\mathbb{E}[V^y(0)]}{db} = w'(0)R - w'(0)0q - w(0)q = w'(0)R > 0. \quad (\text{EC.9})$$

because $0 \in \Omega$, $w'(0) > 0$ and $w(0) = 0$ by definition. In addition,

$$\frac{d\mathbb{E}[V^y(b)]}{db} = w'(b)q \left[\frac{R}{q} - b - \frac{w(b)}{w'(b)} \right] < 0 \quad \text{for } b > b^\circ \quad (\text{EC.10})$$

since $w'(b) > 0$ and $\left(b + \frac{w(b)}{w'(b)}\right)$ is strictly increasing by Assumption 1. Combining all, we see that $\mathbb{E}[V^y(b)]$ is strictly increasing in $[0, b^\circ)$ and strictly decreasing in (b°, ∞) because $\mathbb{E}[V^y(b)]$ is differentiable and hence is continuous in Ω and b° is unique. So, $\mathbb{E}[V^y(b)]$ is unimodal in Ω and its unique optimal solution is $b^* = b^\circ$.

(iv): From the analysis above, the condition for the optimal bid to be positive is

$$b^* > 0 \Leftrightarrow \frac{\mathbb{E}[\min(\mathbf{U}, \frac{D}{q})]}{\mathbb{E}[\mathbf{U}]} > \frac{h}{\rho}. \quad (\text{EC.11})$$

When $D/q > h/\rho$, we have

$$\frac{\mathbb{E}[\min(\mathbf{U}, \frac{D}{q})]}{\mathbb{E}[\mathbf{U}]} \geq \frac{\mathbb{E}[\min(\mathbf{U}, \frac{h}{\rho})]}{\mathbb{E}[\mathbf{U}]} > \frac{h}{\rho}. \quad (\text{EC.12})$$

The first inequality follows from $D/q > h/\rho$ and the second inequality follows from Lemma EC.1 in Online Appendix EC.1 because $h/\rho < 1$. This proves $b^* > 0$ holds if $D/q > h/\rho$. \square

Proof of Lemma 1 (i) The function in Assumption 1 is

$$b + \frac{w(b)}{w'(b)} = b + \frac{1}{\mu(b)}. \quad (\text{EC.13})$$

When $\mu(b)$ is differentiable, the condition of strict monotonicity in Assumption 1 reduces to

$$\frac{d}{db} \left(b + \frac{1}{\mu(b)} \right) > 0 \Leftrightarrow 1 + \frac{-\frac{d}{db}\mu(b)}{\mu(b)^2} > 0 \Leftrightarrow \frac{d\mu(b)}{db} < \mu(b)^2. \quad (\text{EC.14})$$

(ii) By definition $\mu(b)$ is defined over $(0, \infty)$ and $\mu(b) = \frac{w'(b)}{w(b)} > 0$ for all $b > 0$. When \mathbf{Z} has DRHR, $\mu(\tilde{b}) \leq \mu(b)$ for all $\tilde{b} > b > 0$. Working with the basic definition of a decreasing function, we have

$$\mu(\tilde{b}) \leq \mu(b) \Rightarrow \frac{1}{\mu(\tilde{b})} \geq \frac{1}{\mu(b)} \Rightarrow b + \frac{1}{\mu(\tilde{b})} \geq b + \frac{1}{\mu(b)} \Rightarrow (\tilde{b} - b) + b + \frac{1}{\mu(\tilde{b})} > b + \frac{1}{\mu(\tilde{b})} \geq b + \frac{1}{\mu(b)} \quad (\text{EC.15})$$

because $\tilde{b} > b$. Hence, if \mathbf{Z} has DRHR, $b + \frac{w(b)}{w'(b)}$ is strictly increasing for $b > 0$ and Assumption 1 is satisfied.

(iii) Follows when the result in part (ii) is combined with the fact that all distributions with a logconcave cdf have DRHR; see Ross et al. (2005). \square

Proof of Theorem 2 We prove each result in turn:

Effect of D , q and D/q : Given q , ρ , and h , we consider the effect of D in two intervals: In the first interval (or set), $D \in \{\eta : \eta \geq 0, \mathbb{E}[\min(\mathbf{U}, \eta/q) \leq \mathbb{E}[\mathbf{U}]h/\rho]\}$ and in the second interval $D \in \{\eta : \mathbb{E}[\min(\mathbf{U}, \eta/q) > \mathbb{E}[\mathbf{U}]h/\rho]\}$. Suppose D is in the first interval. Then, $b^* = 0$ by Theorem 1. When D starts increasing from 0, $b^* = 0$ remains unchanged as long as D is in the first interval. As D increases, it moves from the first interval to the second one and we have $b^* = b^\circ > 0$ where b° satisfies the first order condition in equation (6) in Theorem 1. D only appears in the term on the right hand side of (6) and that term is nondecreasing in D . Because of Assumption 1, b° that satisfies the first order condition, increases as D increases. Therefore, b^* is nondecreasing in D . The effect of q is the opposite of the effect of D because the parameters D and q appear in the optimality conditions only in the coupled term $\frac{D}{q}$. Therefore b^* is nonincreasing in q . The effect of D/q follows from this discussion.

Effect of r , π and t : The effect of increase in r or increase in π are the same, and their effect is the opposite of an increase in t because these three parameters all appear together as $\rho = (r - t + h + \pi)$ in equation (6) in Theorem 1. We only discuss the effect of ρ here. Note that $\rho > 0$ by definition. Suppose $b^* = 0$, i.e., $0 < \rho \leq \frac{qh\mathbb{E}[\mathbf{U}]}{\mathbb{E}[\min(q\mathbf{U}, D)]}$. Then, any increase in ρ has no effect on b^* as long as $\rho \leq \frac{qh\mathbb{E}[\mathbf{U}]}{\mathbb{E}[\min(q\mathbf{U}, D)]}$. However, as ρ increases further, we get $\rho > \frac{qh\mathbb{E}[\mathbf{U}]}{\mathbb{E}[\min(q\mathbf{U}, D)]}$ and the optimal solution switches from $b^* = 0$ to $b^* = b^\circ > 0$ where b° satisfies the first order condition in equation (6) in Theorem 1. Note that ρ only appears in the term on the right hand side of this equation and that term is increasing in ρ . Because of Assumption 1, b° that satisfies the first order condition, increases as ρ increases. Therefore b^* is nondecreasing in ρ . This proves that b^* is nondecreasing in r and π and nonincreasing in t .

Effect of h and h/ρ : Note that $h/\rho = h/(r - t + h + \pi)$ is nondecreasing in h and $b^* = b^\circ > 0$ when $h = 0$ because $h < \frac{\rho\mathbb{E}[\min(q\mathbf{U}, D)]}{q\mathbb{E}[\mathbf{U}]}$. Then, any increase in h while $h \in [0, \frac{\rho\mathbb{E}[\min(q\mathbf{U}, D)]}{q\mathbb{E}[\mathbf{U}]}]$ retains $b^* = b^\circ$ where b° satisfies the first order condition in equation (6) in Theorem 1. h only appears in the right hand side term in the first order condition, and the term on the right hand side is nonincreasing in h because $\mathbb{E}[\mathbf{U}] \geq 0$. Consequently, b° that satisfies the first order condition, is nonincreasing in h due to Assumption 1. When h increases further and $h > \frac{\rho\mathbb{E}[\min(q\mathbf{U}, D)]}{q\mathbb{E}[\mathbf{U}]}$ then b^* switches from $b^* = b^\circ > 0$ to $b^* = 0$. Further increase in h does not affect $b^* = 0$. Therefore, b^* is nonincreasing in h . The effect of h/ρ is the same and follows from the same discussion. \square

Proof of Theorem 3 Let $\mu(b)$ and $\tilde{\mu}(b)$ be the RHR functions of \mathbf{Z} and $\tilde{\mathbf{Z}}$, respectively. There is a unique optimal bid b^* and \tilde{b}^* corresponding to the problems with \mathbf{Z} and $\tilde{\mathbf{Z}}$, respectively, because \mathbf{Z} and $\tilde{\mathbf{Z}}$ both satisfy Assumption 1. Note that if $b^* = 0$ holds (i.e., $R \leq 0$), then $\tilde{b}^* = 0$ also holds and vice versa. The bid outcome uncertainty affects an optimal bid only if $b^* = b^\circ > 0$ and equation (6) in Theorem 1 is satisfied. When $\mathbf{Z} \geq_{rh} \tilde{\mathbf{Z}}$, we have $\mu(b) \geq \tilde{\mu}(b)$ for all b . So, it follows that $[b + 1/\mu(b)] \leq [b + 1/\tilde{\mu}(b)]$. Then,

$$b^\circ + \frac{1}{\tilde{\mu}(b^\circ)} \geq b^\circ + \frac{1}{\mu(b^\circ)} = \rho \mathbb{E}[\min\left(\mathbf{U}, \frac{D}{q}\right) - h\mathbb{E}[\mathbf{U}]. \quad (\text{EC.16})$$

Because \tilde{b}° is optimal, it has to satisfy equation (6), too. So, $b^\circ \geq \tilde{b}^\circ$ because $b + 1/\tilde{\mu}(b)$ is strictly increasing by Assumption 1. \square

Proof of Theorem 4 We define two functions of scalar $s \geq 0$ as

$$\beta(s) = \rho\mathbb{E}[\min(\mathbf{U}, s)] - h\mathbb{E}[\mathbf{U}], \quad \tilde{\beta}(s) = \rho\mathbb{E}[\min(\tilde{\mathbf{U}}, s)] - h\mathbb{E}[\tilde{\mathbf{U}}]. \quad (\text{EC.17})$$

We have

$$\beta(0) = -h\mathbb{E}[\mathbf{U}], \quad \tilde{\beta}(0) = -h\mathbb{E}[\tilde{\mathbf{U}}] \quad (\text{EC.18})$$

and

$$\beta(s) = (\rho - h)\mathbb{E}[\mathbf{U}], \quad \tilde{\beta}(s) = (\rho - h)\mathbb{E}[\tilde{\mathbf{U}}], \quad \text{for } s \geq 1. \quad (\text{EC.19})$$

Given $s = \frac{D}{q}$, the optimal bids b^* and \tilde{b}^* satisfy

$$\beta\left(\frac{D}{q}\right) > 0 \Leftrightarrow b^* > 0 \quad \text{where} \quad \beta\left(\frac{D}{q}\right) = b^* + \frac{1}{\mu(b^*)}, \quad (\text{EC.20})$$

$$\tilde{\beta}\left(\frac{D}{q}\right) > 0 \Leftrightarrow \tilde{b}^* > 0 \quad \text{where} \quad \tilde{\beta}\left(\frac{D}{q}\right) = \tilde{b}^* + \frac{1}{\mu(\tilde{b}^*)}. \quad (\text{EC.21})$$

Clearly, $b^* \geq \tilde{b}^* > 0$ if and only if $\beta\left(\frac{D}{q}\right) \geq \tilde{\beta}\left(\frac{D}{q}\right)$ because \mathbf{Z} satisfies Assumption 1. If $\mathbb{E}[\mathbf{U}] \geq \mathbb{E}[\tilde{\mathbf{U}}]$, then

$$\beta(1) = (\rho - h)\mathbb{E}[\mathbf{U}] \geq (\rho - h)\mathbb{E}[\tilde{\mathbf{U}}] = \tilde{\beta}(1), \quad (\text{EC.22})$$

$$\beta(0) = -h\mathbb{E}[\mathbf{U}] \leq -h\mathbb{E}[\tilde{\mathbf{U}}] = \tilde{\beta}(0). \quad (\text{EC.23})$$

Note that $\beta(s)$ and $\tilde{\beta}(s)$ are both nondecreasing and concave functions of $s \in [0, \infty)$. Therefore, $\beta(s)$ and $\tilde{\beta}(s)$ intersect only once in the region $s \in (0, 1)$. This is depicted in Figure EC.1. One of the functions must be non-concave for $\beta(s)$ and $\tilde{\beta}(s)$ to intersect more than once in $s \in (0, 1)$ when $\mathbb{E}[\mathbf{U}] \geq \mathbb{E}[\tilde{\mathbf{U}}]$ holds, leading to a contradiction. Therefore, a unique $s^* \in (0, 1)$ exists such that

$$\beta(s^*) = \tilde{\beta}(s^*) \Rightarrow \frac{\mathbb{E}[\min(\mathbf{U}, s^*)] - \mathbb{E}[\min(\tilde{\mathbf{U}}, s^*)]}{\mathbb{E}[\mathbf{U}] - \mathbb{E}[\tilde{\mathbf{U}}]} = \frac{h}{\rho} \quad (\text{EC.24})$$

for $\mathbb{E}[\mathbf{U}] > \mathbb{E}[\tilde{\mathbf{U}}]$.

In Figure EC.1, we define $s^{\tilde{\beta}}$, s^β , and s^* such that

$$\tilde{\beta}(s^{\tilde{\beta}}) = 0, \quad \beta(s^\beta) = 0, \quad \beta(s^*) = \tilde{\beta}(s^*). \quad (\text{EC.25})$$

The graph shows the case for $\beta(s^*) = \tilde{\beta}(s^*) > 0$, i.e.,

$$\frac{\mathbb{E}[\min(\mathbf{U}, s^*)]}{\mathbb{E}[\mathbf{U}]} = \frac{\mathbb{E}[\min(\tilde{\mathbf{U}}, s^*)]}{\mathbb{E}[\tilde{\mathbf{U}}]} > \frac{h}{\rho} \quad (\text{EC.26})$$

which leads to $0 < s^{\tilde{\beta}} < s^\beta < s^*$. In this case, we have

$$0 \geq \tilde{\beta}(s) > \beta(s) \Rightarrow 0 = b^* = \tilde{b}^* \quad \text{for } s \in [0, s^{\tilde{\beta}}), \quad (\text{EC.27})$$

$$\tilde{\beta}(s) > 0 \geq \beta(s) \Rightarrow 0 = b^* < \tilde{b}^* \quad \text{for } s \in [s^{\tilde{\beta}}, s^\beta), \quad (\text{EC.28})$$

$$\tilde{\beta}(s) \geq \beta(s) > 0 \Rightarrow 0 < b^* < \tilde{b}^* \quad \text{for } s \in [s^\beta, s^*), \quad (\text{EC.29})$$

$$0 < \tilde{\beta}(s) < \beta(s) \Rightarrow 0 < \tilde{b}^* < b^* \quad \text{for } s \in [s^*, 1), \quad (\text{EC.30})$$

$$0 < \tilde{\beta}(s) < \beta(s) \Rightarrow 0 < \tilde{b}^* < b^* \quad \text{for } s \in [1, \infty). \quad (\text{EC.31})$$

Therefore, $b^* \leq \tilde{b}^*$ for $s \in [0, s^*)$ and $b^* > \tilde{b}^*$ for $s > s^*$.

In case $\beta(s^*) = \tilde{\beta}(s^*) \leq 0$ (not shown in Figure EC.1), the order of the parameters are $0 < s^* \leq s^\beta < s^{\tilde{\beta}}$. This leads to $0 = b^* = \tilde{b}^*$ for $s \in [0, s^\beta)$, $b^* > \tilde{b}^* = 0$ for $s \in [s^\beta, s^{\tilde{\beta}})$, and $b^* \geq \tilde{b}^* > 0$ for $s \in [s^{\tilde{\beta}}, \infty)$. Combining these conditions, we get $b^* \geq \tilde{b}^*$ for all $s \in [0, 1)$ when $\beta(s^*) = \tilde{\beta}(s^*) \leq 0$. \square

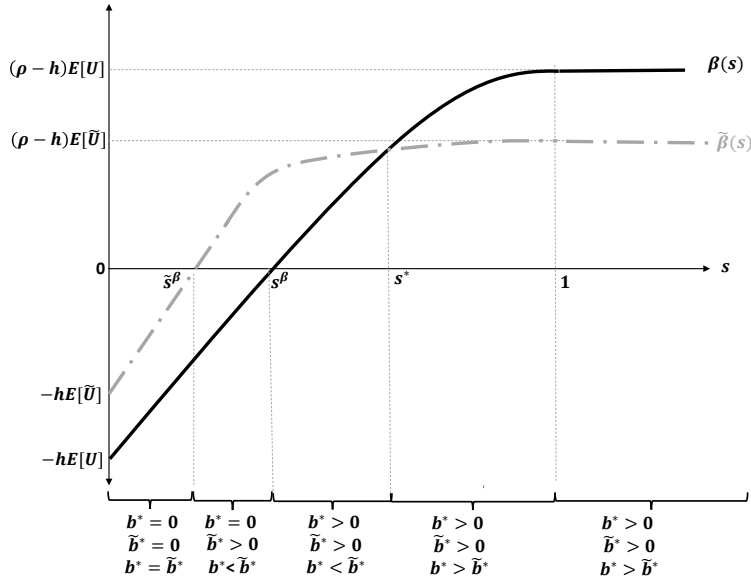


Figure EC.1 The relationship between the optimal bids b^* and \tilde{b}^* in the proof of Theorem 4 when $\beta(s^*) = \tilde{\beta}(s^*) > 0$

Proof of Theorem 5 Define $\phi(u) = \rho q \min\left(u, \frac{D}{q}\right) - hqu$. The optimal bids b^* and \tilde{b}^* satisfy $\mathbb{E}[\phi(\mathbf{U})] = b^* + \frac{1}{\mu(b^*)}$ and $\mathbb{E}[\phi(\tilde{\mathbf{U}})] = \tilde{b}^* + \frac{1}{\mu(\tilde{b}^*)}$. Note that $-\phi(\cdot)$ is a convex function. Since expectation preserves convexity and $\mathbf{U} \leq_{cx} \tilde{\mathbf{U}}$, by the definition of the stochastic convex order we have $-\mathbb{E}[\phi(\mathbf{U})] \leq -\mathbb{E}[\phi(\tilde{\mathbf{U}})] \Leftrightarrow \mathbb{E}[\phi(\mathbf{U})] \geq \mathbb{E}[\phi(\tilde{\mathbf{U}})]$. Hence, $b^* + \frac{1}{\mu(b^*)} \geq \tilde{b}^* + \frac{1}{\mu(\tilde{b}^*)}$. Because \mathbf{Z} has DRHR, $b + \frac{1}{\mu(b)}$ is an increasing function from Lemma 1. As a result, we have $b^* \geq \tilde{b}^*$. \square

Proof of Theorem 6 We prove each part of the theorem in turn:

(i) It is sufficient to show that $\frac{d\mathbb{E}[V^y(b_1, \dots, b_m)]}{db_i}$ changes sign from positive to negative only once for $b_i \in [0, \infty)$.

We know

$$\frac{d\mathbb{E}[V^y(b_1, \dots, b_m)]}{db_i} = \frac{d\mathbb{E}[KP^*(\mathbf{Q}_1^y(\mathbf{b}_1), \dots, \mathbf{Q}_m^y(\mathbf{b}_m))]}{db_i} - h_i q_i \mathbb{E}[\mathbf{U}_i] w'_i(b_i) - q_i w_i(b_i) - b_i q_i w'_i(b_i) \quad (\text{EC.32})$$

where KP^* is the optimal objective value of the knapsack problem in (13). Rearranging the terms, we can also write

$$\frac{d\mathbb{E}[V^y(b_1, \dots, b_m)]}{db_i} = w'_i(b_i) q_i \left[\frac{1}{w'_i(b_i) q_i} \frac{d\mathbb{E}[KP^*(\mathbf{Q}_1^y(\mathbf{b}_1), \dots, \mathbf{Q}_m^y(\mathbf{b}_m))]}{db_i} - \mathbb{E}[\mathbf{U}_i] h_i - b_i - \frac{1}{\mu_i(b_i)} \right]. \quad (\text{EC.33})$$

We know from the proof of Theorem 1 that the first order derivative of the expected profit function of the single-seller problem has the following form: $\frac{d\mathbb{E}[V^y(b)]}{db} = w'(b) q \left[\frac{R}{q} - b - \frac{1}{\mu(b)} \right]$. We will prove that the multi-seller problem's derivative $\frac{d\mathbb{E}[V^y(b_1, \dots, b_m)]}{db_i}$ has the same structure as the single-seller problem's. Once we have the equivalence, then the steps in the proof of Theorem 1 are repeated to prove unimodality with respect to b_i . Thus, we need to prove that the function $A_i(b_{j \neq i})$ defined as

$$A_i(b_{j \neq i}) = \frac{1}{w'_i(b_i) q_i} \frac{d\mathbb{E}[KP^*(\mathbf{Q}_1^y(\mathbf{b}_1), \dots, \mathbf{Q}_m^y(\mathbf{b}_m))]}{db_i} - h_i \mathbb{E}[\mathbf{U}_i] \quad (\text{EC.34})$$

is not a function of b_i but may be a function of b_j where $j \neq i$. Unimodality is proved separately for each i , hence we need $A_i(b_{j \neq i})$ to be a “scalar” with respect to b_i . With the help of $A_i(b_{j \neq i})$, it is easy to see that the form of

$$\frac{d\mathbb{E}[V^y(b_1, \dots, b_m)]}{db_i} = w'_i(b_i)q_i \left[A_i(b_{j \neq i}) - b_i - \frac{1}{\mu_i(b_i)} \right] \quad (\text{EC.35})$$

with respect to b_i is the same as the form of

$$\frac{d\mathbb{E}[V(b)]}{db} = w'(b)q \left[\frac{R}{q} - b - \frac{1}{\mu(b)} \right]$$

with respect to b .

The challenge in the analysis of $A_i(b_{j \neq i})$ arises from the function $\mathbb{E}[KP^*(\mathbf{Q}_1^y(\mathbf{b}_1), \dots, \mathbf{Q}_m^y(\mathbf{b}_m))]$. We write this function in a recursive manner for $m > 1$ as follows:

$$\mathbb{E}[KP^*(\mathbf{Q}_1^y(\mathbf{b}_1), \dots, \mathbf{Q}_m^y(\mathbf{b}_m))] = \mathbb{E}[KP_m^*(Q_1^y, \dots, Q_m^y)] \quad (\text{EC.36})$$

where for a realization of u_1, \dots, u_m of the random variables $\mathbf{U}_1, \dots, \mathbf{U}_m$, respectively, we have

$$KP_m^*(Q_1^y, \dots, Q_m^y) = w_m(b_m)KP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, q_m u_m) + (1 - w_m(b_m))KP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, 0), \quad (\text{EC.37})$$

$$KP_{m-1}^*(Q_1^y, \dots, Q_m^y) = w_{m-1}(b_{m-1})KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, q_{m-1} u_{m-1}, Q_m^y) \\ + (1 - w_{m-1}(b_{m-1}))KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, 0, Q_m^y), \quad (\text{EC.38})$$

$$\vdots \quad (\text{EC.39})$$

$$KP_1^*(Q_1^y, \dots, Q_m^y) = w_1(b_1)KP^*(q_1 u_1, Q_2^y, \dots, Q_m^y) + (1 - w_1(b_1))KP^*(0, Q_2^y, \dots, Q_m^y). \quad (\text{EC.40})$$

This recursion captures all possible bid outcome scenarios for m sellers. (The recursion is similar to finding the expected value of a decision tree.)

For all $i = 1, \dots, m$, we have

$$\frac{d\mathbb{E}[KP^*(\mathbf{Q}_1^y(\mathbf{b}_1), \dots, \mathbf{Q}_m^y(\mathbf{b}_m))]}{db_i} = \mathbb{E} \left[\frac{dKP_m^*(Q_1^y, \dots, Q_m^y)}{db_i} \right]. \quad (\text{EC.41})$$

We start our analysis with $i = m$:

$$\frac{dKP_m^*(Q_1^y, \dots, Q_m^y)}{db_m} = w'_m(b_m) [KP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, q_m u_m) - KP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, 0)] \quad (\text{EC.42})$$

which results in

$$A_m(b_{j \neq m}) = \frac{1}{q_m} [\mathbb{E}[KP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, q_m \mathbf{U}_m)] - \mathbb{E}[KP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, 0)]] - h_m \mathbb{E}[\mathbf{U}_m]. \quad (\text{EC.43})$$

Notice that in the recursive formulation of KP^* , b_m only appears in (EC.37) and KP_i^* for $i < m$ is independent of b_m in all equations from (EC.38) to (EC.40). Therefore, $A_m(b_{j \neq m})$ is independent of b_m and $\mathbb{E}[V^y(b_1, \dots, b_m)]$ is unimodal in b_m , following the steps in the proof of Theorem 1.

Next, we analyze $i = m - 1$.

$$\begin{aligned} \frac{dKP_m^*(Q_1^y, \dots, Q_m^y)}{db_{m-1}} &= w_m(b_m) \frac{dKP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, q_m u_m)}{db_{m-1}} \\ &\quad + (1 - w_m(b_m)) \frac{dKP_{m-1}^*(Q_1^y, \dots, Q_{m-1}^y, 0)}{db_{m-1}} \end{aligned} \quad (\text{EC.44})$$

$$\begin{aligned} &= w_m(b_m) w'_{m-1}(b_{m-1}) (KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, q_{m-1} u_{m-1}, q_m u_m) \\ &\quad - KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, 0, q_m u_m)) \\ &\quad + (1 - w_m(b_m)) w'_{m-1}(b_{m-1}) (KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, q_{m-1} u_{m-1}, 0) \\ &\quad - KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, 0, 0)) \end{aligned} \quad (\text{EC.45})$$

$$\begin{aligned} &= w'_{m-1}(b_{m-1}) [w_m(b_m) KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, q_{m-1} u_{m-1}, q_m u_m) \\ &\quad + (1 - w_m(b_m)) KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, q_{m-1} u_{m-1}, 0) \\ &\quad - w_m(b_m) KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, 0, q_m u_m) \\ &\quad - (1 - w_m(b_m)) KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, 0, 0)]. \end{aligned} \quad (\text{EC.46})$$

Embedding this term in $A_{m-1}(b_{j \neq m-1})$, we see that $A_{m-1}(b_{j \neq m-1})$ is not a function of b_{m-1} as b_{m-1} is not included in the recursive terms KP_j^* for $j = 1, \dots, m - 2$. Therefore, $\mathbb{E}[V^y(b_1, \dots, b_m)]$ is unimodal in b_{m-1} .

For $i = 1, \dots, m - 2$, the expressions gets very involved (hence are omitted) but $A_i(b_{j \neq i})$ is shown to be a function of b_j for $j \neq i$ but not a function of b_i . Therefore, $\mathbb{E}[V^y(b_1, \dots, b_m)]$ is unimodal in b_i for $i = 1, \dots, m$. We illustrate the recursions and derive $A_i(b_{j \neq i})$ for $m = 3$ below.

First, we define

$$KP_3^*(Q_1^y, Q_2^y, Q_3^y) = w_3(b_3) KP_2^*(Q_1^y, Q_2^y, q_3 u_3) + (1 - w_3(b_3)) KP_2^*(Q_1^y, Q_2^y, 0) \quad (\text{EC.47})$$

$$KP_2^*(Q_1^y, Q_2^y, Q_3^y) = w_2(b_2) KP_1^*(Q_1^y, q_2 u_2, Q_3^y) + (1 - w_2(b_2)) KP_1^*(Q_1^y, 0, Q_3^y) \quad (\text{EC.48})$$

$$KP_1^*(Q_1^y, Q_2^y, Q_3^y) = w_1(b_1) KP^*(q_1 u_1, Q_2^y, Q_3^y) + (1 - w_1(b_1)) KP^*(0, Q_2^y, Q_3^y). \quad (\text{EC.49})$$

and we have

$$\frac{dKP_3^*(Q_1^y, Q_2^y, Q_3^y)}{db_1} = w_3(b_3) \frac{dKP_2^*(Q_1^y, Q_2^y, q_3 u_3)}{db_1} + (1 - w_3(b_3)) \frac{dKP_2^*(Q_1^y, Q_2^y, 0)}{db_1} \quad (\text{EC.50})$$

$$\begin{aligned} &= w_3(b_3) \left[w_2(b_2) \frac{dKP_1^*(Q_1^y, q_2 u_2, q_3 u_3)}{db_1} + (1 - w_2(b_2)) \frac{dKP_1^*(Q_1^y, 0, q_3 u_3)}{db_1} \right] \\ &\quad + (1 - w_3(b_3)) \left[w_2(b_2) \frac{dKP_1^*(Q_1^y, q_2 u_2, 0)}{db_1} + (1 - w_2(b_2)) \frac{dKP_1^*(Q_1^y, 0, 0)}{db_1} \right] \end{aligned} \quad (\text{EC.51})$$

$$\begin{aligned} &= w'_1(b_1) w_3(b_3) [w_2(b_2) (KP^*(q_1 u_1, q_2 u_2, q_3 u_3) - KP^*(0, q_2 u_2, q_3 u_3)) \\ &\quad + (1 - w_2(b_2)) (KP^*(q_1 u_1, 0, q_3 u_3) - KP^*(0, 0, q_3 u_3))] \\ &\quad + w'_1(b_1) (1 - w_3(b_3)) [w_2(b_2) (KP^*(q_1 u_1, q_2 u_2, 0) - KP^*(0, q_2 u_2, 0)) \\ &\quad + (1 - w_2(b_2)) (KP^*(q_1 u_1, 0, 0) - KP^*(0, 0, 0))], \end{aligned} \quad (\text{EC.52})$$

$$\frac{dKP_3^*(Q_1^y, Q_2^y, Q_3^y)}{db_2} = w_3(b_3) \frac{dKP_2^*(Q_1^y, Q_2^y, q_3 u_3)}{db_2} + (1 - w_3(b_3)) \frac{dKP_2^*(Q_1^y, Q_2^y, 0)}{db_2}, \quad (\text{EC.53})$$

$$\begin{aligned} &= w'_2(b_2) w_3(b_3) [KP_1^*(Q_1^y, q_2 u_2, q_3 u_3) - KP_1^*(Q_1^y, 0, q_3 u_3)] \\ &\quad + w'_2(b_2) (1 - w_3(b_3)) [KP_1^*(Q_1^y, q_2 u_2, 0) - KP_1^*(Q_1^y, 0, 0)], \end{aligned} \quad (\text{EC.54})$$

$$\frac{dKP_3^*(Q_1^y, Q_2^y, Q_3^y)}{db_3} = w'_3(b_3) [KP_2^*(Q_1^y, Q_2^y, q_3 u_3) - KP_2^*(Q_1^y, Q_2^y, 0)]. \quad (\text{EC.55})$$

When these terms are embedded into $A_i(b_{i \neq j})$, the term $w'_i(b_i)$ in $\mathbb{E} \left[\frac{dKP_3^*(Q_1^y, Q_2^y, Q_3^y)}{db_i} \right]$ cancels out with the term $\frac{1}{w'_i(b_i)}$, leaving no terms in $A_i(b_{i \neq j})$ that are a function of b_i . As this demonstration shows, $A_i(b_{i \neq j})$ is a function of b_j for $j \neq i$. This retains the desired structural form of $\frac{d\mathbb{E}[KP^*(\mathbf{Q}_1^y, \dots, \mathbf{Q}_m^y)]}{db_i}$, making $\mathbb{E}[V^y(b_1, \dots, b_m)]$ a unimodal function of b_i given $b_{j \neq i}$, including $m > 3$.

(ii) We have

$$\frac{d^2 \mathbb{E}[V(b_1, \dots, b_m)]}{db_k db_i} = w'_i(b_i) q_i \frac{dA_i(b_{i \neq j})}{db_k} = \frac{d^2 \mathbb{E}[KP^*(\mathbf{Q}_1^y, \dots, \mathbf{Q}_m^y)]}{db_k db_i} \quad (\text{EC.56})$$

where $A_i(b_{i \neq j})$ and KP_m^* were defined in part (i) above.

KP^* is a special case of the transportation problem formulated in [Karaesmen and Van Ryzin \(2004\)](#). It is known that the objective function of this transportation problem has nonincreasing differences in the supply parameters, i.e., submodular (see [Topkis \(1998\)](#) or [Karaesmen and Van Ryzin \(2004\)](#) and the references therein). In the agent's problem, KP^* is submodular with respect to a pair (Q_i^y, Q_j^y) for $i \neq j$. For instance,

$$KP^*(q_1 u_1, q_2 u_2, Q_3^y, \dots, Q_m^y) - KP^*(0, q_2 u_2, Q_3^y, \dots, Q_m^y) \leq KP^*(q_1 u_1, 0, Q_3^y, \dots, Q_m^y) - KP^*(0, 0, Q_3^y, \dots, Q_m^y) \quad (\text{EC.57})$$

holds with respect to (Q_1^y, Q_2^y) due to nonincreasing differences property. It is also known that a linear combination of two submodular functions, weighted by non-negative scalars, is also submodular. Following our recursion, we see that KP_1^* is submodular with respect to supply as KP_1^* is the convex combination of two submodular KP^* functions in equation [\(EC.40\)](#). Recursively, we see that KP_i^* is submodular with respect to supply as it is the convex combination of two submodular KP_{i-1} functions for all $i > 1$. Therefore, KP_m^* is submodular with respect to supply (Q_i^y, Q_k^y) for any $i \neq k$. Differentiating KP_m^* with respect to b_i and b_k , one gets an expression that is a function of the decreasing differences with respect to supply. Because the expressions get too involved, we illustrate this for b_m and b_{m-1} below.

$$\begin{aligned} \frac{d^2 KP_m^*(Q_1^y, \dots, Q_m^y)}{db_m db_{m-1}} &= w'_{m-1}(b_{m-1}) w'_m(b_m) \{ [KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, q_{m-1} u_{m-1}, q_m u_m) \\ &\quad - KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, q_{m-1} u_{m-1}, 0)] \\ &\quad - [KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, 0, q_m u_m) \\ &\quad - KP_{m-2}^*(Q_1^y, \dots, Q_{m-2}^y, 0, 0)] \} \end{aligned} \quad (\text{EC.58})$$

$$\leq 0 \quad (\text{EC.59})$$

where the inequality follows from KP_{m-2}^* having decreasing differences in supply of m and $m-1$. We can also derive the cross partials for the example with $m=3$:

$$\frac{d^2 KP_3^*(Q_1^y, Q_2^y, Q_3^y)}{db_2 db_3} = w'_3(b_3) \left[\frac{d}{db_2} KP_2^*(Q_1^y, Q_2^y, q_3 u_3) - \frac{d}{db_2} KP_2^*(Q_1^y, Q_2^y, 0) \right] \quad (\text{EC.60})$$

$$= w'_3(b_3) w'_2(b_2) \{ [KP_1^*(Q_1^y, q_2 u_2, q_3 u_3) - KP_1^*(Q_1^y, 0, q_3 u_3)] \\ - [KP_1^*(Q_1^y, q_2 u_2, 0) - KP_1^*(Q_1^y, 0, 0)] \} \quad (\text{EC.61})$$

$$\leq 0 \quad (\text{EC.62})$$

because KP_1^* is submodular and has decreasing differences in (Q_2^y, Q_3^y) . Likewise,

$$\frac{d^2 KP_2^*(Q_1^y, Q_2^y, Q_3^y)}{db_1 db_3} = w'_3(b_3) \left[\frac{d}{db_1} KP_2^*(Q_1^y, Q_2^y, q_3 u_3) - \frac{d}{db_1} KP_2^*(Q_1^y, Q_2^y, 0) \right] \quad (\text{EC.63})$$

$$\leq 0 \quad (\text{EC.64})$$

because

$$\begin{aligned} \frac{dKP_2^*(Q_1^y, Q_2^y, q_3u_3)}{db_1} &= w_2(b_2)w_1'(b_1)[KP^*(q_1u_1, q_2u_2, q_3u_3) - KP^*(0, q_2u_2, q_3u_3)] \\ &\quad + (1 - w_2(b_2))w_1'(b_1)[KP^*(q_1u_1, 0, q_3u_3) - KP^*(0, 0, q_3u_3)], \end{aligned} \quad (\text{EC.65})$$

$$\begin{aligned} \frac{dKP_2^*(Q_1^y, Q_2^y, 0)}{db_1} &= w_2(b_2)w_1'(b_1)[KP^*(q_1u_1, q_2u_2, 0) - KP^*(0, q_2u_2, 0)] \\ &\quad + (1 - w_2(b_2))w_1'(b_1)[KP^*(q_1u_1, 0, 0) - KP^*(0, 0, 0)], \end{aligned} \quad (\text{EC.66})$$

and

$$\frac{dKP_2^*(Q_1^y, Q_2^y, q_3u_3)}{db_1} - \frac{dKP_2^*(Q_1^y, Q_2^y, 0)}{db_1} \leq 0 \quad (\text{EC.67})$$

Interchanging derivative with expectation and taking expectation with respect to the yield variables \mathbf{U}_i , we get $\frac{d^2 \mathbb{E}[KP_m^*(Q_1^y(\mathbf{b}_1), \dots, Q_m^y(\mathbf{b}_m))]}{db_k db_i} \leq 0$ for any $i \neq k$, without loss of generality. Therefore, $\frac{d^2 \mathbb{E}[V(b_1, \dots, b_m)]}{db_k db_i} \leq 0$.

(iii) Follows from (ii). It is known that b_i^* is nonincreasing in b_j if $\frac{\partial^2 \mathbb{E}[V(b_1, \dots, b_m)]}{\partial b_j \partial b_i} \leq 0$; see the supermodularity and parametric monotonicity results in Chapter 10 of Sundaram (1996). \square

Proof of Theorem 7 We prove each result in turn:

Proof of (i): We have $\mathbb{E}[V^y(0, 0)] \geq \mathbb{E}[V^y(0, b_2)]$ for all $b_2 \geq 0$ because $\mathbb{E}[V^y(0, b_2)]$ is nonincreasing in b_2 for $b_1 = 0$ if and only if $R_2 \leq 0$ due to part (i) of Theorem 1. We also have $\mathbb{E}[V^y(0, 0)] \geq \mathbb{E}[V^y(b_1, 0)]$ for $b_1 \geq 0$ because $\mathbb{E}[V^y(b_1, 0)]$ is nonincreasing in b_1 for $b_2 = 0$ if and only if $R_1 \leq 0$ due to part (i) of Theorem 1. We also know that that $\frac{d\mathbb{E}[V^y(0, 0)]}{db_i} \leq 0$ if and only if $R_i \leq 0$ $i = 1, 2$; this follows because $\frac{d\mathbb{E}[V^y(0, 0)]}{db_i} = w_i'(b_i)R_i$. From part (ii) of Theorem 6, we also know $\frac{d\mathbb{E}[V^y(0, b_2)]}{db_1} \leq \frac{d\mathbb{E}[V^y(0, 0)]}{db_1}$ and $\frac{d\mathbb{E}[V^y(b_1, 0)]}{db_2} \leq \frac{d\mathbb{E}[V^y(0, 0)]}{db_2}$ due to the decreasing differences property. The same property leads to $\frac{d\mathbb{E}[V^y(b_1, b_2)]}{db_2} \leq \frac{d\mathbb{E}[V^y(b_1, 0)]}{db_2}$ and $\frac{d\mathbb{E}[V^y(b_1, b_2)]}{db_1} \leq \frac{d\mathbb{E}[V^y(0, b_2)]}{db_1}$ for all $b_1, b_2 \geq 0$. Therefore, the point $(b_1, b_2) = (0, 0)$ is the global maximizer of the problem if and only if $R_i \leq 0$.

Proof of (ii) and (iii): We know from the theory of nonlinear programming and the KKT conditions (see Theorem 4.2.13 of Bazaraa et al. (1993)) that if (b_1^*, b_2^*) locally solves the problem

$$\max \mathbb{E}[V(b_1, b_2)] \quad s.t. \quad b_1, b_2 \geq 0,$$

then, there exists scalars u_1 and u_2 such that

$$\frac{d\mathbb{E}[V(b_1^*, b_2^*)]}{db_i} + u_i = 0, \quad u_i b_i^* = 0, \quad u_i \geq 0 \quad i = 1, 2. \quad (\text{EC.68})$$

These necessary conditions hold for the global optimal solution as a global optimal solution is also a local optimal solution.

(a) When $(b_1^y, 0)$ is optimal: At $(b_1^y, 0)$, the KKT conditions reduce to

$$\frac{d\mathbb{E}[V(b_1^y, 0)]}{db_1} = w_1'(b_1^y)q_1 \left[R_1 - b_1^y - \frac{w_1(b_1^y)}{w_1'(b_1^y)} \right] = 0, \quad u_1 = 0, \quad b_1^y \geq 0, \quad u_1 b_1^y = 0$$

and

$$\frac{d\mathbb{E}[V(b_1^y, 0)]}{db_2} = w_2'(0)[R_2 - w_1(b_1^y)\delta_2] + u_2 = 0, \quad u_2 = -w_2'(0)[R_2 - w_1(b_1^y)\delta_2] \geq 0, \quad b_2 = 0, \quad u_2 b_2 = 0.$$

When $(b_1^y, 0)$ locally or globally solves the problem, then scalars (u_1, u_2) that satisfy KKT conditions exist if and only if $R_1 > 0$ (ensuring $b_1^y > 0$) and $R_2 - w_1(b_1^y)\delta_2 \leq 0$ (ensuring $u_2 \geq 0$).

(b) When $(0, b_2^y)$ is optimal: At $(0, b_2^y)$, the KKT conditions reduce to

$$\frac{d\mathbb{E}[V(0, b_2^y)]}{db_1} = w_1'(0)[R_1 - w_2(b_2^y)\delta_2] + u_1 = 0, u_1 = -w_1'(0)[R_1 - w_2(b_2^y)\delta_2] \geq 0, b_1 = 0, u_1 b_1 = 0$$

and

$$\frac{d\mathbb{E}[V(0, b_2^y)]}{db_2} = w_2'(b_2^y)q_2 \left[R_2 - b_2 - \frac{w_2(b_2^y)}{w_2'(b_2^y)} \right] = 0, u_2 = 0, b_2^y \geq 0, u_2 b_2^y = 0.$$

When $(0, b_2^y)$ locally or globally solves the problem, then scalars (u_1, u_2) that satisfy the KKT conditions exist if and only if $R_2 > 0$ (ensuring $b_2^y > 0$) and $R_1 - w_2(b_2^y)\delta_2 \leq 0$ (ensuring $u_1 \geq 0$).

Proof of (iv): The conditions $R_2 - w_1(b_1^y)\delta_2 > 0$ and $R_1 - w_2(b_2^y)\delta_2 > 0$ reverse the second conditions in parts (ii) and (iii) of this theorem, respectively. Therefore, the solutions $(b_1^y, 0)$ and $(0, b_2^y)$ cannot be optimal. We know $R_1 > 0$ needs to hold for $R_1 - w_2(b_2^y)\delta_2 > 0$ to hold because $\delta_2 \geq 0$; likewise we know $R_2 > 0$ needs to hold for $R_2 - w_1(b_1^y)\delta_2 > 0$. This implies two things: first, $b_i^y \geq 0$ for $i = 1, 2$ and second, $(0, 0)$ cannot be optimal (see the conditions in part (i) of this theorem). This leaves $(b_1^*, b_2^*) > (0, 0)$ as the only candidate for an optimal solution. When (b_1^*, b_2^*) where $b_1^*, b_2^* > 0$ is optimal: At (b_1^*, b_2^*) , the KKT conditions reduce to

$$\frac{d\mathbb{E}[V(b_1^*, b_2^*)]}{db_1} = w_1'(b_1^*)q_1 \left[\frac{R_1 - w_2(b_2^*)\delta_2}{q_1} - b_1^* - \frac{1}{\mu_1(b_1^*)} \right] = 0, b_1^* > 0, u_1 = 0, u_1 b_1^* = 0 \quad (\text{EC.69})$$

and

$$\frac{d\mathbb{E}[V(b_1^*, b_2^*)]}{db_2} = w_2'(b_2^*)q_2 \left[\frac{R_2 - w_1(b_1^*)\delta_2}{q_2} - b_2^* - \frac{1}{\mu_2(b_2^*)} \right] = 0, b_2^* > 0, u_2 = 0, u_2 b_2^* = 0. \quad (\text{EC.70})$$

Scalars $u_1 = u_2 = 0$ satisfy the KKT conditions, (b_1^*, b_2^*) is a stationary point, and $\frac{d\mathbb{E}[V(b_1^*, b_2^*)]}{db_i} = 0$ for $i = 1, 2$ have to hold. From Theorem 6, we know $b_i^* \leq b_i^y$. \square

EC.3. Additional remarks on using ask price distributions with closed and bounded support

We focused on \mathbf{Z} with support $\Omega = [0, \infty)$ in our analysis. Our major results in Section 4 include the uniqueness of the optimal bid and derivation of the optimal bid value for $\Omega = [0, \infty)$. Given that the ask price \mathbf{Z} is a non-negative, continuous random variable, it is possible for \mathbf{Z} to have a support $\Omega \subset [0, \infty)$. We illustrate how the results in Theorem 1 are updated when $\Omega = [\underline{B}, \bar{B}]$ where $\infty > \bar{B} > \underline{B} \geq 0$.

Note that $w(\underline{B}) = 0$ and $w(\bar{B}) = 1$ and $w'(b) > 0$ for all $b \in \Omega$, by definition. Practically speaking, the agent can never win by bidding at or below \underline{B} and is guaranteed to win by bidding at or above \bar{B} . Bidding \underline{B} means the agent loses the bid, hence it is equivalent to not bidding. In addition, the agent's optimal bid would never be more than \bar{B} because bidding $b > \bar{B}$ would increase the procurement cost but not the probability of winning or the expected benefit of meeting the demand. Therefore, the agent chooses a bid $b \in \Omega$.

$$\mathbb{E}[V(b)] = w(b)[R - bq] - \pi D \quad \text{for } b \in \Omega \quad (\text{EC.71})$$

where $R = \rho\mathbb{E}[\min(q\mathbf{U}, D)] - hq\mathbb{E}[\mathbf{U}]$. $\mathbb{E}[V(b)]$ is continuous in b in the set Ω . Notice that this is no different than what we had in Section 3.2. But this time we have additional information

$$\mathbb{E}[V(\underline{B})] = -\pi D, \quad (\text{EC.72})$$

$$\mathbb{E}[V(\bar{B})] = R - \bar{B}q - \pi D. \quad (\text{EC.73})$$

Similar to what we had in the proof of Theorem 1, $\mathbb{E}[V(b)]$ is nonincreasing and $b^* = \underline{B}$ is the unique optimal solution if $R \leq 0$; this is practically equivalent to not bidding because the bid is lost with probability 1. The differences from the results in Theorem 1 appear when $R > 0$ because there are now three possibilities: Suppose Assumption 1 holds. Then, $\mathbb{E}[V(b)]$ is (i) strictly increasing in Ω if $\frac{R}{q} > \left(b + \frac{w(b)}{w'(b)}\right)$ for all $b \in \Omega$, (ii) strictly decreasing in Ω if $\frac{R}{q} < \left(b + \frac{w(b)}{w'(b)}\right)$ for all $b \in \Omega$, or (iii) unimodal in Ω where unique b° solves $\frac{R}{q} = \left(b^\circ + \frac{w(b^\circ)}{w'(b^\circ)}\right)$ for $b^\circ \in \Omega$. In summary, the optimal bid takes one of three possible values $b^* \in \{\underline{B}, b^\circ, \bar{B}\}$ if $R > 0$. An example with $\Omega \subset [0, \infty)$ would be when the ask price \mathbf{Z} follows a continuous uniform distribution on a convex bounded set, e.g., $\mathbf{Z} \sim \text{Uniform}(\underline{B}, \bar{B})$. Note that a continuous uniform distribution that is defined on a convex bounded set is logconcave and by Lemma 1 satisfies Assumption 1. Therefore, when the ask distribution is uniform distributed over $[\underline{B}, \bar{B}]$, the condition for the unimodality of the expected profit function is the same as in Theorem 1 and the optimal bid is $b^* \in \{\underline{B}, b^\circ, \bar{B}\}$.

Based on these observations, it is clear that our analysis easily expands to include $\Omega \subset [0, \infty)$. Note that the insights about the demand-to-supply ratio and the trade agent's critical ratio remain the same. We decided to communicate the important findings and insights without further complicating the notation in the main text, hence the derivation of optimal bid is presented for only $\Omega = [0, \infty)$ in the paper.

References

- Bazaraa MS, Sherali HD, Shetty C (1993) *Nonlinear Programming Theory and Algorithms* (John Wiley & Sons Inc.).
- Karaesmen I, Van Ryzin G (2004) Overbooking with substitutable inventory classes. *Operations Research* 52(1):83–104.
- Ross SM, Shanthikumar JG, Zhu Z (2005) On increasing-failure-rate random variables. *Journal of Applied Probability* 42(3):797–809.
- Sundaram RK (1996) *A First Course in Optimization Theory* (New York University Press).
- Topkis DM (1998) *Supermodularity and Complementarity* (Princeton University Press).