

WEB APPENDIX

A Continuous-Time Model of Subscription and Repeat Purchase

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A. Proofs

A.1. Proof of Theorem 1:

Proof: We will show that the proposed solution $V(v)$ given by (5) and (6) satisfies the HJB equation (4) for all $v \in \mathbb{R}$. The fact that $\lim_{t \rightarrow \infty} \mathbb{E}[e^{-rt}V(v_t)] = 0$ is clear, thus the transversality condition is satisfied. Then it follows by the verification theorem that $V(v)$ is the value function for the optimal stopping problem (3).

For $v < \underline{v}$ it follows from (5) that $V(v) = 0$, and $\frac{1}{2}\sigma^2 V''(v) - rV(v) + \lambda \max(v - p, 0) - c = \lambda \max(v - p, 0) - c \leq \lambda \max(\underline{v} - p, 0) - c \leq c - \frac{\sigma\lambda}{\sqrt{2r}} - c = -\frac{\sigma\lambda}{\sqrt{2r}} < 0$ therefore the HJB (4) is satisfied in this case. For $v \geq \underline{v}$ we divide our consideration into three further cases: (1) $\underline{v} \leq p$, $\underline{v} \leq v \leq p$, (2) $\underline{v} \leq p$, $v > p$, and (3) $v \geq \underline{v} > p$. In the case of $\underline{v} \leq p$, $\underline{v} \leq v \leq p$, we have $V(v) = \frac{c}{2r} \left(e^{+\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} + e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} \right) - \frac{c}{r} \geq 0$ with the minimum at $v = \underline{v}$ with $V(\underline{v}) = 0$. Therefore, $-V(v) \leq 0$, and we check that $\frac{1}{2}\sigma^2 V''(v) - rV(v) + \lambda \max(v - p, 0) - c = \frac{c}{2} \left(e^{+\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} + e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} \right) - \frac{c}{2} \left(e^{+\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} + e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} \right) + c - c = 0$. For the case of $\underline{v} \leq p$, $v > p$, we have $V(v) = \frac{c}{2r} \left(e^{-\frac{\sqrt{2r}}{\sigma}(v+\underline{v}-2p)} + e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} \right) + \frac{\lambda}{r}(v-p) - \frac{c}{r}$. It is easy to check that $V'(v) > -\frac{c}{\sigma} \sqrt{\frac{2}{r}} + \frac{\lambda}{r} > 0$ for all $v > p$, where the last inequality can be rewritten as $\frac{c}{\lambda} \leq \frac{\sigma}{\sqrt{2r}}$ which is equivalent to $\underline{v} \leq p$ by (6). Therefore, $V(v) > V(p) > 0$ for all $v > p$ which means $-V(v) < 0$, and we can check that $\frac{1}{2}\sigma^2 V''(v) - rV(v) + \lambda \max(v - p, 0) - c = \frac{c}{2} \left(e^{-\frac{\sqrt{2r}}{\sigma}(v+\underline{v}-2p)} + e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} \right) - \frac{c}{2} \left(e^{-\frac{\sqrt{2r}}{\sigma}(v+\underline{v}-2p)} + e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} \right) - \lambda(v-p) + c + \lambda(v-p) - c = 0$.

Lastly, we consider the case where $v \geq \underline{v} > p$, we have $V(v) = \frac{\lambda\sigma}{r^{3/2}\sqrt{2}} e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} + \frac{\lambda}{r}(v-p) - \frac{c}{r}$. Note that $\underline{v} = p + \frac{c}{\lambda} - \frac{\sigma}{\sqrt{2r}}$ in this case, therefore $V'(v) = -\frac{\lambda}{r} e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} + \frac{\lambda}{r} \geq -\frac{\lambda}{r} + \frac{\lambda}{r} = 0$ for all $v \geq \underline{v}$. Therefore, it follows that $V(v) \geq V(\underline{v}) = 0$ which means $-V(v) \leq 0$ for all $v \geq \underline{v}$. We can then check that $\frac{1}{2}\sigma^2 V''(v) - rV(v) + \lambda \max(v - p, 0) - c = \frac{\lambda\sigma}{\sqrt{2r}} e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} - \frac{\lambda\sigma}{\sqrt{2r}} e^{-\frac{\sqrt{2r}}{\sigma}(v-\underline{v})} - \lambda(v-p) + c + \lambda(v-p) - c = 0$. Therefore the HJB (4) is also satisfied when $v \geq \underline{v}$, completing the proof.

A.2. Proof of Proposition 1:

Proof: The statement about $\mathbb{P}[v_t \in \text{churner} | v_0]$ is straightforward from the properties of the survivor function $S(t)$. Let us focus on $\mathbb{P}[v_t \in \text{tracker} | v_0]$, the result for $\mathbb{P}[v_t \in \text{buyer} | v_0]$ follows in a similar way. Because $C < 0$, we have that $\mathbb{P}[v_t \in \text{tracker} | v_0] = 2\Phi\left(\frac{v_0 - p - C}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{v_0 - p}{\sigma\sqrt{t}}\right) + \Phi\left(\frac{2C + p - v_0}{\sigma\sqrt{t}}\right) - 1$, therefore:

$$\frac{\partial}{\partial v_0} \mathbb{P}[v_t \in \text{tracker} | v_0] \leq \frac{2}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{v_0 - p - C}{\sigma\sqrt{t}}\right)^2\right)$$

$$-\frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{v_0 - p - 2C}{\sigma\sqrt{t}}\right)^2\right) - \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{v_0 - p}{\sigma\sqrt{t}}\right)^2\right).$$

Since the function $\exp(-x^2)$ is convex for large x , we have that for sufficiently large v_0 the right-hand-side of the inequality must be ≤ 0 , i.e. $\mathbb{P}[v_t \in \text{tracker}|v_0]$ decreases with v_0 for sufficiently large v_0 . On the other hand, at $v_0 = p + C$ we have:

$$\begin{aligned} \frac{\partial}{\partial v_0} \mathbb{P}[v_t \in \text{tracker}|v_0]|_{v_0=p+C} &= -\frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{2p-C}{\sigma\sqrt{t}}\right)^2\right) + \frac{2}{\sigma\sqrt{2\pi t}} \\ &\quad - \frac{1}{\sigma\sqrt{2\pi t}} \exp\left(-\frac{1}{2}\left(\frac{C}{\sigma\sqrt{t}}\right)^2\right) \geq 0. \end{aligned} \quad (26)$$

Now, we note that $\mathbb{P}[v_t \in \text{tracker}|v_0]$ is in fact a function of $v_0 - p$. Therefore, using $\frac{\partial}{\partial p} \mathbb{P}[v_t \in \text{tracker}|v_0] = -\frac{\partial}{\partial v_0} \mathbb{P}[v_t \in \text{tracker}|v_0]$, we have already seen from (26) that when $v_0 \approx C$ (which implies $p \approx 0$) $\mathbb{P}[v_t \in \text{tracker}|v_0]$ decreases with p . If v_0 is sufficiently larger than C and $p \approx 0$ then convexity of $\exp(-x^2)$ implies $\mathbb{P}[v_t \in \text{tracker}|v_0]$ increases with p , but if $p \approx v_0 - C$ then (26) implies that $\mathbb{P}[v_t \in \text{tracker}|v_0]$ decreases with p . Lastly, we check that:

$$\frac{\partial}{\partial C} \mathbb{P}[v_t \in \text{tracker}|v_0] = \frac{2}{\sigma\sqrt{2\pi t}} \left[\exp\left(-\frac{1}{2}\left(\frac{v_0 + p - 2C}{\sigma\sqrt{t}}\right)^2\right) - \exp\left(-\frac{1}{2}\left(\frac{v_0 - p - C}{\sigma\sqrt{t}}\right)^2\right) \right] \leq 0,$$

The inequality follows since $v_0 + p - 2C = v_0 - p - C + (2p - C) \geq v_0 - p - C \geq 0$ where $2p - C > 0$ because $C < 0$.

A.3. Proof of Lemma 1:

Proof: Integrating the PDE (9) with respect to λdt , we obtain an ODE: $\frac{1}{2}\sigma^2 \frac{d^2}{dv^2} \mathcal{P}_1(v|\phi) - \lambda \mathbb{1}_{v \geq p} \mathcal{P}_1(v|\phi) = -\lambda \phi(v)$. Let $\phi(v) := \delta(v - v_0)$, and assume that $v_0 \in [\underline{v}, +\infty) \setminus \{p\}$ then the solution to this ODE takes the form

$$\mathcal{P}_1(v|v_0) = \begin{cases} \frac{1}{\sigma} \sqrt{\frac{\lambda}{2}} e^{-\frac{\sqrt{2\lambda}}{\sigma}|v-v_0|} + Ae^{-\frac{\sqrt{2\lambda}}{\sigma}v} + Be^{+\frac{\sqrt{2\lambda}}{\sigma}v}, & \text{if } v > p \\ -\frac{\lambda}{\sigma^2}|v-v_0| + Cv + D, & \text{if } \underline{v} \leq v < p. \end{cases}$$

From the boundary conditions and the continuously differentiability condition of $\mathcal{P}(v, t|\phi, c, \lambda)$, we can see from (10) that $\mathcal{P}_1$ satisfies $\mathcal{P}_1(\underline{v}|\phi) = \lim_{v \rightarrow +\infty} \mathcal{P}_1(v|\phi) = 0$, and also $\lim_{v \rightarrow p^-} \mathcal{P}_1(v|\phi) = \lim_{v \rightarrow p^+} \mathcal{P}_1(v|\phi)$, $\lim_{v \rightarrow p^-} \frac{d\mathcal{P}_1}{dv} = \lim_{v \rightarrow p^+} \frac{d\mathcal{P}_1}{dv}$ if $\underline{v} < p$. Using these conditions, we can solve for all the integration constants, and Lemma 1 follows.

A.4. Proof of Lemma 2:

Proof: Suppose that the consumer started from an initial valuation $v_0 \geq \underline{v}$, we note that $\mathcal{P}_x$ can be recursively computed by

$$\mathcal{P}_x(v|v_0) = \int_{\max(p, \underline{v})}^{+\infty} \mathcal{P}_1(v|v_{x-1}) \mathcal{P}_{x-1}(v_{x-1}|v_0) dv_{x-1}$$

$$\begin{aligned}
 &= \int_{\max(p, \underline{v})}^{+\infty} \cdots \int_{\max(p, \underline{v})}^{+\infty} \mathcal{P}_1(v|v_{x-1}) \mathcal{P}_1(v_{x-1}|v_{x-2}) \cdots \mathcal{P}_1(v_2|v_1) \mathcal{P}_1(v_1|v_0) dv_1 \cdots dv_{x-1} \\
 &= \int_{\max(p, \underline{v})}^{+\infty} \mathcal{P}_{x-1}(v|v_1) \mathcal{P}_1(v_1|v_0) dv_1.
 \end{aligned} \tag{27}$$

From the ODE satisfied by $\mathcal{P}_1(v|v_0)$ we derived in the proof of Lemma 1, it is possible to show using (27) that each $\mathcal{P}_x$ for $x \geq 2$ satisfies the following: $\frac{1}{2}\sigma^2 \frac{d^2}{dv^2} \mathcal{P}_x(v|v_0) = \lambda \mathbb{1}_{v \geq p} \mathcal{P}_x(v|v_0) - \lambda \mathbb{1}_{v \geq p} \mathcal{P}_{x-1}(v|v_0)$. Let us choose some constant $0 < \xi < 1$ and define the generating function on $(v, v_0, z) \in [\underline{v}, +\infty)^2 \times (-\xi, +\xi)$ by:

$$\mathcal{G}(v, z|v_0) := \sum_{x=1}^{\infty} \mathcal{P}_x(v|v_0) z^x. \tag{28}$$

The choice of ξ is arbitrary within the range, since only the local behavior of $\mathcal{G}$ in a neighborhood of $z = 0$ will be important to us. It is possible to show using Weierstrass M-test that the series (28) along with its first and second derivative series: $\sum_{x=2}^{\infty} \frac{d}{dv} \mathcal{P}_x z^x$, and $\sum_{x=2}^{\infty} \frac{d^2}{dv^2} \mathcal{P}_x z^x$ converges uniformly in v, v_0 , and z on the given domain.²⁰ It follows that the derivatives of (28) can be computed term-by-term, at least twice in v , and infinitely many times with respect to z by the general properties of power series. Therefore, the formula (12) easily follows. For any $v_0 \geq \underline{v}$, since $\mathcal{P}_x(\underline{v}|v_0) = 0$ and $\lim_{v \rightarrow +\infty} \mathcal{P}_x(v|v_0) = 0$ we can conclude by the uniform convergence the boundary conditions $\mathcal{G}(\underline{v}, z|v_0) = 0$ and $\lim_{v \rightarrow +\infty} \mathcal{G}(v, z|v_0) = 0$. Moreover, if $\underline{v} < p$ then for any $v_0 \in [\underline{v}, +\infty) \setminus \{p\}$ since $\mathcal{P}_x(v|v_0)$ and $\frac{d}{dv} \mathcal{P}_x(v|v_0)$ are continuous at $v = p$ we have that $\mathcal{G}(v, z|v_0)$ must also be continuously differentiable at $v = p$. Differentiating (28) using the ODEs for $\mathcal{P}_x$, $x \geq 1$, we find that $\mathcal{G}$ must satisfies the following:

$$\begin{aligned}
 \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial v^2} \mathcal{G}(v, z|v_0) &= (\lambda \mathbb{1}_{v \geq p} \mathcal{P}_1(v|v_0) - \lambda \delta(v - v_0)) z + \sum_{x=2}^{\infty} (\lambda \mathbb{1}_{v \geq p} \mathcal{P}_x(v|v_0) - \lambda \mathbb{1}_{v \geq p} \mathcal{P}_{x-1}(v|v_0)) z^x \\
 &= \lambda(1 - z) \mathbb{1}_{v \geq p} \mathcal{G}(v, z|v_0) - \lambda z \delta(v - v_0).
 \end{aligned} \tag{29}$$

Let us assume that $v_0 \in [\underline{v}, +\infty) \setminus \{p\}$ then the solution to this ODE takes the form

$$\mathcal{G}(v, z|v_0) = \begin{cases} \frac{z}{\sigma} \sqrt{\frac{\lambda}{2(1-z)}} e^{-\frac{\sqrt{2\lambda(1-z)}}{\sigma} |v-v_0|} + A e^{-\frac{\sqrt{2\lambda(1-z)}}{\sigma} v} + B e^{+\frac{\sqrt{2\lambda(1-z)}}{\sigma} v}, & \text{if } v \geq \max(p, \underline{v}) \\ -\frac{\lambda z}{\sigma^2} |v - v_0| + C v + D, & \text{if } \underline{v} \leq v \leq p. \end{cases}$$

Using the boundary conditions $\mathcal{G}(\underline{v}, z|v_0) = 0$, $\lim_{v \rightarrow +\infty} \mathcal{G}(v, z|v_0)$ with additional smooth pasting conditions at $v = p$ if $\underline{v} < p$: $\lim_{v \rightarrow p^-} \mathcal{G}(v, z|v_0) = \lim_{v \rightarrow p^+} \mathcal{G}(v, z|v_0)$, $\lim_{v \rightarrow p^-} \frac{\partial}{\partial v} \mathcal{G}(v, z|v_0) = \lim_{v \rightarrow p^+} \frac{\partial}{\partial v} \mathcal{G}(v, z|v_0)$, we can determine the integration constants, giving the closed-form expression (13).

²⁰ Note that we excluded the $x = 1$ term to avoid the fact that $\frac{d^2}{dv^2} \mathcal{P}_1$ involves $\phi(v) = \delta(v - v_0)$. This does not change the conclusion that we can differentiate (28) term-by-term at least twice in v since we can always differentiate the $x = 1$ term separately before differentiating the $x = 2, 3, \dots$ summation.

A.5. Proof of Proposition 2:

Proof: Denote by $X(T)$ as the number of purchases from time 0 to T by a consumer, then using this notation, we can write: $n_x(v_0) = \mathbb{P}[X(\infty) = x|v_0]$ and $N_x(v_0) = \mathbb{P}[X(\infty) \geq x|v_0]$. By following the convention to set $N_0(v_0) \equiv 1$, we have $n_x(v_0) = N_x(v_0) - N_{x+1}(v_0)$ for $x \in \mathbb{N}^0$. Proposition 2 is a direct application of Lemma 2, which will be introduced below.

In fact, define t_x as the timing of x -th purchase, and following the convention, we set $t_x = +\infty$ if the x -th purchase never took place, we can rewrite $N_x(v_0)$ further as:

$$N_x(v_0) = \mathbb{P} \left[v_{t_x} \geq \max(p, \underline{v}), \inf_{0 \leq t \leq t_x} v_t \geq \underline{v} | v_0 \right] = \int_{\max(p, \underline{v})}^{\infty} \mathcal{P}_x(v|v_0) dv,$$

where $\mathcal{P}_x(v|v_0)$ is introduced in Lemma 2.

Let us define

$$N(v_0, z) \equiv \int_{\max(p, \underline{v})}^{+\infty} \mathcal{G}(v, z|v_0) dv,$$

then it is straightforward to check that we get (15) using the expression of $\mathcal{G}$ as given in Lemma 2. But we can also integrate $\mathcal{G} \equiv \sum_{x=1}^{\infty} \mathcal{P}_x z^x$ term-by-term due to the uniform convergence. We have that

$$N(v_0, z) = \sum_{x=1}^{\infty} \left(\int_{\max(p, \underline{v})}^{\infty} \mathcal{P}_x(v|v_0) dv \right) z^x = \sum_{x=1}^{\infty} N_x(v_0) z^x,$$

from which (14) follows.

Next, we prove the asymptotic properties. First, let us show that for any $v_0 \in [\underline{v}, +\infty)$ we have $\lim_{x \rightarrow \infty} N_x(v_0) = 0$. It is easy to check that $N_1(v_0) < 1$ and that $N_1(v_0) \rightarrow 1$ as $v_0 \rightarrow +\infty$. Therefore,

$$\begin{aligned} N_x(v_0) &= \int_{\max(p, \underline{v})}^{+\infty} N_1(v') \mathcal{P}_{x-1}(v'|v_0) dv' \\ &\leq \left(\sup_{v' \in [\max(p, \underline{v}), +\infty)} N_1(v') \right) \int_{\max(p, \underline{v})}^{+\infty} \mathcal{P}_{x-1}(v'|v_0) dv' = N_{x-1}(v_0). \end{aligned}$$

This means, for any fixed $v_0 \in [\underline{v}, +\infty)$, $\{N_x(v_0)\}_{x=1,2,3,\dots}$ is a decreasing sequence bounded below by zero, hence it must converges to some $N(v_0) \in [0, N_1(v_0)]$. Now, we can equivalently write $N_x(v_0) = \int_{\max(p, \underline{v})}^{+\infty} N_{x-1}(v') \mathcal{P}_1(v'|v_0) dv'$. By interpreting this integral in Lebesgue sense and employ the Dominated Convergence theorem, we find that

$$N(v_0) = \int_{\max(p, \underline{v})}^{+\infty} N(v') \mathcal{P}_1(v'|v_0) dv'. \quad (30)$$

The right-hand-side of the above is actually continuous in v_0 , a fact which can also be deduced from Dominated Convergence theorem, hence $N(v_0)$ is a continuous function and it follows that we can also interpret the integral in the Riemann sense. In the case where $\underline{v} < p$, we can check using Lemma 1 that given any $v \in [p, +\infty)$, we have that $\mathcal{P}_1(v|v_0)$ is an increasing function in

v_0 for $v_0 \in [\underline{v}, p)$, therefore $N(v_0)$ is an increasing function in v_0 over such domain. For v_0 over $[\max(p, \underline{v}), +\infty)$ we can also see from Lemma 1 that $\mathcal{P}_1(v|v_0)$ is symmetric in v and v_0 , hence $\mathcal{P}_1$ also satisfies the same ODE with v replaced by v_0 . This helps us compute the second derivative: $\frac{d^2}{dv_0^2} N(v_0) = \frac{2\lambda}{\sigma^2} \left(\int_{\max(p, \underline{v})}^{+\infty} N(v') \mathcal{P}_1(v'|v_0) dv' - N(v_0) \right) = 0$. It follows that $N(v_0) = Av_0 + B$ for some constants A and B for $v_0 \in [\max(p, \underline{v}), +\infty)$. But since $N(v_0) \in [0, N_1(v_0)] \subset [0, 1]$, we must have $A = 0$. But $N(v_0)$ cannot be a non-zero constant either since (30) would implies $N_1(v_0) = 1$, which is false. Therefore, we conclude that $\lim_{x \rightarrow +\infty} N_x(v_0) = N(v_0) = 0$ for all $v_0 \in [\max(p, \underline{v}), +\infty)$ and hence for all $v_0 \in [\underline{v}, +\infty)$ since $N(v_0)$ is an increasing function over $[\underline{v}, p)$ in the case where $\underline{v} < p$.

To justify the statement on the decay rate of $N_x(v_0)$, we note that $z = 1$ is a singular point in the expression (15) for $N(v_0, z)$ and therefore its power series expansion centered at $z = 0$ given by $N(v_0, z) = \sum_{x=1}^{\infty} N_x(v_0) z^x$ must have the radius of convergence of less than 1. Hence, we know from Cauchy–Hadamard theorem that: $\limsup_{x \rightarrow +\infty} N_x(v_0)^{1/x} \geq 1$, from which we conclude that $N_x(v_0)$ decays to zero sub-exponentially in x .

Since $n_x(v_0) = N_x(v_0) - N_{x+1}(v_0)$, and $\lim_{x \rightarrow +\infty} N_x(v_0) = 0$, we have that $N_x(v_0) = \sum_{x'=x}^{+\infty} n_{x'}(v_0)$. Suppose that $n_x(v_0)$ decays to zero at least exponentially fast, then there exists $\bar{x} \in \mathbb{N}^0$ such that $n_x(v_0) \leq \alpha a^x$ for all $x > \bar{x}$, for some constant $\alpha \in \mathbb{R}_{>0}$ and $0 < a < 1$. It follows that for all $x > \bar{x}$ we have $N_x(v_0) \leq \sum_{x'=x}^{+\infty} \alpha a^{x'} = \alpha a^x \sum_{x'=0}^{+\infty} a^{x'} = \alpha a^x / (1 - a)$, so $N_x(v_0)$ also converges to zero at least exponentially fast, a contradiction. Therefore, $n_x(v_0)$ must decays to zero sub-exponentially, this completes the proof of the proposition.

A.6. Proof of Proposition 3:

Proof: For convenience, we will mainly work with C as defined in (6) which is a monotonically increasing continuously differentiable function of c . First, we consider when C is fixed and show that $p^*(C; v_0)$ is given by the unique stationary point of $\Pi(p, C; v_0)$ for $p \leq v_0 - C$.

If $C \geq 0$ then $p \leq v_0 - C \leq v_0$, therefore

$$\Pi(p, C; v_0) = \frac{\lambda}{s} \left(p + C + \frac{\sigma}{\sqrt{2r}} \right) \left(1 - e^{-\frac{\sqrt{2s}}{\sigma}(v_0 - p - C)} \right)$$

which has a unique stationary point which is the maximum. For the $C < 0$ case, we compute the derivative of $\Pi(p, C; v_0)$ as follows:

$$\frac{\partial \Pi(p, C; v_0)}{\partial p} = \begin{cases} \frac{\lambda}{s} \left(1 - \left(\frac{1 + e^{\frac{2\sqrt{2s}}{\sigma}C}}{2} \right) e^{-\frac{\sqrt{2s}}{\sigma}(v_0 - p)} \right) - \frac{\lambda}{\sigma\sqrt{2s}} \left(1 + e^{\frac{2\sqrt{2s}}{\sigma}C} \right) p e^{-\frac{\sqrt{2s}}{\sigma}(v_0 - p)} \\ \quad - \frac{\lambda}{\sqrt{rs}} e^{-\frac{\sqrt{2s}}{\sigma}(v_0 - p) + \frac{\sqrt{2s} + \sqrt{2r}}{\sigma}C}, & \text{if } p < v_0 \\ \frac{\lambda}{2s} \left(e^{-\frac{\sqrt{2s}}{\sigma}(p - v_0)} - e^{+\frac{\sqrt{2s}}{\sigma}(p - v_0 + 2C)} \right) - \frac{\lambda}{\sigma\sqrt{2s}} p \left(e^{-\frac{\sqrt{2s}}{\sigma}(p - v_0)} + e^{+\frac{\sqrt{2s}}{\sigma}(p - v_0 + 2C)} \right) \\ \quad - \frac{\lambda}{\sqrt{rs}} e^{-\frac{\sqrt{2s}}{\sigma}(v_0 - p) + \frac{\sqrt{2s} + \sqrt{2r}}{\sigma}C}, & \text{if } v_0 < p \leq v_0 - C. \end{cases} \quad (31)$$

We can see that both pieces of derivatives agree at $p = v_0$, hence $\Pi(p, C; v_0)$ is a continuously differentiable function for $p \leq v_0 - C$. Furthermore, we can check that each interval $p < v_0$ and

$v_0 < p \leq v_0 - C$ contains at most one critical point. When $p \leq v_0$, the equation $\partial\Pi(p, C; v_0)/\partial p = 0$ can be written as

$$p = \frac{\sigma}{\sqrt{2s}} \left(\frac{e^{+\frac{\sqrt{2s}}{\sigma}(v_0-p-C)}}{\cosh \frac{\sqrt{2s}}{\sigma} C} - 1 \right) - \frac{\sigma}{\sqrt{2r}} \frac{e^{\frac{\sqrt{2r}}{\sigma} C}}{\cosh \frac{\sqrt{2s}}{\sigma} C}. \quad (32)$$

when $v_0 < p \leq v_0 - C$, the equation $\partial\Pi(p, C; v_0)/\partial p = 0$ can be written as

$$p = \frac{\sigma}{\sqrt{2s}} \tanh \frac{\sqrt{2s}}{\sigma} (v_0 - p - C) - \frac{\sigma}{\sqrt{2r}} \frac{e^{-\frac{\sqrt{2s}}{\sigma}(v_0-p-C)} e^{\frac{\sqrt{2r}-\sqrt{2s}}{\sigma} C}}{\cosh \frac{\sqrt{2s}}{\sigma} (v_0 - p - C)} \quad (33)$$

In both cases, the RHS is a decreasing function in p , whereas the LHS is increasing with p , therefore both equations has at most one solution. It follows that the only critical point of $\Pi(p, C; v_0)$ for $p \leq v_0 - C$ is the maximum point, hence equation $\partial\Pi(p; v_0, c)/\partial p|_{p=p^*} = 0$ implicitly define $p^*(C; v_0)$.

Let us show that p^* is an increasing function in v_0 for a given fixed C . Suppose that $C < 0$ and $p^* < v_0$ then implicitly differentiating (32) gives $1 > \frac{\partial p^*}{\partial v_0} = \frac{e^{\frac{\sqrt{2s}}{\sigma}(v_0-p^*-C)}}{\cosh \frac{\sqrt{2s}}{\sigma} C + e^{\frac{\sqrt{2s}}{\sigma}(v_0-p^*-C)}} \geq 0$.

Suppose that $C < 0$ and $v_0 < p^* \leq v_0 - C$ then implicitly differentiating (33) gives $1 \geq \frac{\partial p^*}{\partial v_0} = \frac{1 + \sqrt{\frac{s}{r}} e^{\frac{\sqrt{2r}-\sqrt{2s}}{\sigma} C}}{\cosh^2 \frac{\sqrt{2s}}{\sigma} (v_0-p^*-C) + 1 + \sqrt{\frac{s}{r}} e^{\frac{\sqrt{2r}-\sqrt{2s}}{\sigma} C}} \geq 0$. For $C \geq 0$ and $p^* \leq v_0 - C \leq v_0$, we also find that $1 > \frac{\partial p^*}{\partial v_0} = 1 / \left(e^{-\frac{\sqrt{2s}}{\sigma}(v_0-p^*-C)} + 1 \right) \geq 0$. It follows from this that p^* is an increasing function of v_0 in all cases with slope bounded above by 1. Therefore, p^* as a function of v_0 intercepts the line $p = v_0$ at most once at some point $v_0 = v^*$ given by (17) which can be found by solving $\partial\Pi(p, C; v_0)/\partial p|_{p^*=v_0=v^*} = 0$.

Similarly, we show that $p^*(C; v_0)$ is a decreasing function in C for a fixed v_0 . When $C < 0$ and $p^* < v_0$ we find by implicitly differentiating (32) that

$$\frac{\partial p^*}{\partial C} = - \frac{e^{\frac{\sqrt{2s}}{\sigma}(v_0-p^*)} + \frac{1}{2} \left(\sqrt{\frac{s}{r}} + 1 \right) e^{\frac{\sqrt{2r}-\sqrt{2s}}{\sigma} C} - \frac{1}{2} \left(\sqrt{\frac{s}{r}} - 1 \right) e^{\frac{\sqrt{2r}+\sqrt{2s}}{\sigma} C}}{\left(e^{\frac{\sqrt{2s}}{\sigma}(v_0-p^*-C)} + \cosh \frac{\sqrt{2s}}{\sigma} C \right) \cosh \frac{\sqrt{2s}}{\sigma} C} \leq 0. \quad (34)$$

When $C < 0$ and $v_0 < p^* \leq v_0 - C$ we find by implicitly differentiating (33) that

$$\frac{\partial p^*}{\partial C} = - \frac{1 + \frac{1}{2} \left(\left(\sqrt{\frac{s}{r}} + 1 \right) - \left(\sqrt{\frac{s}{r}} - 1 \right) e^{-\frac{2\sqrt{2s}}{\sigma}(v_0-p^*-C)} \right) e^{\frac{\sqrt{2r}-\sqrt{2s}}{\sigma} C}}{\cosh^2 \frac{\sqrt{2s}}{\sigma} (v_0 - p^* - C) + 1 + \sqrt{\frac{s}{r}} e^{\frac{\sqrt{2r}-\sqrt{2s}}{\sigma} C}} \leq 0 \quad (35)$$

For $C \geq 0$ and $0 < p \leq v_0 - C \leq v_0$, we also find that $\partial p^*/\partial C = - \left(e^{-\frac{\sqrt{2s}}{\sigma}(v_0-p^*-C)} + 1 \right) / \left(e^{-\frac{\sqrt{2s}}{\sigma}(v_0-p^*-C)} + 1 \right) \leq 0$. Note that $\lim_{C \rightarrow -\infty} v^* = \frac{\sigma}{\sqrt{2s}}$ which means $\lim_{C \rightarrow -\infty} p^* \leq v_0 \iff v_0 \geq \frac{\sigma}{\sqrt{2s}}$. Therefore, if $v_0 < \frac{\sigma}{\sqrt{2s}}$ then by taking the limit of (33) we find that $\lim_{C \rightarrow -\infty} p^* = \frac{\sigma}{\sqrt{2s}}$. Likewise, if $v_0 \geq \frac{\sigma}{\sqrt{2s}}$ then by taking the limit of (32) we find that $\lim_{C \rightarrow -\infty} p^*$ is the solution to $p = \frac{\sigma}{\sqrt{2s}} \left(2e^{\frac{\sqrt{2s}}{\sigma}(v_0-p)} - 1 \right)$.

Next, we consider the problem of maximizing $\Pi(p^*(C; v_0), C; v_0)$ in C for a fixed v_0 . Suppose it is known that $C^* \geq 0$ then the problem is simple since $\Pi(p, C; v_0) = \frac{\lambda}{s} \left(p + \frac{c}{\lambda} \right) \left(1 - e^{-\frac{\sqrt{2s}}{\sigma}(v_0 - \sigma/\sqrt{2r} - p - c/\lambda)} \right)$ is a function of single variable $p + c/\lambda$ when $C \geq 0$, and

any pairs (p^*, c^*) such that $p^* + c^*/\lambda$ maximizes such function is a valid solution. To proceed, let's suppose that $C^* \leq 0$, then the condition satisfied by C^* reads:

$$0 = \frac{d}{dC} \Pi(p^*(C; v_0), C; v_0) = \frac{\partial \Pi}{\partial C} + \frac{\partial \Pi}{\partial p} \Big|_{p=p^*(C; v_0)} \frac{\partial p^*}{\partial C} = \frac{\partial \Pi}{\partial C} \\ \implies p^*(C; v_0) = \left(\frac{\sigma}{\sqrt{2s}} e^{\frac{\sqrt{2s}}{\sigma}(v_0 - p^*(C; v_0) - C)} - \frac{\sigma}{\sqrt{2s}} - \frac{\sigma}{\sqrt{2r}} \right) e^{\frac{\sqrt{2r} - \sqrt{2s}}{\sigma} C} =: q(C; v_0). \quad (36)$$

We note that $\frac{d}{dC} \Pi(p^*(C; v_0), C; v_0) \geq 0 \iff q(C; v_0) \geq p^*(C; v_0)$. When $p^* \leq v_0$ we can see from (32) that $C = 0$ is a solution to $p^*(C; v_0) = q(C; v_0)$. When $p^* > v_0$ we can see from the RHS of (33) that $p^* \leq \frac{\sigma}{\sqrt{2s}}$, therefore $p^*(C; v_0) + C < v_0$ for all sufficiently negative C , hence there exists $\bar{C} \in (-\infty, 0)$ such that $v_0 = p^*(\bar{C}; v_0) + \bar{C}$ and we can check using (33) that $C = \bar{C}$ is a solution to $p^*(C; v_0) = q(C; v_0)$. However, neither $C = 0$ or $C = \bar{C}$ correspond to C^* in general. We will now discuss sufficient conditions for each type of maximum point.

1. Suppose that $r \gg s$. If $p^* \leq v_0$ then for all r sufficiently large $q(C; v_0)$ is an increasing function in C , whereas $p^*(C; v_0)$ is a decreasing function in C as we have shown. If $p^* > v_0$ then for all sufficiently large r , $q(C; v_0)$ is a decreasing function in C with range $(0, -\sigma e^{\frac{\sqrt{2r} - \sqrt{2s}}{\sigma} \bar{C}} / \sqrt{2r}]$ and $\partial q / \partial C|_{C=\bar{C}} \geq -2e^{\frac{\sqrt{2r} - \sqrt{2s}}{\sigma} \bar{C}} \approx 0$, whereas $p^*(\bar{C}; v_0) = -\frac{\sigma}{\sqrt{2s}} e^{\frac{\sqrt{2r} - \sqrt{2s}}{\sigma} \bar{C}}$ and $p^* \rightarrow \sigma / \sqrt{2s}$ as $C \rightarrow -\infty$ and $\partial p^* / \partial C|_{C=\bar{C}} \approx -1/2$. In both cases, we have $q = p^*$ either at $C = 0$ or $C = \bar{C}$ and $q \leq p^*$ everywhere else, so it is optimal to set $C^* = -\infty$.

2. Suppose that $s > r$ and $v_0 \geq \frac{\sigma}{\sqrt{2s}}$. From (17) it follows that $v^* \leq \frac{\sigma}{\sqrt{2s}}$ so the given condition on v_0 ensures $p^*(C; v_0) \leq v_0$ for all $C \leq 0$. We have $\partial p^* / \partial C|_{C=0} = -1$ and

$$\frac{\partial q}{\partial C} \Big|_{C=0} = \sqrt{\frac{s}{r}} - \sqrt{\frac{r}{s}} - \left(1 - \sqrt{\frac{r}{s}} \right) e^{\frac{\sqrt{2s}}{\sigma}(v_0 - p^*(0; v_0))} \\ = \sqrt{\frac{s}{r}} - \sqrt{\frac{r}{s}} - \left(1 - \sqrt{\frac{r}{s}} \right) \left(\frac{\sqrt{2s}}{\sigma} p^*(0; v_0) + \sqrt{\frac{s}{r}} + 1 \right) = \frac{\sqrt{2r} - \sqrt{2s}}{\sigma} p^*(0; v_0) \geq \frac{\sqrt{2r} - \sqrt{2s}}{\sigma} v_0.$$

Where the second equality followed from (32). Therefore, if $\frac{\sqrt{2s} - \sqrt{2r}}{\sigma} v_0 < 1$ then $\partial q / \partial C|_{C=0} > \partial p^* / \partial C|_{C=0}$ and $q < p^*$ for $C < 0$ in some vicinity of 0. We have already argued that $\lim_{C \rightarrow -\infty} p^*(C; v_0)$ is finite, hence the condition $s > r$ ensures $e^{\frac{\sqrt{2s}}{\sigma}(v_0 - p^*) + \frac{\sqrt{2r} - \sqrt{2s}}{\sigma} C} \rightarrow +\infty$ as $C \rightarrow -\infty$, so that $\lim_{C \rightarrow -\infty} q(C; v_0) = +\infty$, therefore the maximum point must be an interior point $-\infty < C^* < 0$.

3. Suppose that $s > r$ and consider a large $v_0 \gg 0$, in particular: $v_0 > \frac{\sigma}{\sqrt{2s}}$. Since $v_0 - p^*(C; v_0)$ is increasing with v_0 , when v_0 is sufficiently large at any fixed $C < 0$ we find that $\frac{q(C; v_0)}{p^*(C; v_0)} \approx e^{\frac{\sqrt{2r} - \sqrt{2s}}{\sigma} C} \cosh \frac{\sqrt{2s}}{\sigma} C > 1$. Moreover, we have already shown in the previous case that $\partial q / \partial C|_{C=0} < \partial p^* / \partial C|_{C=0}$ as long as $\frac{\sqrt{2s} - \sqrt{2r}}{\sigma} v_0 > 1$, and that $\lim_{C \rightarrow -\infty} p^* < \lim_{C \rightarrow -\infty} q = +\infty$. This means $\Pi(p^*(C; v_0), C; v_0)$ is increasing with C and we must have $C^* \geq 0$.

A.7. Proof of Proposition 4:

Proof: For $v_0 \gg 0$ ($v_0 > v^*$) and a fixed C , we have $p^* < v_0$ by Proposition 3. Let $A \equiv \left(e^{\frac{\sqrt{2s}}{\sigma} \max(C, 0)} + e^{\frac{\sqrt{2s}}{\sigma} (2C - \max(C, 0))} \right) / 2$, then after some rearrangement $\partial \Pi(p; v_0, c) / \partial p|_{p=p^*} = 0$

gives:

$$\begin{aligned}
 p^* &= v_0 - \frac{\sigma}{\sqrt{2s}} \ln \left(1 + \frac{\sqrt{2s}}{\sigma} p^* + \frac{c}{\lambda A} e^{\frac{\sqrt{2s}}{\sigma} C} \right) - \frac{\sigma}{\sqrt{2s}} \ln A. \\
 &= \tilde{p}^* - \frac{\sigma}{\sqrt{2s}} \ln \left(1 - \frac{\ln \left(1 + \frac{\sqrt{2s}}{\sigma} p^* + \frac{c}{\lambda A} e^{\frac{\sqrt{2s}}{\sigma} C} \right) + \ln A - \frac{c}{\lambda A} e^{\frac{\sqrt{2s}}{\sigma} C}}{1 + \frac{\sqrt{2s}}{\sigma} v_0} \right) \\
 &\leq \tilde{p}^* - \frac{\sigma}{\sqrt{2s}} \ln \left(1 - \frac{\ln \left(1 + \frac{\sqrt{2s}}{\sigma} v_0 + \frac{c}{\lambda A} e^{\frac{\sqrt{2s}}{\sigma} C} \right) + \ln A - \frac{c}{\lambda A} e^{\frac{\sqrt{2s}}{\sigma} C}}{1 + \frac{\sqrt{2s}}{\sigma} v_0} \right) = \tilde{p}^* + O \left(\frac{\ln v_0}{v_0} \right),
 \end{aligned}$$

where we have used $p^* < v_0$ to obtain the last inequality and the second last line above also shows that $p^* - \tilde{p}^* \geq 0$, therefore $p^* = \tilde{p}^* + O \left(\frac{\ln v_0}{v_0} \right)$ as claimed. Now, we have that $e^{\frac{\sqrt{2s}}{\sigma}(v_0 - p^*)} = \frac{\sqrt{2s}}{\sigma}(v_0 - C) e^{\frac{\sqrt{2s}}{\sigma} C} \cosh \frac{\sqrt{2s}}{\sigma} C \cdot \left(1 + O \left(\frac{\ln v_0}{v_0} \right) \right)$. Recall $q(C; v_0)$ from (36), and using (32) for $p^*(C; v_0) \leq v_0$, the $v_0 \gg 0$ approximation above gives $\frac{p^*(C; v_0) - q(C; v_0)}{v_0 - C} = 1 - e^{\frac{\sqrt{2r} - \sqrt{2s}}{\sigma} C} \cosh \frac{\sqrt{2s}}{\sigma} C + O \left(\frac{\ln v_0}{v_0} \right)$. Therefore, for $v_0 \gg 0$, we either have $p^* > q$ for all $C < 0$, in which case $C^* = 0$, or $p^* < q$ for all $C < 0$, in which case $C^* = -\infty$, or p^* intercepts q exactly once for $C < 0$ at a finite C^* which is given to the $O \left(\frac{\ln v_0}{v_0} \right)$ -order by the negative solution to $e^{\frac{\sqrt{2r} - \sqrt{2s}}{\sigma} C} \cosh \frac{\sqrt{2s}}{\sigma} C = 1$.

A.8. Proof of Lemma 3:

Proof: Let $\psi_\varepsilon : \mathbb{R} \rightarrow [0, 1]$ be a regularized step function $v \mapsto \mathbb{1}_{v \geq p}$. We shall assume that ψ_ε is smooth, it is constant outside $[p - \varepsilon, p + \varepsilon]$, and that $\|\psi'_\varepsilon\|_{L^\infty} \leq C_1/\varepsilon$, $\|\psi''_\varepsilon\|_{L^\infty} \leq C_1/\varepsilon^2$ for some $C_1 > 0$. Let $\mathcal{P}_\varepsilon(v, t|\phi, c, \lambda)$ denotes the solution to the PDE (9) along with the absorbing boundary at $v = \underline{v}$ but with $\psi_\varepsilon(v)$ replacing $\mathbb{1}_{v \geq p}$. Similarly, let

$$\tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda) := e^{-\psi_\varepsilon(v) \cdot \lambda t} \int_{\underline{v}}^{+\infty} G_t(v, v') \phi(v') dv', \quad (37)$$

and $\tilde{\phi}_t(v) := \lim_{\varepsilon \rightarrow 0} \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda)$. We can verify that $\tilde{\mathcal{P}}_\varepsilon$ satisfies:

$$\frac{1}{2} \sigma^2 \frac{\partial^2}{\partial v^2} \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda) - \frac{\partial}{\partial t} \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda) - \lambda \psi_\varepsilon(v) \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda) = R_\varepsilon(v, t)$$

where

$$R_\varepsilon(v, t) := \frac{\sigma^2}{2} \lambda t \left(2\psi'_\varepsilon(v) \frac{\partial}{\partial v} \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda) + \psi''_\varepsilon(v) \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda) + \lambda t \psi'_\varepsilon(v)^2 \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda) \right).$$

Since the support of $\psi_\varepsilon(v) - \mathbb{1}_{v \geq p}$ has measure 2ε , we can show that:

$$\begin{aligned}
 \|\tilde{\phi}_t - \tilde{\mathcal{P}}_\varepsilon(\cdot, t|\phi, c, \lambda)\|_{L^2} &= \left\| \left(e^{\mathbb{1}_{v \geq p} \cdot \lambda t} - e^{-\psi_\varepsilon(v) \cdot \lambda t} \right) \int_{\underline{v}}^{+\infty} G_t(v, v') \phi(v') dv' \right\|_{L^2} \\
 &\leq \sqrt{2\varepsilon} \cdot (1 - e^{-\lambda t}) \cdot \left\| \int_{\underline{v}}^{+\infty} G_t(v, v') \phi(v') dv' \right\|_{L^\infty} \leq \lambda M \sqrt{2\varepsilon} \cdot t, \quad (38)
 \end{aligned}$$

where we used the fact that the maximum density always decreases after diffusion to bound the last factor by $\|\phi\|_{L^\infty} \leq M$. Similarly, we compute that:

$$\|\mathcal{P}(\cdot, t|\phi, c, \lambda) - \mathcal{P}_\varepsilon(\cdot, t|\phi, c, \lambda)\|_{L^2} \leq \lambda \|\mathcal{P}(\cdot, t|\phi, c, \lambda)\|_{L^\infty} t \cdot \sqrt{2\varepsilon} \leq \lambda M \sqrt{2\varepsilon} \cdot t, \quad (39)$$

where the first inequality followed from comparing the PDE for $\mathcal{P}$ and $\mathcal{P}_\varepsilon$. Thus, we shall focus on comparing $\tilde{\mathcal{P}}_\varepsilon$ to $\mathcal{P}_\varepsilon$ for the remainder. We follow the standard energy argument. Let $w(v, t) := \mathcal{P}_\varepsilon(v, t|\phi, c, \lambda) - \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda)$ and define $E(t) := \frac{1}{2} \|w(\cdot, t)\|_{L^2}^2$, then $E(0) = 0$ and:

$$\begin{aligned} \frac{dE(t)}{dt} &= \int_{\underline{v}}^{+\infty} w(v, t) \left(\frac{\sigma^2}{2} \frac{\partial^2}{\partial v^2} - \lambda \psi_\varepsilon(v) \right) w(v, t) dv + \int_{\underline{v}}^{+\infty} w(v, t) R_\varepsilon(v, t) dv \\ &\leq - \int_{\underline{v}}^{+\infty} \left[\frac{\sigma^2}{2} \left(\frac{\partial}{\partial v} w(v, t) \right)^2 dv + \lambda \psi_\varepsilon(v) w(v, t)^2 \right] dv + \|w(\cdot, t)\|_{L^2} \|R_\varepsilon(\cdot, t)\|_{L^2} \\ &\leq \sqrt{2E} \|R_\varepsilon(\cdot, t)\|_{L^2}. \end{aligned} \quad (40)$$

To bounds $\|R_\varepsilon(\cdot, t)\|_{L^2}$, first, we use the bounds on $|\phi|, |\phi'|$ to bounds $|\tilde{\mathcal{P}}_\varepsilon| \leq M$ and $|\frac{\partial}{\partial v} \tilde{\mathcal{P}}_\varepsilon| \leq 2M$ via (37), where integration-by-parts is also needed to bounds the latter. Afterwards, we can use the bounds on $\psi'_\varepsilon, \psi''_\varepsilon$, and the fact that their support has measure 2ε , to conclude that

$$\|R_\varepsilon(\cdot, t)\|_{L^2} \leq \frac{\sigma^2}{2} \lambda t \left(2 \cdot \frac{C_1}{\varepsilon} \cdot 2M + \frac{C_1}{\varepsilon^2} \cdot M + \lambda t \left(\frac{C_1}{\varepsilon} \right)^2 \cdot M \right) \sqrt{2\varepsilon} \leq C_2 M \left(\frac{1}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon^{3/2}} \right) \cdot t$$

where the last inequality holds for all t sufficiently small such that $\lambda t < 1$ and some appropriately defined $C_2 > 0$. It follows from (40) that:

$$\|\mathcal{P}_\varepsilon(v, t|\phi, c, \lambda) - \tilde{\mathcal{P}}_\varepsilon(v, t|\phi, c, \lambda)\|_{L^2} = \sqrt{2E(t)} \leq \frac{C_2 M}{2} \left(\frac{1}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon^{3/2}} \right) \cdot t^2 \quad (41)$$

Combining (38), (39), and (41), we obtain the needed result.

B. Extension to Non-contractual Settings

The non-contractual setting of transactional sales is a widely studied area that attracts a substantial number of research works. In this section, we discuss how our model can be adapted to be used in the non-contractual setting. The key difference is in interpreting the flow cost $c > 0$ as an information tracking cost, rather than a monetary subscription fee.

B.1. Information Tracking Interpretation

Consider a representative consumer who decides whether to keep track of a firm's new product information over time by incurring a constant information tracking cost of $c > 0$ per unit time. This costly process of information tracking can be interpreted as, for example, the consumer actively searching for information about the firm's new product, communicating with friends about the firm, subscribing to the firm's marketing campaigns, or watching the firm's video ads without skipping, etc. Each piece of new information updates the consumer's expected valuation of the firm's product

relative to the outside option at time t , which we denote v_t . In a typical setting such as electronics and fashion where consumers purchase successive generations of new products from the firm, v_t is the consumer's expected valuation of the latest product design at time t ; in a setting such as fast-moving consumer goods where consumers may purchase various products under a brand over time, v_t is the common value of the products under the brand. Following Chernoff (1968), Holmström (1999) and Ke et al. (2022), the consumer's learning process can be modeled using a Bayesian learning framework with an evolving true state where v_t is the consumer's posterior belief of her valuation of the firm's product design.

Specifically, let $\{W_t^X\}_{t=0}^\infty$ and $\{W_t^S\}_{t=0}^\infty$ be independent standard Brownian motions. Suppose that the true state of the world (valuation of product design) X_t evolves according to $dX_t = \sigma dW_t^X$. Consumers learn about the state X_t by observing the signal S_t , governed by $dS_t = X_t dt + \sigma_S dW_t^S$ and hold the belief at any given time t that $X_t \sim \mathcal{N}(v_t, \rho_t^2)$. Consumers' expected valuation at time t is denoted by $v_t = \mathbb{E}[X_t | \mathcal{F}_t^S]$, where $\mathcal{F}_t^S$ is the natural filtration associated with S_t . After time dt , the signal dS_t has been observed, and by applying Bayes' rule to compute the posterior distribution of X_{t+dt} we find that $d\rho_t^2/dt = \sigma^2 - \rho_t^4/\sigma_S^2$ and $dv_t = \rho_t^2/\sigma_S dW_t$ where $\{W_t\}_{t=0}^\infty$ is a standard Brownian motion²¹. If $\sigma = 0$, then we recover the typical Bayesian framework with a fixed unknown true state of the world, where consumers' uncertainty about the true state of the world, ρ_t decreases to zero as they learn more information. In general with $\sigma > 0$, we can see that ρ_t approaches the constant $\sqrt{\sigma\sigma_S}$. In this work, we assume for analytical tractability that either a sufficient amount of time has passed, or that $\rho_0 = \sqrt{\sigma\sigma_S}$. Consequently, $\rho_t = \sqrt{\sigma\sigma_S}$ for $\forall t \in [0, \infty)$ and we recover: $dv_t = \sigma dW_t$. This is the same dynamics as in (1), and most of the theoretical analysis of our model is identical to the contractual setting discussed in the main text.

In reality, consumers may also get a bulk of information from their consumption experience after making a purchase. This would entail jumps in v_t . In Web Appendix D, we extend the basic model in equation (1) by considering a more general optimal learning problem based on a rational inattention framework (Sims 2003, Matějka and McKay 2015, Luo 2017), in which, the consumer can conduct both gradual learning by information tracking and discrete learning by consumption subject to information transmission capacity. We explicitly spell out the conditions under which, equation (1) and the constant information tracking cost $c > 0$ emerge endogenously as the outcome of the optimal learning strategy and the cost of continuous information flow through a channel with limited capacity.

B.2. Estimation of the Model

Most of the details of the estimation procedure remain the same as in the contractual settings; in this section, we will simply elaborate the key differences. A typical transaction history takes the form $H = (x, t_0, t_1, \dots, t_x, T)$. Here, $t_0 = 0$ is the initial transaction time, considered by Schmittlein et al. (1987) to be the time when the consumer becomes "alive". We discretize the time-domain

²¹ Informally, W_t can be thought of as the 'sum' of W_t^X and W_t^S . More precisely, this follows from the standard Kalman–Bucy optimal filtering and Lévy characterization, and we refer to the Web Appendix D for more detail. Equation (1) implies that the model works the same even if $\sigma_S \rightarrow 0$ so that consumers can fully observe the true state of the world. The necessary assumption here is an evolving true product value, not noisy learning of that value.

$[t_0, T]$ into a J -points grid of size Δt , then we approximate the solution to the PDE (9) by computing $\tilde{\phi}_j$ for $j = 0, \dots, J$, with $\tilde{\phi}_0(v) := \phi_0(v|\xi)$, exactly as outlined in §6.2. The model parameters are given by $\theta = (\xi, c, \gamma, \alpha)$. The likelihood function $\mathcal{L}(\theta|H)$ is given by (18). However, since τ is not observable, we cannot make a distinction between a consumer who made no purchase between $(t_x, T]$ due to lack of needs and the one who has dropped-out some time between t_x and T . Therefore, suppose that $t_x \in (s_{j'-1}, s_{j'}]$ for some $j' \leq J$, then we estimate $\mathbb{P}[H|\phi_0, c, \lambda]$ using: $\tilde{P}(H|\phi_0, c, \lambda) := \int_{\underline{v}}^{+\infty} \tilde{\phi}_J(v)dv + \tilde{D}_{j'}$, where

$$\tilde{D}_{j'} := \sum_{j=j'}^{J-1} \left(\int_{\underline{v}}^{+\infty} \tilde{\phi}_j(v)dv - \int_{\underline{v}}^{+\infty} \int_{\underline{v}}^{+\infty} G_{\Delta t}(v, v') \tilde{\phi}_j(v')dv'dv \right),$$

counts the probability that the consumer has dropped-out during $(t_x, T]$ (absorbed by the boundary $v = \underline{v}$). To compute the expected number of repeat transactions during the forecast period $\mathbb{E}[X(T^*)|H, \theta]$, the only adjustment we need to make to (20) is normalization factor for $\phi_T(v'|\theta)$, since conditioning on H no longer guarantees the alive state of the consumer at $t = T$. In particular, we approximate the valuation density at $t = T$ of a consumer who produces the transaction pattern H by $\tilde{\phi}_J(v') / \left(\int_{\underline{v}}^{+\infty} \tilde{\phi}_J(v')dv' + \tilde{D}_{j'} \right)$.

B.3. Benchmark Empirical Studies

We present the estimation and prediction results of our model on the 1/10-th sample CDNOW dataset, and compare with the classic results from Pareto/NBD model. Apart from some differences we have already discussed in §B.2, we adopt the same implementation configurations as in §7.1 for our model. We will assume that consumers, starting from a homogeneous valuation v , have been actively tracking the firm for some unobserved time before t_0 . Therefore, v_0 is distributed according to their expected valuation at the first purchase occasion, which arrives according to the Poisson process. Then v_0 is distributed according to $\phi_0(v_0|\xi) = \mathbb{1}_{v_0 \geq \max(\underline{v}, p)} \mathcal{P}_1(v_0|v, c, \lambda) / \int_{\max(\underline{v}, p)}^{+\infty} \mathcal{P}_1(v'_0|v, c, \lambda)dv'_0$ where $\xi = (v, c, \lambda)$, and $\mathcal{P}_1(v_0|v, c, \lambda)$ has been derived in Lemma 1.

The results are given in Table 2 and Figure 8. We note that Pareto/NBD remains one of the best models on CDNOW dataset, even among modern CLV models such as Fader et al. (2005), Reutterer et al. (2021), Platzer and Reutterer (2016), Xie and Huang (2021). The MAE of our model is also quite comparable to the BG/NBD model (Fader et al. (2005)) with MAE = 0.786 (see Reutterer et al. (2021)).

		v	$\log c$	γ	α	ζ	β	LL	MAE
CDNOW ($N = 2357$)	Our Model	0.174	-4.044	0.889	8.075	—	—	-9288.5	0.788
	Pareto/NBD	—	—	0.553	10.580	0.606	11.656	-9595.0	0.760

Table 2 Parameter estimation results.

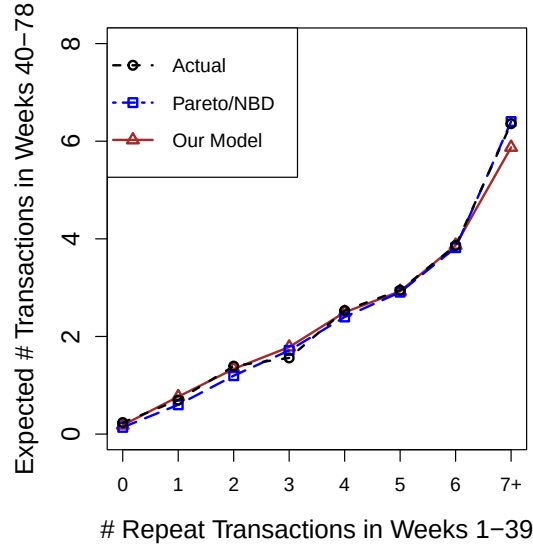


Figure 8 Prediction result for the CDNOW dataset.

C. Extension of Pareto/NBD to Contractual Settings

Pareto/NBD was introduced in Schmittlein et al. (1987) for non-contractual applications. We took the liberty to modify the model to the contractual settings while retaining the essence of the model as much as possible. We assume a consumer makes transactions according to a Poisson process with rate λ as long as her subscription is active. In other words, conditioned on an active subscription and the last transaction t_k , the next transaction is distributed by: $t_{k+1} \sim \lambda e^{-\lambda(t_{k+1}-t_k)}$. The consumer terminates the subscription at a random time τ which is distributed by $\tau \sim \mu e^{-\mu\tau}$, for some drop-out rate μ . Finally, we assume that λ and μ are distributed heterogeneously across the population according to the Gamma distributions:

$$\lambda \sim f(\lambda|\gamma, \alpha) = \frac{\alpha^\gamma \lambda^{\gamma-1} e^{-\lambda\alpha}}{\Gamma(\gamma)}, \quad \mu \sim f(\mu|\zeta, \beta) = \frac{\beta^\zeta \mu^{\zeta-1} e^{-\mu\beta}}{\Gamma(\zeta)}.$$

Therefore, the parameters of the full model are $(\gamma, \alpha, \zeta, \beta)$.

Given the subscription and transaction history $H = (x, t_0, \dots, t_x, \tau, T)$, the likelihood for an individual consumer with parameter λ, μ , is given by:

$$L(\lambda, \mu|H) = \lambda^x \mu^{\mathbb{1}_{\tau \leq T}} e^{-(\lambda+\mu) \min(\tau, T)}.$$

To obtain the population-level likelihood, we integrate the above in λ, μ against the distributions $f(\lambda|\gamma, \alpha)$ and $f(\mu|\zeta, \beta)$, the result is:

$$L(\gamma, \alpha, \zeta, \beta|H) = \frac{\zeta^{\mathbb{1}_{\tau \leq T}} \alpha^\gamma \beta^\zeta}{(\alpha + \min(\tau, T))^{\gamma+x} (\beta + \min(\tau, T))^{\zeta+\mathbb{1}_{\tau \leq T}}} \cdot \frac{\Gamma(\gamma+x)}{\Gamma(\gamma)}.$$

Finally, the expected number of transactions in the forecast period $(T, T + T^*]$ given the transaction history H is:

$$\mathbb{E}[X(T^*)|H, \gamma, \alpha, \zeta, \beta] = \mathbb{1}_{\tau > T} \cdot \frac{(\gamma + x)(\beta + T)}{(\alpha + T)(\zeta - 1)} \left(1 - \left(\frac{\beta + T}{\beta + T + T^*} \right)^{\zeta - 1} \right).$$

Note that this expression is almost identical to that for the standard Pareto/NBD model, except $\mathbb{P}[\text{Alive at } T|H, \gamma, \alpha, \zeta, \beta]$ has been replaced by $\mathbb{1}_{\tau > T}$.

D. Rational Inattention Perspective of The Model

Recall the basic model setup with a representative consumer and a firm offering a single product at a price p . We introduce a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and let $\{W_t^X\}_{t \geq 0}$, and $\{W_t^S\}_{t \geq 0}$ be two mutually independent standard Brownian motions adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. Suppose the ground-truth product utility relative to the outside option for the consumer is given by the process $\{X_t\}_{t \geq 0}$ satisfying $dX_t = \sigma dW_t^X$. We assume that the consumer cannot directly observe X_t but has a prior belief at $t = 0$ that it is normally distributed: $X_0 \sim \mathcal{N}(v_0, \rho_0^2)$. For $t > 0$, the consumer decides how to learn about X_t following rational inattention framework (Sims 2003, Matějka and McKay 2015) through two available channels. Let $\{\mathcal{F}_t^L\}_{t \geq 0}$ be the filtration generated by $\{L_t \equiv (S_t, J_t)\}_{t \geq 0}$, and

$$v_t \equiv \mathbb{E}[X_t | \mathcal{F}_t^L], \quad \rho_t^2 \equiv \mathbb{E}[(X_t - v_t)^2 | \mathcal{F}_t^L],$$

where $\{S_t\}_{t \geq 0}$, and $\{J_t\}_{t \geq 0}$ are two independent observable processes characterizing the two learning channels we will now discuss:

Continuous Learning: The consumer continuously collects information about X_t through various media, such as advertisements, online forums, and social media. The consumer decides the level of attention given to this channel by choosing a progressively measurable process $\{\sigma_{S,t} \in \mathbb{R}_{>0} \cup \infty\}_{t \geq 0}$ and observe the signal $\{S_t\}_{t \geq 0}$ governed by $dS_t = X_t dt + \sigma_{S,t} dW_t^S$. The channel shutdowns if $\sigma_{S,t} = +\infty$ is chosen. Suppose $X_t | \mathcal{F}_t^L \sim \mathcal{N}(v_t, \rho_t^2)$, then by (Liptser and Shiryaev 2013, Theorem 11.1) the belief remains normally distributed after receiving the signal: $X_{t+dt} | \mathcal{F}_{t+dt}^L \sim \mathcal{N}(v_t + dv_t, \rho_t^2 + d\rho_t^2)$, where $dv_t = \frac{\rho_t^2}{\sigma_{S,t}^2} (X_t - v_t) dt + \frac{\rho_t^2}{\sigma_{S,t}} dW_t^S$ and $d\rho_t^2 = \left(\sigma^2 - \frac{\rho_t^4}{\sigma_{S,t}^2} \right) dt$ according to Kalman-Bucy optimal filtering (Liptser and Shiryaev 2013, Theorem 12.1). The information flow rate is given by the expectation of KL-divergence between the distribution of X_t before and after receiving a signal. We assume that the consumer has a limited information process capacity $\chi > 0$, following Luo (2017) the information flow constraint can be written as:

$$\begin{aligned} \mathbb{E}[D_{KL}(X_{t+dt} | \mathcal{F}_{t+dt}^L \| X_{t+dt} | \mathcal{F}_t^L)] &\leq \chi \cdot dt \\ \frac{1}{2} \mathbb{E} \left[\ln \frac{\rho_t^2 + \frac{d\rho_t^2}{dt}}{\rho_t^2 + \sigma^2 dt} + \frac{\rho_t^2 + \sigma^2 dt + \left(\frac{\rho_t^2}{\sigma_{S,t}} dW_t^S \right)^2}{\rho_t^2 + \frac{d\rho_t^2}{dt}} - 1 \right] &\leq \chi \cdot dt \implies \frac{1}{2} \cdot \frac{\rho_t^2}{\sigma_{S,t}^2} \leq \chi. \end{aligned}$$

Subjects to this constraint, there is an upper limit for the attention level the consumer can choose: $\sigma_{S,t}^2 \in [\rho_t^2/2\chi, +\infty]$. Therefore, the continuous learning steady-state uncertainty is $\rho_{ss}^2 = \sigma^2/2\chi$. Let $\kappa_c \geq 0$ be the cost per information flow per time, then $\frac{\kappa_c}{2} \frac{\rho_t^2}{\sigma_{S,t}^2} dt$ is the cost for continuously processing the information.

Discrete Learning: We assume the consumer learns about the firm's product after each consumption occasion with some remaining residue uncertainties σ_C^2 . We shall assume that $\sigma_C^2 \leq \sigma^2/2\chi$, i.e. the consumption occasion to be more informative than learning information from other sources. The consumer controls the channel by choosing a progressively measurable process $\{b_t \in \{0, 1\}\}_{t \geq 0}$, where $b_t = 1$ indicates purchasing if there is a need at t , and $b_t = 0$ indicates not purchasing. From the Bayesian learning perspective (see DeGroot (2005)), the consumer observes X_t after a consumption with variance $\sigma_D^2 = (\sigma_C^{-2} - \rho_{t-}^{-2})^{-1}$ normally distributed noise. Let $J_t \equiv \int_0^t \int_{\mathbb{R}^2} b_{s-} \left(\sqrt{\rho_{s-}^2 - \sigma_C^2} z_1, z_2 \right) J(ds, dz)$ where $J(t, z)$ is a Poisson random measure with $\nu(z) \equiv \mathbb{E}[J(1, z)] = \lambda \cdot \phi(z_1) \cdot \delta(z_2 - 1)$, and ϕ is the standard normal PDF. If $X_{t-} | \mathcal{F}_{t-}^L \sim \mathcal{N}(v_{t-}, \rho_{t-}^2)$ then $X_t | \mathcal{F}_t^L \sim \mathcal{N}(v_t, \rho_t^2)$ after a need arrival at t where $(v_t, \rho_t^2) = (v_{t-}, \rho_{t-}^2) + dJ_t$. Following Sims (2003), the information flows from the consumption at t is given by the KL-divergence:

$$\mathbb{E} [D_{KL}(X_t | \mathcal{F}_t^L \| X_{t-} | \mathcal{F}_{t-}^L)] = \frac{1}{2} \mathbb{E} \left[\ln \frac{\rho_{t-}^2}{\sigma_C^2} + \frac{\sigma_C^2 + (\rho_{t-}^2 - \sigma_C^2) z_1^2}{\rho_{t-}^2} - 1 \right] = \frac{1}{2} \ln \frac{\rho_{t-}^2}{\sigma_C^2}.$$

Let $\kappa_d \geq 0$ be the cost per unit information transmission through consumption, thus we interpret $\frac{\kappa_d}{2} \ln \frac{\rho_{t-}^2}{\sigma_C^2}$ as the cost of exerting mental effort when using an unfamiliar product.

Putting everything together, the state (v_t, ρ_t^2) is right continuous with left limit and evolves according to

$$dv_t = \frac{\rho_{t-}^2}{\sigma_{S,t-}^2} (X_t - v_{t-}) dt + \frac{\rho_{t-}^2}{\sigma_{S,t-}^2} dW_t^S + \int_{\mathbb{R}^2} b_{t-} z_1 \sqrt{\rho_{t-}^2 - \sigma_C^2} J(dt, dz) \quad (42)$$

$$d\rho_t^2 = \left(\sigma^2 - \frac{\rho_{t-}^4}{\sigma_{S,t-}^2} \right) dt + \int_{\mathbb{R}^2} b_{t-} z_2 (\sigma_C^2 - \rho_{t-}^2) J(dt, dz) \quad (43)$$

and we have $X_t | \mathcal{F}_t^L \sim \mathcal{N}(v_t, \rho_t^2)$. The consumer's optimal control problem is,

$$V(v, \rho^2) = \sup_{\{\sigma_{S,t}\}_{t \geq 0}, \{b_t\}_{t \geq 0}} \mathbb{E} \left[\int_0^\infty e^{-rt} \left(\int_{\mathbb{R}^2} b_t z_2 \cdot \left(X_t - p - \frac{\kappa_d}{2} \ln \frac{\rho_t^2}{\sigma_C^2} \right) J(dt, dz) - \frac{\kappa_c}{2} \cdot \frac{\rho_t^2}{\sigma_{S,t}^2} dt \right) | v_0 = v, \rho_0^2 = \rho^2 \right] \quad (44)$$

with the corresponding HJB:

$$0 = \sup_{\sigma_{S,b}} \left[-\frac{\kappa_c}{2} \cdot \frac{\rho^2}{\sigma_S^2} - rV(v, \rho^2) + \left(\sigma^2 - \frac{\rho^4}{\sigma_S^2} \right) V_\rho(v, \rho^2) + \frac{\rho^4}{2\sigma_S^2} V_{vv}(v, \rho^2) \right. \\ \left. + \lambda b \cdot \mathbb{E}_{z_1 \sim \mathcal{N}(0,1)} \left[V(v + \sqrt{\rho^2 - \sigma_C^2} z_1, \sigma_C^2) - V(v, \rho^2) \right] + \lambda b \cdot \left(v - p - \frac{\kappa_d}{2} \ln \frac{\rho^2}{\sigma_C^2} \right) \right]. \quad (45)$$

Since the expression above is linear in $1/\sigma_S^2$, the optimal $\{\sigma_{S,t}\}_{t \geq 0}$ must be a bang-bang control. We have $\sigma_{S,t}^{*2}(v, \rho^2) = \rho_t^2/2\chi$ if $\frac{\rho_t^2}{2}V_{vv} - \rho^2V_\rho - \frac{\kappa_c}{2} \geq 0$, or $\sigma_{S,t}^{*2} = +\infty$ otherwise. Similarly, we have $b^*(v, \rho^2) = 1$ if $v - p - \frac{\kappa_d}{2} \ln \frac{\rho^2}{\sigma_C^2} + \mathbb{E} \left[V(v + \sqrt{\rho^2 - \sigma_C^2} z_1, \sigma_C^2) - V(v, \rho^2) \right] \geq 0$, and $b^*(v, \rho^2) = 0$ otherwise. Given the stochastic jumps, the HJB is an Integro-differential equation, and solving the problem in full generality appears difficult. In our work, we consider the special case of the model under the following two assumptions:

1. $\sigma_C^2 = \sigma^2/2\chi$.
2. $\kappa_d \gg \kappa_c \geq 0$. This is a strong assumption that implies that the consumer lives in an information-rich environment where acquiring new information at any time costs little effort. A relatively high κ_d can also represent the consumer's "preference for gradual learning" (Hébert and Woodford 2023)²².

Under these assumptions, the model can be approximated by the simple pure continuous learning model introduced in the main text. We outline the intuition as follows. Consider a consumer with $\rho_0^2 = \sigma_C^2$ and some $v_0 = v$, e.g. she made her initial purchase at $t = 0$. In this case, we have $b^*(v, \sigma_C^2) = 1_{v \geq p}$. We interpret $\sigma_S^{*2} = \sigma_C^2/2\chi$ as the consumer being 'active', the flow cost is $\frac{\kappa_c}{2} \cdot \frac{\sigma_C^2}{\sigma_S^{*2}} = \kappa_c\chi =: c$, and the consumer becomes 'inactive' when $\sigma_S^{*2} = +\infty$. Since $\sigma_C^2 V_\rho(v, \sigma_C^2) \sim -\frac{\kappa_d \lambda/2}{r+\lambda+2\chi}$ for $v \gg p$ ²³, if $\kappa_d \gg \kappa_c$ then we have $\frac{\rho_t^2}{2}V_{vv} - \rho^2V_\rho - \frac{\kappa_c}{2} \geq 0$, hence $\sigma_S^{*2}(v, \sigma_C^2) = \sigma_C^2/2\chi$ for all $v \gg p$. Since σ_S^{*2} is a bang-bang control, there should exist a threshold $\underline{v}_c$, such that $\sigma_S^{*2} = \sigma_C^2/2\chi$ if $v \geq \underline{v}_c$.

For $v \geq \underline{v}_c$, $V(v, \sigma_C^2)$ satisfies the HJB (45) which reduces to the ODE: $\frac{1}{2}\sigma^2 V''(v) - rV(v) + \lambda \max(v - p, 0) - c = 0$. Effectively, the problem reduces to an optimal stopping problem with the state evolution (42) and (43) simplify to $dv_t = 2\chi(X_t - v_t)dt + \sigma dW_t^S$ and $d\rho_t^2 = 0$ (i.e. ρ_t^2 remains constant at σ_C^2), respectively. We can check that v_t is a Martingale with respect to $\{\mathcal{F}_t^L\}_{t \geq 0}$ with quadratic variation $[v]_t = \sigma^2 \cdot t$. Therefore we have $dv_t = \sigma dW_t$ by Lévy characterization, where $\{W_t\}_{t \geq 0}$ is a standard Brownian motion adapted to $\{\mathcal{F}_t^L\}_{t \geq 0}$, thus recovering (1).

On the other hand, for $v < \underline{v}_c$, the consumer's learning trajectory $t \mapsto (v_t, \rho_t^2) \in \mathbb{R}^2$ leaves the line $\rho^2 = \sigma_C^2$ into the $\rho^2 > \sigma_C^2$ region. Since $\sigma_S^{*2}(v_t, \rho_t^2) = \sigma_C^2/2\chi$ and $b^*(v_t, \rho_t^2) = 1$ for a sufficiently large ρ^2 , the consumer will eventually restart tracking the firm again to resolve the accumulating uncertainties, and we have $V(v, \sigma_C^2) > 0$. The exception is the extreme case where $\kappa_d = +\infty$, and $V(v, \sigma_C^2)$ coincides exactly with the pure continuous learning value function $V(v)$ studied in the main text, in particular: $V(v, \sigma_C^2) = 0$ for $v < \underline{v}_c = \underline{v}$. Given $\varepsilon > 0$, according to Berge's Maximum Theorem, for sufficiently large $\kappa_d \gg \kappa_c$, we have $V(v) \leq V(v, \sigma_C^2) < V(v) + \varepsilon$ and $\underline{v}_c = \underline{v} + O(\varepsilon)$. Therefore, by assuming an appropriate $\kappa_d \gg \kappa_c$, our pure continuous learning model results in the main text are within $O(\varepsilon)$ of the full model, e.g. the expected (discounted) revenue from the

²² We borrowed the concept and terminology from Hébert and Woodford (2023), although our setting is different from theirs which concerned the problem of learning the discrete state of the world to choose the optimal action. It's possible to follow Hébert and Woodford (2023) more closely by combining both continuous and discrete learning channels into one, then measure the information flow using a divergence $D(\cdot \| \cdot)$ which is convex in ρ_t^2 , instead of using the KL-divergence.

²³ Assuming $V(v, \sigma_C^2)$ is approximately linear in v for $v \gg p$, then for $\rho^2 > \sigma_C^2$ over some neighborhood of σ_C^2 we have $b^*(v, \rho^2) = 1$, and since the advantage due to convexity diminishes, the DM set $\sigma_S^{*2}(v, \rho^2 > \sigma_C^2) = \rho_t^2/2\chi$ to decreases ρ_t^2 as quickly as possible. Hence $V(v, \sigma_C^2 + \delta\rho^2) \sim V(v, \sigma_C^2) - \int_0^\infty e^{-rt} \cdot \frac{\kappa_d}{2} \ln \frac{\sigma_C^2 + \delta\rho^2 e^{-2\chi t}}{\sigma_C^2} \cdot \lambda e^{-\lambda t} dt$, from which we may estimate $V_\rho(v, \sigma_C^2)$.

consumer with $v < \underline{v}_c$ is $O(\varepsilon)$ and for practical purposes can be considered as to have dropped out. In reality, dropping out is not an absolute term, but a spectrum of familiarity quantified by ρ_t^2 .

E. Further Remarks on Computation via Heat Diffusion Equation

We remark that $\varepsilon > 0$ is arbitrary in Lemma 3, therefore, we can obtain the sharpest result by choosing ε which minimizes the RHS of (25), although we choose to leave (25) as-is for the ease of presentation. Since some of $\tilde{\phi}_j$ may not be continuously differentiable, we must always first approximate $\tilde{\phi}_j$ with a continuously differentiable function, such that the additional errors to $\mathcal{L}(\theta|H)$ and $\mathbb{E}[X(T^*)|H, \theta]$ are easily understood. How closely we'd like to approximate $\tilde{\phi}$, depends on how large $M > 0$ we can afford, given Δt . In a practical implementation, we replace $[\underline{v}, +\infty)$ with a compact set $[\underline{v}, V]$, for some large $V > 0$, which we further discretize into a finite grid for numerical integration. The discretization effectively acts as an implicit regularization on the magnitude of M . Moreover, on a compact set, the L^2 -bound from Lemma 3 immediately yields the L^1 -bound needed to bound the errors in $\mathcal{L}(\theta|H)$ and $\mathbb{E}[X(T^*)|H, \theta]$. Since we have divided $[t_0, T]$ into J time steps, and the repeated application of Lemma 3 helps bound the errors in each step, and then the errors in $\mathcal{L}(\theta|H)$ and $\mathbb{E}[X(T^*)|H, \theta]$ are the accumulated errors over the J steps, which is of order $O\left(M\left(\frac{1}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon^{3/2}}\right) \cdot T\Delta t + MT\sqrt{\varepsilon}\right)$.

Finally, we note that there are many other ways to compute $\mathbb{P}[H|\phi_0, c, \lambda]$. A possible way is to simulate a large number of random paths then count the fraction of them that match the observed transaction history. Another possibility is to solve the heat diffusion equation (9) using well-established PDE-solving methods such as Finite Difference Method (FDM), or Finite Element Method (FEM). Although FDM and FEM are powerful for studying diffusion over a complex spatial domain, they are constrained by the stability condition which limits the size of the time step ($\Delta t \sim (\Delta v)^2$, see Asadzadeh 2010). This is problematic when considering a simulation of transaction history where we may have a long time horizon, or where other flexibilities in time unit may be needed to match the record data. On the other hand, the random path simulation is computationally inefficient, and stochastic in nature, adding another layer of difficulty in model evaluation. By leveraging the known analytical diffusion solution over the simple spatial domain of our model, the kernel iteration method we proposed combines the flexibility of time discretization choices with the efficiency of a typical numerical PDE solver. What we present in this work is merely a minimal implementation that captures our model specificity. We leave further optimization and refinement via other advanced numerical techniques to future research.