Appendix for "Capacity Allocation and Scheduling in Two-Stage Service Systems with Multi-Class Customers"

EC.1. Proofs of the Results in Section 3

EC.1.1. Proof of Lemma 1

We first prove the "if" part. Suppose $\sum_{i\in\mathcal{I}}\lambda_{c,i}\left(\frac{1}{r_c\mu_{c,i}}+\sum_{j\in\mathcal{J}}\frac{p_{ij}}{r_{s,j}\mu_{s,j}}\right)\le 1$, then let (z',q',n') be

$$z'_{c,i}=\frac{\lambda_{c,i}}{\mu_{c,i}},\quad z'_{s,j}=\frac{1}{\mu_{s,j}}\sum_{i\in\mathcal{I}}\lambda_{c,i}p_{ij},\quad n'_c=\sum_{i\in\mathcal{I}}z'_{c,i},\quad n'_{s,j}=z'_{s,j},$$
$$q'_{c,i}=\frac{\lambda_{c,i}-\mu_{c,i}z'_{c,i}}{\theta_{c,i}}\quad\text{and}\quad q'_{s,j}=\frac{\sum_{i\in\mathcal{I}}\mu_{c,i}z'_{c,i}p_{ij}-\mu_{s,j}z'_{s,j}}{\theta_{s,j}}.$$

We can easily verify that $q'_{c,i}=0$ for all i and $q'_{s,j}=0$ for all j. In addition, one can verify that $(z',q',n')\in\Xi(\lambda_c)$. The objective function at (z',q',n') is 0. As a result, $\bar{L}^*=0$.

Next, we prove the "only if" part. Suppose that an optimal solution is (z^*,q^*,n^*). Then, we have $q^*_{c,i}=0$ for all $i\in\mathcal{I}$ and $q^*_{s,j}=0$ for all $j\in\mathcal{J}$ as $\bar{L}^*=0$. (Recall that we have $w_{c,i}>0$ for $i\in\mathcal{I}$ and $w_{s,j}>0$ for $j\in\mathcal{J}$.) From (23)–(24), we have

$$z^*_{c,i}=\frac{\lambda_{c,i}}{\mu_{c,i}}\quad\text{and}\quad z^*_{s,j}=\frac{1}{\mu_{s,j}}\sum_{i\in\mathcal{I}}\lambda_{c,i}p_{ij}.$$

By (25), (26) and (28),

$$\frac{1}{r_c}\sum_{i\in\mathcal{I}}z^*_{c,i}+\sum_{j\in\mathcal{J}}\frac{z^*_{s,j}}{r_{s,j}}\le 1.$$

Therefore, $\sum_{i\in\mathcal{I}}\lambda_{c,i}(\frac{1}{r_c\mu_{c,i}}+\sum_{j\in\mathcal{J}}\frac{p_{ij}}{r_{s,j}\mu_{s,j}})\le 1$. The proof is hence complete. □

EC.1.2. Proof of Theorem 1

Proof of Part (i). We follow the idea in the proof of Proposition 2.1 in Atar et al. (2011). Let

$$U^m:=\liminf_{t\to\infty}\frac{1}{t}\int_0^t\left(\sum_{i\in\mathcal{I}}w_{c,i}Q^m_{c,i}(s)+\sum_{j\in\mathcal{J}}w_{s,j}Q^m_{s,j}(s)\right)ds.$$

By Fatou's lemma, it suffices to show that

$$\liminf_{m\to\infty}\frac{1}{m}U^m\ge\bar{L}^*,\quad\text{almost surely.}\tag{EC.1}$$

Fix one sample point, and let $\{t_k;k\in\mathbb{N}\}$ be a sequence of real values increasing to infinity, such that

$$U^m=\lim_{k\to\infty}\frac{1}{t_k}\int_0^{t_k}\left(\sum_{i\in\mathcal{I}}w_{c,i}Q^m_{c,i}(s)+\sum_{j\in\mathcal{J}}w_{s,j}Q^m_{s,j}(s)\right)ds.$$

By (2), (9), and (12), the random vector $\int_0^{t_k}Z^m(s)ds/t_k$ is uniformly bounded. Hence, we can choose a subsequence of $\{t_k;k\in\mathbb{N}\}$, still denoted by $\{t_k;k\in\mathbb{N}\}$ without loss of generality, along which random vectors $\int_0^{t_k}Z^m(s)ds/t_k$ converge. Denote the corresponding limiting random vector by $\widehat{Z}^m:=\lim_{k\to\infty}\int_0^{t_k}Z^m(s)ds/t_k$.

We use the following lemma, whose proof is deferred to the end of this section:

Lemma EC.1. *As $t \to \infty$, $X^m(t)/t \to 0$ almost surely.*

After dividing both sides of (5) by t_k, taking $t_k \to \infty$, using Lemma EC.1, and following the same argument as in Atar et al. (2011), we can claim that $\widehat{Q}_c^m := \lim_{t_k \to \infty} \int_0^{t_k} Q_c^m(s)ds/t_k$ exists and

$$0 = \lambda_{c,i}^m - \mu_{c,i}\widehat{Z}_{c,i}^m - \theta_{c,i}\widehat{Q}_{c,i}^m, \qquad \text{for } i \in \mathcal{I}. \tag{EC.2}$$

Similarly, from (10), we can confirm the existence of $\widehat{Q}_s^m := \lim_{t_k \to \infty} \int_0^{t_k} Q_s^m(s)ds/t_k$ and

$$0 = \sum_{i \in \mathcal{I}} p_{ij}\mu_{c,i}\widehat{Z}_{c,i}^m - \mu_{s,j}\widehat{Z}_{s,j}^m - \theta_{s,j}\widehat{Q}_{s,j}^m, \qquad \text{for } j \in \mathcal{J}. \tag{EC.3}$$

From (20) to (22), we have

$$\widehat{Q}^m \geq 0, \quad \widehat{Z}^m \geq 0, \quad \sum_{i \in \mathcal{I}} \widehat{Z}_{c,i}^m \leq N_c^m, \text{ and } \widehat{Z}_{s,j}^m \leq N_{s,j}^m \text{ for } j \in \mathcal{J}. \tag{EC.4}$$

Recalling the definition of feasible region $\Xi(\lambda_c)$ defined below (28), we have

$$\big(\widehat{Z}^m/m, \widehat{Q}^m/m, N^m/m\big) \in \Xi(\lambda_c^m/m)$$

by (12) and (EC.2)–(EC.4). Therefore, we have

$$\begin{aligned} \frac{U^m}{m} &= \sum_{i \in \mathcal{I}} w_{c,i}\frac{\widehat{Q}_{c,i}^m}{m} + \sum_{j \in \mathcal{J}} w_{s,j}\frac{\widehat{Q}_{s,j}^m}{m} \\ &\geq \min_{(z,q,n) \in \Xi(\lambda_c^m/m)} \sum_{i \in \mathcal{I}} w_{c,i}q_{c,i} + \sum_{j \in \mathcal{J}} w_{s,j}q_{s,j}. \end{aligned}$$

From the continuity of the LP problem (29) in its feasible region and (17), we have $\liminf_{m \to \infty} U^m/m \geq \bar{L}^*$, thus establishing (EC.1). □

Proof of Part (ii). We use a similar idea as in the proof of Theorem 2.2 in Atar et al. (2011). From condition (32), there exists $T_m > 0$, such that $\mathbb{E}[\|m^{-1}Q^{m,*}(t) - q^*\|] \leq 2\delta_m$ for $t \geq T_m$. Recall that $\bar{L}^* = \sum_{i \in \mathcal{I}} w_{c,i}q_{c,i}^* + \sum_{j \in \mathcal{J}} w_{s,j}q_{s,j}^*$. Then

$$\begin{aligned} &\mathbb{E}\int_0^T \Big(\sum_{i \in \mathcal{I}} w_{c,i}Q_{c,i}^{m,*}(t)/m + \sum_{j \in \mathcal{J}} w_{s,j}Q_{s,j}^{m,*}(t)/m\Big)dt - T \cdot \bar{L}^* \\ &\leq \big(\max_{i \in \mathcal{I}} w_{c,i} \vee \max_{j \in \mathcal{J}} w_{s,j}\big)\mathbb{E}\Big[\int_0^T \|Q^{m,*}(t)/m - q^*\|dt\Big] \\ &\leq \big(\max_{i \in \mathcal{I}} w_{c,i} \vee \max_{j \in \mathcal{J}} w_{s,j}\big)\Big[\int_0^{T_m} \mathbb{E}[\|m^{-1}Q^{m,*}(t) - q^*\|]dt + 2\delta_m(T - T_m)\Big]. \end{aligned}$$

Dividing both sides by T and letting $T \to \infty$, we obtain

$$\frac{L^m(\psi^{m,*})}{m} \leq \bar{L}^* + 2\big(\max_{i \in \mathcal{I}} w_{c,i} \vee \max_{j \in \mathcal{J}} w_{s,j}\big)\delta_m.$$

As a result, $\limsup_{m \to \infty} L^m(\psi^{m,*})/m \leq \bar{L}^*$. Together with the inequality established in part (i), this result concludes the desired equality. □

Proof of Lemma EC.1. The proof adopts the idea presented in Appendix A.2 in Atar et al. (2011). We will show that for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$, $X^m_{c,i}(t)/t \to 0$ and $X^m_{s,j}(t)/t \to 0$ as $t \to \infty$ almost surely.

By (5), we have

$$X^m_{c,i}(t) = X^m_{c,i}(0) + A_{c,i}(\lambda^m_{c,i} t) - S_{c,i}\left(\mu_{c,i}\int_0^t Z^m_{c,i}(s)ds\right) - G_{c,i}\left(\theta_{c,i}\int_0^t Q^m_{c,i}(s)ds\right).$$

Note that $X^m_{c,i}(t) = Z^m_{c,i}(t) + Q^m_{c,i}(t)$. Hence, by the standard coupling argument (see e.g., the proof of Lemma 3 in Dong et al. (2015)), we can show that the process $X^m_{c,i}$ is stochastically dominated (upper bounded) by the number-in-system (i.e., the total number of customers in the system) process of an $M/M/\infty$ system, in which the arrival rate is $\lambda^m_{c,i}$, the service rate is $\mu_{c,i} \wedge \theta_{c,i}$, and the initial state is $X^m_{c,i}(0)$.

Similarly, by (10), we have

$$X^m_{s,j}(t) = X^m_{s,j}(0) + \sum_{i\in\mathcal{I}} E^m_{ij}(t) - S_{s,j}\left(\mu_{s,j}\int_0^t Z^m_{s,j}(s)ds\right) - G_{s,j}\left(\theta_{s,j}\int_0^t Q^m_{s,j}(s)ds\right).$$

Note that $\sum_{i\in\mathcal{I}} E^m_{ij}(t) \le \sum_{i\in\mathcal{I}}\left(X^m_{c,i}(0) + A_{c,i}(\lambda^m_{c,i} t)\right)$. By using the coupling method, we can show that the process $X^m_{s,j}$ is stochastically dominated by the number-in-system process of an $M/M/\infty$ system, in which the arrival rate is $\sum_{i\in\mathcal{I}} \lambda^m_{c,i}$, the service rate of each server is $\mu_{s,j} \wedge \theta_{s,j}$, and the initial state is $X^m_{s,j}(0) + \sum_{i\in\mathcal{I}} X^m_{c,i}(0)$.

According to the above two upper bound results, the desired result follows if we can show that for any $M/M/\infty$ system, the corresponding number-in-system process, denoted by Y, satisfies $Y(t)/t \to 0$ almost surely as $t \to \infty$. This claim has been proven in Appendix A.2 in Atar et al. (2011) for a more general $G/M/\infty$ system. Their argument is based on considering the system's embedded Markov chain and using a second moment estimate of Y.

For the sake of completeness, we provide an alternative proof below. Suppose the $M/M/\infty$ system has an arrival rate λ, service rate μ, and an initial state $Y(0)$. Given that we deal with almost sure convergence, we can assume that the value of $Y(0)$ is fixed. Note that $Y(t)$ has two independent sources, of which one is from the newly arriving customers and the other is from those initially present in the system. Then, for all $t \ge 0$, $Y(t) - Y(0)$ is bounded by $\tilde{Y}(t)$, which represents the number of new arrivals who are still in the system and is a Poisson distributed random variable with parameter $\lambda(1 - e^{-\mu t})/\mu$ (Eick et al. 1993). For any given $\Delta > 0$, let $\Gamma_k := \max_{t\in[k\Delta,(k+1)\Delta)} Y(t)/t$, $k \in \mathbb{Z}_+$. To establish that $Y(t)/t \to 0$ almost surely as $t \to \infty$, it suffices to show that $\lim_{k\to\infty} \Gamma_k = 0$ almost surely. Note that for any given $\epsilon > 0$, we have

$$\begin{aligned}\sum_{k=1}^{\ell} \mathbb{P}(\Gamma_k > \epsilon) &\le \sum_{k=1}^{\ell} \mathbb{P}\Big(\max_{t\in[k\Delta,(k+1)\Delta)} Y(t) - Y(0) > \epsilon\cdot k\Delta - Y(0)\Big) \\ &\le \sum_{k=1}^{\ell} \mathbb{P}\big(\tilde{Y}(k\Delta) + Z_k > (\epsilon\cdot k\Delta - Y(0))^+\big),\end{aligned} \tag{EC.5}$$

for any positive ℓ, where Z_k denotes the number of arrivals during time interval $[k\Delta,(k+1)\Delta)$ (alternatively referred to as period k), which is Poisson distributed with parameter $\lambda\Delta$. Moreover, because $\tilde{Y}(k\Delta)$ relies on the system evolution before period k while Z_k encodes information about customer arrivals in period k, they are independent from each other. Hence, $\tilde{Y}(k\Delta)+Z_k$ is also Poisson distributed with parameter $\lambda(1-e^{-\mu k\Delta})/\mu+\lambda\Delta$. By Chebyshev's inequality, for any $\epsilon>0$ and sufficiently small Δ,

$$\begin{aligned}
&\sum_{k=0}^{\infty}\mathbb{P}\left(\tilde{Y}(k\Delta)+Z_k>(\epsilon\cdot k\Delta-Y(0))^+\right)\\
&=\sum_{k=1}^{\lfloor Y(0)/(\epsilon\cdot\Delta)\rfloor}\mathbb{P}\left(\tilde{Y}(k\Delta)+Z_k>0\right)+\sum_{k=\lceil Y(0)/(\epsilon\cdot\Delta)\rceil}^{\infty}\mathbb{P}\left(\tilde{Y}(k\Delta)+Z_k>\epsilon\cdot k\Delta-Y(0)\right)\\
&\le\left\lfloor\frac{Y(0)}{\epsilon\cdot\Delta}\right\rfloor+\sum_{k=\lceil Y(0)/(\epsilon\cdot\Delta)\rceil}^{\infty}\frac{(\lambda(1-e^{-\mu k\Delta})/\mu+\lambda\Delta)(\lambda(1-e^{-\mu k\Delta})/\mu+\lambda\Delta+1)}{(\epsilon\cdot k\Delta-Y(0))^2}\\
&\le\left\lfloor\frac{Y(0)}{\epsilon\cdot\Delta}\right\rfloor+\left(\frac{\lambda}{\mu}+\lambda\Delta\right)\left(\frac{\lambda}{\mu}+\lambda\Delta+1\right)\sum_{k=1}^{\infty}\frac{C}{k^2}<\infty,
\end{aligned}$$

for an appropriate constant $C>0$. Hence, by the Borel–Cantelli lemma, we have $Y(t)/t\to 0$ (a.s.) as $t\to\infty$. This completes the proof. □

EC.2. Proofs of the Results in Section 5

EC.2.1. Proof of Lemma 2

(i) If $\phi<J-1$, from the expressions of $\xi_{i,\phi}$ and $\xi_{i,\phi+1}$, one can verify that

$$\xi_{i,\phi}-\xi_{i,\phi+1}=-\left(\frac{w_{s,\phi+1}r_{s,\phi+1}\mu_{s,\phi+1}}{\theta_{s,\phi+1}}-\frac{w_{s,\phi+2}r_{s,\phi+2}\mu_{s,\phi+2}}{\theta_{s,\phi+2}}\right)\chi_{i,\phi+1}\le 0, \tag{EC.6}$$

where the equality uses $\chi_{i,\phi+1}-\chi_{i,\phi}=\frac{p_{i,\phi+1}}{r_{s,\phi+1}\mu_{s,\phi+1}}$, and the inequality is due to $\frac{w_{s,\phi+1}r_{s,\phi+1}\mu_{s,\phi+1}}{\theta_{s,\phi+1}}\ge\frac{w_{s,\phi+2}r_{s,\phi+2}\mu_{s,\phi+2}}{\theta_{s,\phi+2}}$ according to the ordering in (40). If $\phi=J-1$, from (41), one has $\xi_{i,J-1}\le\xi_{i,J}$.

(ii) If $i\in\mathcal{I}_+(\phi)$, then from part (i), $\xi_{i,\phi+1}\ge\xi_{i,\phi}\ge 0$, hence $i\in\mathcal{I}_+(\phi+1)$. □

EC.2.2. Proof of Theorem 2

We solve (38) rigorously. For $\varphi_c\in\Pi_c$, denote by $\varphi_s^*(\varphi_c)$ the optimal solution to the inner optimization of (39), and define

$$\underline{j}(\varphi_c)=\max\left\{j'\in\mathcal{J}:\sum_{i\in\mathcal{I}}\chi_{i,j'}\varphi_{c,i}\le 1\right\}. \tag{EC.7}$$

If the above set is empty, then let $\underline{j}(\varphi_c)=0$. The following lemma, whose proof is omitted, characterizes $\varphi_s^*(\varphi_c)$ using the standard solution to the bin packing problem.

Lemma EC.2. *For any $\varphi_c\in\Pi_c$, we have*

(i) *if $\underline{j}(\varphi_c)=J$, then $\varphi^*_{s,j}(\varphi_c)=\sum_{i\in\mathcal{I}}\varphi_{c,i}p_{ij}$ for all $j\in\mathcal{J}$;*

(ii) *otherwise,*

$$\varphi^*_{s,j}(\varphi_c)=\begin{cases}\sum_{i\in\mathcal{I}}\varphi_{c,i}p_{ij}, & j<\underline{j}(\varphi_c)+1,\\ \left(1-\sum_{i\in\mathcal{I}}\chi_{i,\underline{j}(\varphi_c)}\varphi_{c,i}\right)r_{s,j}\mu_{s,j}, & j=\underline{j}(\varphi_c)+1,\\ 0, & j>\underline{j}(\varphi_c)+1.\end{cases}$$

From Lemma EC.2, it is optimal to fully accommodate the service requirements of the first $\underline{j}(\varphi_c)$ stations in stage 2 given the stage-1 capacity assignment φ_c. If $\underline{j}(\varphi_c)=J$, then for all $j\le J$, $\varphi_{s,j}=\sum_{i\in\mathcal{I}}\varphi_{c,i}p_{ij}$ at optimum. If $\underline{j}(\varphi_c)<J$, then the first $\underline{j}(\varphi_c)$ stations will receive capacity $\sum_{i\in\mathcal{I}}\varphi_{c,i}p_{ij}$, while station $\underline{j}(\varphi_c)+1$ utilizes the remaining resources.

In the second step, we solve the following optimization problem:

$$\max_{\varphi_c\in\Pi_c} \quad \sum_{i\in\mathcal{I}}\left(\frac{w_{c,i}}{\theta_{c,i}}-\sum_{j\in\mathcal{J}}\frac{w_{s,j}}{\theta_{s,j}}p_{ij}\right)\varphi_{c,i}+\sum_{j\in\mathcal{J}}\frac{w_{s,j}}{\theta_{s,j}}\varphi^*_{s,j}(\varphi_c). \tag{EC.8}$$

Denote by φ_c^* the optimal solution to (EC.8). By comparing the objective functions of (39) and (EC.8) with that of (38), one can verify that $(\varphi_c^*,\varphi_s^*(\varphi_c^*))$ is an optimal solution to problem (38).

In the following, we solve problem (EC.8). Noting that the expression of $\varphi_s^*(\varphi_c)$ in problem (EC.8) depends on $\underline{j}(\varphi_c)\in[\mathcal{J}]$, we separate the feasible region Π_c into $J+1$ disjoint sub-regions: $\Pi_c=\cup_{\phi=0}^{J}\Pi_\phi$ with $\Pi_\phi:=\{\varphi_c\in\Pi_c:\underline{j}(\varphi_c)=\phi\}$. Then, we get $J+1$ sub-problems

$$\max_{\varphi_c\in\Pi_\phi} \quad \sum_{i\in\mathcal{I}}\left(\frac{w_{c,i}}{\theta_{c,i}}-\sum_{j\in\mathcal{J}}\frac{w_{s,j}}{\theta_{s,j}}p_{ij}\right)\varphi_{c,i}+\sum_{j\in\mathcal{J}}\frac{w_{s,j}}{\theta_{s,j}}\varphi^*_{s,j}(\varphi_c), \tag{EC.9}$$

for $\phi\in[\mathcal{J}]$. The optimal solution will be the one that attains the maximum of these $J+1$ sub-problems. By substituting the expression of $\varphi^*_{s,j}(\varphi_c)$ in Lemma EC.2 and noticing that $\underline{j}(\varphi_c)=\phi$ for $\varphi_c\in\Pi_\phi$, problem (EC.9) becomes

$$\max_{\varphi_c\in\Pi_\phi}\sum_{i\in\mathcal{I}}\xi_{i,\phi}\varphi_{c,i}+\mathbb{1}\{\phi<J\}\cdot\left(\frac{w_{s,\phi+1}r_{s,\phi+1}\mu_{s,\phi+1}}{\theta_{s,\phi+1}}\right). \tag{EC.10}$$

Note that for $\phi<J$, $\underline{j}(\varphi_c)=\phi$ is equivalent to

$$\sum_{i\in\mathcal{I}}\chi_{i,\phi}\varphi_{c,i}\le 1,\quad\text{and}\quad\sum_{i\in\mathcal{I}}\chi_{i,\phi+1}\varphi_{c,i}>1.$$

In addition, Π_c is a closed set. Hence the closure of Π_ϕ, denoted by $\bar{\Pi}_\phi$, is still a subset of Π_c and can be represented as

$$\bar{\Pi}_\phi=\left\{\varphi_c\in\mathbb{R}^I:\sum_{i\in\mathcal{I}}\chi_{i,\phi}\varphi_{c,i}\le1,\ \sum_{i\in\mathcal{I}}\chi_{i,\phi+1}\varphi_{c,i}\ge1,\ 0\le\varphi_{c,i}\le\lambda_{c,i},\ i\in\mathcal{I}\right\},$$

where the second constraint in the definition of Π_c is suppressed because it can be derived from the constraint $\sum_{i\in\mathcal{I}}\chi_{i,\phi}\varphi_{c,i}\le1$ and the relation $\chi_{i,\phi}\ge1/(r_c\mu_{c,i})$. For $\phi=J$, by using $\chi_{i,J}\ge1/(r_c\mu_{c,i})$ for each i, we have

$$\bar{\Pi}_J=\Pi_J=\left\{\varphi_c\in\mathbb{R}^I:\sum_{i\in\mathcal{I}}\chi_{i,J}\varphi_{c,i}\le1,\ 0\le\varphi_{c,i}\le\lambda_{c,i},\ i\in\mathcal{I}\right\}.$$

We relax the constraint in (EC.10) from Π_ϕ to $\bar{\Pi}_\phi$, that is, for $\phi\in[\mathcal{J}]$, we consider

$$\max_{\varphi_c\in\bar{\Pi}_\phi}\sum_{i\in\mathcal{I}}\xi_{i,\phi}\varphi_{c,i}+\mathbb{1}\{\phi<J\}\cdot\left(\frac{w_{s,\phi+1}r_{s,\phi+1}\mu_{s,\phi+1}}{\theta_{s,\phi+1}}\right). \tag{EC.11}$$

We call (EC.11) the sub-problem ϕ, and denote by $\varphi_c^*(\phi)$ and $\pi^*(\phi)$ the optimal solution and optimal value to sub-problem ϕ, respectively. Proposition EC.2 in Appendix EC.2.2.1 summarizes the solutions to (EC.11), which depend on whether ϕ is in one of the following sets:

$$\begin{gathered}\mathcal{S}_0=\left\{\phi\in[\mathcal{J}]:\sum_{i\in\mathcal{I}}\lambda_{c,i}\chi_{i,\phi+1}<1\right\},\quad\mathcal{S}_1=\mathcal{S}_0^c\cap\left\{\phi\in[\mathcal{J}]:\sum_{i\in\mathcal{I}_+(\phi)}\lambda_{c,i}\chi_{i,\phi+1}<1\right\},\\ \mathcal{S}_2=\mathcal{S}_0^c\cap\left\{\phi\in[\mathcal{J}]:\sum_{i\in\mathcal{I}_+(\phi)}\lambda_{c,i}\chi_{i,\phi}\le1\le\sum_{i\in\mathcal{I}_+(\phi)}\lambda_{c,i}\chi_{i,\phi+1}\right\},\quad\mathcal{S}_3=\mathcal{S}_0^c\cap\left\{\phi\in[\mathcal{J}]:\sum_{i\in\mathcal{I}_+(\phi)}\lambda_{c,i}\chi_{i,\phi}>1\right\},\end{gathered}\tag{EC.12}$$

where $\mathcal{S}_0^c=[\mathcal{J}]\setminus\mathcal{S}_0$ is the complement of $\mathcal{S}_0$. Specifically, if $\phi\in\mathcal{S}_0$, then sub-problem ϕ contains no feasible solutions; if $\phi\in\mathcal{S}_1$ (resp., $\phi\in\mathcal{S}_3$), then the second (resp., first) constraint of sub-problem ϕ is binding at optimum; if $\phi\in\mathcal{S}_2$, the first and second constraints of sub-problem ϕ can be non-binding at optimum. Moreover, $J\in\mathcal{S}_3$ because $\mathcal{I}_+(J)=\mathcal{I}$ and condition (30) is imposed.

We have the following observation, whose proof is omitted:

Lemma EC.3. *Denote by V^* the optimal value of problem* (EC.8). *Then, there exists $\underline{\phi}^*$ such that $\varphi_c^* \in \bar{\Pi}_{\underline{\phi}^*}$, and $V^* = \pi^*(\underline{\phi}^*)$; that is, V^* is the optimal value of the sub-problem $\underline{\phi}^*$.*

We show that ϕ^* defined in (44) can be $\underline{\phi}^*$. Note that ϕ^* can also be represented as

$$\phi^* = \min\{\phi : \phi \in \mathcal{S}_2 \cup \mathcal{S}_3\}.$$

From the intuitive argument, the optimal utilization of resources depends on whether $\phi^* \in \mathcal{S}_2$ or $\mathcal{S}_3$: (i) if $\phi^* \in \mathcal{S}_2$, then station $\phi^* + 1$ in stage 2 exhausts the resources; or (ii) if $\phi^* \in \mathcal{S}_3$ (when $\mathcal{S}_2 = \emptyset$), then one of the classes in $\mathcal{I}_+(\phi^*) \backslash \mathcal{I}_+(\phi^* - 1)$ in stage 1 exhausts the resources.

Using Lemma 2, we can prove the following monotonicity properties of $\pi^*(\phi)$.

Proposition EC.1. (i) *We have $\phi_0 < \phi_1 < \phi_2 < \phi_3$ for $\phi_\ell \in \mathcal{S}_\ell$, with $\ell = 0, 1, 2, 3$.*
(ii) *If $\phi_1 \in \mathcal{S}_1$, then $\pi^*(\phi_1) \le \pi^*(\phi_1 + 1)$.*
(iii) *For $\phi_2, \phi_2' \in \mathcal{S}_2$, $\pi^*(\phi_2) = \pi^*(\phi_2')$.*
(iv) *If $\phi_3 \in \mathcal{S}_3$, then $\pi^*(\phi_3) \le \pi^*(\phi_3 - 1)$.*

The proof of Proposition EC.1 can be found in Appendix EC.2.2.2. Note that Proposition EC.1(i) provides an ordering for the sets in (EC.12). By using Proposition EC.1(ii)–(iv), we show that $\pi(\phi^*) \ge \pi(\phi)$ for any $\phi \in [\mathcal{J}]$, that is, ϕ^* can be $\underline{\phi}^*$:

1. If $\mathcal{S}_2 \neq \emptyset$, then $\phi^* = \phi_2^*$, which is the smallest index in $\mathcal{S}_2$. From Proposition EC.1(ii)–(iii), $\pi^*(\phi^*) = \pi^*(\phi_2) \ge \pi^*(\phi_1)$ for any $\phi_2 \in \mathcal{S}_2$ and $\phi_1 \in \mathcal{S}_1$, and from Proposition EC.1(iii)–(iv), $\pi^*(\phi^*) = \pi^*(\phi_2) \ge \pi^*(\phi_3)$ for any $\phi_2 \in \mathcal{S}_2$ and $\phi_3 \in \mathcal{S}_3$. This is demonstrated in Figure EC.1(a).
2. If $\mathcal{S}_2 = \emptyset$, then $\phi^* = \phi_3^*$, which is the smallest index in $\mathcal{S}_3$ (recall that $J \in \mathcal{S}_3$ so $\mathcal{S}_3 \neq \emptyset$). Then, from Proposition EC.1(ii), $\pi^*(\phi_3^*) \ge \pi^*(\phi_1)$ for any $\phi_1 \in \mathcal{S}_1$, and from Proposition EC.1(iv), $\pi^*(\phi_3^*) \ge \pi^*(\phi_3)$ for any $\phi_3 \in \mathcal{S}_3$. This is demonstrated in Figure EC.1(b).

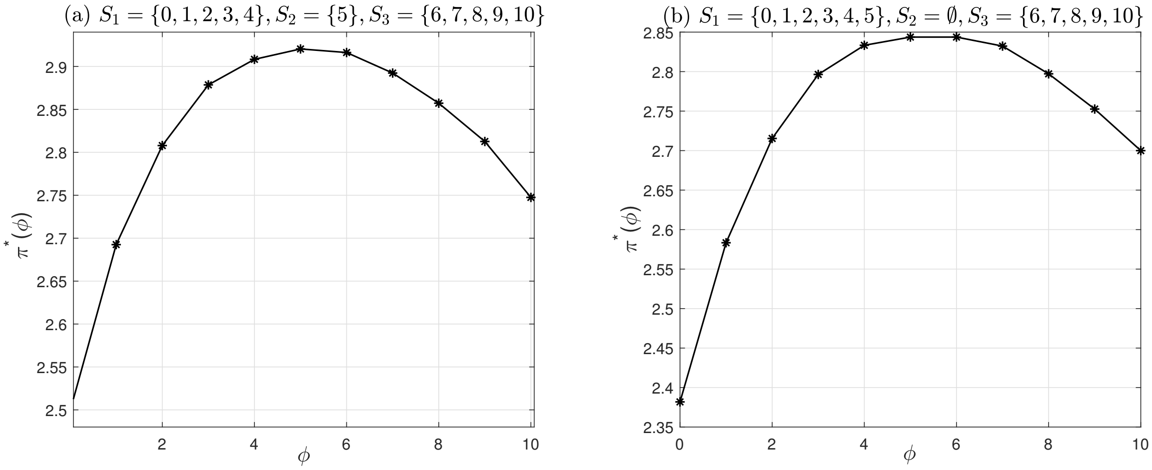
As a result, we use the solution to the sub-problem ϕ^* to finish the proof of Theorem 2. Recall that ϕ^* is the smallest index in $\mathcal{S}_2 \cup \mathcal{S}_3$. We rearrange the indices in $\mathcal{I}$ such that (refer to (EC.14) and Proposition EC.2)

$$\frac{\xi_{1,\phi^*}}{\chi_{1,\phi^*}} \ge \cdots \ge \frac{\xi_{i_+(\phi^*),\phi^*}}{\chi_{i_+(\phi^*),\phi^*}} \ge 0 > \frac{\xi_{i_+(\phi^*)+1,\phi^*}}{\chi_{i_+(\phi^*)+1,\phi^*+1}} \ge \cdots \ge \frac{\xi_{I,\phi^*}}{\chi_{I,\phi^*+1}}. \tag{EC.13}$$

We have the following two cases:

(i) If $\mathcal{S}_2 \neq \emptyset$, then $\phi^* \in \mathcal{S}_2$, φ_c^* is derived by Proposition EC.2(ii), and $\varphi_s^* = \varphi_s^*(\varphi_c^*)$ by Lemma EC.2.
(ii) If $\mathcal{S}_2 = \emptyset$, then $\phi^* \in \mathcal{S}_3$, φ_c^* is derived by Proposition EC.2(iii), and $\varphi_s^* = \varphi_s^*(\varphi_c^*)$ by Lemma EC.2. Note that $i > i_+(\phi^*)$ in (EC.13) is equivalent to $i \in \mathcal{I}_-(\phi^*)$. Because $\varphi_{c,i}^*(\phi^*) = 0$ for $i > i_+(\phi^*)$, the orders in $\mathcal{I}_-(\phi^*)$ do not matter in this case. We can rearrange the indices in $\mathcal{I}_-(\phi^*)$ as in Assumption 3. We now consider the orders in $\mathcal{I}_+(\phi^*)$, that is, $i \le i_+(\phi^*)$ in (EC.13). From the expressions of $\xi_{i,\phi}$ and $\xi_{i,\phi+1}$, we have
$$\frac{\xi_{i,\phi^*}}{\chi_{i,\phi^*}} = \frac{\xi_{i,\phi^*-1}}{\chi_{i,\phi^*}} + \left(\frac{w_{s,\phi^*} r_{s,\phi^*} \mu_{s,\phi^*}}{\theta_{s,\phi^*}} - \frac{w_{s,\phi^*+1} r_{s,\phi^*+1} \mu_{s,\phi^*+1}}{\theta_{s,\phi^*+1}}\right).$$
For $i_1 \in \mathcal{I}_+(\phi^* - 1)$ and $i_2 \in \mathcal{I}_+(\phi^*) \backslash \mathcal{I}_+(\phi^* - 1)$, note that $\xi_{i_1,\phi^*-1} \ge 0 > \xi_{i_2,\phi^*-1}$, hence $\frac{\xi_{i_1,\phi^*}}{\chi_{i_1,\phi^*}} \ge \frac{\xi_{i_2,\phi^*}}{\chi_{i_2,\phi^*}}$. Recall that $i_+(\phi^* - 1) = |\mathcal{I}_+(\phi^* - 1)|$. Therefore, the first $i_+(\phi^* - 1)$ indices in (EC.13) are those in $\mathcal{I}_+(\phi^* - 1)$. Note that because $\mathcal{I}_+(\phi^* - 1) \subseteq \mathcal{I}_+(\phi^*)$ by Lemma 2(i), $i_+(\phi^*) > i_+(\phi^* - 1)$. Because $\mathcal{S}_2 = \emptyset$, one has $i_+(\phi^* - 1) < i_2(\phi^*)$. Then, $\varphi_{c,i}^*(\phi) = \lambda_{c,i}$ for $i \le i_+(\phi^* - 1)$, and the orders in $\mathcal{I}_+(\phi^* - 1)$ do not matter. We rearrange the indices in $\mathcal{I}_+(\phi^* - 1)$ as in Assumption 3. Finally, note that the indices from $i_+(\phi^* - 1) + 1$ to $i_+(\phi^*)$ in (EC.13) are those in $\mathcal{I}_+(\phi^*) \backslash \mathcal{I}_+(\phi^* - 1)$, and the orders in (EC.13) are the same as those in Assumption 3. Hence, we have the solutions, with $i_{\phi^*} = i_2(\phi^*)$, in Theorem 2 (ii).

Figure EC.1 Properties of $\pi^*(\phi)$


(a) $S_1 = \{0,1,2,3,4\}, S_2 = \{5\}, S_3 = \{6,7,8,9,10\}$

(b) $S_1 = \{0,1,2,3,4,5\}, S_2 = \emptyset, S_3 = \{6,7,8,9,10\}$

Note. The parameters are as follows: $I = J = 10$; $\mu_{c,i} = \mu_{s,j} = \theta_{c,i} = \theta_{s,j} = r_c = r_{s,j} = 1$, and $p_{ij} = 1/10$ for all i and j; $w_c^H = w_c^A = (0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 5)$, $w_s^H = w_s^A = 0.5 \times w_c^H$; the total arrival rate is 1.05 for panel (a) and 1.00 for panel (b), with $\lambda_{c,i}$ being identical across all i.

EC.2.2.1. Solutions to Sub-problems (EC.11) Here we establish Proposition EC.2, which solves the sub-problems (EC.11). Recall that for each $\phi \in [\mathcal{J}]$, we separate $\mathcal{I}$ into two disjoint subsets, namely, $\mathcal{I}_+(\phi) = \{i \in \mathcal{I} : \xi_{i,\phi} \geq 0\}$ and $\mathcal{I}_-(\phi) = \{i \in \mathcal{I} : \xi_{i,\phi} < 0\}$, and denote by $i_+(\phi) = |\mathcal{I}_+(\phi)|$. To facilitate the presentation, we rearrange the indices in $\mathcal{I}$ such that

$$\frac{\xi_{1,\phi}}{\chi_{1,\phi}} \geq \cdots \geq \frac{\xi_{i_+(\phi),\phi}}{\chi_{i_+(\phi),\phi}} \geq 0 > \frac{\xi_{i_+(\phi)+1,\phi}}{\chi_{i_+(\phi)+1,\phi+1}} \geq \cdots \geq \frac{\xi_{I,\phi}}{\chi_{I,\phi+1}}. \tag{EC.14}$$

In particular, for the case when $\phi = J$, because $\xi_{i,J} \geq 0$ for all i, we have $\mathcal{I}_+(J) = \mathcal{I}$ and thus arrange the indices in $\mathcal{I}$ such that $\xi_{1,J}/\chi_{1,J} \geq \cdots \geq \xi_{I,J}/\chi_{I,J}$.

Proposition EC.2. *Recall the sets $\mathcal{S}_i$ with $i = 0,1,2,3$ defined in* (EC.12)*, and fix $\phi \in [\mathcal{J}]$. If $\phi \in \mathcal{S}_0$, then there is no feasible solution to problem* (EC.11)*. Otherwise, the optimal solution to problem* (EC.11) *exists and is characterized as follows:*

(i) *If $\phi \in \mathcal{S}_1$, then there exists $i_1(\phi) := \min\{i' \in \mathcal{I} : \sum_{i \leq i'} \lambda_{c,i}\chi_{i,\phi+1} \geq 1\} > i_+(\phi)$, such that $\varphi^*_{c,i}(\phi) = \lambda_{c,i}$ for $i < i_1(\phi)$ and $\varphi^*_{c,i}(\phi) = 0$ for $i > i_1(\phi)$, with $\varphi^*_{c,i_1(\phi)}(\phi)$ being set so that the second constraint in set $\bar{\Pi}_\phi$ is binding, that is, $\sum_{i \in \mathcal{I}} \chi_{i,\phi+1}\varphi^*_{c,i}(\phi) = 1$, or equivalently, $\varphi^*_{c,i_1(\phi)}(\phi) = (1 - \sum_{i < i_1(\phi)} \lambda_{c,i}\chi_{i,\phi+1})/\chi_{i_1(\phi),\phi+1}$.*

(ii) *If $\phi \in \mathcal{S}_2$, then $\varphi^*_{c,i}(\phi) = \lambda_{c,i}$ for $i \in \mathcal{I}_+(\phi)$ and $\varphi^*_{c,i}(\phi) = 0$ for $i \in \mathcal{I}_-(\phi)$.*

(iii) *If $\phi \in \mathcal{S}_3$, then there exists $i_2(\phi) := \min\{i' \in \mathcal{I}_+(\phi) : \sum_{i \leq i'} \lambda_{c,i}\chi_{i,\phi} \geq 1\} \leq i_+(\phi)$, such that $\varphi^*_{c,i}(\phi) = \lambda_{c,i}$ for $i < i_2(\phi)$, $\varphi^*_{c,i}(\phi) = 0$ for $i > i_2(\phi)$, and with $\varphi^*_{c,i_2(\phi)}(\phi)$ being set so that the first constraint in set $\bar{\Pi}_\phi$ is binding, that is, $\sum_{i \in \mathcal{I}} \chi_{i,\phi}\varphi^*_{c,i}(\phi) = 1$, or equivalently, $\varphi^*_{c,i_2(\phi)}(\phi) = (1 - \sum_{i < i_2(\phi)} \lambda_{c,i}\chi_{i,\phi})/\chi_{i_2(\phi),\phi}$.*

Proof of Proposition EC.2. First, if $\phi \in \mathcal{S}_0$, then $\phi < J$ and the second constraint in $\bar{\Pi}_\phi$ cannot hold, and thus problem (EC.11) has no feasible solution. Hence, in the following, we consider $\phi \in \mathcal{S}_0^c$, that is, $\sum_{i \in \mathcal{I}} \lambda_{c,i}\chi_{i,\phi+1} \geq 1$. We first consider the case of $\phi < J$.

1. When $\phi \in \mathcal{S}_1$, that is, $\sum_{i\in\mathcal{I}_+(\phi)} \lambda_{c,i}\chi_{i,\phi+1} < 1$, one must have $\varphi^*_{c,i}(\phi) > 0$ for some $i \in \mathcal{I}_-(\phi)$ to ensure that the second constraint in $\bar{\Pi}_\phi$ holds. Moreover, the second constraint in $\bar{\Pi}_\phi$ must be binding; otherwise, one can always decrease $\varphi^*_{c,i}(\phi) > 0$ for $i \in \mathcal{I}_-(\phi)$ to obtain a larger objective value. Also note that $\chi_{i,\phi} \le \chi_{i,\phi+1}$ and recall that $J \in \mathcal{S}_3$, which means $\phi < J$. Hence, problem (EC.11) is reduced to
$$\begin{aligned}\max_{\varphi_c} &\sum_{i\in\mathcal{I}} \xi_{i,\phi}\varphi_{c,i} + \frac{w_{s,\phi+1}r_{s,\phi+1}\mu_{s,\phi+1}}{\theta_{s,\phi+1}},\\ \text{s.t.} &\sum_{i\in\mathcal{I}} \chi_{i,\phi+1}\varphi_{c,i} = 1, \text{ and } 0 \le \varphi_{c,i} \le \lambda_{c,i}, \text{ for } i \in \mathcal{I}.\end{aligned}$$
Therefore, we arrange the indices in $\mathcal{I}$ such that $\xi_{1,\phi}/\chi_{1,\phi+1} \ge \cdots \ge \xi_{I,\phi}/\chi_{I,\phi+1}$. Then, with $i_1(\phi) := \min\{i' \in \mathcal{I} : \sum_{i\le i'} \lambda_{c,i}\chi_{i,\phi+1} \ge 1\} > i_+(\phi)$, the optimal solution is given by $\varphi^*_{c,i}(\phi) = \lambda_{c,i}$ for $i \le i_1(\phi) - 1$, $\varphi^*_{c,i}(\phi) = 0$ for $i > i_1(\phi)$, with $\varphi^*_{c,i_1(\phi)}(\phi)$ being set so that the constraint of the above problem is binding. This gives Case (i) of Proposition EC.2.
2. When $\phi \notin \mathcal{S}_1$, that is, $\sum_{i\in\mathcal{I}_+(\phi)} \lambda_{c,i}\chi_{i,\phi+1} \ge 1$, we have the following two cases:
   (i) If $\phi \in \mathcal{S}_2$, that is, $\sum_{i\in\mathcal{I}_+(\phi)} \lambda_{c,i}\chi_{i,\phi} \le 1$, then the proposed solution is a feasible solution because $\sum_{i\in\mathcal{I}} \chi_{i,\phi}\varphi^*_{c,i}(\phi) = \sum_{i\in\mathcal{I}_+(\phi)} \chi_{i,\phi}\lambda_{c,i} \le 1$ and $\sum_{i\in\mathcal{I}} \chi_{i,\phi+1}\varphi^*_{c,i}(\phi) = \sum_{i\in\mathcal{I}_+(\phi)} \chi_{i,\phi+1}\lambda_{c,i} \ge 1$. By the definitions of $\mathcal{I}_+(\phi)$ and $\mathcal{I}_-(\phi)$, one can also verify that the value of any other feasible solution cannot be larger than that of the proposed solution. This gives Case (ii) of Proposition EC.2.

   (ii) If $\phi \in \mathcal{S}_3$, that is, $\sum_{i\in\mathcal{I}_+(\phi)} \lambda_{c,i}\chi_{i,\phi} > 1$, then when $\phi < J$, one must have $\varphi^*_{c,i}(\phi) = 0$ for $i \in \mathcal{I}_-(\phi)$. This is because, if $\varphi^*_{c,i}(\phi) > 0$ for some $i \in \mathcal{I}_-(\phi)$, then one can always decrease such $\varphi^*_{c,i}(\phi)$ to obtain a larger objective value (if the second constraint is violated, then one can increase $\varphi^*_{c,i}(\phi)$ for some $i \in \mathcal{I}_+(\phi)$). Moreover, the first constraint in $\bar{\Pi}_\phi$ must be binding; otherwise, if $\sum_{i\in\mathcal{I}_+(\phi)} \chi_{i,\phi}\varphi^*_{c,i}(\phi) < 1$, then there must be $\varphi^*_{c,i}(\phi) < \lambda_{c,i}$ for some $i \in \mathcal{I}_+(\phi)$. Increasing such $\varphi^*_{c,i}(\phi)$ will lead to a larger objective value. Then, the second constraint in $\bar{\Pi}_\phi$ holds because $\chi_{i,\phi+1} \ge \chi_{i,\phi}$. Therefore, problem (EC.11) is equivalent to
$$\begin{aligned}\max_{\varphi_c} &\sum_{i\in\mathcal{I}_+(\phi)} \xi_{i,\phi}\varphi_{c,i} + \frac{w_{s,\phi+1}r_{s,\phi+1}\mu_{s,\phi+1}}{\theta_{s,\phi+1}},\\ \text{s.t.} &\sum_{i\in\mathcal{I}_+(\phi)} \chi_{i,\phi}\varphi_{c,i} = 1, \text{ and } 0 \le \varphi_{c,i} \le \lambda_{c,i}, \text{ for } i \in \mathcal{I}_+(\phi).\end{aligned}$$
   Therefore, we arrange the indices in $\mathcal{I}_+(\phi)$ such that $\xi_{1,\phi}/\chi_{1,\phi} \ge \cdots \ge \xi_{i_+(\phi),\phi}/\chi_{i_+(\phi),\phi}$. Then, with $i_2(\phi) := \min\{i' \in \mathcal{I}_+(\phi) : \sum_{i\le i'} \lambda_{c,i}\chi_{i,\phi} \ge 1\} \le i_+(\phi)$, the optimal solution is given as $\varphi^*_{c,i}(\phi) = \lambda_{c,i}$ for $i \le i_2(\phi) - 1$, $\varphi^*_{c,i}(\phi) = 0$ for $i > i_2(\phi)$, with $\varphi^*_{c,i_2(\phi)}(\phi)$ being set so that the constraint of the above problem is binding. This gives Case (iii) of Proposition EC.2.

Finally, when $\phi = J \in \mathcal{S}_3$, $\mathcal{I}_+(J) = \mathcal{I}$ and $\mathcal{I}_-(J) = \emptyset$. Then, problem (EC.11) becomes
$$\max_{\varphi_c \in \bar{\Pi}_J} \sum_{i\in\mathcal{I}} \varphi_{c,i}\xi_{i,J}. \tag{EC.15}$$
This is a bin packing problem. Therefore, with $i_2(\phi) := \min\{i' \in \mathcal{I} : \sum_{i\le i'} \lambda_{c,i}\chi_{i,J} \ge 1\}$, the optimal solution is given by
$$\varphi^*_c(\phi) = \left(\lambda_{c,1}, \dots, \lambda_{c,i_2(\phi)-1}, \frac{1}{\chi_{i_2(\phi),J}}\left(1 - \sum_{i<i_2(\phi)} \lambda_{c,i}\chi_{i,J}\right), 0, \dots, 0\right).$$
This solution is consistent with Case (iii) of Proposition EC.2. This completes the proof. □

EC.2.2.2. Proof of Proposition EC.1

(i) This can be proved by noting that $\chi_{i,\phi}$ is nondecreasing in ϕ. We prove $\phi_0 < \phi_1$ for illustration, and the other inequalities can be proved similarly. Suppose to the contrary that $\phi_0 \geq \phi_1$, we have $\chi_{i,\phi_1+1} \leq \chi_{i,\phi_0+1}$, hence $\sum_{i\in\mathcal{I}} \lambda_{c,i}\chi_{i,\phi_1+1} \leq \sum_{i\in\mathcal{I}} \lambda_{c,i}\chi_{i,\phi_0+1} < 1$. This contradicts $\sum_{i\in\mathcal{I}} \lambda_{c,i}\chi_{i,\phi_1+1} \geq 1$ as $\phi_1 \in \mathcal{S}_0^c$.

(ii) If $\phi_1 \in \mathcal{S}_1$, then from Proposition EC.2(i), the second constraint in $\bar{\Pi}_{\phi_1}$ must be binding, that is, $\sum_{i\in\mathcal{I}} \chi_{i,\phi_1+1}\varphi^*_{c,i}(\phi_1) = 1$. Then, because $\chi_{i,\phi}$ is nondecreasing in ϕ, we have $\chi_{i,\phi_1+1} \leq \chi_{i,\phi_1+2}$ and $\sum_{i\in\mathcal{I}} \chi_{i,\phi_1+2}\varphi^*_{c,i}(\phi_1) \geq 1$. Hence, $\varphi^*_c(\phi_1) \in \bar{\Pi}_{\phi_1+1}$, that is, $\varphi^*_c(\phi_1)$ is a feasible solution to sub-problem ϕ_1+1. Moreover, we have

$$\begin{aligned}
\pi^*(\phi_1) &= \sum_{i\in\mathcal{I}} \xi_{i,\phi_1}\varphi^*_{c,i}(\phi_1) + \frac{w_{s,\phi_1+1}r_{s,\phi_1+1}\mu_{s,\phi_1+1}}{\theta_{s,\phi_1+1}} \\
&= \sum_{i\in\mathcal{I}} \xi_{i,\phi_1+1}\varphi^*_{c,i}(\phi_1) + \frac{w_{s,\phi_1+1}r_{s,\phi_1+1}\mu_{s,\phi_1+1}}{\theta_{s,\phi_1+1}} \\
&\quad - \left(\frac{w_{s,\phi_1+1}r_{s,\phi_1+1}\mu_{s,\phi_1+1}}{\theta_{s,\phi_1+1}} - \frac{w_{s,\phi_1+2}r_{s,\phi_1+2}\mu_{s,\phi_1+2}}{\theta_{s,\phi_1+2}}\right)\sum_{i\in\mathcal{I}} \chi_{i,\phi_1+1}\varphi^*_{c,i}(\phi_1) \\
&= \sum_{i\in\mathcal{I}} \xi_{i,\phi_1+1}\varphi^*_{c,i}(\phi_1) + \frac{w_{s,\phi_1+1}r_{s,\phi_1+1}\mu_{s,\phi_1+1}}{\theta_{s,\phi_1+1}} \\
&\quad - \left(\frac{w_{s,\phi_1+1}r_{s,\phi_1+1}\mu_{s,\phi_1+1}}{\theta_{s,\phi_1+1}} - \frac{w_{s,\phi_1+2}r_{s,\phi_1+2}\mu_{s,\phi_1+2}}{\theta_{s,\phi_1+2}}\right) \\
&= \sum_{i\in\mathcal{I}} \xi_{i,\phi_1+1}\varphi^*_{c,i}(\phi_1) + \frac{w_{s,\phi_1+2}r_{s,\phi_1+2}\mu_{s,\phi_1+2}}{\theta_{s,\phi_1+2}} \leq \pi^*(\phi_1+1),
\end{aligned}$$

where the second equality follows from (EC.6), and the third equality follows from $\sum_{i\in\mathcal{I}} \chi_{i,\phi_1+1}\varphi^*_{c,i}(\phi_1) = 1$.

(iii) First, we show that if $\mathcal{S}_2 \neq \emptyset$, then all elements in $\mathcal{S}_2$ have the same $\mathcal{I}_+(\phi)$. Without loss of generality, assume that $\phi_2 < \phi_2'$ are in $\mathcal{S}_2$ and $\mathcal{I}_+(\phi_2) \neq \mathcal{I}_+(\phi_2')$. Then,

$$1 \leq \sum_{i\in\mathcal{I}_+(\phi_2)} \lambda_{c,i}\chi_{i,\phi_2+1} \leq \sum_{i\in\mathcal{I}_+(\phi_2)} \lambda_{c,i}\chi_{i,\phi_2'} < \sum_{i\in\mathcal{I}_+(\phi_2')} \lambda_{c,i}\chi_{i,\phi_2'}.$$

The last strict inequality follows from Part (ii) of Lemma 2. Hence, $\phi_2' \notin \mathcal{S}_2$, which is a contradiction. By Proposition EC.2(ii), the solutions to sub-problems ϕ_2 and ϕ_2' are the same. As a result, $\pi^*(\phi_2) = \pi^*(\phi_2')$, for $\phi_2, \phi_2' \in \mathcal{S}_2$.

(iv) If $\phi_3 \in \mathcal{S}_3$, then from Proposition EC.2(iii), the first constraint in $\bar{\Pi}_{\phi_3}$ must be binding, that is, $\sum_{i\in\mathcal{I}} \chi_{i,\phi_3}\varphi^*_{c,i}(\phi_3) = 1$. Then, because $\chi_{i,\phi}$ is nondecreasing in ϕ, we have $\chi_{i,\phi_3-1} \leq \chi_{i,\phi_3}$, and $\sum_{i\in\mathcal{I}} \chi_{i,\phi_3-1}\varphi^*_{c,i}(\phi_3) \leq 1$. Hence, $\varphi^*_c(\phi_3) \in \bar{\Pi}_{\phi_3-1}$, that is, $\varphi^*_c(\phi_3)$ is a feasible solution to sub-problem ϕ_3-1. Moreover, by (EC.6), it holds that $\xi_{i,\phi_3-1} = \xi_{i,\phi_3} - \left(\frac{w_{s,\phi_3}r_{s,\phi_3}\mu_{s,\phi_3}}{\theta_{s,\phi_3}} - \frac{w_{s,\phi_3+1}r_{s,\phi_3+1}\mu_{s,\phi_3+1}}{\theta_{s,\phi_3+1}}\right)\chi_{i,\phi_3}$. Hence, we have

$$\xi_{i,\phi_3} < \left(\frac{w_{s,\phi_3}r_{s,\phi_3}\mu_{s,\phi_3}}{\theta_{s,\phi_3}} - \frac{w_{s,\phi_3+1}r_{s,\phi_3+1}\mu_{s,\phi_3+1}}{\theta_{s,\phi_3+1}}\right)\chi_{i,\phi_3}, \tag{EC.16}$$

for $i \notin \mathcal{I}_+(\phi_3-1)$. Then, we have

$$\begin{aligned}
\pi^*(\phi_3) &= \sum_{i\in\mathcal{I}} \xi_{i,\phi_3}\varphi^*_{c,i}(\phi_3) + \frac{w_{s,\phi_3+1}r_{s,\phi_3+1}\mu_{s,\phi_3+1}}{\theta_{s,\phi_3+1}} \\
&\leq \sum_{i\in\mathcal{I}_+(\phi_3-1)} \xi_{i,\phi_3}\varphi^*_{c,i}(\phi_3) + \frac{w_{s,\phi_3+1}r_{s,\phi_3+1}\mu_{s,\phi_3+1}}{\theta_{s,\phi_3+1}} \\
&\quad + \left(\frac{w_{s,\phi_3}r_{s,\phi_3}\mu_{s,\phi_3}}{\theta_{s,\phi_3}} - \frac{w_{s,\phi_3+1}r_{s,\phi_3+1}\mu_{s,\phi_3+1}}{\theta_{s,\phi_3+1}}\right)\left(1 - \sum_{i\in\mathcal{I}_+(\phi_3-1)} \chi_{i,\phi_3}\varphi^*_{c,i}(\phi_3)\right) \\
&= \sum_{i\in\mathcal{I}_+(\phi_3-1)} \xi_{i,\phi_3-1}\varphi^*_{c,i}(\phi_3) + \frac{w_{s,\phi_3}r_{s,\phi_3}\mu_{s,\phi_3}}{\theta_{s,\phi_3}} \leq \pi^*(\phi_3-1),
\end{aligned}$$

where the inequality follows from $\sum_{i\in\mathcal{I}} \chi_{i,\phi_3}\varphi^*_{c,i}(\phi_3) = 1$ and (EC.16), and the second equality is due to (EC.6). This completes the proof. □

EC.2.3. Proof of Theorem 3

Recall that the indices in $\mathcal{J}$ and $\mathcal{I}$ are ordered as in Assumptions 2 and 3, respectively. Introduce $\mathcal{Z}_c(\cdot) := (\mathcal{Z}_{c,i}(\cdot); i \in \mathcal{I})$ and $\mathcal{Q}_c(\cdot) := (\mathcal{Q}_{c,i}(\cdot); i \in \mathcal{I})$, which are mappings from $\mathbb{R}_+^I$ to $\mathbb{R}_+^I$ and defined as

$$\mathcal{Z}_{c,i}(x) = x_i \wedge \left(N_c^{m,*} - \sum_{k=1}^{i-1} x_k \right)^+, \quad \mathcal{Q}_{c,i}(x) = x_i - \mathcal{Z}_{c,i}(x), \text{ for } x = (x_i; i \in \mathcal{I}).$$

We also introduce $\mathcal{Z}_s(\cdot) := (\mathcal{Z}_{s,j}(\cdot); j \in \mathcal{J})$ and $\mathcal{Q}_s(\cdot) := (\mathcal{Q}_{s,j}(\cdot); j \in \mathcal{J})$, which are mappings from $\mathbb{R}_+^J$ to $\mathbb{R}_+^J$ and defined as

$$\mathcal{Z}_{s,j}(y) = y_j \wedge N_{s,j}^{m,*}, \quad \mathcal{Q}_{c,i}(y) = y_j - \mathcal{Z}_{s,j}(y), \text{ for } y = (y_j; j \in \mathcal{J}).$$

Let $x_c^* = (x_{c,i}^*; i \in \mathcal{I})$ and $x_s^* = (x_{s,j}^*; j \in \mathcal{J})$, in which $x_{c,i}^* = z_{c,i}^* + q_{c,i}^*$ and $x_{s,j}^* = z_{s,j}^* + q_{s,j}^*$. Here, $(z_{c,i}^*, z_{s,j}^*) = (\frac{\varphi_{c,i}^*}{\mu_{c,i}}, \frac{\varphi_{s,j}^*}{\mu_{s,j}})$ according to the definition of φ, and $(q_{c,i}^*, q_{s,j}^*) = (\frac{\lambda_{c,i} - \varphi_{c,i}^*}{\theta_{c,i}}, \frac{\sum_{i \in \mathcal{I}} \varphi_{c,i}^* p_{ij} - \varphi_{s,j}^*}{\theta_{s,j}})$ by (23) and (24). The following lemma can be verified directly, hence its proof is omitted.

Lemma EC.4. (i) *For any $x \in \mathbb{R}_+^I$, we have $\mathcal{Z}_c(x) + \mathcal{Q}_c(x) = x$. Moreover, $\mathcal{Z}_c(mx_c^*) = mz_c^*$, $\mathcal{Q}_c(mx_c^*) = mq_c^*$.*

(ii) *For any $x, x' \in \mathbb{R}_+^I$ and $i \in \mathcal{I}$, there exists a value $\varrho_i(x, x') \in [0,1]$ such that $\mathcal{Q}_{c,i}(x) - \mathcal{Q}_{c,i}(x') = \varrho_i(x, x') \cdot (x_i - x'_i)$. As a result, $\mathcal{Z}_{c,i}(x) - \mathcal{Z}_{c,i}(x') = (1 - \varrho_i(x, x')) \cdot (x_i - x'_i)$.*

(iii) *For any $y \in \mathbb{R}_+^J$, we have $\mathcal{Z}_s(y) + \mathcal{Q}_s(y) = y$. Moreover, $\mathcal{Z}_s(mx_s^*) = mz_s^*$, $\mathcal{Q}_s(mx_s^*) = mq_s^*$.*

(iv) *For any $y, y' \in \mathbb{R}_+^J$ and $j \in \mathcal{J}$, there exists a value $\varrho_j(y, y') \in [0,1]$ such that $\mathcal{Q}_{s,j}(y) - \mathcal{Q}_{s,j}(y') = \varrho_j(y, y') \cdot (y_i - y'_i)$. As a result, $\mathcal{Z}_{s,j}(y) - \mathcal{Z}_{s,j}(y') = (1 - \varrho_j(y, y')) \cdot (y_i - y'_i)$.*

With the above mappings, for the mth system under policy $\psi^{m,*}$, the vectors $Z^m(t) = (Z_c^m(t), Z_s^m(t))$ and $Q^m(t) = (Q_c^m(t), Q_s^m(t))$ can be represented as functions of $X^m(t) = (X_c^m(t), X_s^m(t))$ as follows:

$$Z_c^m(t) = \mathcal{Z}_c(X_c^m(t)), \quad Q_c^m(t) = \mathcal{Q}_c(X_c^m(t)), \quad Z_s^m(t) = \mathcal{Z}_s(X_s^m(t)), \quad Q_s^m(t) = \mathcal{Q}_s(X_s^m(t)).$$

We first assume that there exists δ_m (satisfying $\lim_{m \to \infty} \delta_m = 0$) such that

$$\limsup_{t \to \infty} \mathbb{E}\left[\| m^{-1} X^m(t) - x^* \| \right] \le \delta_m. \tag{EC.17}$$

Let $\mathcal{Q}(\cdot) := (\mathcal{Q}_c(\cdot), \mathcal{Q}_s(\cdot))$. From Lemma EC.4 (ii) and (iv), we have $\|\mathcal{Q}(x) - \mathcal{Q}(x')\| \le \|x - x'\|$ for $x, x' \in \mathbb{R}_+^{I+J}$. This implies

$$\|m^{-1} Q^m(t) - q^*\| = m^{-1} \|\mathcal{Q}(X^m(t)) - \mathcal{Q}(mx^*)\| \le m^{-1} \|X^m(t) - mx^*\| = \|m^{-1} X^m(t) - x^*\|.$$

From (EC.17), we have (32).

The rest of this section is devoted to proving (EC.17). Note that $X^m = (X_c^m, X_s^m)$ is a continuous-time Markov chain with generator

$$\begin{aligned}(\mathcal{L}f)(x,y) = &\sum_{i \in \mathcal{I}} \lambda_{c,i}^m \big(f(x + e_{i,I}, y) - f(x,y)\big) + \sum_{i \in \mathcal{I}} \mu_{c,i} \mathcal{Z}_{c,i}(x) \Big[\sum_{j \in \mathcal{J}} p_{ij} \big(f(x - e_{i,I}, y + e_{j,J}) - f(x,y)\big) \\ &+ \big(1 - \sum_{j \in \mathcal{J}} p_{ij}\big) \big(f(x - e_{i,I}, y) - f(x,y)\big) \Big] + \sum_{i \in \mathcal{I}} \theta_{c,i} \mathcal{Q}_{c,i}(x) \big(f(x - e_{i,I}, y) - f(x,y)\big) \\ &+ \sum_{j \in \mathcal{J}}^{J} \big(\mu_{s,j} \mathcal{Z}_{s,j}(y) + \theta_{s,j} \mathcal{Q}_{s,j}(y)\big) \cdot \big(f(x, y - e_{j,J}) - f(x,y)\big), \quad (x,y) \in \mathbb{N}^I \times \mathbb{N}^J,\end{aligned} \tag{EC.18}$$

for any function f defined on $\mathbb{N}^I \times \mathbb{N}^J$. Here, we use $e_{a,n}$ to denote an n-dimensional column vector with all entries being 0 but its ath entry being 1. The generator $\mathcal{L}$ is well defined because for some $(x,y) \in \mathbb{N}^I \times \mathbb{N}^J$, if $x - e_{i,I} \notin \mathbb{N}^I$ (resp., $y - e_{j,J} \notin \mathbb{N}^J$), then $\mathcal{Z}_{c,i}(x) = \mathcal{Q}_{c,i}(x) = 0$ (resp., $\mathcal{Z}_{s,j}(y) = \mathcal{Q}_{s,j}(y) = 0$).

To proceed, we will use the following Lyapunov function defined on $\mathbb{N}^I \times \mathbb{N}^J$:

$$f^m(x,y) = \sum_{i\in\mathcal{I}} \beta_{c,i}(x_i - mx^*_{c,i})^2 + \sum_{j\in\mathcal{J}} \beta_{s,j}(y_j - mx^*_{s,j})^2,$$

where $\beta_{c,i}$ and $\beta_{s,j}$ are strictly positive constants whose values are to be determined. The following proposition establishes the Foster–Lyapunov drift condition for Markovian process $X^m = (X^m_c, X^m_s)$ regarding the function f^m.

PROPOSITION EC.3. *There exist constants $\beta_{c,i} > 0$, $i \in \mathcal{I}$ and $\beta_{s,j} > 0$, $j \in \mathcal{J}$ such that*

$$(\mathcal{L}f^m)(x,y) \le -a_1 f^m(x,y) + a_2\|(x,y)\| + \epsilon_m m^2, \quad (x,y)\in\mathbb{N}^I\times\mathbb{N}^J, \quad m \ge m_0, \tag{EC.19}$$

where $a_1 > 0$, $a_2 \ge 0$, and m_0 are constants not depending on (x,y) or m, and $\{\epsilon_m; m \in \mathbb{R}_+\}$ is a sequence of positive numbers that is independent of (x,y) and converges to zero.

We will also use the following growth property for $\mathbb{E}[\|X^m(t)\|]$.

LEMMA EC.5. *There exist constants $c_i > 0$, $i = 1,2,3$, such that*

$$\mathbb{E}[\|X^m(t)\|] \le c_1 e^{-c_2 t}\mathbb{E}[\|X^m(0)\|] + c_3 m$$

for all $t \ge 0$ and sufficiently large m.

The proofs of Proposition EC.3 and Lemma EC.5 will be deferred to the end of this section.

Noting that from $\mathbb{E}[\|X^m(0)\|^2] < \infty$ and $\mathbb{E}[\|E^m_c(t)\|^2] < \infty$ for all $t \ge 0$, and $\|X^m(t)\| \le \|X^m_c(0)\| + \|E^m_c(t)\|$, one can obtain $\mathbb{E}[\|X^m(t)\|^2] \le \mathbb{E}[(\|X^m_c(0)\| + \|E^m_c(t)\|)^2] < \infty$, and

$$\begin{aligned}
\mathbb{E}[f^m(X^m_c(t), X^m_s(t))] &= \sum_{i\in\mathcal{I}} \beta_{c,i}\mathbb{E}[(X^m_{c,i}(t) - mx^*_{c,i})^2] + \sum_{j\in\mathcal{J}} \beta_{s,j}\mathbb{E}[(X^m_{s,j}(t) - mx^*_{s,j})^2)] \\
&= \sum_{i\in\mathcal{I}} \beta_{c,i}\mathbb{E}[(X^m_{c,i}(t))^2] + \sum_{j\in\mathcal{J}} \beta_{s,j}\mathbb{E}[(X^m_{s,j}(t))^2] - 2m\sum_{i\in\mathcal{I}} \beta_{c,i}x^*_{c,i}\mathbb{E}[X^m_{c,i}(t)] \\
&\quad - 2m\sum_{j\in\mathcal{J}} \beta_{s,j}x^*_{s,j}\mathbb{E}[X^m_{s,j}(t)] + m^2\left(\sum_{i\in\mathcal{I}} \beta_{c,i}(x^*_{c,i})^2 + \sum_{j\in\mathcal{J}} \beta_{s,j}(x^*_{s,j})^2\right) \\
&\le \beta_{\max}\mathbb{E}[\|X^m(t)\|^2] + (I+J)m^2\beta_{\max}(x^*_{\max})^2 < \infty,
\end{aligned}$$

for all $t \ge 0$, where $\beta_{\max} = (\max_{i\in\mathcal{I}} \beta_{c,i}) \vee (\max_{j\in\mathcal{J}} \beta_{s,j})$ and $x^*_{\max} = (\max_{i\in\mathcal{I}} x^*_{c,i}) \vee (\max_{j\in\mathcal{J}} x^*_{s,j})$, and the last inequality follows from

$$\sum_{i\in\mathcal{I}} \mathbb{E}[(X^m_{c,i}(t))^2] + \sum_{j\in\mathcal{J}} \mathbb{E}[(X^m_{s,j}(t))^2] \le \mathbb{E}\Big[\Big(\sum_{i\in\mathcal{I}} \big|X^m_{c,i}(t)\big| + \sum_{j\in\mathcal{J}} \big|X^m_{s,j}(t)\big|\Big)^2\Big] = \mathbb{E}[\|X^m(t)\|^2].$$

As a result, the process

$$M(\cdot) := f^m(X^m_c(\cdot), X^m_s(\cdot)) - \int_0^{\cdot} (\mathcal{L}f^m)(X^m_c(s), X^m_s(s))ds$$

is a martingale (see, e.g., Theorem 8.3.1. of Øksendal 1998), and we have that for $m \ge m_0$,

$$\begin{aligned}
\mathbb{E}[f^m(X^m_c(t), X^m_s(t))] &= \mathbb{E}[f^m(X^m_c(0), X^m_s(0))] + \mathbb{E}\left[\int_0^t (\mathcal{L}f^m)(X^m_c(s), X^m_s(s))ds\right] \\
&\le \mathbb{E}[f^m(X^m_c(0), X^m_s(0))] \\
&\quad + \mathbb{E}\left[\int_0^t (-a_1 f^m(X^m_c(s), X^m_s(s)) + a_2\|(X^m_c(s), X^m_s(s))\| + \epsilon_m m^2)ds\right],
\end{aligned}$$

where the inequality follows from Proposition EC.3. Combined with Lemma EC.5, the above inequality yields that for any sufficiently large m,

$$\begin{aligned}
&\mathbb{E}[f^m(X_c^m(t),X_s^m(t))]\\
\le\ &\mathbb{E}[f^m(X_c^m(0),X_s^m(0))]+\int_0^t\left(a_2\left(c_1e^{-c_2s}\mathbb{E}[\|X^m(0)\|]+c_3m\right)+\epsilon_m m^2\right)ds\\
&-a_1\int_0^t\mathbb{E}[f^m(X_c^m(s),X_s^m(s))]ds\\
\le\ &\mathbb{E}[f^m(X_c^m(0),X_s^m(0))]+\frac{a_2c_1}{c_2}\mathbb{E}[\|X^m(0)\|]+(a_2c_3m+\epsilon_m m^2)t-a_1\int_0^t\mathbb{E}[f^m(X_c^m(s),X_s^m(s))]ds.
\end{aligned}$$

Then, using the generalized Gronwall's inequality (see, e.g., Viorel 1974), we have

$$\begin{aligned}
\mathbb{E}[f^m(X_c^m(t),X_s^m(t))]\le\ &e^{-a_1t}\left(\mathbb{E}[f^m(X_c^m(0),X_s^m(0))]+\frac{a_2c_1}{c_2}\mathbb{E}[\|X^m(0)\|]\right)\\
&+\frac{1}{a_1}(a_2c_3m+\epsilon_m m^2)(1-e^{-a_1t}).
\end{aligned}$$

Note that there exists a positive constant ϑ that is sufficiently small such that

$$f^m(x,y)=\sum_{i\in\mathcal{I}}\beta_{c,i}x_i^2+\sum_{j\in\mathcal{J}}\beta_{s,j}y_j^2\ge\vartheta\left(\sum_{i\in\mathcal{I}}x_i+\sum_{j\in\mathcal{J}}y_j\right)^2=\vartheta\|(x,y)\|^2,$$

for any $(x,y)\in\mathbb{N}^I\times\mathbb{N}^J$. Let $\delta_m:=\sqrt{\frac{1}{a_1m^2\vartheta}(a_2c_3+\epsilon_m m)}$. Then we have

$$\limsup_{t\to\infty}\mathbb{E}\left[\|m^{-1}X^m(t)-x^*\|^2\right]\le\limsup_{t\to\infty}\frac{1}{m^2\vartheta}\mathbb{E}[f^m(X_c^m(t),X_s^m(t))]\le\delta_m^2.$$

This implies (EC.17). □

Proof of Proposition EC.3. The proof follows a similar argument as that for Lemma 3.1 in Atar et al. (2011). Fix $(x,y)\in\mathbb{N}^I\times\mathbb{N}^J$. For notational simplicity, we write $\zeta_i:=x_i-mx_{c,i}^*$, $\eta_j:=y_j-mx_{s,j}^*$, $\delta_{c,i}:=\theta_{c,i}\wedge\mu_{c,i}$, and $\delta_{s,j}:=\theta_{s,j}\wedge\mu_{s,j}$.

We first prove that

$$\begin{aligned}
(\mathcal{L}f^m)(x,y)\le&\sum_{i\in\mathcal{I}}\beta_{c,i}\left(\lambda_{c,i}^m+(\mu_{c,i}+\theta_{c,i})x_i+2\zeta_i\left(\lambda_{c,i}^m-m\lambda_{c,i}\right)\right)+\sum_{j\in\mathcal{J}}\beta_{s,j}\left(\sum_{i\in\mathcal{I}}\mu_{c,i}x_ip_{ij}+(\mu_{s,j}+\theta_{s,j})y_j\right)\\
&-2\sum_{i\in\mathcal{I}}\beta_{c,i}\delta_{c,i}\zeta_i^2+2\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\mu_{c,i}\beta_{s,j}p_{ij}\zeta_i\eta_j-2\sum_{j\in\mathcal{J}}\beta_{s,j}\delta_{s,j}\eta_j^2.
\end{aligned}\tag{EC.20}$$

Note that

$$\begin{aligned}
f(x+e_{i,I},y)-f(x,y)&=\beta_{c,i}+2\beta_{c,i}\zeta_i,\\
f(x-e_{i,I},y+e_{j,J})-f(x,y)&=\beta_{c,i}+\beta_{s,j}-2\beta_{c,i}\zeta_i+2\beta_{s,j}\eta_j,\\
f(x-e_{i,I},y)-f(x,y)&=\beta_{c,i}-2\beta_{c,i}\zeta_i,\\
f(x,y-e_{j,J})-f(x,y)&=\beta_{s,j}-2\beta_{s,j}\eta_j.
\end{aligned}$$

Thus, from (EC.18), we have

$$\begin{aligned}
(\mathcal{L}f^m)(x,y)=&\sum_{i\in\mathcal{I}}\beta_{c,i}\left(\lambda_{c,i}^m+\mu_{c,i}\mathcal{Z}_{c,i}(x)+\theta_{c,i}\mathcal{Q}_{c,i}(x)\right)+2\sum_{i\in\mathcal{I}}\beta_{c,i}\zeta_i\left(\lambda_{c,i}^m-\mu_{c,i}\mathcal{Z}_{c,i}(x)-\theta_{c,i}\mathcal{Q}_{c,i}(x)\right)\\
&+\sum_{j\in\mathcal{J}}\beta_{s,j}\left(\sum_{i\in\mathcal{I}}\mu_{c,i}\mathcal{Z}_{c,i}(x)p_{ij}+\mu_{s,j}\mathcal{Z}_{s,j}(y)+\theta_{s,j}\mathcal{Q}_{s,j}(y)\right)\\
&+2\sum_{j\in\mathcal{J}}\beta_{s,j}\eta_j\left(\sum_{i\in\mathcal{I}}\mu_{c,i}\mathcal{Z}_{c,i}(x)p_{ij}-\mu_{s,j}\mathcal{Z}_{s,j}(y)-\theta_{s,j}\mathcal{Q}_{s,j}(y)\right).
\end{aligned}\tag{EC.21}$$

Note that $0 \le \mathcal{Z}_c(x) \le x$ and $0 \le \mathcal{Q}_c(x) \le x$ for any $x \in \mathbb{N}^I$. (Here for any two vectors x, $x' \in \mathbb{N}^I$, we say that $x \le x'$ if $x_i \le x'_i$ for any $i \in \mathcal{I}$.) Hence, for the first term on the right-hand side of (EC.21), we have

$$\lambda^m_{c,i} + \mu_{c,i}\mathcal{Z}_{c,i}(x) + \theta_{c,i}\mathcal{Q}_{c,i}(x) \le \lambda^m_{c,i} + (\mu_{c,i} + \theta_{c,i})x_i. \tag{EC.22}$$

Similarly, because $0 \le \mathcal{Z}_s(y) \le y$ and $0 \le \mathcal{Q}_s(y) \le y$ for any $y \in \mathbb{N}^J$, for the third term on the right-hand side of (EC.21), we have

$$\sum_{i\in\mathcal{I}} \mu_{c,i}\mathcal{Z}_{c,i}(x)p_{ij} + \mu_{s,j}\mathcal{Z}_{s,j}(y) + \theta_{s,j}\mathcal{Q}_{s,j}(y) \le \sum_{i\in\mathcal{I}} \mu_{c,i}x_i p_{ij} + (\mu_{s,j} + \theta_{s,j})y_j. \tag{EC.23}$$

For the second term on the right-hand side of (EC.21), using Lemma EC.4 (i) and (ii), we have

$$\begin{aligned} &\lambda^m_{c,i} - \mu_{c,i}\mathcal{Z}_{c,i}(x) - \theta_{c,i}\mathcal{Q}_{c,i}(x) \\ =&\lambda^m_{c,i} - \mu_{c,i}\mathcal{Z}_{c,i}(mx^*_c) - \theta_{c,i}\mathcal{Q}_{c,i}(mx^*_c) - \left[\mu_{c,i}(1-\varrho_i(x,mx^*_c)) + \theta_{c,i}\varrho_i(x,mx^*_c)\right]\zeta_i \\ =&\lambda^m_{c,i} - m\lambda_{c,i} - \left[\mu_{c,i}(1-\varrho_i(x,mx^*_c)) + \theta_{c,i}\varrho_i(x,mx^*_c)\right]\zeta_i \\ \le&\lambda^m_{c,i} - m\lambda_{c,i} - \delta_{c,i}\zeta_i, \end{aligned} \tag{EC.24}$$

where the second equality holds because (q^*_c, z^*_c) satisfies (23), and the inequality holds because $\varrho_i(x, mx^*_c) \in [0,1]$.

Similarly, for the fourth term on the right-hand side of (EC.21), we use Lemma EC.4 to obtain

$$\begin{aligned} &\sum_{i\in\mathcal{I}} \mu_{c,i}\mathcal{Z}_{c,i}(x)p_{ij} - \mu_{s,j}\mathcal{Z}_{s,j}(y) - \theta_{s,j}\mathcal{Q}_{s,j}(y) \\ =&\sum_{i\in\mathcal{I}} \mu_{c,i}\mathcal{Z}_{c,i}(mx^*_c)p_{ij} - \mu_{s,j}\mathcal{Z}_{s,j}(mx^*_s) - \theta_{s,j}\mathcal{Q}_{s,j}(mx^*_s) + \sum_{i\in\mathcal{I}} \mu_{c,i}\varrho_i(x,mx^*_c)\zeta_i p_{ij} \\ &- \Big[\mu_{s,j}(1-\varrho_j(y,mx^*_s)) + \theta_{s,j}\varrho_j(y,mx^*_s)\Big]\eta_j \\ \le&\sum_{i\in\mathcal{I}} \mu_{c,i}\zeta_i p_{ij} - \delta_{s,j}\eta_j, \end{aligned} \tag{EC.25}$$

where the inequality is from (24) and the fact that both $\varrho_i(x, mx^*_c)$ and $\varrho_j(y, mx^*_s)$ are in $[0,1]$.

By combining the upper bound results in (EC.22)–(EC.25) and substituting them back to (EC.21), we have (EC.20).

Next, we bound the terms on the right-hand side of (EC.20) to obtain (EC.19). Fix $a_0 > 0$ such that $a_0 < 2\big(\min_{i\in\mathcal{I}} \delta_{c,i} \wedge \min_{j\in\mathcal{J}} \delta_{s,j}\big)$. Then, there exist $\beta_{c,i}$, $i \in \mathcal{I}$, that are appropriately chosen such that

$$2\beta_{c,i}\delta_{c,i} - \sum_{j\in\mathcal{J}} p_{ij}\mu^2_{c,i} - a_0\beta_{c,i} \ge 0, \qquad \text{for } i \in \mathcal{I}.$$

There also exist $\beta_{s,j}$, $j \in \mathcal{J}$, that are appropriately chosen such that

$$2\beta_{s,j}\delta_{s,j} - \sum_{i\in\mathcal{I}} p_{ij}\beta^2_{s,j} - a_0\beta_{s,j} \ge 0, \qquad \text{for } j \in \mathcal{J}.$$

These imply that

$$\sum_{i\in\mathcal{I}} \Big(2\beta_{c,i}\delta_{c,i} - \sum_{j\in\mathcal{J}} p_{ij}\mu^2_{c,i} - a_0\beta_{c,i}\Big)\zeta^2_i + \sum_{j\in\mathcal{J}} \Big(2\beta_{s,j}\delta_{s,j} - \sum_{i\in\mathcal{I}} p_{ij}\beta^2_{s,j} - a_0\beta_{s,j}\Big)\eta^2_j \ge 0.$$

Together with $2\mu_{c,i}\beta_{s,j}\zeta_i\eta_j \le \mu^2_{c,i}\zeta^2_i + \beta^2_{s,j}\eta^2_j$, the above inequality gives

$$\begin{aligned} &2\sum_{i\in\mathcal{I}} \beta_{c,i}\delta_{c,i}\zeta^2_i - 2\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}} p_{ij}\mu_{c,i}\beta_{s,j}\zeta_i\eta_j + 2\sum_{j\in\mathcal{J}} \beta_{s,j}\delta_{s,j}\eta^2_j \\ &\ge a_0\Big(\sum_{i\in\mathcal{I}} \beta_{c,i}\zeta^2_i + \sum_{j\in\mathcal{J}} \beta_{s,j}\eta^2_j\Big) = a_0 f^m(x,y). \end{aligned} \tag{EC.26}$$

Note that by (17), for any $m_0>1$, there exists a constant $a_2>0$, and a sequence $\{\epsilon_m; m\in\mathbb{R}_+\}$ with $\lim_{m\to\infty}\epsilon_m=0$ such that

$$\begin{aligned}
&\sum_{i\in\mathcal{I}}\beta_{c,i}\big(\lambda^m_{c,i}+(\mu_{c,i}+\theta_{c,i})x_i+2\zeta_i\left(\lambda^m_{c,i}-m\lambda_{c,i}\right)\big)+\sum_{j\in\mathcal{J}}\beta_{s,j}\Big(\sum_{i\in\mathcal{I}}\mu_{c,i}x_ip_{ij}+(\mu_{s,j}+\theta_{s,j})y_j\Big)\\
&=\sum_{i\in\mathcal{I}}\beta_{c,i}\Big(\mu_{c,i}\big(1+\sum_{j\in\mathcal{J}}\beta_{s,j}p_{ij}\big)+\theta_{c,i}\Big)x_i+\sum_{j\in\mathcal{J}}\beta_{s,j}\left(\mu_{s,j}+\theta_{s,j}\right)y_j+m^2\sum_{i\in\mathcal{I}}\beta_{c,i}\left[\frac{\lambda^m_{c,i}}{m^2}+2\left(\frac{x_i}{m}-x^*_{c,i}\right)\left(\frac{\lambda^m_{c,i}}{m}-\lambda_{c,i}\right)\right]\\
&\le a_2\|(x,y)\|+\epsilon_m m^2+\epsilon_m f^m(x,y),
\end{aligned}\tag{EC.27}$$

for any $(x,y)\in\mathbb{N}^I\times\mathbb{N}^J$ and $m\ge m_0$. In the last inequality we use $2|\frac{x_i}{m}-x^*_{c,i}|\le 1+(\frac{x_i}{m}-x^*_{c,i})^2$.

By combining (EC.20), (EC.26) and (EC.27), we have (EC.19) with $a_1\in(0,a_0-\epsilon_m)$. The proof is complete. □

Proof of Lemma EC.5. We first establish a bound for $\mathbb{E}\left[X^m_{c,i}(t)\right]$, $i\in\mathcal{I}$. For any $i\in\mathcal{I}$, from (1), (3)–(5), we have (recall that $\delta_{c,i}:=\theta_{c,i}\wedge\mu_{c,i}$)

$$\begin{aligned}
\mathbb{E}\left[X^m_{c,i}(t)\right]&=\mathbb{E}\left[X^m_{c,i}(0)\right]+\lambda^m_{c,i}t-\theta_{c,i}\int_0^t\mathbb{E}\left[Q^m_{c,i}(s)\right]dt-\mu_{c,i}\int_0^t\mathbb{E}\left[Z^m_{c,i}(s)\right]ds\\
&\le\mathbb{E}\left[X^m_{c,i}(0)\right]+\lambda^m_{c,i}t-\delta_{c,i}\int_0^t\mathbb{E}\left[X^m_{c,i}(s)\right]ds,\quad i\in\mathcal{I}
\end{aligned}$$

for any $t\ge0$, where the inequality is from $X^m_{c,i}(\cdot)=Q^m_{c,i}(\cdot)+Z^m_{c,i}(\cdot)$. Hence, from the generalized Gronwall's inequality (see, e.g., Viorel 1974), for $i\in\mathcal{I}$ and $t\ge0$,

$$\begin{aligned}
\mathbb{E}\left[X^m_{c,i}(t)\right]&\le\mathbb{E}\left[X^m_{c,i}(0)\right]+\lambda^m_{c,i}t-\delta_{c,i}\int_0^t\left(\mathbb{E}\left[X^m_{c,i}(0)\right]+\lambda^m_{c,i}s\right)e^{-\delta_{c,i}(t-s)}ds\\
&=e^{-\delta_{c,i}t}\mathbb{E}[X^m_{c,i}(0)]+\frac{\lambda^m_{c,i}}{\delta_{c,i}}(1-e^{-\delta_{c,i}t}).
\end{aligned}\tag{EC.28}$$

Next we establish a bound for $\sum_{j\in\mathcal{J}}\mathbb{E}\left[X^m_{s,j}(t)\right]$, $j\in\mathcal{J}$. Fix $j\in\mathcal{J}$. From (7) and (8) to (10), we have (recall that $\delta_{s,j}:=\theta_{s,j}\wedge\mu_{s,j}$)

$$\begin{aligned}
\mathbb{E}\left[X^m_{s,j}(t)\right]=&\mathbb{E}\left[X^m_{s,j}(0)\right]+\sum_{i\in\mathcal{I}}p_{ij}\int_0^t\mu_{c,i}\mathbb{E}\left[Z^m_{c,i}(s)\right]ds-\theta_{s,j}\int_0^t\mathbb{E}\left[Q^m_{s,j}(s)\right]ds-\mu_{s,j}\int_0^t\mathbb{E}\left[Z^m_{s,j}(s)\right]ds\\
\le&\mathbb{E}\left[X^m_{s,j}(0)\right]+\sum_{i\in\mathcal{I}}p_{ij}\mu_{c,i}\int_0^t\mathbb{E}\left[X^m_{c,i}(s)\right]ds-\delta_{s,j}\int_0^t\mathbb{E}\left[X^m_{s,j}(s)\right]ds\\
\le&\mathbb{E}\left[X^m_{s,j}(0)\right]+\sum_{i\in\mathcal{I}}\frac{p_{ij}\mu_{c,i}}{\delta_{c,i}}\left[\left(\mathbb{E}[X^m_{c,i}(0)]-\frac{\lambda^m_{c,i}}{\delta_{c,i}}\right)\cdot(1-e^{-\delta_{c,i}t})+\lambda^m_{c,i}t\right]-\delta_{s,j}\int_0^t\mathbb{E}\left[X^m_{s,j}(s)\right]ds\\
\le&\mathbb{E}\left[X^m_{s,j}(0)\right]+\sum_{i\in\mathcal{I}}\frac{p_{ij}\mu_{c,i}}{\delta_{c,i}}\left(\mathbb{E}[X^m_{c,i}(0)]+\lambda^m_{c,i}t\right)-\delta_{s,j}\int_0^t\mathbb{E}\left[X^m_{s,j}(s)\right]ds,
\end{aligned}$$

where the second inequality follows from (EC.28). Hence, by invoking the generalized Gronwall's inequality for $\mathbb{E}[X_{s,j}(t)]$, we have

$$\mathbb{E}\left[X^m_{s,j}(t)\right]\le e^{-\delta_{s,j}t}\Big(\mathbb{E}[X^m_{s,j}(0)]+\sum_{i\in\mathcal{I}}\frac{p_{ij}\mu_{c,i}}{\delta_{c,i}}\mathbb{E}[X^m_{c,i}(0)]\Big)+\sum_{i\in\mathcal{I}}\frac{p_{ij}\mu_{c,i}\lambda^m_{c,i}}{\delta_{c,i}\delta_{s,j}}(1-e^{-\delta_{s,j}t}),\tag{EC.29}$$

for all $j\in\mathcal{J}$ and $t\ge0$. Finally, by combining (EC.28) and (EC.29), we have

$$\begin{aligned}
\mathbb{E}\left[\|X^m(t)\|\right]=&\sum_{i\in\mathcal{I}}\mathbb{E}\left[X^m_{c,i}(t)\right]+\sum_{j\in\mathcal{J}}\mathbb{E}\left[X^m_{s,j}(t)\right]\\
\le&\sum_{i\in\mathcal{I}}\left(e^{-\delta_{c,i}t}+\sum_{j\in\mathcal{J}}e^{-\delta_{s,j}t}\frac{p_{ij}\mu_{c,i}}{\delta_{c,i}}\right)\mathbb{E}[X^m_{c,i}(0)]+\sum_{j\in\mathcal{J}}e^{-\delta_{s,j}t}\mathbb{E}[X^m_{s,j}(0)]\\
&+\sum_{i\in\mathcal{I}}\frac{\lambda^m_{c,i}}{\delta_{c,i}}\left(1-e^{-\delta_{c,i}t}\right)+\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\frac{p_{ij}\mu_{c,i}\lambda^m_{c,i}}{\delta_{c,i}\delta_{s,j}}\left(1-e^{-\delta_{s,j}t}\right)\\
\le&\left(I+J+\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\frac{p_{ij}\mu_{c,i}}{\delta_{c,i}}\right)e^{-\delta_{\min}t}\mathbb{E}[\|X^m(0)\|]+\sum_{i\in\mathcal{I}}\frac{\lambda^m_{c,i}}{\delta_{c,i}}\left(1+\sum_{j\in\mathcal{J}}\frac{p_{ij}\mu_{c,i}}{\delta_{s,j}}\right),
\end{aligned}$$

where $\delta_{\min} := (\min_{i\in\mathcal{I}} \delta_{c,i}) \wedge (\min_{j\in\mathcal{J}} \delta_{s,j})$. This, together with (17), yields the desired result. $\square$

EC.3. Grouping and Pooling

The discussion after Theorem 3 inspires us to consider the following strategy to streamline the operations of the MM system. First, merge multiple customer classes in stage 1 to form (at most) three service groups: $\mathcal{I}_1 := \{i \in \mathcal{I} : \varphi^*_{c,i} = \lambda_{c,i}\}$, $\mathcal{I}_3 := \{i \in \mathcal{I} : \varphi^*_{c,i} = 0\}$, and $\mathcal{I}_2 := \mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_3)$. Here, group $\mathcal{I}_2$ may be empty and contain at most one class only if $\phi^* \in \mathcal{S}_3$. Second, pool the stations in the second stage to form at most three clusters: $\mathcal{J}_1 := \{j \in \mathcal{J} : \varphi^*_{s,j} = \sum_{i\in\mathcal{I}} \varphi^*_{c,i} p_{ij}\}$, $\mathcal{J}_3 := \{j \in \mathcal{J} : \varphi^*_{s,j} = 0\}$, and $\mathcal{J}_2 := \mathcal{J} \setminus (\mathcal{J}_1 \cup \mathcal{J}_3)$. Cluster $\mathcal{J}_2$ may be empty and contain at most one station only if $\phi^* \in \mathcal{S}_2$. Therefore, at least one of $\mathcal{I}_2$ or $\mathcal{J}_2$ is empty.

With this grouping and pooling, we propose the following policy, denoted by $\psi^{m,\mathrm{GP}} = (N^{m,\mathrm{GP}}, \omega^{m,\mathrm{GP}})$, for the resulting mth grouped-pooled (GP) system.

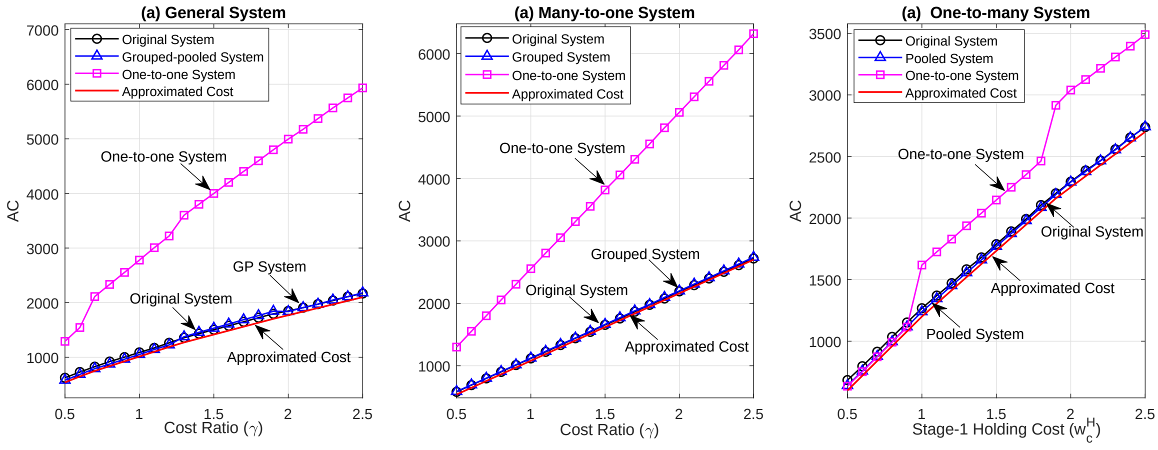
(GP1) The capacity allocation is given by $N^{m,\mathrm{GP}} = (N_c^{m,\mathrm{GP}}, N_{s,\mathcal{J}_1}^{m,\mathrm{GP}}, N_{s,\mathcal{J}_2}^{m,\mathrm{GP}}, N_{s,\mathcal{J}_3}^{m,\mathrm{GP}})$, in which $N_c^{m,\mathrm{GP}} = N_c^{m,*}$ and $N_{s,\mathcal{J}_k}^{m,\mathrm{GP}} = \sum_{j\in\mathcal{J}_k} N_j^{m,*}$ for $k = 1, 2, 3$. (Here, $N_{s,\mathcal{J}_3}^{m,\mathrm{GP}} = 0$.)

(GP2) The scheduling rule $\omega^{m,\mathrm{GP}}$ assigns the highest (resp., lowest) priority to classes in group $\mathcal{I}_1$ (resp., $\mathcal{I}_3$) while adopting FCFS for customers of the classes in the same group.

We use numerical experiments to illustrate that the GP system under $\psi^{m,\mathrm{GP}}$ performs similarly to the original system under $\psi^{m,*}$. We first consider an MM system with five classes in stage 1 and five stations in stage 2, that is, $\mathcal{I} = \mathcal{J} = \{1, \ldots, 5\}$. The parameters for the original system are as follows: the total arrival rate is $\lambda = 400$, with equal arrival rate to each class, that is, $\lambda_{c,i} = \lambda/5$ for $i \in \mathcal{I}$, the service rates are $\mu_{c,i} = 2$ for $i \in \mathcal{I}$ and $\mu_{s,j} = 1$ for $j \in \mathcal{J}$, the abandonment rates are $\theta_{c,i} = \theta_{s,j} = 0.5$ for $i \in \mathcal{I}$ and $j \in \mathcal{J}$, the routing probabilities are $p_{ij} = 1/5$ for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, the total amount of resource is $m = 400$, the conversion rates are $r_c = 1$ and $r_{s,j} = 2$ for $j \in \mathcal{J}$, and the holding and abandonment costs are $w_s^H = w_s^A = (5, 2.5, 1.5, 1, 0.5)$ and $w_c^H = w_c^A = \gamma w_s^H$, respectively, with γ ranging from 0.5 to 2.5 in increments of 0.1. For the GP system, stage-1 classes form at most three service groups and are subsequently attended by at most two stations in stage 2. We also construct a one-to-one system by further combining all stage-1 classes into a single queue and pooling all stage-2 servers into one station. Customers within the same queue are served following the FCFS discipline. From the comparisons in Figure EC.2(a), the average cost incurred under the GP system (blue line) is almost indistinguishable from that incurred under the original system (black line), that is, a well-designed scheme of grouping and pooling can achieve a simulated cost comparable to that of the original system. Moreover, the GP system significantly outperforms the corresponding one-to-one system (magenta line), suggesting that simply grouping all customer classes in stage 1 and pooling all stations in stage 2 to construct a one-to-one system compromises system performance.

The cost efficiency in the GP system is attributed to the combined effect of grouping and pooling. In what follows, we investigate their effects separately by using MO and OM systems, respectively. For the MO system, we keep its stage-1 parameters the same as those in the general system while setting the transition probability as $p_{is} = 1$ and the stage-2 parameters as follows: $\mu_s = 1$, $\theta_s = 0.5$, $r_s = 2$, and $w_s^H = w_s^A = 2.5$. For the OM system, we keep the stage-2 parameters the same as those in the preceding general system while setting the transition probability as $p_{cj} = 1/5$ and the stage-1 parameters as follows: $\lambda = 400$, $\mu_c = 2$, $\theta_c = 0.5$, $r_c = 1$, and $w_c^H = w_c^A = \gamma$, with γ ranging from 0.5 to 2.5. For these two systems, we repeat the grouping and pooling process to construct the corresponding grouped system and pooled system, respectively.

Figure EC.2 Value of Optimal Grouping and Pooling



The results are plotted in Figures EC.2(b) and (c). As shown in panel (b), the grouped system incurs long-run average costs (blue line) that are very close to those in the original system (black line). This is because, for the grouped system, $\mathcal{I}_1$ consists of classes whose service requirements can be fully satisfied. Therefore, the practice of making customers from these particular groups wait in a single line and serving them on a first-come, first-served basis does not significantly affect their overall service fulfillment in the long run. This prevents the queue dedicated to $\mathcal{I}_1$ from building up, resulting in a zero long-run average cost. By contrast, the classes in $\mathcal{I}_3$ receive almost no service capacity in the grouped system (because the service capacity is mainly used for serving customers from the other groups), similar to the original system where these classes are given the least priority. Hence, nearly all customers from these classes eventually abandon the queue, incurring similar abandonment penalties in both systems. For the class (if one exists) in $\mathcal{I}_2$, its steady-state queue length remains unchanged as it receives the same amount of service capacity after grouping, thus resulting in a similar delay-related cost. Analogously, panel (c) reveals that a pooled system incurs a simulated cost comparable to that of the original system. This can be explained using similar arguments. Specifically, the aggregate resources assigned to service types belonging to $\mathcal{J}_1$ (resp., $\mathcal{J}_3$) remain unchanged after pooling, with their service requirements being fully (resp., never) accommodated. For the service type (if one exists) in $\mathcal{J}_2$, it receives the same amount of resource, and thus its long-run average cost is not affected after pooling.

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