Online Appendix: Production rate control in condition-based maintenance planning

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Proof of Lemma 1. The proof originates from Jagannathan (1966). It is reproduced specifically for the purposes of the current study.

(i) We have

$$g_\gamma = P^{\pi_\gamma}(0) - \gamma T^{\pi_\gamma}(0) < P^{\pi_\gamma}(0) - (\gamma - \epsilon)T^{\pi_\gamma}(0) \leq \max_\pi g^\pi(0, \gamma - \epsilon) = g_{\gamma-\epsilon}$$

for any $\epsilon > 0$. Hence, g_γ is decreasing in γ. Furthermore, g_γ is continuous in γ, as it is the maximum of the linear functions $g^\pi(0,\gamma) = P^\pi(0) - \gamma T^\pi(0)$ over all policies π, whose slopes $-T^\pi(0)$ are bounded.

(ii) Let γ satisfy $g_\gamma = 0$. Then, since $0 = g_\gamma = P^{\pi_\gamma}(0) - \gamma T^{\pi_\gamma}(0)$, it follows that

$$\gamma = P^{\pi_\gamma}(0)/T^{\pi_\gamma}(0) = \phi^{\pi_\gamma} \leq \max_\pi \phi^\pi = \phi^*.$$

Furthermore, as π_γ is a maximizing policy for γ, we have

$$0 = g_\gamma = P^{\pi_\gamma}(0) - \gamma T^{\pi_\gamma}(0) \geq P^{\pi^*}(0) - \gamma T^{\pi^*}(0).$$

This implies $\gamma \geq P^{\pi^*}(0)/T^{\pi^*}(0) = \phi^*$. Thus $\gamma = \phi^*$ if $g_\gamma = 0$.

Conversely, suppose $\gamma = \phi^*$. Since $\phi^* = P^{\pi^*}(0)/T^{\pi^*}(0)$, we have

$$g_{\phi^*} = \max_\pi g^\pi(0, \phi^*) \geq g^{\pi^*}(0, \phi^*) = P^{\pi^*}(0) - \phi^* T^{\pi^*}(0) = 0.$$

If $g_{\phi^*} > 0$, then a policy exists with long-term profit strictly exceeding ϕ^*, contradicting the optimality of ϕ^*. Thus $g_\gamma = 0$ if $\gamma = \phi^*$.

The root $g_\gamma = 0$ is unique as g_γ is decreasing in γ. □

Proof of Lemma 2. Let us first observe that the lemma holds for $x \in X_{\text{fail}}$. This follows from the fact that $c(x)$ and $\tau(x)$ are both non-decreasing in x and $g(x,\gamma) = -c(x) - \gamma\tau(x)$ for all $x \in X_{\text{fail}}$. Therefore, we can proceed by induction.

Let $\pi_\gamma = (M, \Gamma)$ be a policy maximizing the parameterized expected cycle profit. Suppose that y is a state such that $g(x,\gamma)$ is non-increasing in x for all $x > y$. We consider the following four cases.

Case 1: $y, y+1 \in M$. Due to the monotonicity of $c(x)$ and $\tau(x)$, we have

$$g(y,\gamma) = -c(y) - \gamma\tau(y) \geq -c(y+1) - \gamma\tau(y+1) = g(y+1,\gamma).$$

Case 2: $y \notin M$ and $y+1 \in M$. It follows from the previous case that

$$g(y,\gamma) \geq -c(y) - \gamma\tau(y) \geq -c(y+1) - \gamma\tau(y+1) = g(y+1,\gamma).$$

Case 3: $y, y+1 \notin M$. By the induction assumption, $g(x,\gamma)$ is non-increasing in x for all $x > y$. Hence, we have

$$g(y+1,\gamma) = (r(y+1,\Gamma(y+1)) - \gamma)\mathbb{E}\{S_{y+1,\Gamma(y+1)}\} + \mathbb{E}\{g(y+1+A_{y+1,\Gamma(y+1)},\gamma)\} \geq g(y+2,\gamma),$$

and, as $A_{y+1,\Gamma(y+1)} > 0$, we also have

$$\mathbb{E}\{g(y+1+A_{y+1,\Gamma(y+1)},\gamma)\} \leq g(y+2,\gamma).$$

Jointly, these imply $(r(y+1,\Gamma(y+1)) - \gamma)\mathbb{E}\{S_{y+1,\Gamma(y+1)}\} \geq 0$. Let us recall that $r(x,\mu)$ is non-increasing in x for any μ. Besides, we have from Assumption 1(i) that $\mathbb{E}\{S_{x,\mu}\}$ is positive and non-increasing in x for any μ. Hence, we can conclude that

$$(r(y,\Gamma(y+1)) - \gamma)\mathbb{E}\{S_{y,\Gamma(y+1)}\} \geq (r(y+1,\Gamma(y+1)) - \gamma)\mathbb{E}\{S_{y+1,\Gamma(y+1)}\}. \tag{EC.1}$$

From Assumption 1(ii), we know that $A_{x,\mu}$ is stochastically non-decreasing in x for any μ. By the induction assumption, $g(x,\gamma)$ is non-increasing in x for all $x > y$. Then it follows from the definition of stochastic dominance that $\mathbb{E}\{g(x+A_{x,\mu},\gamma)\}$ is non-increasing in x. Hence, we have

$$\mathbb{E}\{g(y+A_{y,\Gamma(y+1)},\gamma)\} \geq \mathbb{E}\{g(y+1+A_{y+1,\Gamma(y+1)},\gamma)\}. \tag{EC.2}$$

Finally, we can write

$$\begin{aligned}
g(y,\gamma) &= \max_{\mu\in[0,1]} \{(r(y,\mu) - \gamma)\mathbb{E}\{S_{y,\mu}\} + \mathbb{E}\{g(y+A_{y,\mu},\gamma)\}\} \\
&\geq (r(y,\Gamma(y+1)) - \gamma)\mathbb{E}\{S_{y,\Gamma(y+1)}\} + \mathbb{E}\{g(y+A_{y,\Gamma(y+1)},\gamma)\} \\
&\geq (r(y+1,\Gamma(y+1)) - \gamma)\mathbb{E}\{S_{y+1,\Gamma(y+1)}\} + \mathbb{E}\{g(y+1+A_{y+1,\Gamma(y+1)},\gamma)\} \\
&= g(y+1,\gamma)
\end{aligned}$$

where the second line follows as $\Gamma(y+1) \in [0,1]$, and the third line follows from (EC.1) and (EC.2).

Case 4: $y \in M$ and $y+1 \notin M$. It follows from the previous case that

$$\begin{aligned}
g(y,\gamma) &= -c(y) - \gamma\tau(y) \\
&\geq \max_{\mu\in[0,1]} \{(r(y,\mu) - \gamma)\mathbb{E}\{S_{y,\mu}\} + \mathbb{E}\{g(y+A_{y,\mu},\gamma)\}\} \\
&\geq g(y+1,\gamma).
\end{aligned}$$

In light of the results obtained for the cases we have examined, we can conclude that if there exists a y such that $g(x,\gamma)$ is non-increasing in x for all $x > y$, then $g(x,\gamma)$ is also non-increasing in x for all $x \geq y$. This completes the induction and hence the proof. $\square$

Proof of Corollary 1. Suppose $x \in X_{\text{func}} \setminus M$ and $r(x,\Gamma(x)) - \gamma < 0$. Then,

$$\begin{aligned} g(x,\gamma) &= (r(x,\Gamma(x)) - \gamma)\mathbb{E}\{S_{x,\Gamma(x)}\} + \mathbb{E}\{g(x + A_{x,\Gamma(x)},\gamma)\} \\ &< \mathbb{E}\{g(x + A_{x,\Gamma(x)},\gamma)\} \\ &\leq g(x,\gamma), \end{aligned}$$

where the first inequality holds because $r(x,\Gamma(x)) - \gamma < 0$ and $\mathbb{E}\{S_{x,\Gamma(x)}\} > 0$, and the second inequality follows from Lemma 2. This yields a contradiction. □

Proof of Theorem 1. Let $\pi_\gamma = (M,\Gamma)$ be a maximizing policy that maintains in every state in which maintenance is optimal. That is, $M = \{x \in \mathbb{Z}_{\geq 0} : g(x,\gamma) = -c(x) - \gamma\tau(x)\}$. Consider an interval X_k such that $X_k \cap M \neq \emptyset$. To complete the proof, it is sufficient to show that $y \in X_k \cap M$ implies $y+1 \in X_k \cap M$ for any $y, y+1 \in X_k$.

Suppose $y \in X_k \cap M$. Then we have

$$g(y,\gamma) = -c(y) - \gamma\tau(y) = -c(y+1) - \gamma\tau(y+1) \leq g(y+1,\gamma),$$

where the second equality follows as $y, y+1 \in X_k$. From Lemma 2, we know that $g(y,\gamma) \geq g(y+1,\gamma)$. Thus $g(y+1,\gamma)$ cannot (strictly) exceed $-c(y+1) - \gamma\tau(y+1)$. Hence, we have $y+1 \in M$. □

Proof of Theorem 2. Let us first observe that the lower bound $\underline{\omega}(x,\gamma)$ is well-defined since we know from Corollary 1 that there exists a μ such that $r(x,\mu) - \gamma \geq 0$ for each $x \in X_{\text{func}} \setminus M$. Then the result immediately follows as $r(x,\mu)$ is increasing in μ for any $x \in X_{\text{func}}$. □

For simplicity, in the following we use the shorthand notation

$$g_\mu(x,\gamma) := (r(x,\mu) - \gamma)\mathbb{E}\{S_{x,\mu}\} + \mathbb{E}\{g(x + A_{x,\mu},\gamma)\},$$

which denotes the expected parameterized cycle profit accumulated over the remainder of a cycle as a function of the production rate μ, provided that no maintenance intervention is initiated in state x and optimal maintenance and production control decisions are followed thereafter.

Proof of Theorem 3. In the following, we assume there exists a unique μ_x that satisfies the properties laid out in Assumption 3. However, the proof trivially extends to the case where a compact set of values satisfy these properties.

We first consider the lower bound. Suppose $\Gamma(x) < \mu_x$ for some $x \in X_{\text{func}} \setminus M$. From Corollary 1, we know that $r(x,\Gamma(x)) - \gamma \geq 0$. Then, as $r(x,\mu)$ is increasing in μ, we have that $r(x,\mu) - \gamma$ is non-negative and increasing in μ on $[\Gamma(x),\mu_x]$. From Assumption 3, we know that $\mathbb{E}\{S_{x,\mu}\}$ is non-decreasing in μ on $[0,\mu_x]$. Therefore, the product $(r(x,\mu) - \gamma)\mathbb{E}\{S_{x,\mu}\}$ is increasing in μ on $[\Gamma(x),\mu_x]$. It follows from Lemma 2 that $g(x,\gamma)$ is non-increasing in x and from Assumption 3 that $A_{x,\mu}$ is stochastically non-increasing in μ on $[0,\mu_x]$. Then we have from the definition of

stochastic dominance that $\mathbb{E}\{g(x+A_{x,\mu},\gamma)\}$ is non-decreasing in μ on $[0,\mu_x]$. These jointly imply that $g_\mu(x,\gamma)$ is increasing in μ on $[\Gamma(x),\mu_x]$. However, this is a contradiction as $\Gamma(x)$ must satisfy $g_{\Gamma(x)}(x,\gamma) \geq g_\mu(x,\gamma)$ for all $\mu \in [0,1]$.

Next, we consider the upper bound. Let $x \in X_{\text{func}} \setminus M$. By the definition of $\bar{\omega}(x,\gamma)$, we have $(r(x,\mu)-\gamma)\mathbb{E}\{S_{x,\mu}\} \leq (r(x,\bar{\omega}(x,\gamma))-\gamma)\mathbb{E}\{S_{x,\bar{\omega}(x,\gamma)}\}$ for all $\mu \in [\mu_x,1]$. It follows from Lemma 2 that $g(x,\gamma)$ is non-increasing in x and from Assumption 3 that $A_{x,\mu}$ is stochastically non-decreasing in μ on $[\mu_x,1]$. Then we have from the definition of stochastic dominance that $\mathbb{E}\{g(x+A_{x,\mu},\gamma)\}$ is non-increasing in μ on $[\mu_x,1]$. These jointly imply that $g_\mu(x,\gamma) \leq g_{\bar{\omega}(x,\gamma)}(x,\gamma)$ for all $\mu \in [\bar{\omega}(x,\gamma),1]$. Hence, we can conclude that $g_\mu(x,\gamma)$ has a maximizer in the interval $[\mu_x,\bar{\omega}(x,\gamma)]$. □

Proof of Theorem 4. To prove that $\Gamma(x)$ is non-increasing in $x \in X_{\text{func}} \setminus M$, it suffices to show that $g_{\mu_1}(x_1,\gamma) \geq g_{\mu_2}(x_1,\gamma)$ implies $g_{\mu_1}(x_2,\gamma) \geq g_{\mu_2}(x_2,\gamma)$ for any $\mu_1 \leq \mu_2$ and $x_1 \leq x_2$, such that $\mu_1,\mu_2 \in [0,1]$ and $x_1,x_2 \in X_{\text{func}} \setminus M$. This holds under the sufficient condition

$$g_{\mu_1}(x_1,\gamma) + g_{\mu_2}(x_2,\gamma) - g_{\mu_2}(x_1,\gamma) - g_{\mu_1}(x_2,\gamma) \leq 0.$$

Further, we can assume that μ_1 and μ_2 are such that $\gamma \leq r(\mu_1) < r(\mu_2)$, as it follows from Corollary 1 and Assumption 4 that any optimal policy satisfies $r(\Gamma(x)) - \gamma \geq 0$ for all $x \in X_{\text{func}} \setminus M$.

If we extend the left-hand side of this expression, then collecting the terms involving $r(x,\mu) = r(\mu)$ based on Assumption 4, and canceling the terms including $A_{x,\mu} = A_x$ based on Assumption 5(ii), we obtain

$$\begin{aligned}
&g_{\mu_1}(x_1,\gamma) + g_{\mu_2}(x_2,\gamma) - g_{\mu_1}(x_2,\gamma) - g_{\mu_2}(x_1,\gamma) \\
&\quad = (r(\mu_1)-\gamma)(\mathbb{E}\{S_{x_1,\mu_1}\} - \mathbb{E}\{S_{x_2,\mu_1}\}) + (r(\mu_2)-\gamma)(\mathbb{E}\{S_{x_2,\mu_2}\} - \mathbb{E}\{S_{x_1,\mu_2}\}) \\
&\quad \leq (r(\mu_1)-\gamma)(\mathbb{E}\{S_{x_1,\mu_1}\} - \mathbb{E}\{S_{x_2,\mu_1}\} + \mathbb{E}\{S_{x_2,\mu_2}\} - \mathbb{E}\{S_{x_1,\mu_2}\}) \\
&\quad \leq 0,
\end{aligned}$$

where the inequalities follow as $\gamma \leq r(\mu_1) < r(\mu_2)$, and from Assumption 1(i) and Assumption 5(i). □

Proof of Theorem 5. To prove that $\Gamma(x)$ is non-decreasing in $x \in X_{\text{func}} \setminus M$, it suffices to show that $g_{\mu_1}(x_1,\gamma) \leq g_{\mu_2}(x_1,\gamma)$ implies $g_{\mu_1}(x_2,\gamma) \leq g_{\mu_2}(x_2,\gamma)$ for any $\mu_1 \leq \mu_2$ and $x_1 \leq x_2$, such that $\mu_1,\mu_2 \in [0,1]$ and $x_1,x_2 \in X_{\text{func}} \setminus M$. This holds under the sufficient condition

$$g_{\mu_1}(x_1,\gamma) + g_{\mu_2}(x_2,\gamma) - g_{\mu_2}(x_1,\gamma) - g_{\mu_1}(x_2,\gamma) \geq 0.$$

If we extend the left-hand side of this expression, then collecting the terms involving $r(x,\mu) = r(\mu)$ based on Assumption 4, and canceling the terms including $\mathbb{E}\{S_{x,\mu}\} = \mathbb{E}\{S_\mu\}$ based on Assumption 6(i), we obtain

$$g_{\mu_1}(x_1,\gamma) + g_{\mu_2}(x_2,\gamma) - g_{\mu_2}(x_1,\gamma) - g_{\mu_1}(x_2,\gamma)$$

$$
\begin{aligned}
&= \mathbb{E}\{g(x_1 + A_{x_1,\mu_1}, \gamma)\} + \mathbb{E}\{g(x_2 + A_{x_2,\mu_2}, \gamma)\} - \mathbb{E}\{g(x_1 + A_{x_1,\mu_2}, \gamma)\} - \mathbb{E}\{g(x_2 + A_{x_2,\mu_1}, \gamma)\} \\
&= (\mathbb{P}\{A_{x_1,\mu_1} = L\} - \mathbb{P}\{A_{x_1,\mu_2} = L\})(-g(x_1 + 1, \gamma) + g(x_1 + L, \gamma)) \\
&+ (\mathbb{P}\{A_{x_2,\mu_2} = L\} - \mathbb{P}\{A_{x_2,\mu_1} = L\})(-g(x_2 + 1, \gamma) + g(x_2 + L, \gamma)) \\
&\geq (\mathbb{P}\{A_{x_1,\mu_1} = L\} + \mathbb{P}\{A_{x_2,\mu_2} = L\} - \mathbb{P}\{A_{x_2,\mu_1} = L\} - \mathbb{P}\{A_{x_1,\mu_2} = L\})(-g(x_1 + 1, \gamma) + g(x_1 + L, \gamma)) \\
&\geq 0
\end{aligned}
$$

where the inequalities follow from Lemma 2, Assumption 6(ii), and Assumption 7. □