Appendix

A. Sample Trade Finance Contract

Based on a trade-credit financing facility contract (Law Insider 2002, Section 3: Collateral), we provide examples of a borrower's assets that are held as collateral. While the contract involves a bank as an intermediary, it is not an example of reverse factoring. Rather, the trade credit facility results in a loan to the borrower based on its assets and its business relationship with the supplier. The contract lists all equipment, inventory, accounts, documents, monies, and deposit accounts as collateral. In the interest of brevity, we only provide examples of equipment and inventory. Equipment: All goods now owned or hereafter acquired by the Borrower or in which the Borrower now has or may hereafter acquire any interest, including, but not limited to, all machinery, equipment, furniture, furnishings, fixtures, tools, supplies and motor vehicles of every kind and description, and all additions, accessions, improvements, replacements and substitutions thereto and thereof. Inventory: All inventory now owned or hereafter acquired by the Borrower, including, but not limited to, all raw materials, work in process, finished goods, inventory leased to others or held for lease, merchandise, parts and supplies of every kind and description, including inventory temporarily out of the Borrower's custody or possession, together with all returns on accounts.

The contract further states that "The Bank's security interest in the Collateral shall be a continuing lien and shall include the proceeds and products of the Collateral including, but not limited to, the proceeds of any insurance thereon. Borrower hereby consents to and instructs Bank to file financing statements in all locations deemed appropriate by the Bank from time to time. The security interest granted to Bank in the Collateral shall not secure or be deemed to secure any Indebtedness of the Borrower to the Bank which is, at the time of its creation, subject to the provisions of any state or federal consumer credit or truth-in-lending disclosure statutes."

B. Proof of Lemma 1

We prove Lemma 1 by contradiction. According to the statement of the Lemma, there exists a twice differentiable function $g(y)$, defined over $y \in Y$ such that on every interval over which it is increasing, it is also concave over the same interval. Let $\hat{Y}$ be one such interval. That is, $g'(y) = \partial g(y)/\partial y \geq 0$ and $g''(y) = \partial^2 g(y)/\partial y^2 \leq 0$ for every $y \in \hat{Y}$. We want to show that there exists at most one subset $\hat{Y}$ and that once $g(y)$ starts decreasing, it never increases again. To construct the contradiction, let $\tilde{y} \in Y$ be a point such that $g'(\tilde{y}) = 0$, $g''(\tilde{y}) \geq 0$ and $g'(y) < 0$ for $y \to \tilde{y}^-$ and $g'(y) > 0$ for $y \to \tilde{y}^+$, where $\tilde{y}^-$ denotes the left limit and $\tilde{y}^+$ denotes the right limit. That is, $\tilde{y}$ is a point at which $g(y)$ switches from being a decreasing to an increasing function. Clearly, $g(y)$ is increasing in y for $y > \tilde{y}$ until some other point at which it switches to decreasing again. Because the function is increasing, by the statement of the Lemma, it must also be concave over some range of y beginning at $y = \tilde{y}$. For the purpose of constructing the contradiction, it suffices to note that $g''(y) \leq 0$ at $y \to \tilde{y}^+$. This, combined with the fact that by construction $g''(\tilde{y}) \geq 0$, implies that the function $g(y)$ must have a kink at $\tilde{y}$. The existence of a kink contradicts the fact that $g(y)$ is twice differentiable on the entire domain Y. Thus, the point $\tilde{y}$ must not exist, which implies that $g(y)$ must not increase once it starts decreasing. This also means that there must not be more than one interval $\hat{Y}$ over which $g(y)$ is increasing and concave because for there to be more than one such interval, the function $g(y)$ must switch from being a decreasing function to an increasing function at least once. Finally, if there exists a $y \in \hat{Y}$ such that $g'(y) = 0$, then it must be the case that $g''(y) \le 0$. Put differently, that y is a global maximizer of $g(y)$. ☐

C. Proof of Proposition 1

We show that neither the retailer nor the supplier has an incentive to deviate from the equilibrium strategies of Proposition 1. We omit the argument (q) of $\bar{p}(q)$ in this proof to keep notation compact.

Proposition 1 states that the retailer offers $\bar{p}$ if $x \ge (\bar{p} - \alpha)^+/r$ and 0, otherwise. The supplier accepts an offer of $\bar{p}$ and verifies an offer of 0. Her posterior belief is that $x \ge (\bar{p} - \alpha)^+/r$ if the retailer offers $\bar{p}$ and $x < (\bar{p} - \alpha)^+/r$ if the retailer offers 0. We consider the retailer's strategy first. Suppose that the retailer observes a realized demand $x < (\bar{p} - \alpha)^+/r$. Then, he cannot offer $\bar{p}$. Because the supplier verifies any offer $< \bar{p}$, the retailer's terminal wealth is $(\alpha + (1-\beta)r\min\{x,q\} - \ell)^+$, which is independent of p. In this case, his payment offer does not matter. The equilibrium exists when $p = 0$ and he has no reason to deviate, justifying our assumption that he offers 0. Thus, $\gamma^*(0|x) = 1$ if $x < (\bar{p} - \alpha)^+/r$. Suppose that the retailer observes $x \ge (\bar{p} - \alpha)^+/r$. In this case, he can offer any $p \le \alpha + rx$. If $p \ge \bar{p}$, the supplier accepts the payment offer with probability 1 and the retailer makes $(\alpha + r\min\{x,q\} - p)$. Because this amount is decreasing in p, the retailer will not offer $> \bar{p}$. If the retailer offers $p < \bar{p}$, his offer is verified. We next show that the retailer's terminal wealth is higher upon offering $\bar{p}$ than $p < \bar{p}$. The retailer's terminal wealth is $(\alpha + r\min\{x,q\} - \bar{p})$ at $p = \bar{p}$ and $(\alpha + (1-\beta)r\min\{x,q\} - \ell)^+$ at $p < \bar{p}$. If $(\alpha + (1-\beta)r\min\{x,q\} - \ell)^+ = 0$, then $0 \le (\alpha + r\min\{x,q\} - \bar{p})$ because $x \ge (\bar{p} - \alpha)^+/r$. If $(\alpha + (1-\beta)r\min\{x,q\} - \ell)^+ > 0$, then $(\alpha + (1-\beta)r\min\{x,q\} - \ell) \le (\alpha + r\min\{x,q\} - \ell) \le (\alpha + r\min\{x,q\} - \bar{p})$, where the last inequality follows from the fact that $\bar{p} \le \ell$. Thus, the retailer will not unilaterally deviate from the equilibrium strategy of Proposition 1.

We next turn to the supplier's strategy. If the supplier observes $p = \bar{p}$, she knows that $x \ge (\bar{p} - \alpha)^+/r$. Therefore, $\mu(\xi|\bar{p}) = \frac{\gamma(\bar{p}|\xi)f(\xi)d\xi}{\int_{z=\frac{(\bar{p}-\alpha)^+}{r}}^{\infty}\gamma(\bar{p}|z)f(z)dz}$ and the amount she expects to make from verifying the sales is

$$\int_{(\bar{p}-\alpha)^+/r}^{\infty}[\min\{\ell, \alpha + (1-\beta)r\min\{\xi,q\}\}]\frac{\gamma(\bar{p}|\xi)f(\xi)}{\int_{z=\frac{(\bar{p}-\alpha)^+}{r}}^{\infty}\gamma(\bar{p}|z)f(z)dz}d\xi. \tag{20}$$

Moreover, because the retailer offers $\bar{p}$ with probability 1, $\gamma(\bar{p}|\xi) = 1$ and $\int_{z=\frac{(\bar{p}-\alpha)^+}{r}}^{\infty}\gamma(\bar{p}|z)f(z)dz = \int_{\frac{(\bar{p}-\alpha)^+}{r}}^{\infty}f(z)dz = 1 - F(\frac{(\bar{p}-\alpha)^+}{r})$. Thus, Equation (20) can be simplified to

$$\int_{(\bar{p}-\alpha)^+/r}^{\infty}[\min\{\ell, \alpha + (1-\beta)r\min\{\xi,q\}\}]\frac{f(\xi)}{1 - F(\frac{(\bar{p}-\alpha)^+}{r})}d\xi.$$

The supplier's expected profit then equals $\pi_{S,2}(v(\bar{p})) = (1 - v(\bar{p}))\bar{p} + v(\bar{p})\int_{(\bar{p}-\alpha)^+/r}^{\infty}[\min\{\ell, \alpha + (1-\beta)r\min\{\xi,q\}\}]\frac{f(\xi)}{1-F(\frac{(\bar{p}-\alpha)^+}{r})}d\xi$. Upon taking derivative with respect to $v(\bar{p})$, we obtain

$$\frac{\partial \pi_{S,2}(v(\bar{p}))}{\partial v(\bar{p})} = -\bar{p} + \int_{(\bar{p}-\alpha)^+/r}^{\infty}[\min\{\ell, \alpha + (1-\beta)r\min\{\xi,q\}\}]\frac{f(\xi)}{1 - F(\frac{(\bar{p}-\alpha)^+}{r})}d\xi = 0,$$

where the second equality follows from the definition of $\bar{p}$ (see Equation (11)). We assume that if the supplier is indifferent between accepting the payment offer and verifying the sales, she will accept the offer. Thus, the supplier accepts the offer $\bar{p}$. Upon observing $p = 0$, her posterior belief would be that $x < (\bar{p} - \alpha)^+/r$. In this case, $\mu(\xi|0) = \frac{\gamma(0|\xi)f(\xi)d\xi}{\int_{z=0}^{\frac{(\bar{p}-\alpha)^+}{r}} \gamma(0|z)f(z)dz} = \frac{f(\xi)d\xi}{F(\frac{(\bar{p}-\alpha)^+}{r})}$ because $\gamma(0|\xi) = 1$ and her expected profit from verifying the realized sales is

$$\int_0^{(\bar{p}-\alpha)^+/r} [\min\{\ell, \alpha + (1-\beta)r\min\{\xi, q\}\}] \frac{f(\xi)}{F(\frac{(\bar{p}-\alpha)^+}{r})} d\xi. \tag{21}$$

Her expected profit is $\pi_{S,2}(v(0)) = (1 - v(0)) \cdot 0 + v(0) \int_0^{(\bar{p}-\alpha)^+/r} [\min\{\ell, \alpha + (1-\beta)r\min\{\xi, q\}\}] \frac{f(\xi)}{F(\frac{(\bar{p}-\alpha)^+}{r})} d\xi$. The derivative of $\pi_{S,2}(v(0))$ with respect to $v(0)$ is

$$\frac{\partial \pi_{S,2}(v(0))}{\partial v(0)} = \int_0^{(\bar{p}-\alpha)^+/r} [\min\{\ell, \alpha + (1-\beta)r\min\{\xi, q\}\}] \frac{f(\xi)}{F(\frac{(\bar{p}-\alpha)^+}{r})} d\xi > 0,$$

which implies that the supplier's optimal strategy is $v(0) = 1$. Therefore, neither the retailer nor the supplier has an incentive to deviate from the equilibrium strategy of Proposition 1.

We conclude this proof by showing that $\bar{p}$ of Equation (11) is the unique threshold in a single-threshold PBE. We prove this claim by contradiction. Suppose that there exists a single-threshold PBE with threshold $\hat{p} \neq \bar{p}$. Consider first the case where $\hat{p} < \bar{p}$. If the retailer offers $\hat{p}$, then the supplier should accept it. However, at $\hat{p}$, $\frac{\partial \pi_{S,2}(v(\hat{p}))}{\partial v(\hat{p})} > 0$ because $\hat{p} < \bar{p}$. The supplier thus has an incentive to deviate and verify an offer of $\hat{p}$, which implies that $\hat{p}$ cannot be an equilibrium. Consider $\hat{p} > \bar{p}$. Then, the supplier should verify any offer $< \hat{p}$. However, $\frac{\partial \pi_{S,2}(v(p))}{\partial v(p)} < 0$ for $\bar{p} < p \leq \hat{p}$ and therefore the supplier should accept any offer above $\bar{p}$. Knowing this, the retailer will never offer more than $\bar{p}$ because his profit decreases in p. Thus, $\hat{p} > \bar{p}$ cannot be an equilibrium either. ☐

D. Existence of a n−threshold PBE

The following constraints need to be simultaneously satisfied for a $n-$threshold PBE to exist.

$$p_n = \int_{x_n}^{\infty} \min\{\ell, \alpha + r(1-\beta)\min\{\xi, q\}\} \frac{f(\xi)}{1 - F(x_n)} d\xi, \text{ and} \tag{22}$$

$$p_k = \int_{x_k}^{x_{k+1}} \min\{\ell, \alpha + r(1-\beta)\min\{\xi, q\}\} \frac{f(\xi)}{F(x_{k+1}) - F(x_k)} d\xi, \text{ and} \tag{23}$$

$$\begin{aligned} &\alpha + r\min\{x_{k+1}, q\} - p_{k+1} \geq v_k[(\alpha + r(1-\beta)\min\{x_{k+1}, q\} - \ell)^+] \\ &+(1 - v_k)(\alpha + r\min\{x_{k+1}, q\} - p_i), \text{ for each } \ k \in \{1, \ldots, n-1\}. \end{aligned} \tag{24}$$

Constraints (22) and (23) ensure that the supplier is indifferent between accepting the payment offer and verifying the realized sales, while constraints (24) ensure that the retailer offers p_{k+1} when demand is sufficiently high (i.e., $x \geq x_{k+1}$). Next, we prove that the supplier can never make strictly more than what she makes under the single-threshold equilibrium of Proposition 1, i.e., $E[\min\{\alpha + (1-\beta)r\min\{X, q\}, \ell\}]$.

Lemma 2. *The supplier's maximum expected profit in Stage 2 is equal to* $E[\min\{\alpha + (1-\beta)r\min\{X, q\}, \ell\}]$.

Proof: Per the Revelation Principle, for any verification strategy chosen by the supplier, there exists an equivalent direct mechanism in which the retailer reports the truth and the supplier, being the firm with less information, pays informational rent to learn the realized demand. In expectation, the informational rent cannot be less than $E[\min\{\alpha + r\min\{X, q\}, \ell\}] - E[\min\{\alpha + (1-\beta)r\min\{X, q\}, \ell\}]$, where the first term is the supplier's expected profit if she knew the demand realization and the second term is her expected profit if she had to pay to obtain that information. Any policy that results in the informational rent being exactly equal to the above difference is an optimal policy. Such a policy would result in the supplier's expected profit being equal to $E[\min\{\alpha + (1-\beta)r\min\{X, q\}, \ell\}]$. Hence proved. ☐

E. A comparison of the retailer's terminal wealth and the supplier's expected profit versus $\boldsymbol{\beta}$ with fixed Stage-1 decisions

We first show that the retailer's terminal wealth is higher under $\beta > 0$. For this to hold, it suffices to show that $\bar{p}$ decreases in β for fixed $(w, \bar{\ell})$ and q. We show in Online Appendix F that if $\bar{p} \le \alpha$, then $\bar{p} = \ell$, which is independent of β. This is intuitive because if the retailer has sufficient assets to pay $\bar{p}$, then there will be no court action and the value of β will be irrelevant. Henceforth, we shall deal with the case for which $\bar{p} \ge \alpha$. We re-write Equation (11) as $\bar{p}\bar{F}(\frac{\bar{p}-\alpha}{r}) = \int_{\frac{\bar{p}-\alpha}{r}}^{\infty} \min\{\ell, \alpha + (1-\beta)r\xi, \alpha + (1-\beta)rq\} f(\xi)d\xi$, and perform implicit differentiation with respect to β to get

$$\begin{aligned}
&\frac{\partial \bar{p}}{\partial \beta}\bar{F}\Big(\frac{\bar{p}-\alpha}{r}\Big) - \frac{\bar{p}}{r}\frac{\partial \bar{p}}{\partial \beta} f\Big(\frac{\bar{p}-\alpha}{r}\Big) = -\frac{1}{r}\frac{\partial \bar{p}}{\partial \beta} f\Big(\frac{\bar{p}-\alpha}{r}\Big)\min\{\ell, \alpha + (1-\beta)(\bar{p}-\alpha), \alpha + (1-\beta)rq\} \\
&\quad + \int_{\frac{\bar{p}-\alpha}{r}}^{\infty} \frac{\partial \min\{\ell, \alpha + (1-\beta)r\xi, \alpha + (1-\beta)rq\}}{\partial \beta} f(\xi)d\xi \\
&\Rightarrow \frac{\partial \bar{p}}{\partial \beta}\bar{F}\Big(\frac{\bar{p}-\alpha}{r}\Big)\Big[1 - \frac{1}{r}h\Big(\frac{\bar{p}-\alpha}{r}\Big)\Big(\bar{p} - \min\{\ell, \alpha + (1-\beta)(\bar{p}-\alpha), \alpha + (1-\beta)rq\}\Big)\Big] = \\
&\quad \int_{\frac{\bar{p}-\alpha}{r}}^{\infty} \frac{\partial \min\{\ell, \alpha + (1-\beta)r\xi, \alpha + (1-\beta)rq\}}{\partial \beta} f(\xi)d\xi \\
&\Rightarrow \frac{\partial \bar{p}}{\partial \beta}\bar{F}\Big(\frac{\bar{p}-\alpha}{r}\Big)\Big[1 - \beta\Big(\frac{\bar{p}-\alpha}{r}\Big)h\Big(\frac{\bar{p}-\alpha}{r}\Big)\Big] = \int_{\frac{\bar{p}-\alpha}{r}}^{\infty} \frac{\partial \min\{\ell, \alpha + (1-\beta)r\xi, \alpha + (1-\beta)rq\}}{\partial \beta} f(\xi)d\xi. \qquad (25)
\end{aligned}$$

Recall that $h(\cdot)$ denotes the hazard rate of the demand distribution and it equals $f(\cdot)/\bar{F}(\cdot)$. Additionally, we have used the fact that $\bar{p} \le \ell$ to simplify $[1 - \frac{1}{r}h(\frac{\bar{p}-\alpha}{r})(\bar{p} - \min\{\ell, \alpha + (1-\beta)(\bar{p}-\alpha), \alpha + (1-\beta)rq\})] = [1 - \beta(\frac{\bar{p}-\alpha}{r})h(\frac{\bar{p}-\alpha}{r})]$. Since $\min\{\ell, \alpha + (1-\beta)r\xi, \alpha + (1-\beta)rq\}$ is decreasing in β, it follows that $\frac{\partial \min\{\ell, \alpha + (1-\beta)r\xi, \alpha + (1-\beta)rq\}}{\partial \beta} \le 0$. Therefore, the right hand side of (25) is negative. Moreover, from the definition of $\bar{p}$, $1 - \beta(\frac{\bar{p}-\alpha}{r})h(\frac{\bar{p}-\alpha}{r}) \ge 0$ (see Equation 12). Thus, the multiplier of $\frac{\partial \bar{p}}{\partial \beta}$ on the left hand side of (25) is non-negative. This implies that $\frac{\partial \bar{p}}{\partial \beta} \le 0$.

Next, we show that $\pi_{R,1}(q)$ increases in β for a fixed q. The retailer makes 0 if $x \le \frac{\bar{p}-\alpha}{r}$ and $[\alpha + r\min\{x, q\} - \bar{p}]$, otherwise. Thus, $\pi_{R,1}(q) = \int_{\frac{\bar{p}-\alpha}{r}}^{\infty} (\alpha + r\min\{\xi, q\} - \bar{p})dF(\xi)$ and $\frac{\partial \pi_{R,1}(q)}{\partial \beta} = -\frac{\partial \bar{p}(q)}{\partial \beta}\bar{F}(\frac{\bar{p}(q)-\alpha}{r}) \ge 0$, because $\frac{\partial \bar{p}(q)}{\partial \beta} \le 0$. For fixed $(w, \bar{\ell})$ and q, $\pi_{R,1}(q)$ increases in β. Let q^0 denote the optimal order quantity under $\beta = 0$ and let q^β denote the optimal order quantity under $\beta > 0$ for given contract terms $(w, \bar{\ell})$. Then, $\pi_{R,1}(q^\beta | \beta > 0) \ge \pi_{R,1}(q^0 | \beta > 0) \ge \pi_{R,1}(q^0 | \beta = 0)$. That is, for a given set of contract terms, the retailer's expected terminal wealth is higher in the setting with $\beta > 0$ as compared to the setting with $\beta = 0$.

The supplier's expected profit for fixed q, w and $\bar{\ell}$ is $E[\min\{\ell, \alpha + (1-\beta)r\min\{X, q\}\}]$, as shown in Lemma 2. Thus, for given Stage-1 decisions, the supplier's expected profit decreases in β. The intuition is that supplier pays more in informational rents. □

F. Proof of Proposition 2

At this stage, the loan amount is $\ell = wq$ because the retailer either ordered his unconstrained amount, or $\bar{\ell}/w$. Using this fact, we re-write the equation for $\bar{p}$ (Equation 11) as

$$\bar{p} = E\Big[\min\{wq, \alpha + (1-\beta)rX, \alpha + (1-\beta)rq\} | X \ge \big(\frac{\bar{p}-\alpha}{r}\big)^+\Big]. \qquad (26)$$

We determine $\bar{p}$ in each range of ℓ (induced by q) and show that it is a continuous and increasing function of q. We omit the argument (q) from $\bar{p}$ to keep notation compact. Because ℓ is a continuous and increasing function of q, this immediately implies that $\bar{p}$ is also a continuous and increasing function of ℓ. We also prove in the sequel that Equation (26) has a unique solution in the interval $(0,\ell]$ if the inequality $1-\beta(\frac{\bar{p}-\alpha}{r})^+h((\frac{\bar{p}-\alpha}{r})^+)\geq 0$ holds. Lastly, we show that $\bar{p}$ has at most one kink at $\ell=\alpha+(1-\beta)rq$. We consider one range of ℓ at a time.

Range 1: $\ell\leq\alpha$. The RHS of Equation (26) simplifies to wq and (26) reduces to $\bar{p}=wq$. The first derivative of $\bar{p}$ with respect to q is $\frac{\partial\bar{p}}{\partial q}=w\geq 0$, and $\bar{p}$ is therefore increasing in q. At the boundary of this range, we have $\bar{p}=\alpha$, and $\frac{\partial\bar{p}}{\partial q}|_{\ell\to\alpha^-}=w$, where α^- denotes the left limit.

Range 2: $\alpha<\ell\leq\alpha+(1-\beta)rq$. In this range, $\bar{p}\geq\alpha$. Moreover, because $\ell\leq\alpha+(1-\beta)rq$, the third term in the min function of (26) is dominated by the first term and $\bar{p}$ is such that

$$\bar{p}=E\Big[\min\{wq,\alpha+(1-\beta)rX\}|X\geq\frac{\bar{p}-\alpha}{r}\Big]. \tag{27}$$

We re-write Equation (27) as $\frac{\bar{p}\bar{F}(\bar{p}/r-\alpha/r)-\int_{(\bar{p}-\alpha)/r}^{\infty}\min\{wq,\alpha+(1-\beta)r\xi\}dF(\xi)}{1-F(\bar{p}/r-\alpha/r)}=0$. Let $\Phi(p)$ be defined as $\Phi(p):=p\bar{F}(p/r-\alpha/r)-\int_{(p-\alpha)/r}^{\infty}\min\{wq,\alpha+(1-\beta)r\xi\}dF(\xi)$. Because at $p=wq$, $wq\bar{F}(wq/r-\alpha/r)\geq\int_{(wq-\alpha)/r}^{\infty}\min\{wq,\alpha+(1-\beta)r\xi\}dF(\xi)$, it follows that $\Phi(wq)\geq 0$. Similarly, at $p=\alpha\leq wq$, $\alpha\leq\int_0^{\infty}\min\{wq,\alpha+(1-\beta)r\xi\}dF(\xi)$. Therefore, $\Phi(\alpha)\leq 0$. This means that there exists at least one value of p for which $\Phi(p)=0$. Taking the derivative of $\Phi(p)$ with respect to p, we obtain $\frac{\partial\Phi(p)}{\partial p}=\bar{F}(\frac{p-\alpha}{r})(1-\beta\frac{(p-\alpha)}{r}h(\frac{p-\alpha}{r}))$. Therefore, $\Phi(p)$ increases in p if $(1-\beta\frac{(p-\alpha)}{r}h(\frac{p-\alpha}{r}))\geq 0$ and decreases in p otherwise. From Assumption 8, $h(\xi)$ is increasing in ξ for any arbitrary $\xi\geq 0$. Therefore, $(\frac{p-\alpha}{r})h(\frac{p-\alpha}{r})$ is increasing in p for $p\geq\alpha$. This implies that once $\Phi(p)$ starts decreasing, it continues to decrease. Note that at $p=\alpha$, $\Phi(p)$ is increasing, or else $\Phi(wq)$ would be negative, which contradicts our earlier finding that $\Phi(wq)\geq 0$. This implies that there are at most two values of $\bar{p}$ that solve Equation (27). The first occurs in the interval $[\alpha,wq]$, and the second, if it occurs, is strictly greater than wq. We are interested in the first because the retailer never offers to pay more than his debt. That is realized in the range in which $\Phi(p)$ is increasing. Thus, $\bar{p}$ of interest to us is the unique solution of Equation (27), such that $1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r})\geq 0$. Next, upon taking the derivative of $\bar{p}$ with respect to q, we find that $\bar{p}$ is increasing in q as shown below.

$$\frac{\partial\bar{p}}{\partial q}=\frac{w\bar{F}(\frac{wq-\alpha}{(1-\beta)r})}{\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}\geq 0. \tag{28}$$

Moreover, at $\ell=\alpha$, $\bar{p}=\alpha$ and the derivative simplifies to $\frac{\partial\bar{p}}{\partial q}|_{\ell\to\alpha^+}=w$, where α^+ denotes the right limit. At the boundary of Range 1, we had earlier concluded that $\frac{\partial\bar{p}}{\partial q}|_{\ell\to\alpha^-}=w$. Therefore, $\bar{p}$ is continuous, smooth, and increasing in q for $\ell\leq\alpha+(1-\beta)rq$. As $\ell\longrightarrow\alpha+(1-\beta)rq$ from the left, $\frac{\partial\bar{p}}{\partial q}|_{\ell\to(\alpha+(1-\beta)rq)^-}=\frac{w\bar{F}(\frac{wq-\alpha}{(1-\beta)r})}{\bar{F}(\bar{p}(\frac{wq-\alpha}{(1-\beta)r})/r-\alpha/r)[1-\beta(\bar{p}(\frac{wq-\alpha}{(1-\beta)r})-\alpha)h(\bar{p}(\frac{wq-\alpha}{(1-\beta)r})/r-\alpha/r)/r]}$. We shall use this observation to prove the fact that $\bar{p}$ has a kink in ℓ as we analyze the last range of ℓ.

Range 3: $\ell>\alpha+(1-\beta)rq$. This range exists only if $w>(1-\beta)r$. Because $\ell>\alpha+(1-\beta)rq$, the first term in the min function of (26) is dominated by the third term and $\bar{p}$ solves

$$\bar{p}=\alpha+(1-\beta)rE\big[\min\{X,q\}|X\geq\frac{\bar{p}-\alpha}{r}\big]. \tag{29}$$

Using arguments similar to those presented for Range 2, it can be proved that $\bar{p}$ always exists, that it is the unique solution to Equation (29) and that $(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))\geq 0$ ensures that $\bar{p}\leq wq$. We omit the details because the arguments are similar to those presented for Range 2. The derivative of $\bar{p}$ with respect to q is

$$\frac{\partial\bar{p}}{\partial q}=\frac{(1-\beta)r\bar{F}(q)}{\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}\geq 0. \tag{30}$$

At $\ell=\alpha+(1-\beta)rq$, $\bar{p}=E[\min\{wq,\alpha+(1-\beta)rX\}|X\geq\frac{\bar{p}-\alpha}{r}]$, which is the same as (27). Thus, $\bar{p}$ is continuous at $\ell=\alpha+(1-\beta)rq$. However, it may have a kink at the right limit of ℓ in this range, which we explore next. The derivative of $\bar{p}$ with respect to q as $\ell\longrightarrow[\alpha+(1-\beta)rq]$ from the right is $\frac{\partial\bar{p}}{\partial q}|_{\ell\to(\alpha+(1-\beta)rq)^+}=\frac{(1-\beta)r\bar{F}(\frac{wq-\alpha}{(1-\beta)r})}{\bar{F}(\bar{p}(\frac{wq-\alpha}{(1-\beta)r})/r-\alpha/r)[1-\beta(\bar{p}(\frac{wq-\alpha}{(1-\beta)r})-\alpha)h(\bar{p}(\frac{wq-\alpha}{(1-\beta)r})/r-\alpha/r)/r]}=\frac{(1-\beta)r}{w}\frac{\partial\bar{p}}{\partial q}|_{\ell\to(\alpha+(1-\beta)rq)^-}\leq\frac{\partial\bar{p}}{\partial q}|_{\ell\to(\alpha+(1-\beta)rq)^-}$, because $w>(1-\beta)r$. Thus, $\bar{p}$ has at most one kink at $\ell=\alpha+(1-\beta)rq$. The kink disappears if $w=(1-\beta)r$. □

G. Proof of Proposition 3

The retailer has a positive terminal wealth only if he can pay the settlement amount, which occurs when demand exceeds $\frac{(\bar{p}-\alpha)^+}{r}$. If the retailer is verified (which occurs with probability $F((\bar{p}-\alpha)^+/r)$), then he loses all of his assets and his terminal wealth is zero. Therefore, his expected terminal wealth in Stage 1 is

$$\pi_{R,1}(q)=\int_{\frac{(\bar{p}(q)-\alpha)^+}{r}}^{\infty}(\alpha+r\min\{\xi,q\}-\bar{p}(q))dF(\xi).$$

Recall that we are dealing with the case where $w\leq(1-\beta)r$. This implies that there are two ranges of ℓ to consider: $\ell\leq\alpha$ and $\ell>\alpha$. We analyze each of them one at a time. We do not impose any loan limit in this analysis and therefore $\ell=wq$. In this proof, we omit the argument (w) of q_1^* and q_2^*.

Range 1: $\ell\leq\alpha$. In this case, $\bar{p}=wq$ and $\pi_{R,1}(q)=\alpha+rE[\min\{X,q\}]-wq$. This function is concave in q because it is the sum of a constant, a concave and a linear function. It has a unique maximum, which is obtained by setting $\frac{\partial\pi_{R,1}(q)}{\partial q}=0$. The maximum is $q_1^*=\bar{F}^{-1}(\frac{w}{r})$.

Range 2: $\ell\geq\alpha$. In this case, $\bar{p}\geq\alpha$, where $\bar{p}$ solves Equation (27). We apply Lemma 1 to argue that $\pi_{R,1}(q)$ is unimodal in q. In particular, we show that whenever $\pi_{R,1}(q)$ increases in q, it does so at a decreasing rate.

$$\frac{\partial\pi_{R,1}(q)}{\partial q}=r\bar{F}(q)-\frac{w\bar{F}(\frac{wq-\alpha}{(1-\beta)r})}{(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}\geq 0\Rightarrow -r\leq-\frac{w\bar{F}\left(\frac{wq-\alpha}{(1-\beta)r}\right)}{\bar{F}(q)(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}. \tag{31}$$

$$\begin{aligned}\frac{\partial^2\pi_{R,1}(q)}{\partial q^2}&=-rf(q)+\frac{\frac{w^2}{(1-\beta)r}f\left(\frac{wq-\alpha}{(1-\beta)r}\right)}{(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}-\frac{w\bar{F}\left(\frac{wq-\alpha}{(1-\beta)r}\right)\frac{\beta}{r}\frac{\partial\bar{p}}{\partial q}[h(\frac{\bar{p}-\alpha}{r})+\frac{(\bar{p}-\alpha)}{r}h'(\frac{\bar{p}-\alpha}{r})]}{(1-\beta\frac{(p-\alpha)}{r}h(\frac{p-\alpha}{r}))^2}\\&\leq-\frac{w\bar{F}\left(\frac{wq-\alpha}{(1-\beta)r}\right)\left[h(q)-\frac{w}{(1-\beta)r}h\left(\frac{wq-\alpha}{(1-\beta)r}\right)\right]}{(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}-\frac{w\bar{F}\left(\frac{wq-\alpha}{(1-\beta)r}\right)\frac{\beta}{r}\frac{\partial\bar{p}}{\partial q}[h(\frac{\bar{p}-\alpha}{r})+\frac{(\bar{p}-\alpha)}{r}h'(\frac{\bar{p}-\alpha}{r})]}{(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))^2}\leq 0.\end{aligned} \tag{32}$$

We use $h'(\xi)$ to denote the derivative of $h(\xi)$ with respect to ξ. The first inequality in (32) follows from the inequality in (31), whereas the second inequality follows from the fact that $h(\cdot)$ is increasing (Assumption 8) and that $\frac{\partial\bar{p}}{\partial q}\geq 0$. We utilize the aforementioned property of $h(\cdot)$ and the fact that $w\leq(1-\beta)r$ to argue that $h(q)\geq\frac{w}{(1-\beta)r}h(\frac{wq-\alpha}{(1-\beta)r})$. Together, these arguments serve to prove that Lemma 1 applies and therefore any point at which $\frac{\partial \pi_{R,1}(q)}{\partial q} = 0$ is a global maximizer of $\pi_{R,1}(q)$. The optimal order quantity in this range is q_2^* such that $r\bar{F}(q_2^*)(1-\beta\frac{(\bar{p}(q_2^*)-\alpha)}{r}h(\frac{\bar{p}(q_2^*)-\alpha}{r})) = w\bar{F}(\frac{wq_2^*-\alpha}{(1-\beta)r})$, where $\bar{p}(q_2^*)$ solves Equation (27) where $q = q_2^*$.

We have so far determined how $\pi_{R,1}(q)$ changes in q in each range of the loan when $w \le (1-\beta)r$. In the proof of Proposition 2, we proved that $\bar{p}$ is a continuous and smooth function of q at the boundary of the two ranges mentioned above. Therefore, $\pi_{R,1}(q)$ is also a continuous and smooth function at the boundary of these two ranges, i.e., at $\bar{p} = \ell = \alpha$. If $w \le (1-\beta)r$, then $\pi_{R,1}(q)$ is unimodal. The overall optimal order quantity $q^*(w|w \le (1-\beta)r)$ can then be found by evaluating the derivative of $\pi_{R,1}(q)$ at this boundary. If $\frac{\partial \pi_{R,1}(q)}{\partial q} \le 0$ at $\ell = \alpha$, then the optimal solution is in the first range. If $\frac{\partial \pi_{R,1}(q)}{\partial q} \ge 0$ at $\ell = \alpha$, then the optimal solution is in the second range. Next, we show that there exists a unique threshold of w that allows us to re-write the condition $\frac{\partial \pi_{R,1}(q)}{\partial q} = 0$ at $\ell = \alpha$. Let $\bar{w}_{21}$ be the unique solution to $\bar{w}_{21}q_1^*(\bar{w}_{21}) = \alpha$. That is, $\bar{w}_{21}$ is such that $\bar{w}_{21} = r\bar{F}(\alpha/\bar{w}_{21})$. If $w \le \bar{w}_{21}$, then $wq_1^* \ge \alpha$ and the optimal solution is q_2^*. If $w \ge \bar{w}_{21}$, the optimal solution is q_1^*. Under the assumption that $\alpha \le cq^{FB}$ (Assumption 3), $\bar{w}_{21}$ is unique in the range $[c, r]$ because of the following facts: (i) $r\bar{F}(\alpha/w)$ is increasing and concave in w, (ii) at $w = r$, $r\bar{F}(\frac{\alpha}{r}) < r$, and (iii) at $w = c$, $r\bar{F}(\frac{\alpha}{c}) > c$. Using the threshold $\bar{w}_{21}$, we can write the condition for $q^*(w|w \le (1-\beta)r)$ as

$$q^*(w|w \le (1-\beta)r) = \begin{cases} q_2^*, \text{ if } w \le \bar{w}_{21}, \\ q_1^*, \text{ if } w \ge \bar{w}_{21}. \end{cases} \tag{33}$$

We emphasize that for each α, the threshold $\bar{w}_{21}$ is obtained as a solution to the expression $\bar{w}_{21} = r\bar{F}(\alpha/\bar{w}_{21})$, which does not lead to an explicit expression for $\bar{w}_{21}$. Hence proved. □

H. Proof of Proposition 4

If $w > (1-\beta)r$, the retailer may find it optimal to order $q > q^{FB}$, where q^{FB} is the first-best order quantity. If that happened, the supplier would realize lower profits. The intuition behind this argument is straightforward. The total supply chain profit decreases in q for $q > q^{FB}$. If the retailer's expected terminal wealth is greater when $q > q^{FB}$ than when $q = q^{FB}$, then the supplier must be making lower profits at $q > q^{FB}$ than at $q = q^{FB}$. This implies that given an arbitrary set of parameters, if there exist contract terms that induce the retailer to order exactly q^{FB} and others that induce the retailer to order $> q^{FB}$, the supplier will always realize a higher expected profit under the former set of terms. We state the above observation formally in Lemma 3.

Lemma 3. *Given two sets of contract terms such that the first set induces the retailer to order the first-best order quantity and the second set induces the retailer to order strictly more than the first-best order quantity, the supplier always realizes higher profits under the first set of contract terms.*

Proof: We show that the supplier's profit is always lower at $q > q^{FB}$ than at q^{FB}. Recall that $q^{FB} = \bar{F}^{-1}(c/r)$ is the first-best order quantity. That is, q^{FB} maximizes $\pi_{R+S}(q)$ when there are no frictions, i.e., when $\beta = 0$. In the presence of positive verification costs, the supply chain profit (defined as the sum of the retailer's terminal wealth and the supplier's profit) for any q is given by $\pi_{R+S}(q) = \alpha + rE[\min\{X, q\}] - cq - \beta\int_0^{(\bar{p}(q)-\alpha)^+/r} r\xi dF(\xi)$. The first two terms denote the assets and the revenue from expected sales, the third term is the cost of producing q units and the fourth term denotes the expected loss due to verification. When $\beta = 0$, $\pi_{R+S}(q)$ decreases in q for $q > q^{FB}$ which follows from the optimality of q^{FB}. When $\beta > 0$, $\frac{\partial \pi_{R+S}(q)}{\partial q} = r\bar{F}(q) - c - \frac{\beta}{r}(\bar{p}(q)-\alpha)^+\frac{\partial \bar{p}(q)}{\partial q}f((\bar{p}(q)-\alpha)^+/r)$. Because $\frac{\partial \bar{p}(q)}{\partial q} \ge 0$, $\forall q$, it holds that for $q > q^{FB}$,

$\frac{\partial \pi_{R+S}(q)}{\partial q} < 0$. This implies that $\pi_{R+S}(q) < \pi_{R+S}(q^{FB})$ for $q > q^{FB}$. Suppose that the retailer finds it optimal to choose $q > q^{FB}$. Then this implies that $\pi_{R,1}(q) > \pi_{R,1}(q^{FB})$. The expected profit of the supplier is then $\pi_{S,1}(q) = \pi_{R+S}(q) - \pi_{R,1}(q) < \pi_{R+S}(q^{FB}) - \pi_{R,1}(q^{FB}) = \pi_{S,1}(q^{FB})$. Thus, the supplier's expected profit will always be higher at q^{FB} than at any $q > q^{FB}$. $\square$

To construct the proof of Proposition 4, we show that if the retailer's order quantity is such that $\ell \leq \alpha + (1-\beta)rq$, then the structure of his optimal order quantity is the same as that when $w \leq (1-\beta)r$. We then show that the retailer's expected terminal wealth is unimodal if he chooses an order quantity such that $\ell > \alpha + (1-\beta)rq$ and derive his optimal order quantity. We consider each range of ℓ separately. Moreover, as in the proof of Proposition 3, $\ell = wq$.

Range 1: $\ell \leq \alpha$. As before, the optimal order quantity is q_1^* because $\pi_{R,1}(q)$ is concave in q.

Range 2: $\alpha < \ell \leq \alpha + (1-\beta)rq$. The second derivative of the retailer's expected terminal wealth is

$$\begin{aligned}\frac{\partial^2 \pi_{R,1}(q)}{\partial q^2} &= -rf(q) + \frac{\frac{w^2}{(1-\beta)r} f\left(\frac{wq-\alpha}{(1-\beta)r}\right)}{(1-\beta\frac{(p-\alpha)}{r}h(\frac{p-\alpha}{r}))} - \frac{w\bar{F}\left(\frac{wq-\alpha}{(1-\beta)r}\right)\frac{\beta}{r}\frac{\partial \bar{p}}{\partial q}[h(\frac{\bar{p}-\alpha}{r}) + \frac{\bar{p}-\alpha}{r}h'(\frac{\bar{p}-\alpha}{r})]}{(1-\beta\frac{(p-\alpha)}{r}h(\frac{p-\alpha}{r}))^2} \\ &\leq -\frac{w\bar{F}\left(\frac{wq-\alpha}{(1-\beta)r}\right)\left[h(q) - \frac{w}{(1-\beta)r}h\left(\frac{wq-\alpha}{(1-\beta)r}\right)\right]}{(1-\beta\frac{(p-\alpha)}{r}h(\frac{p-\alpha}{r}))} - \frac{w\bar{F}\left(\frac{wq-\alpha}{(1-\beta)r}\right)\frac{\beta}{r}\frac{\partial \bar{p}}{\partial q}[h(\frac{\bar{p}-\alpha}{r}) + \frac{\bar{p}-\alpha}{r}h'(\frac{\bar{p}-\alpha}{r})]}{(1-\beta\frac{(p-\alpha)}{r}h(\frac{p-\alpha}{r}))^2}. \qquad (34)\end{aligned}$$

However, because $w > (1-\beta)r$, we cannot conclude that (34) is non-positive. Without unimodality, we can only claim that the retailer will choose an order quantity that lies either at one of the two boundaries of this range, or an interior solution. The interior solution, if it exists, has the same functional form as q_2^* in Proposition 3, i.e., q_2^* solves $r\bar{F}(q_2^*)(1-\beta\frac{(\bar{p}(q_2^*)-\alpha)}{r}h(\frac{\bar{p}(q_2^*)/r-\alpha}{r})) = w\bar{F}(\frac{wq_2^*-\alpha}{(1-\beta)r})$ and $\frac{\partial^2 \pi_{R,1}(q)}{\partial q^2} \leq 0$. The boundary solution is either q such that $\ell = \alpha$ (i.e., $q = \alpha/w$) or q such that $\ell = \alpha + (1-\beta)rq$ (i.e., $q = \frac{\alpha}{w-(1-\beta)r}$). Note that $\alpha/w \leq q_2^* \leq \frac{\alpha}{w-(1-\beta)r}$. Among the three possibilities, we can rule out the possibility that the optimal q equals $\frac{\alpha}{w-(1-\beta)r}$ using the following arguments. If the derivative of $\pi_{R,1}(q)$ at $q = \frac{\alpha}{w-(1-\beta)r}$ is negative, then the optimal order quantity must be $< \frac{\alpha}{w-(1-\beta)r}$. If the derivative is positive, then the optimal order quantity must be $> \frac{\alpha}{w-(1-\beta)r}$ because $\frac{\partial \pi_{R,1}(q)}{\partial q}|_{q \to (\frac{\alpha}{w-(1-\beta)r})^-} \leq \frac{\partial \pi_{R,1}(q)}{\partial q}|_{q \to (\frac{\alpha}{w-(1-\beta)r})^+}$. That is, the retailer's expected terminal wealth is increasing at $q = (\frac{\alpha}{w-(1-\beta)r})$. Thus, it is sufficient to compare $q = \alpha/w$ and q_2^*.

We next argue that the order quantity that maximizes the retailer's terminal wealth depends on w. For each candidate q, we calculate the retailer's expected terminal wealth as follows:

$$\begin{aligned} &q = \alpha/w \Rightarrow \pi_{R,1}(\alpha/w) = \alpha + rE[\min\{X, \alpha/w\}] - \alpha = rE[\min\{X, \alpha/w\}], \\ &q = q_2^* \Rightarrow \pi_{R,1}(q_2^*) = \alpha\bar{F}\left(\frac{\bar{p}(q_2^*)-\alpha}{r}\right) + r\int_{\frac{\bar{p}-\alpha}{r}}^{q_2^*} \xi dF(\xi) + rq_2^*\bar{F}(q_2^*) - \bar{p}(q_2^*)\bar{F}\left(\frac{\bar{p}(q_2^*)-\alpha}{r}\right). \end{aligned}$$

Taking the derivative of $\pi_{R,1}(\cdot)$ with respect to w we get $\frac{\partial \pi_{R,1}(\alpha/w)}{\partial w} = -\frac{r}{w}(\frac{\alpha}{w})\bar{F}(\frac{\alpha}{w})$, and $\frac{\partial \pi_{R,1}(q_2^*)}{\partial w} = -\frac{\partial \bar{p}}{\partial w}\bar{F}(\frac{\bar{p}-\alpha}{r}) = -\frac{q_2^*\bar{F}(\frac{wq_2^*-\alpha}{r(1-\beta)})}{1-\beta\frac{(\bar{p}(q_2^*)-\alpha)}{r}h(\frac{\bar{p}(q_2^*)-\alpha}{r})} = -\frac{r}{w}q_2^*\bar{F}(q_2^*)$, where the last equality follows from the definition of q_2^*. In Range 2, the supplier can prevent the retailer from choosing $q > q^{FB}$ via her choice of w. Following Lemma 3, she will therefore choose w such that $q_2^* \leq q^{FB}$. The fact that $q \leq q^{FB}$ and that $1 - q^{FB}h(q^{FB}) \geq 0$ (Assumption 9) implies that $-q\bar{F}(q)$ decreases in q. This allows us to order the derivatives of $\pi_{R,1}(q)$ such that $\frac{\partial \pi_{R,1}(q_2^*)}{\partial w} \leq \frac{\partial \pi_{R,1}(\alpha/w)}{\partial w}$. The ordering and monotonicity of $\frac{\partial \pi_{R,1}(\cdot)}{\partial w}$ imply that there exists a unique value of w at which the retailer will switch from q_2^* to α/w. The threshold of w is such that $\pi_{R,1}(q_2^*) = \pi_{R,1}(\alpha/w)$. Because $\pi_{R,1}(q)$ is continuous and smooth in q, this threshold must be such that at this threshold $q_2^* = \alpha/w$. The only value of w at which this equality holds is $\bar{w}_{21}$. Recall that $\bar{w}_{21}$ is such that $q_1^*(\bar{w}_{21}) = q_2^*(\bar{w}_{21}) = \alpha/\bar{w}_{21}$. Therefore, in Range 2, the retailer will choose q_2^* if $w \leq \bar{w}_{21}$, and α/w if $w \geq \bar{w}_{21}$. Combining both Range 1 and Range 2, the retailer's optimal order quantity is q_2^* if $w \leq \bar{w}_{21}$ and q_1^* if $w \geq \bar{w}_{21}$.

We have shown that the structure of the optimal order quantity is the same so long as $\ell \leq \alpha + (1-\beta)rq$, regardless of whether $w \leq (1-\beta)r$ or $w > (1-\beta)r$. This structure is shown in Equation (33).

Range 3: $\ell > \alpha + (1-\beta)rq$. The retailer's expected terminal wealth in this range is similar to that in Range 2, with the difference that $\bar{p}$ solves Equation (29). Utilizing the arguments below, and by applying Lemma 1, we conclude that $\pi_{R,1}(q)$ is unimodal in q.

$$\begin{aligned}
\frac{\partial \pi_{R,1}(q)}{\partial q} &= r\bar{F}(q) - \frac{r(1-\beta)\bar{F}(q)}{1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r})} \geq 0 \Rightarrow 1 - \frac{1-\beta}{1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r})} \geq 0,\\
\frac{\partial^2 \pi_{R,1}(q)}{\partial q^2} &= -rf(q) + \frac{r(1-\beta)f(q)}{1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r})} - \frac{(1-\beta)\bar{F}(q)\beta\frac{\partial\bar{p}}{\partial q}[h(\frac{\bar{p}-\alpha}{r}) + \frac{(\bar{p}-\alpha)}{r}h'(\frac{\bar{p}-\alpha}{r})]}{(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))^2}\\
&= -rf(q)\Big[1 - \frac{1-\beta}{1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r})}\Big] - \frac{(1-\beta)\bar{F}(q)\beta\frac{\partial\bar{p}}{\partial q}[h(\frac{\bar{p}-\alpha}{r}) + \frac{(\bar{p}-\alpha)}{r}h'(\frac{\bar{p}-\alpha}{r})]}{(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))^2} \leq 0.
\end{aligned}$$

The optimal order quantity is q_3^* such that $(\frac{\bar{p}(q_3^*)-\alpha}{r})h(\frac{\bar{p}(q_3^*)-\alpha}{r}) = 1$, where $\bar{p}(q_3^*)$ solves Equation (29) when $q = q_3^*$. Note that q_3^* is independent of w because $\bar{p}(q)$ is independent of w when $\ell > \alpha + (1-\beta)rq$. We next show that $q_3^* > q^{FB}$, the first-best order quantity. This will be true only if the derivative of $\pi_{R,1}(q)$ is increasing in q at $q = q^{FB}$. That is, if $1 - \frac{\bar{p}(q^{FB})-\alpha}{r}h(\frac{\bar{p}(q^{FB})-\alpha}{r}) \geq 0$. For every q, we know that $\bar{p}(q) \leq wq \leq rq$. Thus, $\bar{p}(q^{FB}) \leq rq^{FB}$, which implies that $\frac{\bar{p}(q^{FB})-\alpha}{r} \leq q^{FB}$. Because $qh(q)$ is increasing in q, it must be the case that $q^{FB}h(q^{FB}) \geq \frac{\bar{p}(q^{FB})-\alpha}{r}h(\frac{\bar{p}(q^{FB})-\alpha}{r})$, and therefore $1 - q^{FB}h(q^{FB}) \leq 1 - \frac{\bar{p}(q^{FB})-\alpha}{r}h(\frac{\bar{p}(q^{FB})-\alpha}{r})$. Following the assumption that the optimal supply is unitary elastic (Assumption 9), $1 - q^{FB}h(q^{FB}) \geq 0$, this implies that $1 - \frac{\bar{p}(q^{FB})-\alpha}{r}h(\frac{\bar{p}(q^{FB})-\alpha}{r}) \geq 0$. These arguments complete the proof of Proposition 4. ☐

I. Proof of Theorem 1

The retailer's optimal order quantity is one of either $q_1^*(w)$, $q_2^*(w)$ or q_3^*. Recall that q_3^* is independent of w, whereas $q_1^*(w)$ and $q_2^*(w)$ depend on w. We first argue that $\pi_{R,1}(q_1^*(w)) \geq \alpha$ and $\pi_{R,1}(q_2^*(w)) \geq \alpha$. We then determine α such that $\pi_{R,1}(q_3^*) \geq \alpha$. This allows us to determine a range of α over which q_3^* is feasible. We also identify wholesale price thresholds in each range of α corresponding to which the global optimal order quantity can be identified.

If the retailer chooses $q_1^*(w)$, his expected terminal wealth is $\pi_{R,1}(q_1^*(w)) = \alpha + rE[\min\{X, q_1^*(w)\}] - wq_1^*(w) = \alpha + r\int_0^{q_1^*(w)} \xi dF(\xi) \geq \alpha$, where the second equality follows from the definition of $q_1^*(w)$. Recall that $q_1^*(w)$ and $q_2^*(w)$ belong to the left peak of the retailer's expected terminal wealth, whereas q_3^* belongs to the right peak (see the right panel of Figure 3 for an illustration of the two peaks). This means that if the retailer chooses $q_2^*(w)$, his expected terminal wealth has to be at least as high as that when he chooses $q_1^*(w)$, implying that $\pi_{R,1}(q_2^*(w)) \geq \pi_{R,1}(q_1^*(w)) \geq \alpha$.

Next, we show that q_3^* is a candidate optimal order quantity only when α is less than a threshold α^H. At q_3^*, $\pi_{R,1}(q_3^*) = \beta r\int_{\frac{\bar{p}(q_3^*)-\alpha}{r}}^{q_3^*} \xi dF(\xi) + \beta rq_3^*\bar{F}(q_3^*)$, where q_3^* is such that $\frac{\bar{p}(q_3^*)-\alpha}{r}h(\frac{\bar{p}(q_3^*)-\alpha}{r}) = 1$, and $\bar{p}(q)$ solves

$\frac{\bar{p}(q)-\alpha}{r}\bar{F}(\frac{\bar{p}(q)-\alpha}{r}) = (1-\beta)\int_{\frac{\bar{p}(q)-\alpha}{r}}^{q} \xi dF(\xi) + (1-\beta)q\bar{F}(q)$. We show that $\pi_{R,1}(q_3^*)$ is a constant with respect to α by using a change of variable. Let $u \doteq \frac{\bar{p}(q)-\alpha}{r}$. Then, for any q, u solves

$$u\bar{F}(u) = (1-\beta)\int_{u}^{q} \xi dF(\xi) + (1-\beta)rq\bar{F}(q), \tag{35}$$

q_3^* is such that $u(q_3^*)h(u(q_3^*)) = 1$ and $\pi_{R,1}(q_3^*) = \beta r \int_{u(q_3^*)}^{q_3^*} \xi dF(\xi) + \beta r q_3^* \bar{F}(q_3^*)$. Upon taking implicit differentiation of Equation (35) with respect to α, we find that $\frac{\partial u}{\partial \alpha} = 0$. That is, u is a constant with respect to α. This implies that q_3^* and $\pi_{R,1}(q_3^*)$ are also constants with respect to α. There exists a unique value of α at which α and $\pi_{R,1}(q_3^*)$ intersect, because the latter is a constant and the former is α itself. Let α^H denote such value. That is,

$$\alpha^H = \pi_{R,1}(q_3^*) = \beta r \int_{u(q_3^*)}^{q_3^*} \xi dF(\xi) + \beta r q_3^* \bar{F}(q_3^*). \tag{36}$$

Then, if $\alpha \le \alpha^H$, $\pi_{R,1}(q_3^*) \ge \alpha$ and q_3^* is a feasible choice for the retailer. If $\alpha > \alpha^H$, then $\pi_{R,1}(q_3^*) < \alpha$ and the retailer will never find it optimal to order q_3^*. In such case, the optimal order quantity follows the structure shown in Proposition 3. That is, the retailer orders $q_2^*(w)$ if $w \le \bar{w}_{21}$ and $q_1^*(w)$, otherwise.

We next consider the case where $\alpha \le \alpha^H$. The retailer can choose between $q_1^*(w)$, $q_2^*(w)$ and q_3^*. At $w = r$, $q_2^*(r) = q_1^*(r) = 0$ and the retailer's expected terminal wealth is α. If the retailer chooses q_3^*, he makes more than α because $\pi_{R,1}(q_3^*) \ge \alpha$ for $\alpha \le \alpha^H$. This implies that at $w = r$, the retailer will order q_3^*. Moreover, the proof of Proposition 3 showed that the retailer's expected terminal wealth decreases in w at the optimal order quantity. That is, $\pi_{R,1}(q_2^*(w))$ and $\pi_{R,1}(q_1^*(w))$ decrease in w, whereas $\pi_{R,1}(q_3^*)$ is independent of w. This implies that there exists a value of wholesale price above which the retailer's expected terminal wealth is higher at q_3^* and below which it is higher at $q < q_3^*$. The jump to q_3^* may occur at $q_2^*(w)$ or at $q_1^*(w)$. Which one of these two serves as a jumping point depends on the magnitude of α. Let $\bar{w}_{23}$ (respectively, $\bar{w}_{13}$) denote the threshold of wholesale price at which the jump to q_3^* occurs at $q_2^*(w)$ (respectively, $q_1^*(w)$). Because of the monotonicity of $\pi_{R,1}(q_1^*(w))$ and $\pi_{R,1}(q_2^*(w))$ and the independence of $\pi_{R,1}(q_3^*)$ with respect to w, $\bar{w}_{23}$ and $\bar{w}_{13}$ are unique. Note that $\bar{w}_{23}$ and $\bar{w}_{13}$ are functions of α because the retailer's terminal wealth $\pi_{R,1}(\cdot)$ depends on α. As α increases, $\bar{w}_{23}$ increases (proved in the sequel), while $\bar{w}_{21}$ decreases (proved in the sequel). Thus, there exists a unique value of α, denoted by α^L, at which $\bar{w}_{23} = \bar{w}_{21}$. That is, α^L is such that

$$\bar{w}_{23}(\alpha^L) = \bar{w}_{21}(\alpha^L),$$

$$\text{where } \bar{w}_{23}(\alpha) = w(\alpha):\ \pi_{R,1}(q_2^*(w(\alpha))) = \pi_{R,1}(q_3^*) \text{ and } \bar{w}_{21}(\alpha) = w(\alpha):\ q_2^*(w(\alpha)) = q_1^*(w(\alpha)). \tag{37}$$

Note that at α^L, $\bar{w}_{23} = \bar{w}_{13}$ also holds because at $\bar{w}_{21}$, $q_1^*(\bar{w}_{21}) = q_2^*(\bar{w}_{21})$. If $\alpha \le \alpha^L$, then the jump to q_3^* occurs at $q_2^*(w)$. In this case, the retailer orders $q_2^*(w)$ if $w \le \bar{w}_{23}$, and q_3^*, otherwise. If $\alpha^L \le \alpha \le \alpha^H$, then the jump to q_3^* occurs at $q_1^*(w)$, in which case the retailer orders $q_2^*(w)$ if $w \le \bar{w}_{21}$, $q_1^*(w)$ if $\bar{w}_{21} \le w \le \bar{w}_{13}$, and q_3^*, otherwise.

We conclude this proof by showing that $\bar{w}_{23}$ increases in α and that $\bar{w}_{21}$ decreases in α.

$$\begin{aligned}
\pi_{R,1}(q_2^*(\bar{w}_{23})) &= \pi_{R,1}(q_3^*) \\
\Rightarrow \frac{d\pi_{R,1}(q_2^*(\bar{w}_{23}))}{d\alpha} &= \frac{d\pi_{R,1}(q_3^*)}{d\alpha} \\
\Rightarrow \frac{\partial \pi_{R,1}(q_2^*(\bar{w}_{23}))}{\partial \alpha} + \frac{\partial \pi_{R,1}(q_2^*(\bar{w}_{23}))}{\partial w}\frac{\partial \bar{w}_{23}}{\partial \alpha} &= 0.
\end{aligned}$$

Because $\frac{\partial \pi_{R,1}(q_2^*(\bar{w}_{23}))}{\partial w} \le 0$, it suffices to show that $\frac{\partial \pi_{R,1}(q_2^*(\bar{w}_{23}))}{\partial \alpha} \ge 0$. This holds because $\frac{\partial \pi_{R,1}(q_2^*(\bar{w}_{23}))}{\partial \alpha} = (1 - \frac{\partial \bar{p}(q_2^*)}{\partial \alpha})\bar{F}(\frac{\bar{p}(q_2^*)-\alpha}{r}) \ge 0 \Leftrightarrow \frac{\partial \bar{p}(q_2^*)}{\partial \alpha} \le 1$, which is always true.

Recall that $\bar{w}_{21}$ is such that $\bar{w}_{21} = r\bar{F}(\frac{\alpha}{\bar{w}_{21}})$ (see Proposition 3). Upon taking implicit differentiation with respect to α, we get

$$\frac{\partial \bar{w}_{21}}{\partial \alpha} = -\frac{r}{\bar{w}_{21}} f\Big(\frac{\alpha}{\bar{w}_{21}}\Big) + \frac{r\alpha}{\bar{w}_{21}^2} f\Big(\frac{\alpha}{\bar{w}_{21}}\Big)\frac{\partial \bar{w}_{21}}{\partial \alpha}$$

$$\Leftrightarrow \frac{\partial \bar{w}_{21}}{\partial \alpha}\Big[1 - \frac{r\alpha}{\bar{w}_{21}^2} f\Big(\frac{\alpha}{\bar{w}_{21}}\Big)\Big] = -\frac{r}{\bar{w}_{21}} f\Big(\frac{\alpha}{\bar{w}_{21}}\Big) \tag{38}$$

$$\Leftrightarrow \frac{\partial \bar{w}_{21}}{\partial \alpha}\Big[1 - \frac{\alpha}{\bar{w}_{21}} h\Big(\frac{\alpha}{\bar{w}_{21}}\Big)\Big] = -\frac{r}{\bar{w}_{21}} f\Big(\frac{\alpha}{\bar{w}_{21}}\Big), \tag{39}$$

where the step from (38) to (39) follows from the definition of $\bar{w}_{21}$. The facts that $\alpha \le cq^{FB} \le \bar{w}_{21}cq^{FB}$ (Assumption 3) and that $1 - q^{FB}h(q^{FB}) \ge 0$ (Assumption 9) imply that $1 - \frac{\alpha}{\bar{w}_{21}}h(\frac{\alpha}{\bar{w}_{21}}) \ge 0$. Therefore, $\frac{\partial \bar{w}_{21}}{\partial \alpha} \le 0$. □

J. Proof of Proposition 5

For each combination of α and w as given in Table 2 of Theorem 1, we first fix w and determine whether the supplier would want to impose a loan limit to prevent the retailer from ordering his unconstrained optimal quantity. Then, we determine either potential candidate values of optimal w, or a range of values from which the supplier will pick the optimal w. We present the proof for each combination of α and w separately. For the purposes of proving Proposition 5, we introduce notation $\bar{q}$ and $\bar{q}^*$ in this section only. To keep notation compact, in some places we omit the argument (w) of q_1^* and q_2^*.

Part 1: $\boldsymbol{\alpha \le \alpha^L}$

We will argue that the supplier finds it optimal to impose a loan limit such that the retailer orders less than his unconstrained optimal order quantity. In addition, the optimal wholesale price is $\ge \bar{w}_{23}$.

J.1. Part 1a: $\boldsymbol{w \le \bar{w}_{23}}$

We show that the supplier's optimal loan limit increases in w. The retailer's unconstrained optimal order quantity is $q_2^*(w)$. The supplier's expected profit is $\pi_{S,1}(w) = E[\min\{wq_2^*(w), \alpha + (1-\beta)rX\}] - cq_2^*(w)$. The supplier's decision to impose a loan limit is equivalent to her decision of imposing a limit on the retailer's order quantity. Let $\bar{q}$ denote the supplier's order quantity limit. Then, the supplier's expected profit is $\pi_{S,1}(\bar{q}) = E[\min\{w\bar{q}, \alpha + (1-\beta)rX\}] - c\bar{q}$. It is straightforward to see that $\pi_{S,1}(\bar{q})$ is concave in $\bar{q}$. Thus, the supplier's optimal order quantity limit $\bar{q}^*$ solves $w\bar{F}(\frac{w\bar{q}^* - \alpha}{(1-\beta)r}) = c$. If $\bar{q}^* \ge q_2^*$, then the retailer orders q_2^*. Else, the retailer will order $\bar{q}^*$. From the supplier's perspective, her profit is higher if the retailer orders $\bar{q}^*$ as compared to q_2^*. Therefore, if the supplier does not find it optimal to choose $w < \bar{w}_{23}$ when the retailer orders $\bar{q}^*$, then she will also not find it optimal to do so when the retailer orders q_2^*. We argue that the supplier does not find it optimal to choose $w < \bar{w}_{23}$ by showing that her expected profit is increasing in w if the retailer orders $\bar{q}^*$ next. The suppliers' expected profit is $\pi_{S,1}(w) = E[\min\{w\bar{q}^*(w), \alpha + (1-\beta)rX\}] - c\bar{q}^*(w)$. Upon taking the derivative with respect to w, we obtain

$$\frac{\partial \pi_{S,1}(w)}{\partial w} = \Big[\bar{q}^* + w\frac{\partial \bar{q}^*(w)}{\partial w}\Big]\bar{F}\Big(\frac{w\bar{q}^* - \alpha}{(1-\beta)r}\Big) - c\frac{\partial \bar{q}^*(w)}{\partial w} = \bar{q}\bar{F}\Big(\frac{w\bar{q}^* - \alpha}{(1-\beta)r}\Big) \ge 0, \tag{40}$$

where the second equality follows from the definition of $\bar{q}^*$. Because the supplier's expected profit increases in w, she will choose $w = \bar{w}_{23}$ in this range.

J.2. Part 1b: $\boldsymbol{w \geq \bar{w}_{23}}$

A consequence of the fact that the supplier does not find it optimal to choose a wholesale price in the range $w < \bar{w}_{23}$ is that she will choose an optimal wholesale price in this range. That is, $w^* \geq \bar{w}_{23}$. The retailer's optimal unconstrained order quantity is q_3^*, which results in the supplier's expected profit to be $\pi_{S,1}(w) = \alpha + (1-\beta)rE[\min\{X, q_3^*\}] - cq_3^*$. We show that the supplier will impose a loan limit such that the retailer orders less than q_3^*. We prove this result by showing that the supplier's expected profit decreases in $\bar{q}$ at $\bar{q} = q_3^*$. The supplier's expected profit as a function of $\bar{q}$ is $\pi_{S,1}(\bar{q}) = \alpha + (1-\beta)rE[\min\{X, \bar{q}\}] - c\bar{q}$, which is concave in $\bar{q}$. The derivative with respect to $\bar{q}$ is $\frac{\partial \pi_{S,1}(\bar{q})}{\partial \bar{q}} = (1-\beta)r\bar{F}(\bar{q}) - c$. Recall that the first-best order quantity q^{FB} is such that $r\bar{F}(q^{FB}) = c$. Then, it is straightforward to see that $\pi_{S,1}(\bar{q})$ is decreasing in $\bar{q}$ at $\bar{q} = q^{FB}$. Because $q_3^* \geq q^{FB}$ (see Proposition 4 and its proof in Online Appendix H) and $\pi_{S,1}(\bar{q})$ is concave in $\bar{q}$, it directly follows that $\pi_{S,1}(\bar{q})$ is decreasing in $\bar{q}$ at $\bar{q} = q_3^*$. Thus, the supplier will want to impose a loan limit such that the retailer orders less than q_3^*. Such loan limit may not be less than q^{FB} because of the challenges highlighted prior to Proposition 5. Moreover, the supplier's choice of the optimal loan limit depends on w. For these reasons, we cannot analytically determine the values of $\bar{\ell}^*$ and w^*.

Part 2: $\boldsymbol{\alpha^L \leq \alpha \leq \alpha^H}$

We will argue that the supplier finds it optimal to impose a loan limit such that the retailer orders less than his unconstrained optimal order quantity. In addition, the optimal wholesale price is $\geq \bar{w}_{13}$.

J.3. Part 2a: $\boldsymbol{w \leq \bar{w}_{21}}$

The retailer's unconstrained optimal order quantity is q_2^*. Following similar arguments to those presented in Part 1a, we can show that the supplier's expected profit at the optimal loan limit increases in w. This means that the supplier will want to choose a wholesale price as high as possible, i.e., $w = \bar{w}_{21}$.

J.4. Part 2b: $\boldsymbol{\bar{w}_{21} \leq w \leq \bar{w}_{13}}$

We show that the supplier's optimal loan limit increases in w. The retailer's unconstrained optimal order quantity is q_1^*, i.e., the newsvendor amount. The supplier's expected profit is $\pi_{S,1}(w) = (w-c)q_1^*$. If the supplier imposes a limit on the retailer's optimal order quantity, her profit would be $\pi_{S,1}(\bar{q}) = (w-c)\bar{q}$. It is clear that $\pi_{S,1}(\bar{q})$ increases in $\bar{q}$. However, for $w \geq \bar{w}_{21}$, $q_1^* \leq \alpha/w$ (see Proposition 3 and its proof in Appendix G). This implies that if the supplier were to limit the retailer's order quantity, she would have to choose $\bar{q} \leq \alpha/w$. Because $\pi_{S,1}(\bar{q})$ increases in $\bar{q}$, the supplier will set $\bar{q}^* = \alpha/w$. The retailer will order α/w only if $\alpha/w \leq q_1^*$, which does not happen for $w \geq \bar{w}_{21}$. Still, for the purpose of this proof, it suffices to show that $\pi_{S,1}(\bar{q}^*)$ increases in w, which implies that the supplier's expected profit is maximized in this range when $w = \bar{w}_{13}$. It directly follows that $\pi_{S,1}(\bar{q}^*, \bar{w}_{13}) \geq \pi_{S,1}(\bar{q}^*, w) \geq \pi_{S,1}(q_1^*, w)$, thus implying that the supplier will not want to be in this range. Upon substituting $\bar{q}^*$ into $\pi_{S,1}(\bar{q})$, we obtain $\pi_{S,1}(\bar{q}^*) = (w-c)\frac{\alpha}{w} = \alpha - \frac{c\alpha}{w}$. The derivative of $\pi_{S,1}(\bar{q}^*)$ with respect to w is $c\alpha/w^2 \geq 0$.

J.5. Part 2c: $\boldsymbol{w \geq \bar{w}_{13}}$

The retailer's unconstrained optimal order quantity is q_3^*. Following similar arguments to those presented in Part 1b, we can show that the supplier will find it optimal to impose a loan limit such that the retailer orders less than q_3^*. In addition, the optimal wholesale price is in this range because the supplier's expected profit increases in w for $w < \bar{w}_{13}$. Thus, $w^* \geq \bar{w}_{13}$.

Part 3: $\alpha \geq \alpha^H$

We will argue that the supplier does not impose any loan limit and that there exists a unique wholesale price that maximizes the supplier's expected profit in each range of w. However, because the supplier's expected profit is not smooth in w, we cannot determine the global optimal wholesale price analytically.

J.6. Part 3a: $w \leq \bar{w}_{21}$

The retailer's unconstrained optimal order quantity is q_2^*. The supplier's optimal order quantity limit is the same as in Part 1a, i.e., $\bar{q}^*$ solves $w\bar{F}(\frac{w\bar{q}^*-\alpha}{(1-\beta)r}) = c$. The retailer orders the minimum of $\bar{q}^*$ and q_2^*. We show that $\bar{q}^*$ increases in w and that q_2^* decreases in w. This allows us to conclude that there exists a unique value of w below which the retailer orders $\bar{q}^*$ and above which he orders q_2^*. Upon taking implicit differentiation of $\bar{q}^*$ with respect to w, we get

$$\frac{\partial \bar{q}^*}{\partial w}\Big[\frac{w^2}{(1-\beta)r}h\Big(\frac{w\bar{q}^*-\alpha}{(1-\beta)r}\Big)\Big] = 1 - \frac{w\bar{q}^*}{(1-\beta)r}h\Big(\frac{w\bar{q}^*-\alpha}{(1-\beta)r}\Big). \tag{41}$$

Following Lemma 3, it is straightforward to argue that the supplier will not induce the retailer to order more than the first-best order quantity. That is, $\bar{q}^* \leq q^{FB}$. Because $q^{FB}h(q^{FB}) \leq 1$ (from Assumption 9), it follows that $1 - \frac{w\bar{q}^*}{(1-\beta)r}h(\frac{w\bar{q}^*-\alpha}{(1-\beta)r}) \geq 0$. Thus, $\frac{\partial \bar{q}^*}{\partial w} \geq 0$. We take the derivative of q_2^* with respect to w and get

$$\frac{\partial q_2^*}{\partial w} = \frac{\beta\frac{q_2^*\bar{F}(q_2^*)^2}{w\bar{F}(\frac{\bar{p}-\alpha}{r})}[rh(\frac{\bar{p}-\alpha}{r}) + (\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})] + \bar{F}(\frac{wq_2^*-\alpha}{(1-\beta)r})(1-\frac{wq_2^*}{r(1-\beta)}h(\frac{wq_2^*-\alpha}{(1-\beta)r}))}{-\beta\frac{\bar{F}(q_2^*)^2}{\bar{F}(\frac{\bar{p}-\alpha}{r})}[rh(\frac{\bar{p}-\alpha}{r}) + (\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})] - w\bar{F}(\frac{wq_2^*-\alpha}{(1-\beta)r})(h(q_2^*) - \frac{w}{(1-\beta)r}h(\frac{wq_2^*-\alpha}{(1-\beta)r}))}. \tag{42}$$

The denominator equals $\frac{\partial^2\pi_{R,1}(q_2^*)}{\partial q^2}$ and it is negative because $\pi_{R,1}$ is concave in q at q_2^*. This implies that

$$\beta\frac{\bar{F}(q_2^*)^2}{\bar{F}(\frac{\bar{p}-\alpha}{r})}\Big[rh(\frac{\bar{p}-\alpha}{r}) + (\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})\Big] + w\bar{F}\Big(\frac{wq_2^*-\alpha}{(1-\beta)r}\Big)\Big(h(q_2^*) - \frac{w}{(1-\beta)r}h\Big(\frac{wq_2^*-\alpha}{(1-\beta)r}\Big)\Big) \geq 0. \tag{43}$$

We multiply both sides by q_2^*/w, which does not change the sign of the inequality. Thus,

$$\frac{q_2^*}{w}\beta\frac{\bar{F}(q_2^*)^2}{\bar{F}(\frac{\bar{p}-\alpha}{r})}\Big[rh(\frac{\bar{p}-\alpha}{r}) + (\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})\Big] + \bar{F}\Big(\frac{wq_2^*-\alpha}{(1-\beta)r}\Big)\Big(q_2^*h(q_2^*) - \frac{wq_2^*}{(1-\beta)r}h\Big(\frac{wq_2^*-\alpha}{(1-\beta)r}\Big)\Big) \geq 0. \tag{44}$$

Because $q_2^* \leq q^{FB}$ and $1 \geq q^{FB}h(q^{FB})$ (Assumption 9), it follows that the numerator of Equation (42) is greater than Inequality (44) and therefore positive. Thus, $\frac{\partial q_2^*(w)}{\partial w} \leq 0$.

The retailer orders $\bar{q}^*$ if w is sufficiently low and $q_2^*(w)$, otherwise. In addition, if the retailer orders $\bar{q}^*$, the supplier's expected profit increases in w, as shown in Part 1a. This implies that the supplier will never want to be in the range of w where the retailer orders $\bar{q}^*$. Thus, the supplier does not impose any loan limit. We next determine the supplier's optimal wholesale price in this range. Because q_2^* is decreasing in w, there exists a one-to-one mapping between w and q_2^*. We show that there exists a unique order quantity that maximizes the supplier's expected profit. It then follows that the optimal wholesale price will be the one such that the retailer finds it optimal to choose the order quantity that maximizes the supplier's expected profit. Let $w(q)$ denote the inverse of $q_2^*(w)$. That is, for any q, $w(q)$ solves

$$r\bar{F}(q)\Big(1-\beta\frac{\bar{p}-\alpha}{r}h(\frac{\bar{p}-\alpha}{r})\Big) = w(q)\bar{F}\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big). \tag{45}$$

Note that $\frac{\partial w(q)}{\partial q} = \Big[\frac{\partial q_2^*(w)}{\partial w}\Big]^{-1} \leq 0$. Using the definition of $\bar{p}$ (see Equation 27), we re-write the supplier's expected profit as a function of q as $\pi_{S,1}(q) = E[\min\{w(q)q, \alpha + (1-\beta)rX\}] - cq$. In order to show that

$\pi_{S,1}(q)$ is concave in q it suffices to show that $w(q)q$ is concave in q. This is the case because the $\min\{\cdot,\cdot\}$ operator is an increasing and concave function of its arguments. If its arguments are concave, then the first term of $\pi_{S,1}(q)$ is concave in q. The last term is linear in q and the difference of a concave and a linear term is concave. Note that $w(q)q$ is the loan amount.

$$\begin{aligned}\frac{\partial(w(q)q)}{\partial q} &= \frac{\partial w(q)}{\partial q}q + w(q)\\ &= \frac{w(q)(1-qh(q))}{\frac{\beta q\bar{F}(q)}{r\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}[rh(\frac{\bar{p}-\alpha}{r})+(\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})]+(1-\frac{w(q)q}{(1-\beta)r}h(\frac{w(q)q-\alpha}{(1-\beta)r}))} \geq 0. \qquad (46)\end{aligned}$$

Next, we want to show that the second derivative of $w(q)q$ with respect to q is negative. Let $a = w(q)(1-qh(q))$, $b = \frac{\beta q\bar{F}(q)}{r\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}[rh(\frac{\bar{p}-\alpha}{r}) + (\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})]$, and $d = (1-\frac{w(q)q}{(1-\beta)r}h(\frac{w(q)q-\alpha}{(1-\beta)r}))$. Using this notation, we can re-write $\frac{\partial(w(q)q)}{\partial q}$ as $\frac{\partial(w(q)q)}{\partial q} = \frac{a}{b+d}$. To argue that $\frac{\partial^2(w(q)q)}{\partial q^2} \leq 0$, we need to show that

$$\frac{\partial a}{\partial q}b + \frac{\partial a}{\partial q}d - \frac{\partial b}{\partial q}a - \frac{\partial d}{\partial q}a \leq 0. \qquad (47)$$

We first show that $\frac{\partial a}{\partial q} \leq 0$ and $\frac{\partial b}{\partial q} \geq 0$, which implies that the first and third terms are negative. Then, we show that $\frac{\partial a}{\partial q}d - \frac{\partial d}{\partial q}a \leq 0$ as well.

$$\frac{\partial a}{\partial q} = \frac{\partial(w(q)(1-qh(q)))}{\partial q} = \frac{\partial w(q)}{\partial q}(1-qh(q)) - w(q)h(q) - w(q)qh'(q) \leq 0. \qquad (48)$$

$$\frac{\partial b}{\partial q} = \frac{\partial}{\partial q}\Big[\frac{\beta q\bar{F}(q)}{r\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}\Big[rh(\frac{\bar{p}-\alpha}{r}) + (\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})\Big]\Big]. \qquad (49)$$

We compute the total derivative of $\bar{p}$ with respect to q.

$$\frac{d\bar{p}}{dq} = \frac{\partial\bar{p}}{\partial q} + \frac{\partial\bar{p}}{\partial w}\frac{\partial w(q)}{\partial q} = \frac{\bar{F}(\frac{w(q)q-\alpha}{(1-\beta)r})}{\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}\frac{\partial(w(q)q)}{\partial q} \geq 0. \qquad (50)$$

This implies that $[rh(\frac{\bar{p}-\alpha}{r}) + (\bar{p}-\alpha)h'(\frac{\bar{p}-\alpha}{r})]$ increases in q. The term $\frac{\beta q\bar{F}(q)}{r\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}$ also increases in q because $q\bar{F}(q)$ increases in q for $q \leq q^{FB}$ and $\frac{1}{r\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}$ increases in q because $\bar{p}$ increases in q. Putting these together, it must be the case that $\frac{\partial b}{\partial q} \geq 0$, thus implying that $\frac{\partial a}{\partial q}b - \frac{\partial b}{\partial q}a \leq 0$. Next, we show that $\frac{\partial a}{\partial q}d - \frac{\partial d}{\partial q}a \leq 0$.

$$\begin{aligned}\frac{\partial a}{\partial q}d - \frac{\partial d}{\partial q}a = &\Big[\frac{\partial w(q)}{\partial q}(1-qh(q)) - wh(q) - wqh'(q)\Big]\Big(1-\frac{w(q)q}{(1-\beta)r}h\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big)\Big)\\ &+ w(1-qh(q))\Big[\frac{\frac{\partial w(q)}{\partial q}q+w}{(1-\beta)r}h\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big) + \frac{wq(\frac{\partial w(q)}{\partial q}q+w)}{(1-\beta)^2r^2}h'\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big)\Big]. \qquad (51)\end{aligned}$$

We use the fact that $(1-qh(q)) \leq (1-\frac{w(q)q}{(1-\beta)r}h(\frac{w(q)q-\alpha}{(1-\beta)r}))$ to simplify the above and re-write it as

$$\begin{aligned}&\frac{\partial w(q)}{\partial q}\Big[(1-qh(q)) + \frac{wq}{(1-\beta)r}h\Big(\frac{w(q)q-\alpha}{r(1-\beta)r}\Big) + \frac{w^2q^2}{(1-\beta)^2r^2}h'\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big)\Big]\\ &- w\Big[\frac{\partial}{\partial q}(qh(q)) - \frac{\partial}{\partial q}\Big(\frac{wq}{(1-\beta)r}h\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big)\Big)\Big] \leq 0. \qquad (52)\end{aligned}$$

The first term is negative because $\frac{\partial w(q)}{\partial q} \leq 0$. Because $qh(q) \geq \Big(\frac{wq}{(1-\beta)r}h\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big)\Big)$ and $h(\cdot)$ is an increasing convex function (Assumption 8), it follows that $\frac{\partial}{\partial q}(qh(q)) - \frac{\partial}{\partial q}\Big(\frac{wq}{(1-\beta)r}h\Big(\frac{w(q)q-\alpha}{(1-\beta)r}\Big)\Big) \geq 0$. Thus, the second term is also negative. We have therefore shown that the supplier's expected profit is concave in q. These arguments allow us to conclude that there exists a unique order quantity that maximizes the supplier's expected profit. The supplier will choose the wholesale price that induces the retailer to order that exact amount. Let w_2^* denote the optimal wholesale price in this range. Then, w_2^* and $q_2^*(w_2^*)$ are such that

$$w_2^*(1-q_2^*h(q_2^*))\bar{F}\Big(\frac{w_2^*q_2^*-\alpha}{(1-\beta)r}\Big)=c\Big[\frac{\beta q_2^*\bar{F}(q_2^*)[h(\frac{\bar{p}-\alpha}{r})+\frac{\bar{p}-\alpha}{r}h'(\frac{\bar{p}-\alpha}{r})]}{\bar{F}(\frac{\bar{p}-\alpha}{r})(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r}))}+\Big(1-\frac{w_2^*q_2^*}{(1-\beta)r}h\Big(\frac{w_2^*q_2^*-\alpha}{(1-\beta)r}\Big)\Big)\Big] \tag{53}$$

$$\text{and } r\bar{F}(q_2^*)\Big(1-\beta\frac{(\bar{p}-\alpha)}{r}h(\frac{\bar{p}-\alpha}{r})\Big)=w_2^*\bar{F}\Big(\frac{w_2^*q_2^*-\alpha}{(1-\beta)r}\Big), \tag{54}$$

where we suppress the argument (w_2^*) of q_2^* to keep the notation compact.

J.7. Part 3b: $w \geq \bar{w}_{21}$

The retailer's unconstrained optimal order quantity is q_1^*. As shown in Part 2b, the supplier's expected profit increases in the loan limit. Thus, she will not want to constrain the retailer's optimal order quantity. We show that there exists a unique value of the wholesale price that maximizes the supplier's expected profit using the same technique as in Part 3a. Let $w(q)$ denote the inverse of q_1^*. That is, $w(q)$ solves $r\bar{F}(q)=w(q)$. The supplier's expected profit is $\pi_{S,1}(q)=\big(w(q)-c\big)q$. We show that the loan $w(q)q$ is concave in q.

$$\frac{\partial(w(q)q)}{\partial q}=\frac{\partial w(q)}{\partial q}q+w(q)=-rqf(q)+w=r\bar{F}(q)(1-qh(q))\geq 0. \tag{55}$$

The second derivative is $\frac{\partial^2(w(q)q)}{\partial q^2}=-rf(q)(1-qh(q))-r\bar{F}(q)(h(q)+qh'(q))\leq 0$. Because $w(q)q$ is concave in q, $\pi_{S,1}(q)$ is concave in q. This implies that there exists a unique wholesale price that maximizes the supplier's expected profit in this range of w. Such wholesale price is denoted by w_1^*, where w_1^* and $q_1^*(w_1^*)$ are such that

$$r\bar{F}(q_1^*)(1-q_1^*h(q_1^*))=c \text{ and } r\bar{F}(q_1^*)=w_1^*. \tag{56}$$

Once again, we suppress the argument w_1^* of q_1^* to keep the notation compact.

Recall that if $\beta=0$, then the retailer orders q_2^* if $w\leq\bar{w}_{21}$ and q_1^*, otherwise (see the discussion at the end of Section 5.1). Following a similar analysis as the one presented for Part 3, it holds that the supplier does not impose any loan limits and chooses the optimal wholesale prices (either w_1^*, or w_2^* at $\beta=0$) that induce the retailer to choose her preferred order quantity. ☐

K. Additional Numerical Experiments

We repeated our experiments from Section 6.1 assuming that the demand distribution is Exponential with parameter $\lambda=1/50$. This results in the same mean demand as for the uniform $[0,100]$ distribution. All other parameters are the same as in Section 6.1. Figure 10 shows the in-equilibrium supplier's expected profit, the retailer's expected terminal wealth and the supply chain profit versus $\alpha\in[0,365]$ for $\beta=0$ and $\beta=0.3$. We limit α to be at most 365 (instead of 480) because $cq^{FB}=365$ for the Exponential(1/50) case. We confirm that the supplier can realize higher profits (in Region II) from incurring positive verification costs when the retailer's assets are sufficiently high, but not too high. Similarly, there exist values of α in Range II over which the retailer's terminal wealth and the supply chain profit are higher under $\beta>0$ compared to $\beta=0$.

Figure 11 shows the in-equilibrium supplier's expected profit, the retailer's expected terminal wealth and the supply chain profit for $\beta\in[0,0.5]$ and $\alpha=160$. We confirm that the supplier makes higher profits from positive verification costs if β is sufficiently high but not too high. In addition, the supply chain profit can be higher for β in the mid-range. In this example, the retailer's terminal wealth is slightly higher at $\beta=0.05$ compared to $\beta=0$.

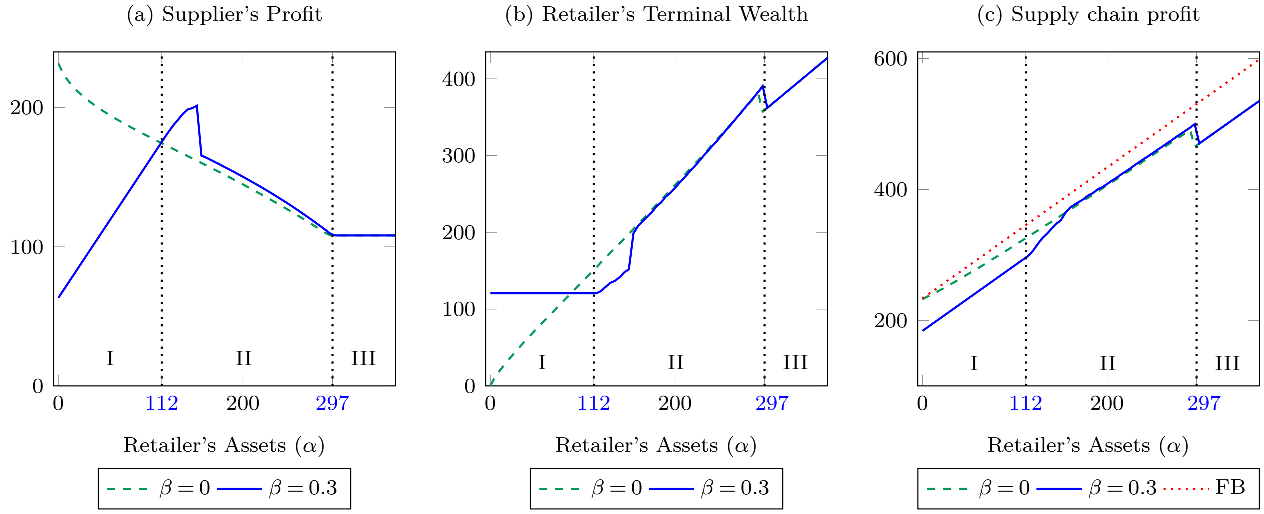

Figure 10 **Supplier's Profit, Retailer's Terminal Wealth and Supply chain profit vs α for $\beta = 0$ and $\beta = 0.3$.**

Notes: Vertical dotted lines define three regions. Region I ($0 \leq \alpha < 112$): $\pi_S^\beta < \pi_S^0$. Region II ($112 < \alpha < 297$): $\pi_S^\beta > \pi_S^0$. Region III ($297 \leq \alpha \leq 365$): $\pi_S^\beta = \pi_S^0$. There exist values of α in Region II over which the retailer's terminal wealth and the supply chain profit are higher under $\beta > 0$ than $\beta = 0$. The retailer's terminal wealth and the supply chain profit show an increasing pattern because they include α. Parameters: $X \sim Exp(1/50), r = 20, c = 8, \alpha \in [0, 365], \beta = \{0, 0.3\}$. Contract parameters $w^*, \bar{\ell}^*$ are optimally chosen. Optimal order quantity is evaluated at $w^*, \bar{\ell}^*$.

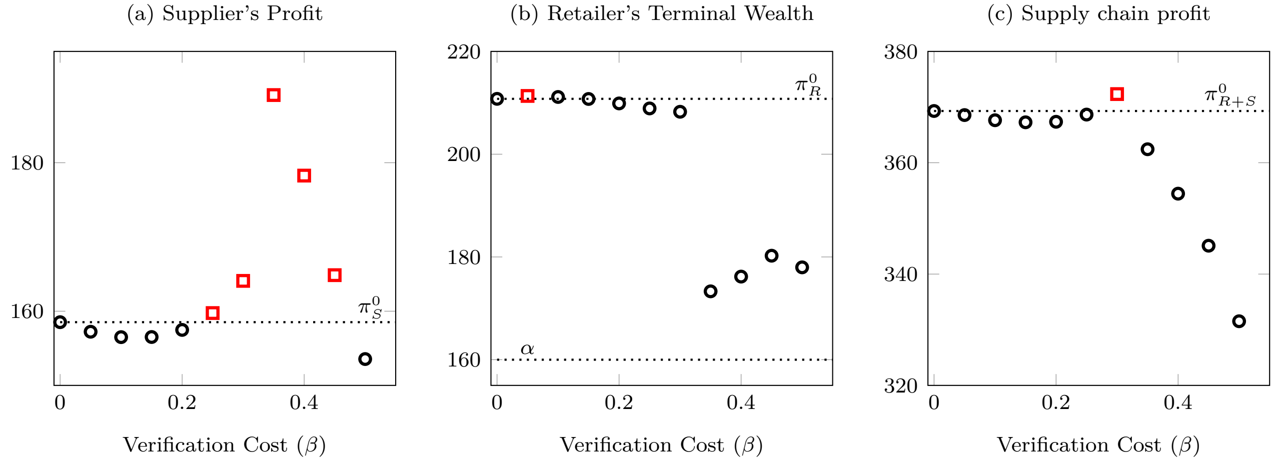

Figure 11 **Supplier's Profit, Retailer's Terminal Wealth and Supply chain profit vs β for $\alpha = 160$.**

Notes: In (a), the dotted line equals π_S^0, which is the supplier's expected profit when $\beta = 0$. The squares show values of β for which $\pi_S^\beta > \pi_S^0$. In (b), the top horizontal dotted line equals π_R^0 and the bottom equals α. The square shows a value of β for which $\pi_R^\beta > \pi_R^0$. The retailer's terminal wealth is always $\geq \alpha$, $\forall \beta$. In (c), the horizontal dotted line equals π_{R+S}^0. The square shows a value of β for which $\pi_{R+S}^\beta > \pi_{R+S}^0$. Parameters: $X \sim Exp(1/50), r = 20, c = 8, \alpha = 160, \beta \in [0, 0.5]$. Contract parameters $w^*, \bar{\ell}^*$ are optimally chosen. Optimal order quantity is evaluated at $w^*, \bar{\ell}^*$.

References

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