Online Appendix: Scope Contracts to Coordinate Assortment Planning in Omnichannel Retail Supply Chains. Aslani & Alp

EC.1. Summary of Notation

Table EC.1 Summary of Notations.

Notation	Definition
$\mathcal{A}$	Set of all attributes in the product type
$A_{x\|M}$	Utility of purchasing product x
$\bar{A}_{x\|M}$	Expected utility of purchasing product x
C	Capacity of the physical store
$d_{k,l(k)}$	Inaccuracy in assessing the level $l(k)$ of attribute k online
$D_{x\|M}$	Inaccuracy in assessing product x online, given M
$\Bbbk$	The attribute whose levels are considered to be inaccurately assessed ($d_{\Bbbk} \neq 0$)
K_x^r, $K_{y\|M}^m$	Keep probability for product $x \in M$ (r) or for product $y \in X \setminus M$, given M (m)
$l(k)$	Level index in attribute k
$L(k)$	Set of all possible levels for attribute k
M	Decision variable; assortment in the physical store
N	Total number of combinations of levels of all attributes but $\Bbbk$: $N = \prod_{k\in\mathcal{A},k\neq\Bbbk} \|L(k)\|$
$P_{x\|M}^r$, $P_{y\|M}^m$	Purchasing probability for product $x \in M$ (r) or for product $y \in X \setminus M$ (m)
$\mathcal{P}_x^r$, $\mathcal{P}_{x\|M}^m$	Marginal profit of product x sold in the physical (r) or the online channel (m)
$Q_{\mathcal{D}}$	Set of potential optimal assortments under the decentralized setting
$Q'_{\mathcal{D}}$	Set of potential optimal decentralized assortments if $d_{\Bbbk,l(\Bbbk)} = d_{\Bbbk} > 0, \forall l(\Bbbk) \in L(\Bbbk)$
$Q_{\mathcal{C}}$	Set of potential optimal selections of levels of $\Bbbk$ in the centralized setting
r	Disutility of the return process ($r < 0$)
R_x^r, $R_{y\|M}^m$	Return probability for product $x \in M$ (r) or for product $y \in X \setminus M$, given M (m)
$S(k)$	Set of levels of attribute k showcased
$\widetilde{u}_{k,l(k)}$	Part-worth attractiveness of level $l(k)$ of attribute k physically assessed
$\widetilde{U}_x$	Attractiveness of a product $x \in X$
$\overline{U}_x$	Expected utility of a product $x \in X$
U_x	Utility of product x physically assessed
$U_{x\|M}$	Utility of product x given M, assessed physically or online
v	Value of selling a returned product in a second market
w_x	Wholesale price of product x
$\mathcal{W}(M)$	Customer welfare when assortment M is showcased in the physical store
x_k	Level of attribute k in product x
X	Universal set of all products
β	Price sensitivity of utility ($\beta \leq 0$)
γ	Fraction of money refunded upon return ($0 \leq \gamma \leq 1$)
δ_i, λ_i	Discount rates on the wholesale price offered for any product with $x_{\Bbbk} = i$
ε_x	Error term in the utility assessment of the product x
$\varepsilon_{x,\text{keep}}$	Error term in capturing utilities associated with keeping the product x
$\varepsilon_{x,\text{return}}$	Error term in capturing utilities associated with returning the product x
η	Sensitivity to the initial inaccuracy in keeping decision of products purchased online
μ	Inverse scale parameter of the Gumbel distribution in purchasing decision
μ'	Inverse scale parameter of the Gumbel distribution in keep-or-return decision
π_x	Retail price of the product x
$\Pi_{\mathcal{C}}^T$	Total ORSC's expected profit under the centralized structure
$\Pi_{\mathcal{D}}^m$	Manufacturer's expected profit under the decentralized structure
$\Pi_{\mathcal{D}}^r$	Retailer's expected profit under the decentralized structure
$[n]_{k:i}$	Product with the n^{th} highest attractiveness among those with level i of attribute k

EC.2. Scope Contracts Under Identical Product Prices

A particular case common in many practical settings is where products differ in attribute levels that do not affect price (Gaur and Honhon 2006). For example, clothing retailers sell basic t-shirts in different colors and sizes at identical prices. In such cases, the scope contract terms can be simplified compared to the form in Theorem 1.

Corollary EC.1. *Suppose that $\pi_x = \pi_y = \pi$ and $w_x = w_y = w$. In Theorem 1, the discount rates will be defined such that*

$$\{\delta_i = 0\}\ and \{\lambda_i = 0\}, \qquad if\ i \notin S^*_{\mathcal{C}}(\Bbbk)$$
$$\left\{\frac{\alpha_1 \Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}}) - \alpha_2}{w^*_{\mathcal{D}}} \le \sum_{x \in M^*_{\mathcal{SC}}: x_{\Bbbk} = i} (1-\delta_i) e^{\bar{A}_x/\mu} \le \frac{\alpha_3 - \alpha_1 \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}})}{w^*_{\mathcal{D}}}\right\} and \{\lambda_i = 1\},\ if\ i \in S^*_{\mathcal{C}}(\Bbbk).$$

In case $\alpha_3 - \alpha_1 \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) < 0$, $\mathcal{SC} = \vec{\lambda}$ with a lump sum payment of $\mathcal{L}$ coordinates the ORSC where $\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}}) - \Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|w^*_{\mathcal{D}}, \vec{\lambda}) \le \mathcal{L} \le \Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\lambda}) - \Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}})$.

EC.3. Scope Contracts with Multiple Inaccurate Attributes

When there are multiple non-digital attributes that are inaccurately assessed online, Lemma 1 is no longer valid for the decentralized setting. In Lemma 1, when there is one inaccurate attribute, the minimum capacity required to showcase $S_{\mathcal{D}}(\Bbbk)$ is $C = |S_{\mathcal{D}}(\Bbbk)|$. That is, for each selected level, exactly one product—the one with the highest $\widetilde{U}_x$—must be included in the assortment. But this condition does not hold when multiple attributes are inaccurately assessed online. We discuss and address this issue below.

First, let number the levels of an attribute $k \in \mathcal{A}$ as $1, 2, \ldots, |L(k)|$, with 1 and $|L(k)|$ corresponding to the levels with the highest and lowest part-worth attractiveness values, $\widetilde{u}_{k,l(k)}$, respectively. For simplicity and without loss of generality, we assume that products in the ORSC consist of two inaccurate attributes, denoted by $\Bbbk$ and $\Bbbk'$, while the levels of all other attributes are accurately assessed. Additionally, we define $[n]_{\Bbbk:l(\Bbbk),\Bbbk':l(\Bbbk')}$ as the product with the n^{th} highest attractiveness value that includes level $l(\Bbbk)$ of $\Bbbk$ and level $l(\Bbbk')$ of $\Bbbk'$. For example, $[4]_{\Bbbk:2,\Bbbk':1}$ denotes the product with the fourth-highest attractiveness value, which includes level 2 of attribute $\Bbbk$ and level 1 of attribute $\Bbbk'$.

Assuming that $S_{\mathcal{D}}(\Bbbk)$ and $S_{\mathcal{D}}(\Bbbk')$ are predetermined, the minimum capacity to represent them is $C = \max\{|S_{\mathcal{D}}(\Bbbk)|, |S_{\mathcal{D}}(\Bbbk')|\}$. As an illustrative example, suppose $S_{\mathcal{D}}(\Bbbk) = \{1, 2\}$ and $S_{\mathcal{D}}(\Bbbk') = \{1, 2\}$. In this case, to ensure representing the selected levels of both $\Bbbk$ and $\Bbbk'$, showcasing at least 2 products is necessary, corresponding to $\max\{2, 2\}$. Given $C = 2$, the question becomes which two products should be selected. Evidently, for each combination of levels of $\mathbb{k}$ and $\mathbb{k}'$, the product with the highest $\widetilde{U}_x$ is preferred. Hence, the candidate products are $[1]_{\mathbb{k}:1,\mathbb{k}':1}$, $[1]_{\mathbb{k}:1,\mathbb{k}':2}$, $[1]_{\mathbb{k}:2,\mathbb{k}':1}$, and $[1]_{\mathbb{k}:2,\mathbb{k}':2}$.

Among the candidate products, $S_{\mathcal{D}}(\mathbb{k})$ and $S_{\mathcal{D}}(\mathbb{k}')$ can be represented by either $M^1 = \{[1]_{\mathbb{k}:1,\mathbb{k}':1}, [1]_{\mathbb{k}:2,\mathbb{k}':2}\}$ or $M^2 = \{[1]_{\mathbb{k}:1,\mathbb{k}':2}, [1]_{\mathbb{k}:2,\mathbb{k}':1}\}$. By definition, M^1 includes the product with the highest $\widetilde{U}_x$ among the candidates, but it also includes the product with the lowest $\widetilde{U}_x$. In contrast, M^2 consists of two products with intermediate $\widetilde{U}_x$ values—neither the highest nor the lowest. Therefore, unlike the case with a single non-digital attribute, the assortment cannot be determined solely by selecting products with the highest attractiveness. Instead, the resulting profits from different candidate assortments must be compared.

We define $\mathcal{M}_{\mathcal{D}}$ as the feasibility set containing all potential assortments that represent both $S_{\mathcal{D}}(\mathbb{k})$ and $S_{\mathcal{D}}(\mathbb{k}')$. In other words, for any $M \in \mathcal{M}_{\mathcal{D}}$, we have $\bigcup_{x\in M}\{x_{\mathbb{k}}\} = S_{\mathcal{D}}(\mathbb{k})$ and $\bigcup_{x\in M}\{x_{\mathbb{k}'}\} = S_{\mathcal{D}}(\mathbb{k}')$. The assortments in $\mathcal{M}_{\mathcal{D}}$ should be compared to select the optimal one that maximizes the retailer's profit. Notably, some assortments in $\mathcal{M}_{\mathcal{D}}$ can be easily eliminated. For instance, if two assortments differ by only one product, the assortment containing the product with the lower $\widetilde{U}_x$ is clearly suboptimal. Furthermore, for any $C \geq a$, where $a = |S_{\mathcal{D}}(\mathbb{k})| \times |S_{\mathcal{D}}(\mathbb{k}')| \prod_{k\in\mathcal{A},\, k\neq \mathbb{k},\mathbb{k}'} |L(k)|$, the optimal assortment includes all products formed by every combination of levels in $S_{\mathcal{D}}(\mathbb{k})$ and $S_{\mathcal{D}}(\mathbb{k}')$, totaling a products. Adding more products beyond this will alter the representation of $S_{\mathcal{D}}(\mathbb{k})$ and $S_{\mathcal{D}}(\mathbb{k}')$. Lemma EC.1 formally states this discussion.

LEMMA EC.1. *In the decentralized setting, suppose that* $S_{\mathcal{D}}(\mathbb{k})$ *and* $S_{\mathcal{D}}(\mathbb{k}')$ *are predetermined. For any* C *satisfying* $\max\{|S_{\mathcal{D}}(\mathbb{k})|, |S_{\mathcal{D}}(\mathbb{k}')|\} \leq C < a$, *where* $a = |S_{\mathcal{D}}(\mathbb{k})| \times |S_{\mathcal{D}}(\mathbb{k}')| \prod_{k\in\mathcal{A},\, k\neq \mathbb{k},\mathbb{k}'} |L(k)|$, *let* $\mathcal{M}_{\mathcal{D}}$ *be the feasibility set containing all assortments that represent* $S_{\mathcal{D}}(\mathbb{k})$ *and* $S_{\mathcal{D}}(\mathbb{k}')$. *The assortment in* $\mathcal{M}_{\mathcal{D}}$ *that yields the highest* $\Pi^r_{\mathcal{D}}$ *is optimal under the decentralized setting. Also, for any* $C \geq a$, *the optimal assortment consists of all products formed by every combination of levels in* $S_{\mathcal{D}}(\mathbb{k})$ *and* $S_{\mathcal{D}}(\mathbb{k}')$.

Lemma EC.1 introduces additional complexity in the design of the scope contract. In the original scope contract, discount rates are offered based on the levels of a single inaccurate attribute—for example, all products with a certain level receive the same discount. Considering $S^*_{\mathcal{C}}(\mathbb{k})$, the manufacturer knows the assortment that the retailer will select under the contract given Lemma 1, i.e., $M^*_{\mathcal{SC}}$, and can design the contract and set discount rates accordingly. However, with multiple inaccurate attributes, the choice of $M^*_{\mathcal{SC}}$ among potentially multiple assortments that represent $S^*_{\mathcal{C}}(\Bbbk)$ and $S^*_{\mathcal{C}}(\Bbbk')$ should be done through Lemma EC.1. Notably, this choice depends on the discount rates offered, as the profit margins change with varying rates, which may result in $M^*_{\mathcal{SC}}$ changing dynamically. To address this, we modify the contract to ensure coordination remains achievable.

Showcasing any assortment that represents $S^*_{\mathcal{C}}(\Bbbk)$ and $S^*_{\mathcal{C}}(\Bbbk')$ guarantees coordination. According to Lemma 2, multiple such assortments may exist. In this modified mechanism—termed the "prime scope contract", $\mathcal{SC}'$—the manufacturer arbitrarily selects one such assortment, $M^*_{\mathcal{C}}$, as the assortment under contract, i.e., $M^*_{\mathcal{SC}'}$, and offers discount rates only on products in this set. Unlike the original scope contract, discount rates are not applied to all products containing certain attribute levels but are limited to those in $M^*_{\mathcal{SC}'}$. While this form of the contract reduces the retailer's flexibility in selecting its optimal assortment, it still ensures coordination of the ORSC. The distribution of additional profit between the parties can be managed using discount rates as a lever.

Definition EC.1. A prime scope contract, $\mathcal{SC}'$, is defined as a contract that incentivizes buyers to purchase a certain variety of products in the same transaction. It is given by

$$\mathcal{SC}' = [\theta_1, \theta_2, ..., \theta_{|X|}]$$

where θ_x such that $0 \le \theta_x \le 1$ is the discount rate applied to product $x \in X$. If $\theta_x = 0$, then the product x is sold at the regular price.

Theorem EC.1. *Suppose that $M^*_{\mathcal{SC}'}$ is an assortment representing $S^*_{\mathcal{C}}(\Bbbk)$ and $S^*_{\mathcal{C}}(\Bbbk')$, selected by the manufacturer for the contract. For a given problem instance, let $\alpha'_1 = 1 + \sum_{x \in M^*_{\mathcal{SC}'}} e^{\bar{A}_x/\mu} + \sum_{x \in X \setminus M^*_{\mathcal{SC}'}} e^{(\bar{A}_{x|M^*_{\mathcal{SC}'}})/\mu}$, $\alpha'_2 = \sum_{x \in X \setminus M^*_{\mathcal{SC}'}} \mathcal{P}^m_{x|M^*_{\mathcal{SC}'}} e^{(\bar{A}_{x|M^*_{\mathcal{SC}'}})/\mu}$, and $\alpha'_3 = \sum_{x \in M^*_{\mathcal{SC}'}} \mathcal{P}^r_x e^{\bar{A}_x/\mu}$. Let $\vec{\theta} = [\theta_1, \theta_2, ..., \theta_{|X|}]$ and $\vec{\tau} = [\tau, \tau, ..., \tau_{|X|}]$ be such that*

$$\{\theta_x = 0\} \text{ and } \{\tau_x = 0\}, \qquad \text{if } x \notin M^*_{\mathcal{SC}'}$$

$$\left\{\alpha'_1 \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}}) - \alpha'_2 \le \sum_{x \in M^*_{\mathcal{SC}'}} (1-\theta_x) w^*_{x,\mathcal{D}} e^{\bar{A}_x/\mu} \le \alpha'_3 - \alpha'_1 \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}})\right\} \text{ and } \{\tau_x = 1\},$$
$$\text{if } x \in M^*_{\mathcal{SC}'}. \tag{EC.1}$$

If $\alpha'_3 - \alpha'_1 \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) \ge 0$, then $\mathcal{SC}' = \vec{\theta}$ coordinates the ORSC. Otherwise, $\mathcal{SC} = \vec{\tau}$ with a lump sum payment of $\mathcal{L}$ from the manufacturer to the retailer coordinates the ORSC where $\mathcal{L}$ is any value that satisfies

$$\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) - \Pi^r_{\mathcal{SC}'}(M^*_{\mathcal{SC}'}|\boldsymbol{w}^*_{\mathcal{D}}, \vec{\tau}) \le \mathcal{L} \le \Pi^m_{\mathcal{SC}'}(\boldsymbol{w}^*_{\mathcal{D}}, \vec{\tau}) - \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}}),$$

where $\Pi^r_{\mathcal{SC}'}(M^*_{\mathcal{SC}'}|\boldsymbol{w}^*_{\mathcal{D}},\vec{\tau}) = \sum_{x\in M^*_{\mathcal{SC}'}} P^r_{x|M^*_{\mathcal{SC}'}}\left(\mathcal{P}^r_x-(1-\tau_x)w^*_{x,\mathcal{D}}\right)$ *and* $\Pi^m_{\mathcal{SC}'}(\boldsymbol{w}^*_{\mathcal{D}},\vec{\tau}) = \sum_{x\in X\setminus M^*_{\mathcal{SC}'}} P^m_{x|M^*_{\mathcal{SC}'}}\mathcal{P}^m_{x|M^*_{\mathcal{SC}'}} + \sum_{x\in M^*_{\mathcal{SC}'}}(1-\tau_x)w^*_{x,\mathcal{D}}P^r_{x|M^*_{\mathcal{SC}'}}$.

The prime scope contract in Theorem EC.1 incentivizes the retailer to showcase the assortment $M^*_{\mathcal{SC}'}$ selected by the manufacturer, ensuring that $S^*_{\mathcal{C}}(\mathbb{k})$ and $S^*_{\mathcal{C}}(\mathbb{k}')$ are represented. Similar to the original scope contract, there may be infinitely many θ_x values that coordinate the ORSC, each leading to different profit distributions between the parties.

Additionally, to avoid challenges in practical implementation, the prime scope contract can be designed with a single discount rate applicable to all $x\in M^*_{\mathcal{SC}'}$. Suppose that $\alpha'_3 - \alpha'_1\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}})\geq 0$ in Theorem EC.1, and $\mathcal{SC}'=\vec{\theta}=[\theta_1,\theta_2,...,\theta_{|X|}]$ coordinates the ORSC. Consider $\vec{\theta'}=[\theta'_1,\theta'_2,...,\theta'_{|X|}]$ as another discount vector. Then, for a σ' that satisfies

$$\frac{\alpha'_1\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}})-\alpha'_2}{\sum_{x\in M^*_{\mathcal{SC}'}} w^*_{x,\mathcal{D}}e^{\bar{A}_x/\mu}} \leq 1-\sigma' \leq \min\left\{1,\frac{\alpha'_3-\alpha'_1\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|w^*_{\mathcal{D}})}{\sum_{x\in M^*_{\mathcal{SC}'}} w^*_{x,\mathcal{D}}e^{\bar{A}_x/\mu}}\right\},$$

$\mathcal{SC}'=\vec{\theta'}$ where $\theta'_x=\sigma'$ if $x\in M^*_{\mathcal{SC}'}$ and $\theta'_x=0$ otherwise, also coordinates this ORSC. Additionally, for any specific $\mathcal{SC}'=\vec{\theta}$, let $\overline{\sigma'}=1-\frac{\sum_{x\in M^*_{\mathcal{SC}'}}(1-\theta_x)w^*_{x,\mathcal{D}}e^{\bar{A}_x/\mu}}{\sum_{x\in M^*_{\mathcal{SC}'}}w^*_{x,\mathcal{D}}e^{\bar{A}_x/\mu}}$. Then, $\mathcal{SC}'=\vec{\theta}$ is equivalent to $\mathcal{SC}'=\vec{\theta'}$ where $\theta'_x=\overline{\sigma'}$ if $x\in M^*_{\mathcal{SC}'}$ and $\theta'_x=0$ otherwise.

EC.3.1. Optimal Assortments in Presence of Multiple Inaccurate Attributes

Considering $\mathbb{k}$ and $\mathbb{k}'$ as attributes inaccurately assessed, identifying the optimal levels of these attributes for showcasing is not sufficient for determining the optimal decentralized assortment. According to Lemma EC.1, if the levels are predetermined, we must also construct the corresponding feasibility set $\mathcal{M}_{\mathcal{D}}$ and evaluate all assortments within this set to find the one that maximizes the retailer's profit. This added layer of evaluation significantly increases the computational complexity.

Given the available capacity C, we must explore different combinations of attribute levels for $\mathbb{k}$ and $\mathbb{k}'$, while ensuring that the total number of distinct levels does not exceed capacity. Specifically, we can select up to C levels from each attribute, and we evaluate combinations involving 1, 2, ..., up to $\min\{C,|L(\mathbb{k})|\}$ levels of $\mathbb{k}$ and similarly for $\mathbb{k}'$. For each such pair of level selections, we form the feasibility set $\mathcal{M}_{\mathcal{D}}$ and compare the resulting assortments to identify the optimal one.

Under the centralized setting, the process is less complicated. As established in Lemma 2, any assortment that represents the selected levels of the inaccurate attributes is optimal.

Therefore, once a combination of levels is chosen, there is no need to compare multiple assortments for that selection. This eliminates a substantial portion of the computational burden. As a result, the centralized approach reduces the problem to comparing one representative assortment per combination of selected levels. Procedure 4 remains applicable in this setting, with the modification that it must include an additional counter parameter to account for the number of levels selected for attribute $\Bbbk'$, alongside those for $\Bbbk$.

EC.4. Proofs

Lemma 1. First, we define $[n]_{\Bbbk:i}$ as the product with the n^{th} highest $\widetilde{U}_x$ among those containing level i of attribute $\Bbbk$. Also, let $N = \prod_{k\in\mathcal{A},k\neq\Bbbk} |L(k)|$ be the number of possible combinations of levels of all other attributes for each level of $\Bbbk$; i.e., the number of products in X containing each level of $\Bbbk$. Without loss of generality, suppose that $S^*_{\mathcal{D}}(\Bbbk) = \{1, 2, \ldots, m\}$. The capacity to successfully display $S^*_{\mathcal{D}}(\Bbbk)$ should be $m \le C$, and the retailer can showcase at most $m \times N$ products; otherwise, the representation of $S^*_{\mathcal{D}}(\Bbbk)$ will alter. We first prove that when $C = m$, selecting the product with the highest $\widetilde{U}_x$ value corresponding to each level in $S^*_{\mathcal{D}}(\Bbbk)$ yields a higher expected profit for the retailer than any other selection. Define the optimal assortment as $M^*_m = \{[1]_{\Bbbk:i} \mid i \in S^*_{\mathcal{D}}(\Bbbk)\}$, and consider an alternative assortment $M'_m = M^*_m \cup \{[2]_{\Bbbk:t}\} \setminus \{[1]_{\Bbbk:t}\}$, where for the level $t \le m$, $[1]_{\Bbbk:t}$ is replaced by $[2]_{\Bbbk:t}$. The retailer's expected profits for M^*_m and M'_m are given as:

$$\Pi^r_{\mathcal{D}}(M^*_m) = \frac{\displaystyle\sum_{i=1,i\neq t}^{m} \left(\mathcal{P}^r_{[1]_{\Bbbk:i}} - w_{[1]_{\Bbbk:i}}\right) e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu} + \quad \left(\mathcal{P}^r_{[1]_{\Bbbk:t}} - w_{[1]_{\Bbbk:t}}\right) e^{(\bar{A}_{[1]_{\Bbbk:t}})/\mu}}{1 + \sum_{x\in X, x_{\Bbbk}\in S^*_{\mathcal{D}}(\Bbbk)} e^{\bar{A}_x/\mu} + \sum_{x\in X, x_{\Bbbk}\notin S^*_{\mathcal{D}}(\Bbbk)} e^{(\bar{A}_{x|M^*_m})/\mu}}, \tag{EC.2}$$

$$\Pi^r_{\mathcal{D}}(M'_m) = \frac{\displaystyle\sum_{i=1,i\neq t}^{m} \left(\mathcal{P}^r_{[1]_{\Bbbk:i}} - w_{[1]_{\Bbbk:i}}\right) e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu} + \quad \left(\mathcal{P}^r_{[2]_{\Bbbk:t}} - w_{[2]_{\Bbbk:t}}\right) e^{(\bar{A}_{[2]_{\Bbbk:t}})/\mu}}{1 + \sum_{x\in X, x_{\Bbbk}\in S^*_{\mathcal{D}}(\Bbbk)} e^{\bar{A}_x/\mu} + \sum_{x\in X, x_{\Bbbk}\notin S^*_{\mathcal{D}}(\Bbbk)} e^{(\bar{A}_{x|M'_m})/\mu}}. \tag{EC.3}$$

Given that both M^*_m and M'_m represent $S^*_{\mathcal{D}}(\Bbbk)$, the denominators in (EC.2) and (EC.3) are the same. Therefore, to prove $\Pi^r_{\mathcal{D}}(M^*_m) > \Pi^r_{\mathcal{D}}(M'_m)$, it suffices to show that the numerator of (EC.2) is greater than that of (EC.3). Writing this condition, we get

$$\left(\mathcal{P}^r_{[1]_{\Bbbk:t}} - w_{[1]_{\Bbbk:t}}\right) e^{(\bar{A}_{[1]_{\Bbbk:t}})/\mu} > \left(\mathcal{P}^r_{[2]_{\Bbbk:t}} - w_{[2]_{\Bbbk:t}}\right) e^{(\bar{A}_{[2]_{\Bbbk:t}})/\mu}.$$

Given the assumption that $(\mathcal{P}_x^r - w_x)\, e^{\bar{A}_x/\mu}$ is strictly increasing in $\widetilde{U}_x$, and $\widetilde{U}_{[1]_{\Bbbk:t}} > \widetilde{U}_{[2]_{\Bbbk:t}}$, the inequality holds true. A similar argument can be applied to any other alternative M'_m, proving the optimality of M^*_m.

We now demonstrate that if $m < C \leq m \times N$, beyond the products $[1]_{\Bbbk:i}$ $\forall i \in S^*_{\mathcal{D}}(\Bbbk)$, the remaining capacity, i.e., $C - m$, should be allocated to the products with the highest $\widetilde{U}_x$ values, among those for which $x_{\Bbbk} \in S^*_{\mathcal{D}}(\Bbbk)$. Let M^*_{C-m} be this set and M'_{C-m} represent an alternative set of $C - m$ products that also represents $S^*_{\mathcal{D}}(\Bbbk)$ but differs from $M^*_{C-m\times n}$ in at least one product.

Define the full assortments as $M^*_C = M^*_m \cup M^*_{C-m}$ and $M'_C = M^*_m \cup M'_{C-m}$. To establish that $\Pi^r_{\mathcal{D}}(M^*_C) > \Pi^r_{\mathcal{D}}(M'_C)$, it suffices to show:

$$\sum_{x \in M^*_{C-m}} (\mathcal{P}_x^r - w_x)\, e^{\bar{A}_x/\mu} > \sum_{x \in M'_{C-m}} (\mathcal{P}_x^r - w_x)\, e^{\bar{A}_x/\mu}.$$

This inequality holds by construction, since M^*_{C-m} contains the $C - m$ products with the highest $\widetilde{U}_x$ values (excluding the top $[1]_{\Bbbk:i}$ products already chosen for each $i \in S^*_{\mathcal{D}}(\Bbbk)$). Any deviation from this selection—as in M'_{C-m}—necessarily replaces at least one higher-utility product with a lower-utility alternative, thereby reducing the expected profit.

Notably, when $m \leq C \leq m \times N$, the available capacity should be fully utilized. This is because once $S^*_{\mathcal{D}}(\Bbbk)$ is showcased, the inaccuracies associated with these attribute levels are eliminated from the online channel. Thus, showcasing additional products with these selected levels does not provide any new information. As a result, the retailer prefers to utilize the available capacity to expand its product offering and generate additional profit. However, when $C > m \times N$, the retailer showcases exactly $m \times N$ items, leaving the excess capacity $C - m \times N$ unutilized to preserve the optimal representation of $S^*_{\mathcal{D}}(\Bbbk)$. Q.E.D.

Lemma 2. Under the centralized setting, the total expected profit of the ORSC accounts for the combined profits of both the retailer and the manufacturer. As a result, the specific channel through which a product is sold becomes irrelevant, provided that the showcased products reveal $S^*_{\mathcal{C}}(\Bbbk)$. Once a particular attribute level is selected, any product containing that level can be included in the assortment without affecting the total profit. In the following, we formally show this.

Without loss of generality, suppose that $S^*_{\mathcal{C}}(\Bbbk) = \{1, 2, \ldots, m\}$. We aim to show that any assortment M such that $\bigcup_{x \in M}\{x_{\Bbbk}\} = S^*_{\mathcal{C}}(\Bbbk)$ is optimal. Let M^1 and M^2 be two such assortments that differ in at least one product. Although $M^1 \neq M^2$, since both represent $S^*_{\mathcal{C}}(\Bbbk)$, the total expected profit of the ORSC for M^1 and M^2 are the same and can be written as the following

$$\Pi^T_{\mathcal{C}}(M^1) = \Pi^T_{\mathcal{C}}(M^2) = \frac{\sum_{x\in X, x_\Bbbk \in S^*_{\mathcal{C}}(\Bbbk)} \mathcal{P}^r_x e^{\bar{A}_x/\mu} + \sum_{x\in X, x_\Bbbk \notin S^*_{\mathcal{C}}(\Bbbk)} \mathcal{P}^m_x e^{\bar{A}_{x|M^*}/\mu}}{1 + \sum_{x\in X, x_\Bbbk \in S^*_{\mathcal{C}}(\Bbbk)} e^{\bar{A}_x/\mu} + \sum_{x\in X, x_\Bbbk \notin S^*_{\mathcal{C}}(\Bbbk)} e^{\bar{A}_{x|M^*}/\mu}},$$

which completes the proof. Q.E.D.

Theorem 1. Under the $\mathcal{SC} = \vec{\delta}$, neither of the parties should be worse off compared to their decentralized expected profits. Therefore, we must have

$$\Pi^r_{\mathcal{SC}}(M^*_{\mathcal{C}}|\boldsymbol{w}^*_{\mathcal{D}}, \vec{\delta}) \geq \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}), \tag{EC.4a}$$

$$\Pi^m_{\mathcal{SC}}(\boldsymbol{w}^*_{\mathcal{D}}, \vec{\delta}) \geq \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}}). \tag{EC.4b}$$

Substituting the corresponding profit functions into (EC.4), we will get

$$\frac{\alpha_3 - \sum_{x\in M^*_{\mathcal{C}}: x_\Bbbk = i}(1-\delta_i) w^*_{x,\mathcal{D}} e^{\bar{A}_x/\mu}}{\alpha_1} \geq \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}),$$
$$\frac{\alpha_2 + \sum_{x\in M^*_{\mathcal{C}}: x_\Bbbk = i}(1-\delta_i) w^*_{x,\mathcal{D}} e^{\bar{A}_x/\mu}}{\alpha_1} \geq \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}})$$

simplifying which, condition (8) will be derived.

Notably, the upper bound of (8) must be greater than or equal to its lower bound. Writing this condition, we will get

$$\alpha_3 + \alpha_2 \geq \alpha_1 \left(\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) + \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}}) \right).$$

Substituting α_1, α_2, and α_3, this inequality turns into $\Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}}) \geq \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) + \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}})$, which always holds by the definition of the centralized total profit.

Moreover, since $\sum_{x\in M^*_{\mathcal{C}}: x_\Bbbk = i}(1-\delta_i) w^*_{x,\mathcal{D}} e^{\bar{A}_x/\mu}$ obtains a nonnegative value, it is necessary that the upper bound be nonnegative. Otherwise, if the upper bound is negative, i.e., $\alpha_3 - \alpha_1 \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) < 0$, then no $\vec{\delta}$ can compensate for the retailer's loss. In such cases, a one-time payment $\mathcal{L}$ from the manufacturer to the retailer can be introduced using an alternative discount rate $\mathcal{SC} = \vec{\lambda}$, ensuring

$$\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) \leq \Pi^r_{\mathcal{SC}}(M^*_{\mathcal{C}}|\boldsymbol{w}^*_{\mathcal{D}}, \vec{\lambda}) + \mathcal{L}.$$

The amount of this payment should also guarantee that the manufacturer is not worse off compared to its decentralized profit (i.e., $\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}}) \leq \Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\lambda}) - \mathcal{L}$). This defines the valid interval for $\mathcal{L}$ within the theorem.

Note that since $\Pi_{\mathcal{C}}^{T}(M_{\mathcal{C}}^{*}) \geq \Pi_{\mathcal{D}}^{r}(M_{\mathcal{D}}^{*}|\boldsymbol{w}_{\mathcal{D}}^{*}) + \Pi_{\mathcal{D}}^{m}(\boldsymbol{w}_{\mathcal{D}}^{*})$, the retailer's loss is never greater than the additional profit generated by the system. Hence, either $\mathcal{SC} = \vec{\delta}$ or $\mathcal{SC} = \vec{\lambda}$ with $\mathcal{L}$ successfully coordinates the ORSC for any problem instance. Q.E.D.

Theorem 2. We prove the theorem in two parts, each corresponding to one of the items in its statement.

(i) Suppose that the multi-rate discount structure $\mathcal{SC} = \vec{\delta} = [\delta_1, \delta_2, ..., \delta_{|L(\mathbb{k})|}]$ coordinates the ORSC. Let define a single-rate discount structure $\vec{\delta}' = [\delta_1', \delta_2', ..., \delta_{|L(\mathbb{k})|}']$, where $\delta_i' = \sigma, \forall i \in S_{\mathcal{C}}^{*}(\mathbb{k})$ and $\delta_i' = 0$ otherwise. This means that instead of applying different discount rates δ_i for different attribute levels, we apply a uniform discount rate σ to all levels in $S_{\mathcal{C}}^{*}(\mathbb{k})$.

Substituting $\vec{\delta}'$ into the coordination condition (8), we obtain

$$\alpha_1 \Pi_{\mathcal{D}}^{m}(\boldsymbol{w}_{\mathcal{D}}^{*}) - \alpha_2 \leq 1 - \sigma \leq \sum_{x \in M_{\mathcal{SC}}^{*}} w_{x,\mathcal{D}}^{*} e^{\bar{A}_x} \leq \alpha_3 - \alpha_1 \Pi_{\mathcal{D}}^{r}(M_{\mathcal{D}}^{*}|\boldsymbol{w}_{\mathcal{D}}^{*}). \tag{EC.5}$$

Since this inequality is identical to the one satisfied by the multi-rate contract $\mathcal{SC} = \vec{\delta}$, it follows that the single-rate discount structure also coordinates the ORSC. Rearranging the terms in (EC.5), the upper and lower bounds in (9) will be derived. Note that since the single discount rate σ cannot exceed 1, we accordingly adjust the upper bound in (13).

To show that any single discount rate σ satisfying (EC.5) can be represented by a multi-rate $\vec{\delta}$, we must prove the existence of a set of δ_i such that

$$(1 - \sigma) \sum_{x \in M_{\mathcal{SC}}^{*}} w_{x,\mathcal{D}}^{*} e^{\bar{A}_x/\mu} = \sum_{x \in M_{\mathcal{SC}}^{*}: x_{\mathbb{k}} = i} (1 - \delta_i) w_{x,\mathcal{D}}^{*} e^{\bar{A}_x/\mu}.$$

By setting $\delta_i = \sigma$ for $i \in S_{\mathcal{C}}^{*}(\mathbb{k})$ and $\delta_i = 0$ otherwise, we obtain a multi-rate structure that satisfies this equation. So, for any valid σ, there exists at least one $\vec{\delta}$ that coordinates the ORSC, establishing equivalence.

(ii) We now define $\overline{\sigma} = 1 - \frac{\sum_{x \in M_{\mathcal{SC}}^{*}: x_{\mathbb{k}} = i} (1-\delta_i) w_{x,\mathcal{D}}^{*} e^{\bar{A}_x/\mu}}{\sum_{x \in M_{\mathcal{SC}}^{*}} w_{x,\mathcal{D}}^{*} e^{\bar{A}_x/\mu}}$ which represents the weighted average discount rate applied to the retailer's assortment. We aim to prove that substituting $\overline{\sigma}$ into the profit functions yields the same retailer's and manufacturer's profits as the original multi-rate contract, i.e.,

$$\Pi_{\mathcal{SC}}^{r}(M_{\mathcal{SC}}^{*}|\boldsymbol{w}_{\mathcal{D}}^{*}, \overline{\sigma}) = \Pi_{\mathcal{SC}}^{r}(M_{\mathcal{SC}}^{*}|\boldsymbol{w}_{\mathcal{D}}^{*}, \vec{\delta}),$$

$$\Pi_{\mathcal{SC}}^{m}(\boldsymbol{w}_{\mathcal{D}}^{*}, \overline{\sigma}) = \Pi_{\mathcal{SC}}^{m}(\boldsymbol{w}_{\mathcal{D}}^{*}, \vec{\delta}).$$

Expanding either of these equations, we obtain:

$$\sum_{x\in M^*_{\mathcal{SC}}}(1-\overline{\sigma})w^*_{x,\mathcal{D}}e^{\bar{A}_x/\mu}=\sum_{x\in M^*_{\mathcal{SC}}:x_{\Bbbk}=i}(1-\delta_i)w^*_{x,\mathcal{D}}e^{\bar{A}_x/\mu}. \tag{EC.6}$$

By substituting the expression for $\overline{\sigma}$, the left-hand side of (EC.6) simplifies to $\sum_{x\in M^*_{\mathcal{SC}}:x_{\Bbbk}=i}(1-\delta_i)w^*_{x,\mathcal{D}}e^{\bar{A}_x/\mu}$, which matches the right-hand side. Thus, $\overline{\sigma}$ preserves the profit structure of the multi-rate contract, proving that any $\mathcal{SC}=\vec{\delta}$ can be represented by an equivalent single-rate contract $\mathcal{SC}=\vec{\delta'}$, where $\delta'_i=\overline{\sigma}$ if $i\in S^*_{\mathcal{C}}(\Bbbk)$, and $\delta'_i=0$ otherwise. Q.E.D.

Proposition 1. The optimal assortment is determined by comparing the potential optimal assortments in feasibility set $Q_{\mathcal{D}}$, obtained by Procedures 2. This comparison results in the terms outlined in the theorem. Importantly, the selected assortment $M^*_{\mathcal{D}}(\boldsymbol{w})$ determines $S^*_{\mathcal{D}}(k)$ for all $k\in\mathcal{A}$, including $k=\Bbbk$. Q.E.D.

Corollary 1. As mentioned earlier, we suppose the levels of attribute $\Bbbk$ are numbered from 1 to $|L(\Bbbk)|$, where level 1 has the highest $\widetilde{u}_{\Bbbk,l(\Bbbk)}$ and level $|L(\Bbbk)|$ has the lowest $\widetilde{u}_{\Bbbk,l(\Bbbk)}$. To facilitate the proof, assume that each level of $\Bbbk$ corresponds to exactly one product. This assumption does not reduce generality, since if multiple products share the same level, Lemma 1 ensures the optimal selection among them.

Given $|S_{\mathcal{C}}(\Bbbk)|=\zeta$, we define an assortment $M^*=\{[1]_{\Bbbk:1},[1]_{\Bbbk:2},...,[1]_{\Bbbk:\zeta}\}$. Moreover, considering $t<\zeta<t'$, we define an alternative assortment $M'=\{[1]_{\Bbbk:1},[1]_{\Bbbk:2},...,[1]_{\Bbbk:t-1},[1]_{\Bbbk:t+1},...,[1]_{\Bbbk:\zeta},[1]_{\Bbbk:t'}\}$ that differs from M^* in one product and assume that $\widetilde{U}_{[1]_{\Bbbk:t}}>\widetilde{U}_{[1]_{\Bbbk:t'}}$. The retailer's expected profits for these assortments are

$$\Pi^r_{\mathcal{D}}(M^*)=\frac{\sum_{i=1}^{\zeta}\left(\mathcal{P}^r_{[1]_{\Bbbk:i}}-w_{[1]_{\Bbbk:i}}\right)e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu}}{1+\sum_{i=1}^{\zeta}e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu}+\sum_{i=\zeta+1}^{|L(\Bbbk)|}e^{(\bar{A}_{[1]_{\Bbbk:i}|M^*})/\mu}}, \tag{EC.7}$$

$$\Pi^r_{\mathcal{D}}(M^*)=\frac{\sum_{i=1,i\neq t}^{\zeta}\left(\mathcal{P}^r_{[1]_{\Bbbk:i}}-w_{[1]_{\Bbbk:i}}\right)e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu}+\left(\mathcal{P}^r_{[1]_{\Bbbk:t'}}-w_{[1]_{\Bbbk:it'}}\right)e^{(\bar{A}_{[1]_{\Bbbk:t'}})/\mu}}{1+\sum_{i=1,i\neq t}^{\zeta}e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu}+e^{(\bar{A}_{[1]_{\Bbbk:t'}})/\mu}+\sum_{i=\zeta+1,i\neq t'}^{|L(\Bbbk)|}e^{(\bar{A}_{[1]_{\Bbbk:i}|M'})/\mu}+e^{(\bar{A}_{[1]_{\Bbbk:t}|M'})/\mu}}. \tag{EC.8}$$

Note that in these profit terms, for $j\notin M^*,M'$, we have $e^{(\bar{A}_{j|M^*})/\mu}=e^{(\bar{A}_{j|M'})/\mu}$, because it was assumed that each level of $\Bbbk$ is represented by one product; therefore, no other products reveal accurate information for j.

Let $\mathcal{H} = \sum_{i=1,i\neq t}^{\zeta} \left(\mathcal{P}^r_{[1]_{\Bbbk:i}} - w_{[1]_{\Bbbk:i}}\right) e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu}$ and $\mathcal{G} = 1 + \sum_{i=1,i\neq t}^{\zeta} e^{(\bar{A}_{[1]_{\Bbbk:i}})/\mu} + \sum_{i=\zeta+1,i\neq t'}^{|L(\Bbbk)|} e^{(\bar{A}_{[1]_{\Bbbk:i}|M^*})/\mu}$. Then, (EC.7) and (EC.8) can be written as

$$\Pi^r_{\mathcal{D}}(M^*) = \frac{\mathcal{H} + \left(\mathcal{P}^r_{[1]_{\Bbbk:t}} - w_{[1]_{\Bbbk:t}}\right) e^{(\bar{A}_{[1]_{\Bbbk:t}})/\mu}}{\mathcal{G} + e^{(\bar{A}_{[1]_{\Bbbk:t}})/\mu} + e^{(\bar{A}_{[1]_{\Bbbk:t'}|M^*})/\mu}}, \tag{EC.9}$$

$$\Pi^r_{\mathcal{D}}(M') = \frac{\mathcal{H} + \left(\mathcal{P}^r_{[1]_{\Bbbk:t'}} - w_{[1]_{\Bbbk:t'}}\right) e^{(\bar{A}_{[1]_{\Bbbk:t'}})/\mu}}{\mathcal{G} + e^{(\bar{A}_{[1]_{\Bbbk:t'}})/\mu} + e^{(\bar{A}_{[1]_{\Bbbk:t}|M'})/\mu}}. \tag{EC.10}$$

We aim to show that $\Pi^r_{\mathcal{D}}(M^*) > \Pi^r_{\mathcal{D}}(M')$. Since we set $\widetilde{U}_{[1]_{\Bbbk:t}} > \widetilde{U}_{[1]_{\Bbbk:t'}}$, the numerator of $\Pi^r_{\mathcal{D}}(M^*)$ in (EC.9) is strictly greater than that of $\Pi^r_{\mathcal{D}}(M')$ in (EC.10). Given that $d_{\Bbbk,t} = d_{\Bbbk,t'} = d_{\Bbbk}$, if $d_{\Bbbk} > 0$, we have $e^{(\bar{A}_{[1]_{\Bbbk:t}})/\mu} + e^{(\bar{A}_{[1]_{\Bbbk:t'}|M^*})/\mu} < e^{(\bar{A}_{[1]_{\Bbbk:t'}})/\mu} + e^{(\bar{A}_{[1]_{\Bbbk:t}|M'})/\mu}$. Therefore, the denominator of $\Pi^r_{\mathcal{D}}(M^*)$ is smaller than that of $\Pi^r_{\mathcal{D}}(M')$, which approves that $\Pi^r_{\mathcal{D}}(M^*) > \Pi^r_{\mathcal{D}}(M')$. Notably, however, if $d_{\Bbbk} < 0$, the denominator of $\Pi^r_{\mathcal{D}}(M^*)$ is not necessarily smaller than that of $\Pi^r_{\mathcal{D}}(M')$. Thus, selecting the levels with the highest $\widetilde{u}_{\Bbbk,l(\Bbbk)}$ will not be always optimal in this case. Q.E.D.

Proposition 2. Under the centralized setting, any level of $L(\Bbbk)$ may appear in $S^*_{\mathcal{C}}(\Bbbk)$. Thus, all subsets of levels in $Q_{\mathcal{C}}$ must be compared. Furthermore, by Lemma 2, multiple assortments M can represent each subset of levels. Consequently, selecting an arbitrary assortment for each subset in $Q_{\mathcal{C}}$ and comparing their resulting profits determines the optimal selection. Q.E.D.

Corollary EC.1. When products are horizontally differentiated, the optimal wholesale prices in $\boldsymbol{w}^*_{\mathcal{D}}$ are equal across all products, i.e., $w^*_{\mathcal{D}}$. Substituting this uniform wholesale price, along with the uniform retail price π, into the general condition of Theorem 1, the coordination mechanism simplifies to the form stated in Corollary EC.1. Thus, the corollary follows as a direct special case of the general scope contract. Q.E.D.

Lemma EC.1. If $C < a$, determining the optimal assortment for given $S_{\mathcal{D}}(\Bbbk)$ and $S_{\mathcal{D}}(\Bbbk')$ requires comparing the profitability of potential assortments in $\mathcal{M}_{\mathcal{D}}$. However, when $C \geq a$, the optimal assortment becomes straightforward, consisting of all products formed by every combination of levels in $S_{\mathcal{D}}(\Bbbk)$ and $S_{\mathcal{D}}(\Bbbk')$. Adding more products beyond this point alters the representation of at least one of $S_{\mathcal{D}}(\Bbbk)$ or $S_{\mathcal{D}}(\Bbbk')$.

Once $S_{\mathcal{D}}(\Bbbk)$ and $S_{\mathcal{D}}(\Bbbk')$ are represented, inaccuracies associated with these levels are eliminated in the online channel. Thus, adding more products with these levels does not alter the provided information. Consequently, the retailer utilizes available capacity as much as possible (up to a products), provided that the representation of $S_{\mathcal{D}}(\mathbb{k})$ and $S_{\mathcal{D}}(\mathbb{k}')$ remains unchanged, enabling an expanded assortment and increased profit. Q.E.D.

Theorem EC.1. The proof is similar to the proof of Theorem 1. Under $\mathcal{SC} = \vec{\theta}$, neither of the parties should be worse off compared to their decentralized setting. Therefore, we must have

$$\Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|\boldsymbol{w}^*_{\mathcal{D}}, \vec{\theta}) \geq \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) \tag{EC.11a}$$

$$\Pi^m_{\mathcal{SC}}(\boldsymbol{w}^*_{\mathcal{D}}, \vec{\theta}) \geq \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}}). \tag{EC.11b}$$

Substituting the corresponding profit functions into (EC.11) and simplifying, condition (EC.1) will be derived.

Notably, the upper bound of (EC.1) must be greater than or equal to its lower bound. Writing this condition, we will get

$$\alpha'_3 + \alpha'_2 \geq \alpha'_1 \Bigg(\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) + \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}}) \Bigg).$$

Substituting α'_1, α'_2, and α'_3, this inequality turns into $\Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}}) \geq \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) + \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}})$, which always holds by the definition of the centralized total profit.

Moreover, since $\sum_{x \in M^*_{\mathcal{SC}}} (1-\theta_x) w^*_{x,\mathcal{D}} e^{\bar{A}_x/\mu}$ obtains a nonnegative value, it is necessary that the upper bound be nonnegative. Otherwise, if the upper bound is negative, i.e., $\alpha'_3 - \alpha'_1 \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) < 0$, no $\vec{\theta}$ can compensate for the retailer's loss. In such cases, a one-time payment $\mathcal{L}$ from the manufacturer to the retailer can be introduced using an alternative discount rate $\mathcal{SC} = \vec{\tau}$, ensuring

$$\Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) \leq \Pi^r_{\mathcal{SC}}(M^*_{\mathcal{SC}}|\boldsymbol{w}^*_{\mathcal{D}}, \vec{\tau}) + \mathcal{L}.$$

The value of this payment should also guarantee that the manufacturer is not worse off compared to its decentralized profit (i.e., $\Pi^m_{\mathcal{D}}(w^*_{\mathcal{D}}) \leq \Pi^m_{\mathcal{SC}}(w^*_{\mathcal{D}}, \vec{\tau}) - \mathcal{L}$). This defines the valid interval for $\mathcal{L}$ within the theorem.

Notably, since $\Pi^T_{\mathcal{C}}(M^*_{\mathcal{C}}) \geq \Pi^r_{\mathcal{D}}(M^*_{\mathcal{D}}|\boldsymbol{w}^*_{\mathcal{D}}) + \Pi^m_{\mathcal{D}}(\boldsymbol{w}^*_{\mathcal{D}})$, the retailer's loss is never greater than the additional profit generated by the system. Hence, either $\mathcal{SC} = \vec{\theta}$ or $\mathcal{SC} = \vec{\tau}$ with $\mathcal{L}$ successfully coordinates the ORSC for any problem instance. Q.E.D.

EC.5. List of Products and Their Attribute Levels

Table EC.2 List of products based on their attractiveness and their corresponding attribute levels.

Product	$[1]_{3:1}$	$[2]_{3:1}$	$[3]_{3:1}$	$[4]_{3:1}$	$[5]_{3:1}$	$[6]_{3:1}$	$[1]_{3:2}$	$[2]_{3:2}$	$[3]_{3:2}$	$[4]_{3:2}$	$[5]_{3:2}$	$[6]_{3:2}$	$[1]_{3:3}$	$[2]_{3:3}$	$[3]_{3:3}$	$[4]_{3:3}$	$[5]_{3:3}$	$[6]_{3:3}$	$[1]_{3:4}$	$[2]_{3:4}$	$[3]_{3:4}$	$[4]_{3:4}$	$[5]_{3:4}$	$[6]_{3:4}$
$\mathcal{A}_1$ level	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
$\mathcal{A}_2$ level	1	1	1	2	2	2	1	1	1	2	2	2	1	1	1	2	2	2	1	1	1	2	2	2
$\mathcal{A}_3$ level	1	1	1	1	1	1	2	2	2	2	2	2	3	3	3	3	3	3	4	4	4	4	4	4

References

Gaur V, Honhon D (2006) Assortment planning and inventory decisions under a locational choice model. *Management Science* 52(10):1528–1543.