Please Customers or Prevent Wastage? Replenishment and Issuance Policy for a Perishable Product with Age-sensitive Demand: Proofs and Computational Results

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EC.1. Lemma 1

Part 1: $\frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,i}} \leq \frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,i+1}}$ is equivalent to showing $\pi_t(\mathbf{x}_t + \delta \mathbf{e}_i) \leq \pi_t(\mathbf{x}_t + \delta \mathbf{e}_{i+1})$, where $\delta > 0$ is infinitesimal and $\mathbf{e}_i$ is an $n-1$ dimensional vector whose i^{th} element is 1 and all other elements are 0.

Consider two systems. Let $\pi_t(\mathbf{x}_t + \delta \mathbf{e}_i)$ be the profit for system 1, which follows the optimal policy, and let $\pi_t^P(\mathbf{x}_t + \delta \mathbf{e}_{i+1})$ be the profit for system 2, which follows policy P. Policy P mimics the replenishment and issuing policy of system 1 with one change. For any sample demand path, whenever the extra δ unit in system 1 is used to satisfy demand, the extra δ unit in system 2 is also used to satisfy the same demand. Moreover, in case the extra δ unit in system 1 perishes, the extra δ unit in system 2 also perishes. Clearly, the goodwill and outdating costs of system 2 are not more than that of system 1. Furthermore, all the other units incur the same goodwill or outdating cost in both systems. Therefore, $\pi_t(\mathbf{x}_t + \delta \mathbf{e}_{i+1})^P \geq \pi_t(\mathbf{x}_t + \delta \mathbf{e}_i)$, and so $\frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,i}} \leq \frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,i+1}}$.

Next, proving $\frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,n-1}} \leq c$ is equivalent to proving $\pi_t(\mathbf{x}_t + \delta \mathbf{e}_{n-1}) - \pi_t(\mathbf{x}_t) \leq c\delta$. Let $\pi_t(\mathbf{x}_t) = \max_{q_t \geq 0}\{-cq_t + v_t(\mathbf{x}_t, q_t)\}$, where $v_t(\mathbf{x}_t, q_t) = -\theta x_{t,1} + \mathbf{E}_{D_t}\{\max_{\mathbf{0} \leq \mathbf{z}_t \leq \tilde{\mathbf{x}}_t, \sum_{i=1}^{n} z_{t,i} \leq D_t} u_t(\mathbf{z}_t, \tilde{\mathbf{x}}_t)\}$. The above argument to prove $\frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,i}} \leq \frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,i+1}}$ can also be used to prove $\frac{\partial v_t(\mathbf{x}_t, q_t)}{\partial x_{t,1}} \leq ... \leq \frac{\partial v_t(\mathbf{x}_t, q_t)}{\partial x_{t,n-1}} \leq \frac{\partial v_t(\mathbf{x}_t, q_t)}{\partial q_t}$. Let q_t^* be the optimal order quantity for $\mathbf{x}_t + \delta \mathbf{e}_{n-1}$. Then, $\pi_t(\mathbf{x}_t + \delta \mathbf{e}_{n-1}) - \pi_t(\mathbf{x}_t) \leq -cq_t^* + v_t(\mathbf{x}_t + \delta \mathbf{e}_{n-1}, q_t^*) + c(q_t^* + \delta) - v_t(\mathbf{x}_t, q_t^* + \delta) = v_t(\mathbf{x}_t + \delta \mathbf{e}_{n-1}, q_t^*) + c\delta - v_t(\mathbf{x}_t, q_t^* + \delta) \leq c\delta$, where the first inequality follows since $q_t^* + \delta$ may not be the optimal order quantity for $\mathbf{x}_t$, and the second inequality follows from $\frac{\partial v_t(\mathbf{x}_t, q_t)}{\partial x_{t,n-1}} \leq \frac{\partial v_t(\mathbf{x}_t, q_t)}{\partial q_t}$.

Lastly, to show that $-\theta \leq \frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,1}}$, define $\pi_t(\mathbf{x}_t)$ and $\pi_t^P(\mathbf{x}_t + \delta \mathbf{e}_1)$ as before, where policy P is such that system 2, which follows policy P, mimics the replenishment and issuing policy of system 1, which follows the optimal policy, and the extra δ unit in system 2 is not used to fulfill any demand and perishes. Clearly, $\pi_t(\mathbf{x}_t + \delta \mathbf{e}_1) - \pi_t(\mathbf{x}_t) \geq \pi_t^P(\mathbf{x}_t + \delta \mathbf{e}_1) - \pi_t(\mathbf{x}_t) = -\theta\delta$. Hence, $\frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,1}} \geq -\theta$.

Part 2: We first show that $v_t(\mathbf{x}_t + \delta \mathbf{e}_1, q_t) \leq v_t(\mathbf{x}_t, q_t + \delta(1 - \xi_1))$, where $\delta > 0$ is infinitesimal. Define $v_t(\mathbf{x}_t + \delta \mathbf{e}_1, q_t)$ as the profit for system 1, which follows the optimal policy, and

$v_t^P(\mathbf{x}_t, q_t+\delta(1-\xi_1))$ as the profit for system 2, which follows policy P. Policy P is such that system 2 mimics the replenishment and issuing policy of system 1 with one change. For any sample demand path, if the extra δ unit in system 1 is used to satisfy demand, the extra $(1-\xi_1)\delta$ unit in system 2 is used to fulfill $(1-\xi_1)\delta$ portion of the same demand. The rest of the corresponding demand $(\xi_1\delta)$ is left unfulfilled in system 2. Thus, the revenue generated in the two systems net of the goodwill cost is the same. Additionally, if the extra δ unit in system 1 perishes, the extra unit in system 2 also perishes, so the outdating cost of system 2 is at most the outdating cost of system 1. Furthermore, every unit apart from the extra δ unit in system 1 and $(1-\xi_1)\delta$ unit in system 2 incurs the same goodwill or outdating cost in both systems. Therefore, $v_t(\mathbf{x}_t, q_t+\delta(1-\xi_1)) \geq v_t(\mathbf{x}_t+\delta\mathbf{e}_1, q_t)$. Let q_t^* be the optimal order quantity for $\pi_t(\mathbf{x}_t+\delta\mathbf{e}_1)$. Thus, $\pi_t(\mathbf{x}_t+\delta\mathbf{e}_1)-\pi_t(\mathbf{x}_t) \leq -cq_t^*+v_t(\mathbf{x}_t+\delta\mathbf{e}_1, q_t^*)+c(q_t^*+\delta(1-\xi_1))-v_t(\mathbf{x}_t, q_t^*+\delta(1-\xi_1)) = v_t(\mathbf{x}_t+\delta\mathbf{e}_1, q_t^*)+c(1-\xi_1)\delta - v_t(\mathbf{x}_t, q_t^*+\delta(1-\xi_1)) \leq c(1-\xi_1)\delta$. The first inequality follows since $q_t^*+\delta(1-\xi_1)$ may not be the optimal order quantity for $\mathbf{x}_t$. The second inequality follows since $v_t(\mathbf{x}_t+\delta\mathbf{e}_1, q_t) \leq v_t(\mathbf{x}_t, q_t+\delta(1-\xi_1))$.

EC.2. Theorem 1

We use x_t instead of $\mathbf{x}_t$, and let $z_t = z_{t,1}$. When $d_t < x_t+q_t$, $u_t(\mathbf{z}_t, \tilde{\mathbf{x}}_t) = (-r\xi_1+\theta)z_t + rd_t + \alpha\pi_{t+1}(q_t+z_t-d_t)$ and so $\frac{\partial u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)}{\partial z_t} = -r\xi_1+\theta+\alpha\frac{\partial \pi_{t+1}(q_t+z_t-d_t)}{\partial z_t}$. Since π_{t+1} is concave (using Theorem 2), $u_t(\mathbf{z}_t, \tilde{\mathbf{x}}_t)$ is concave in z_t. Three cases arise: $\alpha\frac{\partial\pi_{t+1}(x)}{\partial x} - r\xi_1+\theta < 0 \quad \forall x \geq 0$ (Case 1), $\alpha\frac{\partial\pi_{t+1}(x)}{\partial x} - r\xi_1+\theta > 0 \quad \forall x \geq 0$ (Case 2), and $\alpha\frac{\partial\pi_{t+1}(x)}{\partial x} - r\xi_1+\theta = 0$ for some $x = \lambda_t \geq 0$ (Case 3). We define M_t as $0, \infty$, and λ_t in Cases 1, 2, and 3, respectively.

In Case 1, z_t^* should be as small as possible, so $z_t^* = \min((d_t-q_t)^+, x_t)$. In Case 2, z_t^* should be as large as possible, so $z_t^* = \min(x_t, d_t)$. In Case 3, whenever $q_t \leq \lambda_t$, z_t^* should again be maximized and so $z_t^* = \min(x_t, d_t)$. Conversely, whenever $q_t > \lambda_t$, z_t^* should be chosen so that $q_t+z_t^*-d_t$ is as close as possible to λ_t, so $z_t^* = \min((d_t-(q_t-\lambda_t))^+, x_t)$.

Thus, in general, we can express $z_t^* = \min((D_t-(q_t-M_t)^+)^+, x_t) = \min((D_t+M_t-q_t)^+, D_t, x_t)$. Using the last expression, $\pi_t(x_t) = \max_{q_t\geq 0}\mathbf{E}\big\{-cq_t + r(1-\xi_1)\min((D_t+M_t-q_t)^+, D_t, x_t) + r\min\big(q_t, [D_t-\min((D_t+M_t-q_t)^+, D_t, x_t)]^+\big) - \theta\big(x_t - \min((D_t+M_t-q_t)^+, D_t, x_t)\big)^+ + \alpha\pi_{t+1}\big(q_t - \min\big(q_t, [D_t-\min((D_t+M_t-q_t)^+, D_t, x_t)]^+\big)\big)\big\}$. Let $g_t(q_t, x_t)$ be the maximand in the above formulation. Also, let $K = \min((D_t+M_t-q_t)^+, D_t, x_t)$. Using the definition of K, $g_t(q_t, x_t) = \mathbf{E}\big\{-cq_t + r(1-\xi_1)K + r\min(q_t, (D_t-K)^+) - \theta(x_t-K)^+ + \alpha\pi_{t+1}(q_t - \min(q_t, (D_t-K)^+))\big\}$.

We next show that $\frac{\partial g_t(q_t,x_t)}{\partial q_t}$ is decreasing in x_t. We will drop subscript t in the rest of the proof. Fix $D=d$. First, let $x \geq d$. Suppose that $q<M$. Then, $K=d$ and so $g_t(q,x)=-cq+r(1-\xi_1)d-\theta(x-d)+\alpha\pi_{t+1}(q)$ and $\frac{\partial g_t(q,x)}{\partial q}=-c+\alpha\frac{\partial \pi_{t+1}(q)}{\partial q}$. Thus, $\frac{\partial g_t(q,x)}{\partial q}$ is independent of x. Suppose next that $M \leq q < M+d$. Then, $K=M+d-q$ and so $g_t(q,x)=-cq+r(1-\xi_1)(M+d-q)+r(q-M)-\theta(x-M-d+q)+\alpha\pi_{t+1}(M)$ and $\frac{\partial g_t(q,x)}{\partial q}=-c+r\xi_1-\theta$. Once again, $\frac{\partial g_t(q,x)}{\partial q}$ is independent of x. Finally, let $q \geq M+d$. Then, $K=0$ and so $g_t(q,x)=-cq+rd-\theta x+\alpha\pi_{t+1}(q-d)$, and $\frac{\partial g_t(q,x)}{\partial q}=-c+\alpha\frac{\partial \pi_{t+1}(q-d)}{\partial q}$. Once again, $\frac{\partial g_t(q,x)}{\partial q}$ is independent of x. Hence, we can conclude that $\frac{\partial g_t(q,x)}{\partial q}$ is independent of x when $x \geq d$.

Suppose now that $x<d$. Let $q<d-x$. Then, $K=x$ and so $g_t(q,x)=-cq+r(1-\xi_1)x+rq+\alpha\pi_{t+1}(0)$ and $\frac{\partial g_t(q,x)}{\partial q}=-c+r$. Thus, $\frac{\partial g_t(q,x)}{\partial q}$ is independent of x. Suppose now that $d-x \leq q < M+d-x$. Then, $K=x$ and so $g_t(q,x)=-cq+r(1-\xi_1)x+r(d-x)+\alpha\pi_{t+1}(q+x-d)$ and $\frac{\partial g_t(q,x)}{\partial q}=-c+\frac{\partial \pi_{t+1}(q+x-d)}{\partial q}$. Thus, $\frac{\partial g_t(q,x)}{\partial q}$ decreases with x since π_{t+1} is concave. Next, let $M+d-x \leq q < M+d$. Then, $K=M+d-q$ and so $g_t(q,x)=-cq+r(1-\xi_1)(M+d-q)+r(q-M)-\theta(x-M-d+q)+\alpha\pi_{t+1}(M)$ and $\frac{\partial g_t(q,x)}{\partial q}=-c+r\xi_1-\theta$. Thus, $\frac{\partial g_t(q,x)}{\partial q}$ is independent of x. Finally, let $q \geq M+d$. Here, $K=0$ and so $g_t(q,x)=-cq+rd-\theta x+\alpha\pi_{t+1}(q-d)$ and $\frac{\partial g_t(q,x)}{\partial q}=-c+\alpha\frac{\partial \pi_{t+1}(q-d)}{\partial q}$. Thus, $\frac{\partial g_t(q,x)}{\partial q}$ is independent of x. Hence, $\frac{\partial g_t(q,x)}{\partial q}$ decreases with x when $x<d$.

It can be seen that $\frac{\partial g_t(q,x)}{\partial q}$ is continuous at all the break points, so $\frac{\partial g_t(q,x)}{\partial q}$ decreases with x. Since $\pi(x)=\max_{q\geq 0} g_t(q,x)$, q^* decreases with x (using implicit function theorem).

EC.3. Example 1

As $\xi_1<\frac{\theta+\alpha s}{r}$, in period T, it is optimal to issue the oldest inventory first, then fresh inventory, and finally inventory of remaining lifetime of two periods. Therefore, $\pi_T(x_{T,1},x_{T,2})=\max_{q_T\geq 0}\mathbf{E}\{-cq_T+(1-\xi_1)r\min(D_T,x_{T,1})+r\min((D_T-x_{T,1},)^+,q_T)+(1-\xi_2)r\min((D_T-x_{T,1}-q_T)^+,x_{T,2})-\theta(x_{T,1}-D_T)^+\}$. Substituting the parameter values and computing the expectations, we get $\pi_T(x_{T,1},x_{T,2})=\max_{q_T\geq 0}\{-0.04q_T^2-0.044x_{T,1}^2-0.036x_{T,2}^2+3q_T+6.8x_{T,1}+7.2x_{T,2}-0.08q_Tx_{T,1}-0.072q_Tx_{T,2}-0.072x_{T,1}x_{T,2}\}$. Solving for q_T^*, we get $q_T^*=(37.5-x_{T,1}-0.9x_{T,2})^+$.

Recall that $u_{T-1}(\mathbf{z}_{T-1},\tilde{\mathbf{x}}_{T-1})=r(1-\xi_1)z_{T-1,1}+r(1-\xi_2)z_{T-1,2}+rz_{T-1,3}+\pi_T(30-z_{T-1,2},20-z_{T-1,3})$. Using the above expression for π_T, $u_{T-1}(\mathbf{z}_{T-1},\tilde{\mathbf{x}}_{T-1})=250.8-0.044z_{T-1,2}^2-0.036z_{T-1,3}^2+4.48z_{T-1,2}+4.4z_{T-1,3}-0.072z_{T-1,2}z_{T-1,3}$ when $x_{T,1}+(1-\xi_2)x_{T,2}>37.5$, which leads to $\frac{\partial u_{T-1}}{\partial z_{T-1,2}}=4.48-0.088z_{T-1,2}-0.072z_{T-1,3}$ and $\frac{\partial u_{T-1}}{\partial z_{T-1,3}}=4.4-0.072z_{T-1,2}-0.072z_{T-1,3}$.

The above expressions imply that it is optimal to issue inventory of remaining lifetime of two periods till $z_{T-1,2}=5$ due to greater marginal value. At this point, the marginal values of the two inventories become equal. Subsequently, it is optimal to issue fresh inventory, though marginal values of both inventories remain equal, till $x_{T,1}+(1-\xi_2)x_{T,2}=37.5$ or $z_{T-1,3}=\frac{55}{9}=6.1$.

When $x_{T,1}+(1-\xi_2)x_{T,2}\le 37.5$, $u_{T-1}(\mathbf{z}_{T-1},\tilde{\mathbf{x}}_{T-1})=255.21-0.004z_{T-1,2}^2-0.0036z_{T-1,3}^2+3.64z_{T-1,2}+3.644z_{T-1,3}$. Thus, $\frac{\partial u_{T-1}}{\partial z_{T-1,2}}=3.64-0.008z_{T-1,2}$ and $\frac{\partial u_{T-1}}{\partial z_{T-1,3}}=3.644-0.0072z_{T-1,3}$. Thus, it is optimal to issue $z_{T-1,2}$ and $z_{T-1,3}$ in the ratio of $0.9:1$ till $x_{T-1,3}$ is completely exhausted, which will require demand of 26.4 units. When the fresh inventory is exhausted, 12.5 units of $x_{T-1,2}$ are left, which are issued in the end.

EC.4. Proposition 1

Consider the following constrained optimization problem: $\max u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)=\theta z_{t,1}+r\sum_{i=1}^n(1-\xi_i)z_{t,i}+\alpha\pi_{t+1}(\tilde{x}_{t,2}-z_{t,2},\dots,\tilde{x}_{t,n}-z_{t,n})$ s.t. $\mathbf{0}\le\mathbf{z}_t\le\tilde{\mathbf{x}}_t,\sum_{i=1}^n z_{t,i}\le d_t$. Since $u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)$ is concave (using Theorem 2) and constraints are affine, the KKT conditions are necessary and sufficient. The corresponding Lagrangian function is $u_t^1(\mathbf{z}_t,\tilde{\mathbf{x}}_t)=\theta z_{t,1}+r\sum_{i=1}^n(1-\xi_i)z_{t,i}+\alpha\pi_{t+1}(\tilde{x}_{t,2}-z_{t,2},\dots,\tilde{x}_{t,n}-z_{t,n})-\lambda_0(\sum_{i=1}^n z_{t,i}-d_t)-\sum_{i=1}^n\lambda_i(z_{t,i}-\tilde{x}_{t,i})+\sum_{i=1}^n\lambda_i^1 z_{t,i}$.

For the optimum $\mathbf{z}_t^*$, the complementary slackness conditions are as follows: $\lambda_0(\sum_{i=1}^n z_{t,i}^*-d_t)=0$ and $\lambda_j(z_{t,j}^*-\tilde{x}_{t,j})$ and $\lambda_j^1 z_{t,j}^*=0$ for $j\in\{1...,n\}$. Suppose $z_{t,i}^*\in(0,\tilde{x}_i)$ for $i=j,k$, then $\lambda_j=\lambda_k=\lambda_j^1=\lambda_k^1=0$. Now, using the stationarity conditions, at $\mathbf{z}_t^*$, $\frac{\partial u_t^1(\mathbf{z}_t,\tilde{\mathbf{x}}_t)}{\partial z_{t,j}}=\frac{\partial u_t^1(\mathbf{z}_t,\tilde{\mathbf{x}}_t)}{\partial z_{t,k}}=0$ or $\frac{\partial u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)}{\partial z_{t,j}}-\lambda_0=\frac{\partial u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)}{\partial z_{t,k}}-\lambda_0$ or $\frac{\partial u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)}{\partial z_{t,j}}=\frac{\partial u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)}{\partial z_{t,k}}$.

EC.5. Theorem 2

Suppose, by the way of induction, that $\pi_{t+1}(\mathbf{x}_{t+1})$ is concave. It is easy to see that $u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)$ is jointly concave in $(\mathbf{z}_t,\tilde{\mathbf{x}}_t)$ since the first two terms in the expression for u_t are linear and the last term is jointly concave by induction.

Consider the function $\psi(\tilde{\mathbf{x}}_t,d_t)=\max_{\mathbf{0}\le\mathbf{z}_t\le\tilde{\mathbf{x}}_t,\sum_{i=1}^n z_{t,i}\le d_t}u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)$. Since each constraint is a half space, the feasible set for each constraint is a convex set. Furthermore, since an intersection of half spaces is a convex set, the feasible space for the above maximization problem is a convex set. This implies that $\psi(\tilde{\mathbf{x}}_t,d_t)$ is concave in $\tilde{\mathbf{x}}_t$ for any demand realization d_t. Since concavity is preserved under expectation, $\mathbf{E}_{D_t}\psi(\tilde{\mathbf{x}}_t,D_t)=\mathbf{E}_{D_t}\max_{\mathbf{0}\le\mathbf{z}_t\le\tilde{\mathbf{x}}_t,\sum_{i=1}^n z_{t,i}\le d_t}u_t(\mathbf{z}_t,\tilde{\mathbf{x}}_t)$ is also concave in $\tilde{\mathbf{x}}_t$. Define $g_t(\tilde{\mathbf{x}}_t)=-c\tilde{x}_{t,n}+\mathbf{E}_{D_t}\psi(\tilde{\mathbf{x}}_t,D_t)$. Clearly, g is concave in $\tilde{\mathbf{x}}_t$. Since concavity is preserved under maximization, $\pi_t(\mathbf{x}_t)=\max_{q_t\ge 0}g_t(\mathbf{x}_t,q_t)$, is also concave.

EC.6. Theorem 3

LEMMA EC.1. 1. *Suppose that the optimal issuance sequence is fixed to* $(i_1, ..., i_n)$*, where* $i_j \in \{1, .., n\}$ *such that* $i_1 = 1$ *and* $i_n = n$*. If* $\pi_{t+1}(x_{t+1,i_2-1}, ..., x_{t+1,i_n-1})$ *is anti-multimodular, then* $\pi_t(x_{t,i_1}, ..., x_{t,i_{n-1}})$ *is anti-multimodular.*

2. *Suppose that the optimal issuance sequence is fixed to* $(i_1, ..., i_n)$*, where* $i_j \in \{1, .., n\}$ *such that* $i_1 = n$ *and* $i_n = 1$*. If* $\pi_{t+1}(x_{t+1,i_1-1}, ..., x_{t+1,i_{n-1}-1})$ *is anti-multimodular, then* $\pi_t(x_{t,i_1}, ..., x_{t,i_{n-1}})$ *is anti-multimodular.*

Proof: We prove the result here for Part 1; the proof is similar (and hence omitted) for Part 2. Let $\mathbf{x}_t = (x_{t,i_1}, ..., x_{t,i_{n-1}})$, where $(i_1, ..., i_n)$ is the optimal issuance sequence such that $i_1 = 1$ and $i_n = n$, and let $\mathbf{x}'_t = (x_{t,i_1}, ..., x_{t,i_{n-1}}, q_t)$. Also, let $\mathbf{z}'_t = (z_{t,i_1}, ..., z_{t,i_n})$. We will use z_{t,i_n} and $z_{t,n}$ interchangeably as $i_n = n$ by assumption. Similarly, we will use q_t, $x_{t,n}$ and x_{t,i_n} interchangeably.

Let $g_t(\mathbf{x}'_t) = \max_{\mathbf{z}'_t \in \mathcal{O}(\mathbf{x}'_t, d_t)} u_t(\mathbf{x}'_t, \mathbf{z}'_t)$, where $u_t(\mathbf{x}'_t, \mathbf{z}'_t) = -\theta(x_{t,i_1} - z_{t,i_1}) + r\sum_{j=1}^{n}(1 - \xi_{i_j})z_{t,i_j} + \alpha\pi_{t+1}(x_{t,i_2} - z_{t,i_2}, ..., x_{t,i_n} - z_{t,i_n})$ and $\mathcal{O}(\mathbf{x}'_t, d_t) = \left\{\mathbf{z}'_t : \sum_{j=1}^{n} z_{t,i_j} \le d_t, 0 \le z_{t,i_j} \le x_{t,i_j} \text{ for } 1 \le j \le n\right\}$. Therefore, $\pi_t(\mathbf{x}_t) = \max_{q_t \ge 0}\{-cq_t + \mathbf{E}g_t(\mathbf{x}'_t)\}$. Note that the order of arguments for π_{t+1}, $(i_2, ..., i_n)$, may represent a different order than the argument of π_t, $(i_1, ..., i_{n-1})$, because inventories have aged by one period.

Define a new constraint set, $\mathcal{T}(\mathbf{x}'_t, d_t) = \left\{\mathbf{z}'_t : z_{t,i_k} \le x_{t,i_k}, \sum_{j=1}^{k} z_{t,i_j} \le d_t \text{ for } 1 \le k \le n\right\}$. Now, it is optimal to satisfy the demand as much as possible; that is, $\sum_{j=1}^{n} z_{t,i_j} = \min(\sum_{j=1}^{n} x_{t,i_j}, d_t)$. Further, the optimal issuance sequence being fixed to $(i_1, ..., i_n)$ implies that given the constraints, $0 \le z_{t,i_k} \le x_{t,i_k}$ $\forall k \in \{1, ..., n\}$ and $\sum_{j=1}^{n} z_{t,i_j} \le d_t$, the optimal issued quantities are equal to $z^*_{t,i_j} = \min(x_{t,i_j}, d_t - \sum_{k=1}^{j-1} z^*_{t,i_k})$. The optimal sequence being fixed to $(i_1, ..., i_{n-1}, n)$ also implies that it is preferable to issue z_{t,i_1} over z_{t,i_2}, z_{t,i_2} over z_{t,i_3} and so on. Thus, it is optimal to issue inventory from a bucket in the order sequence only when all the preceding buckets have been completely issued.

However, without the non-negativity constraints, it is possible that a set of preceding buckets are filled at the expense of a subsequent bucket while maintaining $z_{t,i_k} \le x_{t,i_k}$ $\forall k \in \{1, ..., n\}$ and $\sum_{j=1}^{n} z_{t,i_j} \le d_t$. In this scenario, the optimal solution could be $\sum_{j=1}^{k} z_{t,i_j} > d_t$ for some k and $z_{t,i_m} < 0$ for some $m > k$. However, the constraint set $\sum_{j=1}^{k} z_{t,i_j} \le d_t$ for $1 \le k \le n$ ensures that such scenarios do not occur. Hence, the solution to $\mathcal{T}(\mathbf{x}'_t, d_t)$ is same as the solution to $\mathcal{O}(\mathbf{x}_t^{'}, d_t)$ and is equal to $z^*_{t,i_j} = \min(x_{t,i_j}, d_t - \sum_{k=1}^{j-1} z^*_{t,i_k})$. Therefore, it does not matter whether the maximization is subjected to $\mathcal{O}(\mathbf{x}_t^{'}, d_t)$ or $\mathcal{T}(\mathbf{x}_t^{'}, d_t)$.

Let $\bar{z}_{t,i_j} = z_{t,i_j} - x_{t,i_j}$ for $1 \le j \le n$. Also, let $\bar{\mathbf{z}}_t = (\bar{z}_{t,i_1}, ..., \bar{z}_{t,i_n})$ and $\bar{\mathbf{x}}_t = (x_{t,i_n} \ldots, x_{t,i_1})$. Using these definitions, $g_t(\mathbf{x}_t^{'})$ can be written as $g_t(\mathbf{x}_t^{'}) = \max_{\bar{\mathbf{z}}_t \in \mathcal{T}(\mathbf{x}_t^{'}, d_t)} u_t(\bar{\mathbf{x}}_t, \bar{\mathbf{z}}_t)$ where $u_t(\bar{\mathbf{x}}_\mathbf{t}, \bar{\mathbf{z}}_t) = \max\left\{\theta \bar{z}_{t,i_1} + r\sum_{j=1}^{n}(1 - \xi_{i_j})(\bar{z}_{t,i_j} + x_{t,i_j}) + \alpha\pi_{t+1}(-\bar{z}_{t,i_1}, ..., -\bar{z}_{t,i_n})\right\}$ and $\mathcal{T}(\mathbf{x}_t^{'}, d_t) = \left\{\bar{z}_{t,i_j} \le 0 \text{ for } 1 \le j \le n, \sum_{j=1}^{k}(\bar{z}_{t,i_j} + x_{t,i_j}) \le d_t \text{ for } 1 \le k \le n\right\}$.

The above constraint set is clearly a polyhedron satisfying anti-multimodularity conditions (Li and Yu 2014). Furthermore, in the above expression, the first two terms are linear and the third term is anti-multimodular since $\pi_{t+1}(x_{t+1,i_2-1}, ..., x_{t+1,i_n-1})$ is assumed to be anti-multimodular and inventories of remaining lifetimes $(i_2, ..., i_n)$ after aging by one period have remaining lifetimes of $(i_2 - 1, ..., i_n - 1)$, respectively. Hence, $u_t(\bar{\mathbf{x}}_t, \bar{\mathbf{z}}_t)$ is anti-multimodular, which implies that $g_t(\mathbf{x}_t^{'})$ is anti-multimodular, using Theorem 1 (i) and Lemma 2 (vi) of Li and Yu (2014).

Furthermore, $\mathbf{E}g_t(\mathbf{x}_t^{'})$ is anti-multimodular from Lemma 2 (iv) of Li and Yu (2014). Since $\pi_t(\mathbf{x}_t) = \max_{q_t \ge 0}\{-cq_t + \mathbf{E}g_t(\mathbf{x}_t^{'})\}$, $\pi_t(\mathbf{x}_t)$ is anti-multimodular directly using Theorem 1 (ii) in Li and Yu (2014). □

Using Lemma EC.1, if the optimal issuance sequence is FIFO (or $(1, ..., n)$) and π_{t+1} is anti-multimodular in $(x_{t+1,1}, ..., x_{t+1,n-1})$, then π_t is anti-multimodular in $(x_{t,1}, ..., x_{t,n-1})$. Similarly, if the optimal issuance sequence is LIFO (or $(n, ..., 1)$) and π_{t+1} is anti-multimodular in $(x_{t+1,n-1}, ..., x_{t+1,1})$, then π_t is anti-multimodular $(x_{t,n-1}, ..., x_{t,1})$, which also implies that π_t is anti-multimodular in $(x_{t,1}, ..., x_{t,n-1})$ (from Lemma 2 (ii) of Li and Yu (2014)). Hence, for both these cases, anti-multimodular can easily be established through induction. The second result is a direct consequence of the anti-multimodularity (Li and Yu 2014).

EC.7. Proposition 2

Part 1: We prove the result by contradiction. Consider any sample path of demands, $\{d_1, ..., d_T\}$. Suppose that the optimal policy, P^o, is such that it is optimal to sell a unit with remaining lifetime of j periods (unit u_j) instead of a unit with remaining lifetime of i periods (unit u_i), where $i > j$, in some period t.

Consider an alternate policy (P^a) that is same as P^o with one exception: The period of sale of unit u_j is interchanged with that of unit u_i. Suppose that u_i is sold in period

$t_1 > t$ in policy P^o. If $t_1 - t \geq j$, then u_j would expire before sale in policy P^a. In that case, we let a unit of demand go unsatisfied in P^a in period t_1. Let $m = t_1 - t$. The following possibilities arise.

- $0 < t_1 - t < j$. In this case, u_j is sold in P^a and u_i is sold in P^o. Then, $\pi_t^a - \pi_t^o = (\xi_j - \xi_i)r - \alpha^m(\xi_{j-m} - \xi_{i-m})r \geq (\xi_j - \xi_i)r - (\xi_{j-m} - \xi_{i-m})r \geq 0$, where the last inequality follows since $\xi_j - \xi_i \geq \xi_{j-m} - \xi_{i-m}$ by assumption.
- $j \leq t_1 - t < i$. In this case, u_j expires before sale in P^a, but u_i is sold in P^o. Then, $\pi_t^a - \pi_t^o = (\xi_j - \xi_i)r - \alpha^m(1 - \xi_{i-m})r - \alpha^{j-1}\theta \geq (\xi_j - \xi_i)r - \alpha^m(1 - \xi_{i-j})r - \alpha^{j-1}\theta \geq (\xi_j - \xi_i)r - (1 - \xi_{i-j})r - \theta \geq 0$. The first inequality holds since $m \geq j$ and hence $\xi_{i-m} \geq \xi_{i-j}$. The last inequality follows since $(\xi_j - \xi_i) - (1 - \xi_{i-j}) - \frac{\theta}{r} \geq 0$ by assumption.
- $t_1 - t \geq i$. In this case, u_i expires in P^o, and u_j expires in P^a. Therefore, $\pi_t^a - \pi_t^o = (\xi_j - \xi_i)r + (\alpha^{i-1} - \alpha^{j-1})\theta \geq 0$.
- $t_1 = T + 1$ and $t_1 - t < j$. In this case, u_j is salvaged in P^a and u_i is salvaged in P^o, so $\pi_t^a - \pi_t^o = (\xi_j - \xi_i)r \geq 0$.
- $t_1 = T + 1$ and $j \leq t_1 - t < i$. In this case, u_i is salvaged in P^o but u_j has expired by then, so $\pi_t^a - \pi_t^o = (\xi_j - \xi_i)r - \alpha^m s - \alpha^{j-1}\theta = (\xi_j - \xi_i)r - s - \theta \geq 0$, where the last inequality follows since $(\xi_j - \xi_i) \geq \frac{s+\theta}{r}$ by assumption.

Hence, by contradiction, P^o cannot be optimal. Since this is true for any demand path, under given assumptions, LIFO is optimal.

Part 2: We prove subpart (a) by proving a more general result: if $\xi_i = \xi_{i+1}$ for any $i \in \{1, ..., n-1\}$, issuing inventory with remaining lifetime i periods before inventory with remaining lifetime $i+1$ periods is optimal. Using Lemma 1, the marginal value of a unit with remaining lifetime i periods is more than the marginal value of a unit with remaining lifetime $i-1$ periods. Therefore, when $i \geq 2$, issuing a unit with remaining lifetime i periods instead of a unit with remaining lifetime $i+1$ periods does not impact the profit in the current period while it increases the future profit. When $i = 1$, issuing a unit with remaining lifetime one period instead of a unit with remaining lifetime two periods increases the current period profit by θ while the maximum possible decrease in the future profit is θ as $\frac{\partial \pi_t(\mathbf{x}_t)}{\partial x_{t,1}} \geq -\theta$. Therefore, it is optimal to issue inventory with remaining lifetime i periods before inventory with remaining lifetime $i+1$ periods. Thus, if $\xi_1 = ... = \xi_{n-1} = 0$, issuing inventory in the FIFO order is optimal. Subpart (b) follows directly from Lemma 1.

EC.8. Theorem 4

Given $\tilde{\mathbf{x}}_t$, $\zeta(d_t) = \max_{\mathbf{0} \le \mathbf{z}_t \le \tilde{\mathbf{x}}_t, \sum_{i=1}^n z_{t,i} \le d_t} u_t(\mathbf{z}_t, \tilde{\mathbf{x}}_t)$ is concave in d_t. Suppose that an infinitesimal increase in demand, δ, is satisfied with the oldest inventory. Then, $\zeta(d_t + \delta) = \zeta(d_t) + ((1-\xi_1)r + \theta)\delta$. This rate is the same regardless of the quantity of the oldest inventory $x_{t,1}$. Since $\zeta(d_t)$ is concave in d_t, once its marginal rate of increase with respect to d_t equals $(1-\xi_1)r + \theta$, the best one can do is maintain that rate; continuously issuing the oldest inventory till it is depleted ensures this.

EC.9. Proposition 3

Clearly, $\tilde{u}_t$ is a linear function of $\mathbf{z}_t$ such that $\frac{\partial \tilde{u}_t}{\partial z_{t,1}} = \theta + r(1-\xi_1) =: a_1$ and $\frac{\partial \tilde{u}_t}{\partial z_{t,i}} = r(1-\xi_i) - \alpha\gamma c(1-\xi_{i-1}) =: a_i$ for $i \in \{2,..,n\}$. Since the derivatives are independent of the state $(\tilde{\mathbf{x}}_t)$ and the period index (t), the issuance policy is a bucket policy that remains the same throughout the planning horizon. Suppose $a_{i_1} \ge .... \ge a_{i_n}$ and $i_k = n$, that is, the $x_{t,i_1}, ..., x_{t,i_{k-1}}$ are issued before q_t and the rest are issued after. Let $D_{t,i_1} = D_t$ and $D_{t,i_j} = (D_{t,i_{j-1}} - x_{t,i_{j-1}})^+$ for $j \in \{2,...,n\}$. Then,

$$\pi_t^{AIOP}(\mathbf{x}_t) = \max_{q_t \ge 0}\{-cq_t + \mathbf{E}_{D_t}\{\sum_{j=1}^n r(1-\xi_{i_j})\min(x_{t,i_j}, D_{t,i_j}) + \sum_{j=1, i_j \ne 1}^n \alpha\gamma c(1-\xi_{i_j-1})(x_{t,i_j} - D_{t,i_j})^+ - \theta(x_{t,1} - D_{t,1})^+\}\}.$$

Let $\tilde{U}_t(\mathbf{x}_t, q_t) = -cq_t + \mathbf{E}_{D_t}\{\sum_{j=1}^n r(1-\xi_{i_j})\min(x_{i_j}, D_{t,i_j}) + \sum_{j=1,i_j\ne 1}^n \alpha\gamma c(1-\xi_{i_j-1})(x_{t,i_j} - D_{t,i_j})^+ - \theta(x_{t,1} - D_{t,1})^+\}$. Since $\frac{\partial E\min(x_{t,i_j}, D_{t,i_j})}{\partial q_t} = \begin{cases} 0, & j \in \{1,...,k-1\} \\ 1 - \Phi(\sum_{\ell=1}^k x_{t,i_\ell}), & j = k \\ \Phi(\sum_{\ell=1}^{j-1} x_{t,i_\ell}) - \Phi(\sum_{\ell=1}^{j} x_{t,i_\ell}), & j > k \end{cases}$ and $\frac{\partial E(x_{t,i_j} - D_{t,i_j})^+}{\partial q_t} = \begin{cases} 0, & j \in \{1,...,k-1\} \\ \Phi(\sum_{\ell=1}^k x_{t,i_\ell}), & j = k \\ \Phi(\sum_{\ell=1}^{j} x_{t,i_\ell}) - \Phi(\sum_{\ell=1}^{j-1} x_{t,i_\ell}), & j > k \end{cases}$, $\frac{\partial \tilde{U}_t(\mathbf{x}_t, q_t)}{\partial q_t} = -c + r(1-\xi_n) - a_{i_k}\Phi(\sum_{\ell=1}^k x_{t,i_\ell}) - \sum_{j=k+1}^n a_{i_j}(\Phi(\sum_{\ell=1}^j x_{t,i_\ell}) - \Phi(\sum_{\ell=1}^{j-1} x_{t,i_\ell}))$. Furthermore, $\frac{\partial^2 \tilde{U}_t(\boldsymbol{x}_t, q_t)}{\partial x_{t,i_j}\partial q_t} = \begin{cases} -\sum_{m=k}^{n-1}(a_{i_m} - a_{i_{m+1}})\phi(\sum_{\ell=1}^m x_{t,i_\ell}) - a_{i_n}\phi(\sum_{\ell=1}^n x_{t,i_\ell}) & j \in \{1,...,k-1\} \\ -\sum_{m=j}^{n-1}(a_{i_m} - a_{i_{m+1}})\phi(\sum_{\ell=1}^m x_{t,i_\ell}) - a_{i_n}\phi(\sum_{\ell=1}^n x_{t,i_\ell}) & j \ge k \end{cases}$. Hence, $\frac{\partial^2 \tilde{U}_t(\boldsymbol{x}_t, q_t)}{\partial x_{t,i_j}\partial q_t} \le 0$ as $a_{i_1} \ge .... \ge a_{i_n}$. Therefore, the order quantity monotonically decreases with inventory of any remaining lifetime.

EC.10. Theorem 5

LEMMA EC.2. *Suppose that $\xi_m = \xi_{m+1} = ... = \xi_n = 0$ for $1 \le m \le n$ and $\xi_{m-1} > 0$ when $m > 1$. If $r > \frac{\alpha\gamma c + \max\{\theta,0\}}{\xi_{m-1}}$, the AIOP issues units with remaining lifetime m periods before older units. Additionally, inventory with remaining lifetime of $m+1$ or more periods in issued in the FIFO order. Moreover, inventory with remaining lifetime of m periods is either issued before inventory with remaining lifetime $m+1$ periods (policy $\mathcal{I}_1$) or after inventory with remaining lifetime n periods (policy $\mathcal{I}_2$). Specifically, if $m > 1$ or $\theta + \alpha\gamma c \ge 0$, $\mathcal{I}_1$ is chosen. Otherwise, $\mathcal{I}_2$ is chosen.*

Proof: Recall from the proof of Proposition 3, $\frac{\partial \tilde{u}_t}{\partial z_{t,1}} = \theta + r(1-\xi_1) =: a_1$ and $\frac{\partial \tilde{u}_t}{\partial z_{t,i}} = r(1-\xi_i) - \alpha\gamma c(1-\xi_{i-1}) =: a_i$ for $i \in \{2,..,n\}$. Clearly, if $m > 1$, $a_{m-1} \le r(1-\xi_{m-1}) + \max\{\theta, 0\}$ and $a_m \ge r - \alpha c\gamma$. Therefore, if $m > 1$, $a_m \ge a_{m-1}$ if $r - \alpha\gamma c > r(1-\xi_{m-1}) + \max\{\theta, 0\}$ or $r > \frac{\alpha\gamma c + \max\{\theta,0\}}{\xi_{m-1}}$. Thus, if $r > \frac{\alpha\gamma c + \max\{\theta,0\}}{\xi_{m-1}}$, the *AIOP* issues units with remaining lifetime m periods before any older units.

Now, if $m > 1$, $a_m = r - \alpha c\gamma(1-\xi_{m-1}) \ge a_{m+1} = ... = a_n = r - \alpha c\gamma$, so the *AIOP* follows policy $\mathcal{I}_1$. If $m = 1$, $a_1 = r + \theta$. Clearly, when $\theta + \alpha\gamma c \ge 0$, $a_1 \ge a_2 = \ldots = a_n$, and, once gain, the *AIOP* follows policy $\mathcal{I}_1$. Otherwise, if $\theta + \alpha\gamma c < 0$, $a_1 < a_2 = \ldots = a_n$, the *AIOP* follows policy $\mathcal{I}_2$. $\square$

Let $\bar{D} = \sup\{x : \Phi(x) < 1\} < \infty$, and consider the *AIOP* policy. From Proposition 3, $\frac{\partial \tilde{U}_t(\mathbf{x}_t,q_t)}{\partial q_t} = -c + r(1-\xi_n) - a_{i_k}\Phi(\sum_{\ell=1}^{k} x_{t,i_\ell}) - \sum_{j=k+1}^{n} a_{i_j}(\Phi(\sum_{\ell=1}^{j} x_{t,i_\ell}) - \Phi(\sum_{\ell=1}^{j-1} x_{t,i_\ell}))$, where $i_k = n$. We next show that as $r \to \infty$, $\frac{\partial \tilde{U}_t(\mathbf{x}_t,q_t)}{\partial q_t} \ge 0$. Suppose, without loss of generality, $\xi_m = \xi_{m+1} = ... = \xi_n = 0$, where $1 \le m \le n$, and $\xi_{m-1} > 0$ when $m > 1$. If $r > \frac{\alpha\gamma c + \max\{\theta,0\}}{\xi_{m-1}}$, *AIOP* either chooses $\mathcal{I}_1$ or $\mathcal{I}_2$ as the issuing policy (Lemma EC.2). We show that as $r \to \infty$, $\frac{\partial \tilde{U}_t(\mathbf{x}_t,q_t)}{\partial q_t} > 0$ if $\Phi(\sum_{i=m}^{n} x_{t,i}) < 1$ under policy $\mathcal{I}_1$ and $m > 1$. The proofs for policy $\mathcal{I}_2$ or when $m = 1$ are similar and hence omitted.

Under $\mathcal{I}_1$, let the inventory with remaining lifetime of $m-1$ or fewer periods be issued in the order $(i_1, ..., i_{m-1})$. Thus, $\frac{\partial \tilde{U}_t(\mathbf{x}_t,q_t)}{\partial q_t} = -c + r - a_n\Phi(\sum_{i=m}^{n} x_{t,i}) - \sum_{j=1}^{m-1} a_{i_j}(\Phi(\sum_{i=m}^{n} x_{t,i} + \sum_{\ell=1}^{j} x_{t,i_\ell}) - \Phi(\sum_{i=m}^{n} x_{t,i} + \sum_{\ell=1}^{j-1} x_{t,i_\ell})) \ge -c + r - r\Phi(\sum_{i=m}^{n} x_{t,i}) - a_{i_1}\sum_{j=1}^{m-1}(\Phi(\sum_{i=m}^{n} x_{t,i} + \sum_{\ell=1}^{m-1} x_{t,i_\ell}) - \Phi(\sum_{i=m}^{n} x_{t,i})) = -c + r - r\Phi(\sum_{i=m}^{n} x_{t,i}) - a_{i_1}(\Phi(\sum_{i=1}^{n} x_{t,i}) - \Phi(\sum_{i=m}^{n} x_{t,i})) \ge -c + r - r\Phi(\sum_{i=m}^{n} x_{t,i}) - (r(1-\xi_{m-1}) + \max\{\theta, 0\})(\Phi(\sum_{i=1}^{n} x_{t,i}) - \Phi(\sum_{i=m}^{n} x_{t,i})) \ge -c - \max\{\theta, 0\} + r - r\Phi(\sum_{i=m}^{n} x_{t,i}) - r(1-\xi_{m-1})(\Phi(\sum_{i=1}^{n} x_{t,i}) - \Phi(\sum_{i=m}^{n} x_{t,i})) = -c - \max\{\theta, 0\} + r - r[\Phi(\sum_{i=1}^{n} x_{t,i}) - \xi_{m-1}(\Phi(\sum_{i=1}^{n} x_{t,i}) - \Phi(\sum_{i=m}^{n} x_{t,i}))]$, where the first inequality follows since $a_{i_1} \ge .... \ge a_{i_m}$, the second inequality follows since $a_{i_1} \le r(1-\xi_{m-1}) + \max\{\theta, 0\}$, and the last inequality follows since $\Phi(\sum_{i=1}^n x_{t,i}) - \Phi(\sum_{i=m}^n x_{t,i}) \geq 0$. Therefore, $\frac{\partial \tilde{U}_t(\mathbf{x}_t,q_t)}{\partial q_t} \geq -c - \max\{\theta, 0\} + r - rS$ where $S = \Phi(\sum_{i=1}^n x_{t,i}) - \xi_{m-1}(\Phi(\sum_{i=1}^n x_{t,i}) - \Phi(\sum_{i=m}^n x_{t,i}))$. We now show that as $r \to \infty$, $\frac{\partial \tilde{U}_t(\mathbf{x}_t,q_t)}{\partial q_t} > 0$ if $\Phi(\sum_{i=m}^n x_{t,i}) < 1$, which is equivalent to proving $S < 1$ if $\Phi(\sum_{i=m}^n x_{t,i}) < 1$.

To see this, observe that either $\Phi(\sum_{i=m}^n x_{t,i}) < \Phi(\sum_{i=1}^n x_{t,i})$ or $\Phi(\sum_{i=m}^n x_{t,i}) = \Phi(\sum_{i=1}^n x_{t,i})$. In the first case, $S = \Phi(\sum_{i=1}^n x_{t,i}) - \xi_{m-1}(\Phi(\sum_{i=1}^n x_{t,i}) - \Phi(\sum_{i=1}^n x_{t,i})) < \Phi(\sum_{i=1}^n x_{t,i}) \leq 1$, where the first inequality follows since $\xi_{m-1} > 0$. In the second case, $S = \Phi(\sum_{i=1}^n x_{t,i}) = \Phi(\sum_{i=m}^n x_{t,i}) < 1$.

Therefore, as $r \to \infty$, the order quantity under $AIOP$ should be such that $\Phi(\sum_{i=m}^n x_{t,i}) = 1$ or $\sum_{i=m}^n x_{t,i} = \bar{D}$. Consequently, the expected profit in any period t must be at least $r\mu - c\bar{D} - \theta\bar{D}$ if $\theta \geq 0$ and at least $r\mu - c\bar{D}$ if $\theta < 0$. On the other hand, the expected profit under the optimal policy in any period t cannot be more than $r\mu$. Therefore, $\lim_{r\to\infty} \frac{\pi_t^{AIOP}(\mathbf{x}_t)}{\pi_t(\mathbf{x}_t)} = 1$.

We now prove a stronger result: $\lim_{r\to\infty} \pi_t(\mathbf{x}_t) - \pi_t^{AIOP}(\mathbf{x}_t) = 0$ when $\theta \geq 0$. From Proposition 2, FIFO is optimal if $m = 1$. Further, from the proof of Proposition 2, if $\xi_i = \xi_{i+1}$ for any $i \in \{1, ..., n-1\}$, it is optimal to issue inventory with remaining lifetime i periods before inventory with remaining lifetime $i+1$ periods. Therefore, it is optimal to issue inventory with m or more periods remaining lifetime in the FIFO order. Moreover, if $m > 1$, from Lemma 1, part 1, it is suboptimal to issue the oldest inventory before completely issuing the inventory with remaining lifetime m or more periods if $r\xi_1 - \theta \geq \alpha c$ or $r \geq \frac{\alpha c + \theta}{\xi_1}$. Similarly, it is suboptimal to issue inventory with remaining lifetime i periods, where $1 < i < m$, before completely issuing inventory with remaining lifetime m or more periods if $r\xi_i \geq \alpha c + \alpha\theta$ or $r \geq \alpha\frac{c+\theta}{\xi_i}$. Therefore, when r is large enough, it is optimal to issue inventory with remaining lifetime m or more periods in the FIFO order and then the rest of the inventory.

Define $w_t(\mathbf{x}_t, q_t) = -cq_t - \theta x_{t,1} + \mathbf{E}_{D_t}\big\{\max_{\mathbf{0}\leq \mathbf{z}_t \leq \tilde{\mathbf{x}}_t, \sum_{i=1}^n z_{t,i} \leq D_t} u_t(\mathbf{z}_t, \tilde{\mathbf{x}}_t)\big\}$, so $\pi_t(\mathbf{x}_t) = \max_{q_t \geq 0}\big\{w_t(\mathbf{x}_t, q_t)\big\}$. From Lemma 1, the marginal value of any inventory lies in $[-\theta, c]$. Therefore, $w_t(\mathbf{x}_t, q_t) \leq w^A(\mathbf{x}_t, q_t)$, where $w^A(\mathbf{x}_t, q_t)$ the profit obtained if all the inventory remaining after fulfilling demand in period t is salvaged at unit price c; that is, $w^A(\mathbf{x}_t, q_t) = -cq_t - \theta x_{t,1} + \mathbf{E}_{D_t} \max_{\mathbf{0}\leq \mathbf{z}_t \leq \tilde{\mathbf{x}}_t, \sum_{i=1}^n z_{t,i} \leq D_t}\big\{\theta z_{t,1} + r\sum_{i=1}^n (1-\xi_i) z_{t,i} + \alpha c \sum_{i=2}^n (\tilde{x}_{t,i} - z_{t,i}) + \alpha\pi_{t+1}(\mathbf{0})\big\}$. Similarly, $w_t(\mathbf{x}_t, q_t) \geq w^B(\mathbf{x}_t, q_t)$, where $w^B(\mathbf{x}_t, q_t)$ is the profit obtained if all the inventory remaining after fulfilling demand in period t is outdated at unit cost of θ; that is, $w^B(\mathbf{x}_t, q_t) = -cq_t - \theta x_{t,1} + \mathbf{E}_{D_t} \max_{\mathbf{0}\leq \mathbf{z}_t \leq \tilde{\mathbf{x}}_t, \sum_{i=1}^n z_{t,i} \leq D_t}\big\{\theta z_{t,1} + r\sum_{i=1}^n (1-\xi_i) z_{t,i} - \alpha\theta \sum_{i=2}^n (\tilde{x}_{t,i} - z_{t,i}) + \alpha\pi_{t+1}(\mathbf{0})\big\}$. The optimal issuing policy in both scenarios A and B is a threshold policy, where inventory with remaining lifetime m or more periods is issued in the FIFO order and the remaining inventory is issued in LIFO order. Therefore, $w_t(\mathbf{x}_t, q_t+\delta) - w_t(\mathbf{x}_t, q_t) \geq w^B(\mathbf{x}_t, q_t+\delta) - w^A(\mathbf{x}_t, q_t)$ where $\delta > 0$ is infinitesimal. This implies that $\frac{\partial w_t(\mathbf{x}_t, q_t)}{\partial q_t} \geq -c + r(1 - \Phi(\sum_{i=m}^n x_{t,i})) - \sum_{j=1}^{m-1}(r(1-\xi_{m-j}) + c + \theta)(\Phi(\sum_{i=m-j}^n x_{t,i}) - \Phi(\sum_{i=m-j+1}^n x_{t,i})) \geq -c + r(1-\Phi(\sum_{i=m}^n x_{t,i})) - \sum_{j=1}^{m-1}(r(1-\xi_{m-1}) + c + \theta)(\Phi(\sum_{i=m-j}^n x_{t,i}) - \Phi(\sum_{i=m-j+1}^n x_{t,i})) = -c + r(1 - \Phi(\sum_{i=m}^n x_{t,i})) - (r(1-\xi_{m-1}) + c + \theta)(\Phi(\sum_{i=1}^n x_{t,i}) - \Phi(\sum_{i=m}^n x_{t,i})) = -c + r - r[\Phi(\sum_{i=1}^n x_{t,i}) - (\xi_{m-1} - \frac{c+\theta}{r})(\Phi(\sum_{i=1}^n x_{t,i}) - \Phi(\sum_{i=m}^n x_{t,i}))]$. Similar to the *AIOP*, it can be seen that as $r \to \infty$, $\frac{\partial w_t(\mathbf{x}_t, q_t)}{\partial q_t} > 0$ if $\Phi(\sum_{i=m}^n x_{t,i}) < 1$. Therefore, as $r \to \infty$, the optimal order quantity is such that $\Phi(\sum_{i=m}^n x_{t,i}) = 1$ or $\sum_{i=m}^n x_{t,i} = \bar{D}$.

Therefore, as $r \to \infty$, both the *AIOP* and optimal policy order inventory such that $\sum_{i=m}^n x_{t,i} = \bar{D}$. Additionally, if $\theta \geq 0$, using Lemma EC.2, the *AIOP* and optimal policies prescribe similar issuing policy, where inventory with remaining lifetime m or more periods is issued in the FIFO order and then the rest of the inventory is issued. Together, these imply that inventory with remaining lifetime more than m periods remains unsold in both policies. As a consequence, as $r \to \infty$, the inventory dynamics in the optimal policy and the *AIOP* are same such that inventory with more than m period lifetime is never sold and ends up outdating. Hence, $\lim_{r\to\infty} \pi_t(\mathbf{x}_t) - \pi_t^{AIOP}(\mathbf{x}_t) = 0$.

EC.11. Theorem 6

Scenario 1: Consider a clairvoyant who can see the demand in advance. Clearly, the expected profit for an inventory system operated by the clairvoyant will be $\pi^{Cl}(\mathbf{0}) = (r-c)\mu\sum_{t=1}^T \alpha^{t-1}$. Consider now a system under policy P. For this system, the expected profit in any period t is at least $\mathbf{E}\{r\min\{\mu, D_t\} - c\mu - \theta\mu\}$ if $\theta > 0$ as the revenue obtained is at least $\mathbf{E}r\min\{\mu, D_t\}$, purchasing cost is $c\mu$, and outdating cost cannot be more than $\theta\mu$. In contrast, if $\theta \leq 0$, the expected profit in any period t is at least $\mathbf{E}\{r\min\{\mu, D_t\} - c\mu\}$. Therefore, $\frac{\pi^P(\mathbf{0})}{\pi(\mathbf{0})} \geq \frac{\mathbf{E}\{r\min\{\mu,D_t\}-c\mu-\theta\mu\}\sum_{t=1}^T\alpha^{t-1}}{(r-c)\mu\sum_{t=1}^T\alpha^{t-1}} \geq \frac{r\frac{\mu}{2}-c\mu-\theta\mu}{(r-c)\mu} = \frac{1}{2}\frac{1-2\frac{c+\theta}{r}}{1-\frac{c}{r}}$ if $\theta > 0$. However, if $\theta \leq 0$, $\frac{\pi^P(\mathbf{0})}{\pi(\mathbf{0})} \geq \frac{\mathbf{E}\{r\min\{\mu,D_t\}-c\mu\}\sum_{t=1}^T\alpha^{t-1}}{(r-c)\mu\sum_{t=1}^T\alpha^{t-1}} \geq \frac{r\frac{\mu}{2}-c\mu}{(r-c)\mu} = \frac{1}{2}\frac{1-2\frac{c}{r}}{1-\frac{c}{r}}$.

Scenario 2: We show this result using a sample demand path analysis. We match units between the *LIFO* and optimal policies, where the units ordered in both systems are numbered as follows. Let the units ordered in period t in the optimal and LIFO policies be numbered as $\{(t,1)^{OPT}, (t,2)^{OPT}, ..., (t,q_t)^{OPT}\}$ and $\{(t,1)^{LIFO}, (t,2)^{LIFO}, ..., (t,q_t)^{LIFO}\}$, respectively. A unit in one policy corresponds to a unit in another policy if both are used to satisfy the same demand unit. Additionally, they are same-numbered if their numbers are identical: for example, $(t,2)^{OPT}$ and $(t,2)^{LIFO}$ are same-numbered units.

A unit u^{OPT} under the optimal policy is either sold, outdated, or remains unsold at the end of the horizon. If it is sold, four outcomes are possible for the same demand unit under the $LIFO$ policy: 1) satisfied by a unit with the same remaining lifetime as u^{OPT}, 2) satisfied by a unit with lower remaining lifetime than u^{OPT}, 3) satisfied by a unit with greater remaining lifetime than u^{OPT}, and 4) not satisfied. If u^{OPT} is outdated or remains unsold at the end of the horizon, two outcomes are possible for the same-numbered unit under the $LIFO$ policy: 1) used for satisfying a demand unit and 2) outdated or remains unsold at the end of the horizon. Therefore, we can divide all the units purchased in the optimal policy into six sets.

- A demand unit is satisfied in the two policies such that the unit used in the $LIFO$ policy has the same remaining lifetime as the optimal policy. Let $\mathcal{A}_1$ be the set of such units.
- A demand unit is satisfied in the two policies such that the unit used in the $LIFO$ policy has lower remaining lifetime than the optimal policy. Let $\mathcal{A}_2$ be the set of such units.
- A demand unit is satisfied in the two policies such that the unit used in the $LIFO$ policy has greater remaining lifetime than the optimal policy. Let $\mathcal{A}_3$ be the set of such units.
- A demand unit is satisfied in the OPT policy but not in the LIFO policy. Let $\mathcal{A}_4$ be the set of such units under the optimal policy.
- A unit ordered in the optimal policy is either outdated or remains unsold at the end of the horizon, whereas the same-numbered unit is sold in the $LIFO$ policy. Let $\mathcal{A}_5$ be the set of such units.
- A unit ordered in the optimal policy is either outdated or remains unsold at the end of the horizon. The same outcome occurs for the same-numbered unit in the LIFO policy. Let $\mathcal{A}_6$ be the set of such units.

Clearly, the sets $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$ and $\mathcal{A}_6$ are mutually exclusive. Furthermore, for any set $\mathcal{A}_i$, we use superscript $LIFO$ to denote the same-numbered units under the $LIFO$ policy. For example, units in $\mathcal{A}_2^{LIFO}$ represent the same-numbered units as in $\mathcal{A}_2$ but under the $LIFO$ policy. We use $\mathcal{A}_i$ and $\mathcal{A}_i^{OPT}$, $i \in \{1,...,6\}$ interchangeably. Before proceeding to the main proof, we prove four claims.

Claim 1: Let $S(\mathcal{A})$ represent the revenue net of the goodwill cost obtained from selling units in set $\mathcal{A}$. Then, $S(\mathcal{A}_2^{LIFO}) \geq S(\mathcal{A}_2^{OPT})$.

Proof: Consider a unit $(t+k,j)^{OPT} \in \mathcal{A}_2$, where $k \geq 1$, and let the same demand unit be satisfied by $(t,i)^{LIFO}$ in the LIFO policy. $(t,i)^{LIFO}$ being sold in the $LIFO$ policy implies that $(t+k,j)^{LIFO}$ is already sold. Moreover, it must be sold with a remaining lifetime at least the unit $(t+k,j)^{OPT}$. Hence, for every unit in $\mathcal{A}_2^{OPT}$, the same-numbered unit in $\mathcal{A}_2^{LIFO}$ has been sold at same or younger age in the $LIFO$ policy, which implies that $S(\mathcal{A}_2^{LIFO}) \geq S(\mathcal{A}_2^{OPT})$. □

Claim 2: $K^{OPT} - K^{LIFO} \leq K^{LIFO} - S(\mathcal{A}_5^{LIFO})$, where K^P represents the revenue net of the goodwill cost obtained from selling units in policy P.

Proof: Consider a unit $(t,i)^{OPT} \in \mathcal{A}_4$. Since, when this unit is sold, inventory is not available in the $LIFO$ policy, $(t,i)^{LIFO}$ is already sold in or before period t. Moreover, $(t,i)^{LIFO}$ must be sold with a remaining lifetime greater than or equal to the unit $(t,i)^{OPT}$. Therefore, for every unit in $\mathcal{A}_4^{OPT}$, the same-numbered unit in $\mathcal{A}_4^{LIFO}$ is sold at the same or greater remaining lifetime. Thus, $S(\mathcal{A}_4^{LIFO}) \geq S(\mathcal{A}_4^{OPT})$.

Now, $K^{OPT} - K^{LIFO} = S(\mathcal{A}_1^{OPT}) + S(\mathcal{A}_2^{OPT}) + S(\mathcal{A}_3^{OPT}) + S(\mathcal{A}_4^{OPT}) - S(\mathcal{A}_1^{LIFO}) - S(\mathcal{A}_2^{LIFO}) - S(\mathcal{A}_3^{LIFO}) - S(\mathcal{A}_4^{LIFO}) - S(\mathcal{A}_5^{LIFO}) \leq S(\mathcal{A}_1^{OPT}) + S(\mathcal{A}_3^{OPT}) - S(\mathcal{A}_1^{LIFO}) - S(\mathcal{A}_3^{LIFO}) - S(\mathcal{A}_5^{LIFO}) \leq S(\mathcal{A}_1^{OPT}) + S(\mathcal{A}_3^{OPT}) - S(\mathcal{A}_5^{LIFO}) \leq K^{LIFO} - S(\mathcal{A}_5^{LIFO})$, where the first inequality follows since $S(\mathcal{A}_2^{LIFO}) \geq S(\mathcal{A}_2^{OPT})$ and $S(\mathcal{A}_4^{LIFO}) \geq S(\mathcal{A}_4^{OPT})$, and the last inequality follows $S(\mathcal{A}_1^{LIFO}) + S(\mathcal{A}_3^{LIFO}) \geq S(\mathcal{A}_1^{OPT}) + S(\mathcal{A}_1^{OPT})$ given the definition of $\mathcal{A}_1$ and $\mathcal{A}_3$. □

Claim 3: Let Θ^P be the total units outdated under policy P, and let $\Delta\Theta = \Theta^{LIFO} - \Theta^{OPT}$. Then, $\Delta\Theta \geq -||\mathcal{A}_5||$.

Proof: Since $s = -\theta$, the units leftover at the end of horizon can be considered to be outdated units. Thus, $\Delta\Theta = \Theta^{LIFO} - ||\mathcal{A}_5|| - ||\mathcal{A}_6|| \geq -||\mathcal{A}_5||$ since $\Theta^{LIFO} \geq ||\mathcal{A}_6||$.□

Claim 4: For every outdated unit under the $LIFO$ policy, the same-numbered unit in the optimal policy is either outdated or sold. Let the set of latter units be denoted by $\bar{\mathcal{A}}$. Then, $||\bar{\mathcal{A}}|| \leq ||\mathcal{A}_3||$.

Proof: For each unit $u \in \bar{\mathcal{A}}$, the corresponding demand unit is satisfied in the LIFO policy. Since the same-numbered unit of u is outdated under the $LIFO$ policy, the demand unit must be satisfied by a younger unit in the $LIFO$ policy. Therefore, $\bar{\mathcal{A}} \subseteq \mathcal{A}_3$ or $||\bar{\mathcal{A}}|| \leq ||\mathcal{A}_3||$. Now, $\Delta\Theta = \Theta^{LIFO} - \Theta^{OPT} = ||\bar{\mathcal{A}}|| + ||\mathcal{A}_6|| - \Theta^{OPT} \leq ||\mathcal{A}_3|| + ||\mathcal{A}_6|| - \Theta^{OPT} = ||\mathcal{A}_3|| - ||\mathcal{A}_5|| \leq ||\mathcal{A}_3||$. □

Let $\theta \leq 0$. If $\Delta\Theta \geq 0$, the outdating revenue obtained is greater in the $LIFO$ policy than the optimal policy and so, $R^{OPT}(\mathbf{0}) - R^{LIFO}(\mathbf{0}) = K^{OPT} - K^{LIFO} + \theta\Delta\Theta \leq K^{LIFO} + \theta\Delta\Theta \leq K^{LIFO} \leq R^{LIFO}(\mathbf{0})$, where the first inequality follows using Claim 2 and since $S(\mathcal{A}_5) \geq 0$. If $\Delta\Theta < 0$, $R^{OPT}(\mathbf{0}) - R^{LIFO}(\mathbf{0}) = K^{OPT} - K^{LIFO} + \theta\Delta\Theta \leq K^{LIFO} - S(\mathcal{A}_5^{LIFO}) + \theta\Delta\Theta \leq K^{LIFO} - S(\mathcal{A}_5^{LIFO}) - \theta||\mathcal{A}_5|| \leq K^{LIFO} - r(1-\xi_1)||\mathcal{A}_5|| - \theta||\mathcal{A}_5|| \leq K^{LIFO} \leq R^{LIFO}$, where the first inequality follows from Claim 2, the second inequality follows since $\Delta\Theta \geq -||\mathcal{A}_5||$, and the third inequality follows since per unit revenue from any sold unit is at least $r(1-\xi_1)$. The fourth inequality follows since $\theta > -r(1-\xi_1)$. Hence, if $\theta \leq 0$, $\frac{R^{LIFO}(\mathbf{0})}{R^{OPT}(\mathbf{0})} \geq \frac{1}{2}$.

Consider now the case $\theta \geq 0$. If $\Delta\Theta \leq 0$, $R^{OPT}(\mathbf{0}) - R^{LIFO}(\mathbf{0}) = K^{OPT} - K^{LIFO} + \theta\Delta\Theta \leq K^{LIFO} + \theta\Delta\Theta = R^{LIFO}(\mathbf{0}) + \theta\Theta^{LIFO} + \theta\Delta\Theta \leq R^{LIFO}(\mathbf{0}) + \theta\Theta^{LIFO} = R^{LIFO}(\mathbf{0}) + R^{LIFO}(\mathbf{0}) - K^{LIFO} \leq 2R^{LIFO}(\mathbf{0})$, where the first inequality follows from Claim 2 and since $S(\mathcal{A}_5) \geq 0$. If $\Delta\Theta > 0$, $R^{OPT}(\mathbf{0}) - R^{LIFO}(\mathbf{0}) = K^{OPT} - K^{LIFO} + \theta\Delta\Theta \leq K^{LIFO} + \theta\Delta\Theta \leq K^{LIFO} + \theta||\mathcal{A}_3|| \leq K^{LIFO} + r(1-\xi_1)||\mathcal{A}_3|| \leq K^{LIFO} + K^{LIFO} = 2K^{LIFO} = 2R^{LIFO}(\mathbf{0}) + 2\theta\Theta^{LIFO} = 2R^{LIFO}(\mathbf{0}) + 2(R^{LIFO}(\mathbf{0}) - K^{LIFO}) \leq 4R^{LIFO}(\mathbf{0})$, where the first inequality follows using Claim 2 and since $S(\mathcal{A}_5) \geq 0$, the second inequality follows since $\Delta\Theta \leq ||\mathcal{A}_3||$ and the third inequality follows since $\theta \leq r(1-\xi_1)$. The fourth inequality follows since the $LIFO$ policy earns more revenue net of the goodwill cost from the units in set $\mathcal{A}_3$ than the optimal policy. Therefore, if $\theta \leq 0$, $R^{OPT}(\mathbf{0}) - R^{LIFO}(\mathbf{0}) \leq 4R^{LIFO}(\mathbf{0})$ or $\frac{R^{LIFO}(\mathbf{0})}{R^{OPT}(\mathbf{0})} \geq \frac{1}{5}$.

EC.12. Lemma 2

The proof is similar to the proof of Lemma 1 and hence omitted.

EC.13. Theorem 7

Part 1: Using Lemma 2, the marginal value of inventory of any remaining lifetime remains higher than that of older inventory even after clearance. Thus, if it is optimal to clear inventory of remaining lifetime i, then all the older inventory should also be cleared.

Part 2: Suppose, by the way of induction, that $\pi_{t+1}(\mathbf{x}_{t+1})$ is concave. Using the same argument as in Theorem 2, $v_t(\mathbf{x}_t, q_t)$ is concave in its arguments. This further implies that $\max_{q_t \geq 0, \mathbf{0} \leq \mathbf{x}_t \leq \mathbf{y}_t} v_t(\mathbf{x}_t, q_t)$ is concave in $\mathbf{y}_t$ since the feasible set for maximization is a convex set, as it is an intersection of half spaces. This implies that $\pi_t(\mathbf{y}_t)$ is concave in $\mathbf{y}_t$.

Part 3: Suppose that it is optimal to salvage an infinitesimal increase δ in $y_{t,i}$, for $i \in \{1, ..., n-1\}$. This implies that $\pi_t(\mathbf{y}_t + \delta\mathbf{e}_i) = s\delta + \pi_t(\mathbf{y}_t)$ or $\frac{\partial \pi_t(\mathbf{y}_t)}{\partial y_{t,i}} = s$, where $\mathbf{e}_i$ is an $n-1$ dimensional vector whose i^{th} element is 1 and all other elements are zero. Since

$\pi_t(\mathbf{y}_t)$ is concave in $y_{t,i}$, $\frac{\partial \pi_t(\mathbf{y}_t)}{\partial y_{t,i}}$ is decreasing in $y_{t,i}$. Therefore, once this rate hits s, the best one can do is maintain that rate, that is, continue salvaging.

Part 4: Assuming $\pi_{t+1}(\mathbf{y}_t)$ to be anti-multimodular, Lemma EC.1 shows that $-cq_t + v_t(\mathbf{x}_t, q_t)$ is anti-multimodular. Therefore, we only need to show that $\pi_t(\mathbf{y}_t) = s\sum_{i=1}^{n-1} y_{t,i} + \max_{q_t \geq 0, \mathbf{0} \leq \mathbf{x}_t \leq \mathbf{y}_t}\{-cq_t + v_t(\mathbf{x}_t, q_t)\}$ is anti-multimodular in $\mathbf{y}_t$. The same functional equation appears in Theorem 3, part (i) in Li and Yu (2014), where the anti-multiomodularity is established by relaxing the constraints $\mathbf{0} \leq \mathbf{x}_t \leq \mathbf{y}_t$ to $\sum_{k=i}^{n-1} x_k \leq \sum_{k=i}^{n-1} y_k$ for $i \in \{1, ...n-1\}$ without any loss of optimality. The same relaxation can be applied in our model as well since the clearance variables are ordered. To see this, suppose $x_i > y_i$ for some $i \in \{1, ...n-2\}$, then there must exist a $j > i$ such that $x_j < y_j$. However, such a solution cannot be optimal as $v_t(\mathbf{x}_t, q_t)$ can be increased by decreasing x_i and increasing x_j (Lemma 2). In other words, the set $\mathbf{0} \leq \mathbf{x}_t \leq \mathbf{y}_t$ can be relaxed to $\sum_{k=i}^{n-1} x_k \leq \sum_{k=i}^{n-1} y_k$ without changing the optimal solution. This leads to anti-multimodularity of $\pi_t(\mathbf{y}_t)$ where the rest of the proof is similar to Lemma EC.1 and hence omitted. The proof of the second part is similar to the proofs of part 2 and part 3 of Theorem 3 of Li and Yu (2014), respectively, and hence omitted.

EC.14. Comparison with Other Heuristics

In this section, we compare the AIOP heuristic with the age threshold and protection level policies for longer product lifetimes ($n = 3-7$). We use a sample path analysis where we consider 10000 sample demand paths. This is in contrast to the experiments in Section 5, where all the possible states are enumerated and backward recursion is used to obtain profit for all the policies, including the optimal policy. For both age threshold and protection level policies, we consider the FOQ and OUL replenishment policies. The parameters for all the policies are identified in the same manner as described in Subsection 5.2.

We parameterize the age-sensitivity factors as follows. We consider five functions to generate age-sensitivity factor such that for each function, age-sensitivity factor of the oldest inventory is 0.6 and of the fresh inventory is 0. We input n uniformly distributed values ($\{0, \frac{1}{n-1}, \frac{2}{n-1}, ..., \frac{n-2}{n-1}, 1\}$) into the following five functions: λ^3 (cubic), λ^2 (square), λ (linear), $\lambda^{\frac{1}{2}}$ (square root), $\lambda^{\frac{1}{3}}$ (cube root). We then scale the function such that it lies between 0 and 0.6. For example, when the function is linear, the age-sensitivity vector is $(0.6, 0.3, 0)$ when n is 3 and $(0.6, 0.4, 0.2, 0)$ when n is 4. Furthermore, we consider the following parameters (100 instances): $c = 5$, $\alpha = 0.98$, $s = 0$, $D_t \sim \{Unif[1,3], Unif[0.6, 3.4]\}$,

$T=50$, and $x_{1,i}=0$ for all $i=1,\dots,n-1$. In addition to the insights listed in Subsection 5.2, a few more observations are as follows.

1. Adding a protection level has no effect when the replenishment policy is OUL.
2. $AIOP$ resulted in the least optimality gap in 56 instances. In the remaining 44 instances, age threshold with OUL policy generated the least optimality gap.
3. Adding a protection level increases the profit for 32 instance when the replenishment policy is FOQ, however, the increase is only 0.07 % on average.
4. Regardless of the issuance policy, FOQ policy performs better than OUL only for six instances. This happens when age-sensitivity is high and old inventory is not valuable. For all these instances, $AIOP$ performed the best.
5. $AIOP$ performs better than the age threshold with OUL policy in all 20 instances when $\boldsymbol{\xi}$ is *cubic* or *quadratic*, 12 instances when $\boldsymbol{\xi}$ is *linear*, 2 instances when $\boldsymbol{\xi}$ is *square root*, and 2 instances when $\boldsymbol{\xi}$ is *cube root*. Therefore, age threshold with OUL policy performs better than the AIOP heuristic when the age-sensitivity effect is weak.

$\boldsymbol{\xi}$	$(1,2,3)$		$(1,3,2)$		$(2,1,3)$		$(2,3,1)$		$(3,1,2)$		$(3,2,1)$	
$r=10$	$\theta=0.5$	$\theta=2.5$	$\theta=0.5$	$\theta=2.5$	$\theta=0.5$	$\theta=2.5$	$\theta=0.5$	$\theta=2.5$	$\theta=0.5$	$\theta=2.5$	$\theta=0.5$	$\theta=2.5$
$(0,0)$	0	0	1.92	2.44	2.56	3.18	4.6	5.5	6.76	7.83	7.64	8.92
$(0.1,0)$	0.06	0	2.98	3.35	1.44	2.15	3.08	4.23	6.5	7.65	6.88	8.27
$(0.2,0)$	0.18	0	3.95	4.05	0.46	1.14	1.76	2.86	6.27	7.38	6.24	7.64
$(0.3,0)$	0.8	0.06	5.18	4.75	0.04	0.3	0.95	1.77	6.44	7.2	6.12	7.17
$(0.4,0)$	1.84	0.75	6.73	5.83	0.02	0.01	0.5	1.12	6.79	7.37	6.2	7.09
$(0.5,0)$	2.88	1.63	8.1	7.09	0.08	0.01	0.08	0.77	7.03	7.75	6.25	7.28
$(0.1,0.1)$	1.28	0.91	0.51	0.87	3.13	3.36	4.33	4.93	4.06	5.23	5.02	6.39
$(0.2,0.1)$	0.22	0.09	0.82	1.23	1.15	1.68	1.99	3.01	3.01	4.49	3.4	5.14
$(0.3,0.1)$	0.24	0.09	1.96	2.24	0.3	0.9	0.82	1.91	2.96	4.49	2.91	4.75
$(0.4,0.1)$	1.02	0.2	3.62	3.12	0.14	0.19	0.22	0.91	3.3	4.38	2.88	4.3
$(0.5,0.1)$	2.17	0.93	5.43	4.42	0.45	0.12	0.1	0.5	3.94	4.75	3.24	4.42
$(0.2,0.2)$	3.48	2.65	0.11	0.16	4.76	4.45	5.24	5.4	2.17	3.39	3.19	4.62
$(0.3,0.2)$	1.89	1.21	0.19	0.17	2.46	2.35	2.66	3.04	0.96	2.3	1.36	3
$(0.4,0.2)$	1.3	0.26	0.9	0.38	1.08	0.71	0.86	1.1	0.22	1.43	0.1	1.67
$(0.5,0.2)$	1.89	0.49	2.62	1.61	0.98	0.29	0.38	0.39	0.72	1.67	0.21	1.54
$(0.3,0.3)$	5.87	4.77	0.18	0.02	6.71	6.06	6.65	6.5	0.71	2.06	1.78	3.35
$(0.4,0.3)$	4.95	3.32	1.01	0.09	5.18	4.05	4.72	4.19	0.03	0.93	0.4	1.62
$(0.5,0.3)$	4.88	2.09	2.48	0.26	4.54	2.29	3.73	2.17	0.21	0.05	0.02	0.26
$(0.4,0.4)$	9.03	6.97	1.32	0.1	9.52	7.84	8.92	7.8	0	0.76	1.03	2.06
$(0.5,0.4)$	8.6	5.63	2.75	0.48	8.62	6.07	7.6	5.79	0.03	0.01	0.34	0.68
$(0.5,0.5)$	12.38	9.36	3.14	0.76	12.63	9.91	11.6	9.46	0	0	0.97	1.27
$r=15$												
$(0,0)$	0	0	1.13	1.5	1.55	2.02	2.91	3.69	4.59	5.68	5.14	6.39
$(0.1,0)$	0.08	0	2.17	2.4	0.59	1.06	1.55	2.38	4.24	5.35	4.34	5.64
$(0.2,0)$	0.55	0.07	3.46	3.22	0.06	0.2	0.61	1.2	4.19	5.03	3.91	4.97
$(0.3,0)$	1.52	0.81	5.05	4.54	0.04	0.01	0.17	0.62	4.49	5.23	3.89	4.88
$(0.4,0)$	2.74	1.69	6.8	5.94	0.35	0.03	0	0.26	4.97	5.54	4.1	4.93
$(0.5,0)$	4.21	2.71	8.66	7.32	0.96	0.21	0	0	5.59	5.85	4.49	5.03
$(0.1,0.1)$	1.35	1.04	0.19	0.33	2.45	2.56	3.05	3.53	2.21	3.31	2.81	4.07
$(0.2,0.1)$	0.39	0.17	0.41	0.56	0.6	0.87	0.8	1.5	0.93	2.26	1.03	2.56
$(0.3,0.1)$	1.03	0.31	2.07	1.65	0.4	0.22	0.18	0.51	1.19	2.12	0.85	2.04
$(0.4,0.1)$	2.33	1.16	4.16	3.26	0.85	0.28	0.17	0.18	1.92	2.54	1.21	2.12
$(0.5,0.1)$	3.61	2.29	6.14	5.04	1.38	0.68	0.18	0.17	2.58	3.19	1.56	2.49
$(0.2,0.2)$	3.6	2.94	0.18	0.03	4.31	4.02	4.23	4.37	0.55	1.63	1.18	2.45
$(0.3,0.2)$	3.29	1.76	1.22	0.12	3.26	2.17	2.76	2.17	0.04	0.41	0.12	0.72
$(0.4,0.2)$	3.9	1.68	3.02	1.14	3.17	1.4	2.26	1.04	0.4	0.14	0	0.02
$(0.5,0.2)$	4.77	2.43	4.86	2.82	3.33	1.5	1.92	0.73	0.83	0.6	0	0.1
$(0.3,0.3)$	6.97	5.19	1.48	0.19	7.35	5.91	6.65	5.68	0	0.27	0.66	1.13
$(0.4,0.3)$	7.08	4.84	3.09	1.27	6.87	5	5.74	4.46	0.06	0	0.11	0.3
$(0.5,0.3)$	7.65	5.1	4.96	2.74	6.86	4.69	5.34	3.75	0.51	0.18	0	0
$(0.4,0.4)$	10.74	8.42	3.47	1.57	10.89	8.84	9.63	8.09	0	0	0.66	0.86
$(0.5,0.4)$	10.8	8.2	5.14	2.89	10.47	8.18	8.81	7.15	0.09	0	0.07	0.27
$(0.5,0.5)$	14.38	11.72	5.63	3.3	14.34	11.92	12.67	10.73	0	0	0.63	0.85

Table EC.1 Optimality Gap of Different Bucket Policies for $n=3$. $D_t\sim Unif[0,2]$, $c=5$, $h=0.5$, $\alpha=0.98$, $s=0$ **and** $T=10$.