Online Supplement: Robust Capacity Planning with General Upgrading

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Appendix A: Proofs

A.1. Proof of Proposition 1

Proof. By strong duality, the dual second-stage problem (2) satisfies

$$
\begin{aligned}
f(\mathbf{d},\mathbf{x}) = \min_{\boldsymbol{\alpha},\boldsymbol{\beta}} \quad & \left\{ \sum_{i\in[n]} \alpha_i d_i + \sum_{i\in[n]} \beta_i x_i \right\}, && \text{(A.1)}\\
\text{s.t.} \quad & \alpha_i - r_i \geq \max_{j\in[i]} \{-\beta_j - s_j\}, && \forall i \in [n]\\
& \alpha_i, \beta_i \geq 0, && \forall i \in [n].
\end{aligned}
$$

Let $I_i = -\max_{j\in[i]}\{-\beta_j - s_j\}$. We can see that $-I_i = \max\{-I_{i-1}, -\beta_i - s_i\}$. Then Problem (A.1) is equivalent to

$$
\begin{aligned}
f(\mathbf{d},\mathbf{x}) = \min_{\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{I}} \quad & \left\{ \sum_{i\in[n]} \alpha_i d_i + \sum_{i\in[n]} \beta_i x_i \right\},\\
\text{s.t.} \quad & \alpha_i - r_i \geq -I_i, && \forall i \in [n]\\
& -I_i \geq -\beta_i - s_i, && \forall i \in [n]\\
& -I_i \leq -I_{i+1}, && \forall i \in [n-1]\\
& \alpha_i, \beta_i \geq 0, && \forall i \in [n]\\
& I_i \in \mathbb{R}, && \forall i \in [n].
\end{aligned}
$$

We observe an optimal solution must satisfy that $\alpha_i = (r_i - I_i)^+$ and $\beta_i = (I_i - s_i)^+$, which implies that

$$
\begin{aligned}
f(\mathbf{d},\mathbf{x}) = \min_{\mathbf{I}} \quad & \left\{ \sum_{i\in[n]} d_i (r_i - I_i)^+ + \sum_{i\in[n]} x_i (I_i - s_i)^+ \right\}, && \text{(A.2)}\\
\text{s.t.} \quad & I_i \geq I_{i+1}, && \forall i \in [n-1]\\
& I_i \in \mathbb{R}, && \forall i \in [n].
\end{aligned}
$$

By the facts that $r_1 \geq \cdots \geq r_n$, $s_1 \geq \cdots \geq s_n$ and $r_i \geq s_i$, the optimal solution of problem (A.2) must satisfy $I_i \in [s_i, r_i]$ for all $i \in [n]$, so that $(r_i - I_i)^+ = r_i - I_i$ and $(I_i - s_i)^+ = I_i - s_i$. Therefore, problem (A.2) is equivalent to the following problem:

$$
\begin{aligned}
f(\mathbf{d},\mathbf{x}) = \min_{\mathbf{I}} \quad & \left\{ \sum_{i\in[n]} d_i (r_i - I_i) + \sum_{i\in[n]} x_i (I_i - s_i) \right\},\\
\text{s.t.} \quad & I_i \geq I_{i+1}, && \forall i \in [n-1]\\
& I_i \in [s_i, r_i], && \forall i \in [n].
\end{aligned}
$$

Using the variable transformations $z_i = r_i - I_i$ for all $i \in [n]$ and adding slack variables w_0,..., w_n to remove the inequality constraints, we obtain Proposition 1. $\square$

A.2. Proof of Corollary 1

Proof. Using the variable transformation $\omega_i = a_i - z_i$ and eliminating the slack variable $\mathbf{w}$, we observe problem (5) is equivalent to the following problem:

$$f(\mathbf{d},\mathbf{x}) = \min_{\mathbf{z}\in\Lambda_{z\omega}} \left\{ \sum_{i\in[n]} (d_i - x_i) z_i \right\} + \sum_{i\in[n]} a_i x_i, \tag{A.3}$$

where

$$\Lambda_{z\omega} = \left\{ \mathbf{z},\boldsymbol{\omega} \middle| \begin{array}{ll} \omega_i + z_i = a_i, & \forall i \in [n] \\ \omega_i + z_{i+1} \geq b_i, & \forall i \in [n-1] \\ \omega_i \geq 0, z_i \geq 0, & \forall i \in [n] \end{array} \right\}.$$

It follows from strong duality that the dual problem (A.3) admits the relaxed one-level upgrading problem (6). $\square$

A.3. Proof of Proposition 2

Proof. For the ambiguity set $\mathcal{F}$, let θ, τ_i and β_i be the dual variables associated with the constraints in $\mathcal{F}$. That is,

$$\begin{array}{lll} \inf & \mathbb{E}_{\mathbb{P}}[f(\tilde{\mathbf{d}},\mathbf{x})] & \\ \text{s.t.} & \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu} & (\cdots \text{dual variable } \boldsymbol{\tau} \in \mathbb{R}^n) \\ & \mathbb{E}_{\mathbb{P}}(\tilde{d}_i^2) \leq \mu_i^2 + \sigma_i^2, \quad \forall i \in [n] & (\cdots \text{dual variable } \beta_i \in \mathbb{R}) \\ & \mathbb{P}(\tilde{\mathbf{d}} \in \mathcal{W}) = 1. & (\cdots \text{dual variable } \theta \in \mathbb{R}) \end{array}$$

Then, for a given $\mathbf{x}$, the dual of (4) is

$$\begin{aligned} \max_{\boldsymbol{\beta}\leq \mathbf{0},\theta,\boldsymbol{\tau}} \quad & \theta + \sum_{i\in[n]} \left(\tau_i \mu_i + \beta_i(\mu_i^2 + \sigma_i^2)\right) \\ \text{s.t.} \quad & \theta + \sum_{i\in[n]} \left(\tau_i d_i + \beta_i d_i^2\right) \leq f(\mathbf{d},\mathbf{x}), \forall \mathbf{d} \in \mathcal{W}. \end{aligned} \tag{A.4}$$

Note that the constraint in problem (A.4) is equivalent to

$$\theta \leq \min_{\mathbf{d}\in\mathcal{W}} \left\{ f(\mathbf{d},\mathbf{x}) - \sum_{i\in[n]} (\tau_i d_i + \beta_i d_i^2) \right\},$$

which can be further written as

$$\begin{aligned} \theta &\leq \min_{\mathbf{d}\in\mathcal{W}} \min_{\mathbf{z},\mathbf{w}\in\Lambda_0} \left\{ \sum_{i\in[n]} (d_i - x_i) z_i + \sum_{i\in[n]} a_i x_i - \sum_{i\in[n]} (\tau_i d_i + \beta_i d_i^2) \right\} \\ &= \min_{\mathbf{z},\mathbf{w}\in\Lambda_0} \min_{\mathbf{d}\in\mathcal{W}} \left\{ \sum_{i\in[n]} (d_i - x_i) z_i + \sum_{i\in[n]} a_i x_i - \sum_{i\in[n]} (\tau_i d_i + \beta_i d_i^2) \right\} \\ &= \min_{\mathbf{z},\mathbf{w}\in\Lambda_0} \sum_{i\in[n]} \min_{d_i\in\mathcal{W}_i} \left\{ (d_i - x_i) z_i + a_i x_i - (\tau_i d_i + \beta_i d_i^2) \right\} \\ &= \min_{\mathbf{z},\mathbf{w}\in\Lambda_0} \sum_{i\in[n]} h_i(z_i,\tau_i,\beta_i,x_i) + \sum_{i\in[n]} a_i x_i. \end{aligned}$$

In the optimal solution, it must hold that $\theta = \min\limits_{\mathbf{z},\mathbf{w}\in\Lambda_0} \sum_{i\in[n]} h_i(z_i,\tau_i,\beta_i,x_i) + \sum_{i\in[n]} a_i x_i$. Substituting θ back into the objective of (A.4), we obtain the result of the proposition. $\square$

A.4. Proof of Lemma 1

Proof. On the one hand, it is sufficient to show that under the condition $a_{i+1} = b_i$, for $i \in [n-1]$, any $(\mathbf{z}, \mathbf{w}) \in \mathcal{E}(\Lambda_0)$ satisfies $w_i z_i = 0$ and $w_i + z_i > 0$, for $i \in [n]$. To see this, under this condition, the network (see Figure 3) representing Λ_0 is uncapacitated. By Theorem 1 in **?**, at most one of the two inflows of each node, i.e., z_i and w_i, is positive in an extreme flow. Hence, for any $(\mathbf{z}, \mathbf{w}) \in \mathcal{E}(\Lambda_0)$, we must have $w_i z_i = 0$. Then, by $\delta_i > 0$, the constraint $w_i + z_i = \delta_i + z_{i+1}$ implies $w_i + z_i > 0$.

On the other hand, if there exists $1 \le k_1 < k_2 < ... < k_M \le n$ for some $M \in [0, n]$ such that $z_{k_1} = \cdots = z_{k_M} = 0$ and $w_j = 0$ for any $k_l < j < k_{l+1}$ and $0 \le l \le M$. Then we obtain $2n+1$ equality constraints. One can easily check that the resulting system of linear equations has the unique solution $(\mathbf{z}, \mathbf{w})$ and hence the binding constraints are linearly independent, which implies that $(\mathbf{z}, \mathbf{w}) \in \mathcal{E}(\Lambda_0)$. $\square$

A.5. Proof of Proposition 3

Proof. We denote the optimal value of problem (12) by Π'. Taking the dual of problem (11), we obtain the following dual formulation:

$$\begin{aligned} \max_{\boldsymbol{\lambda}} \quad & -\lambda_0 + \lambda_{n+1} \\ \text{s.t.} \quad & \lambda_i - \lambda_j + c_{i,j}(\boldsymbol{\tau}, \boldsymbol{\beta}, \mathbf{x}) \ge 0, \forall i \in [0, n], j \in [i+1, n+1]. \end{aligned}$$

Then, we can rewrite problem (3) with $\mathcal{W}_i = \mathbb{R}_+$ as

$$\begin{aligned} \Pi'' = \max_{\mathbf{x} \ge \mathbf{0}, \boldsymbol{\beta} \le \mathbf{0}, \boldsymbol{\tau}, \boldsymbol{\lambda}} \quad & -\lambda_0 + \lambda_{n+1} + \sum_{i \in [n]} \left(\tau_i \mu_i + \beta_i (\sigma_i^2 + \mu_i^2) \right) + \sum_{i \in [n]} (a_i - c_i) x_i \\ \text{s.t.} \quad & \lambda_i - \lambda_j + \sum_{t \in [i+1, \min\{j, n\}]} \min_{d_t \in \mathbb{R}_+} \left\{ (d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2) \right\} \ge 0, \forall i \in [0, n], j \in [i+1, n+1]. \end{aligned} \tag{A.5}$$

On the one hand, for an optimal solution $(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\tau}, \boldsymbol{\lambda})$ of (A.5), if $\beta_t < 0$, then the objective is a quadratic convex function of d_t. The unconstrained minimizer is obtained at $\frac{\pi_{i,j,t} - \tau_t}{2\beta_t}$. So the constrained minimizer in $\mathbb{R}_+$ is $\frac{\min\{\pi_{i,j,t} - \tau_t, 0\}}{2\beta_t}$, given $\beta_t < 0$. Therefore, we have $\min_{d_t \in \mathbb{R}_+} ((d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2) + \lambda_i - \lambda_j) = \frac{\left(\min(\pi_{i,j,t} - \tau_t, 0)\right)^2}{4\beta_t} - x_t \pi_{i,j,t} + \lambda_i - \lambda_j$. If $\beta_t = 0$, then we have $\min_{d_t \in \mathbb{R}_+} ((d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2) + \lambda_i - \lambda_j) = \begin{cases} -\infty, & \pi_{i,j,t} < \tau_t \\ -x_t \pi_{i,j,t} + \lambda_i - \lambda_j, & \pi_{i,j,t} \ge \tau_t. \end{cases}$, and hence, in the optimal solution it must hold $\pi_{i,j,t} \ge \tau_t$ and $\min_{d_t \in \mathbb{R}_+} ((d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2) + \lambda_i - \lambda_j) = -x_t \pi_{i,j,t} + \lambda_i - \lambda_j$. We can construct a feasible solution $(\mathbf{x}', \boldsymbol{\beta}', \boldsymbol{\tau}', \boldsymbol{\lambda}', \boldsymbol{\gamma}', \boldsymbol{\xi}')$ to problem (12), where $\mathbf{x}' = \mathbf{x}$, $\boldsymbol{\beta}' = \boldsymbol{\beta}$, $\boldsymbol{\tau}' = \boldsymbol{\tau}$, $\boldsymbol{\lambda}' = \boldsymbol{\lambda}$, $\gamma'_{i,j,t} = \min(\pi_{i,j,t} - \tau_t, 0)$, $\xi'_{i,j,t} = \frac{(\gamma_{i,j,t})^2}{4\beta_t}$ for $\beta_t < 0$, and $\xi'_{i,j,t} = \gamma'_{i,j,t} = 0$ for $\beta_t = 0$, which achieves the same objective value as Π''. Thus we have $\Pi' \ge \Pi''$. On the other hand, for an optimal solution $(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\tau}, \boldsymbol{\lambda}, \boldsymbol{\gamma}, \boldsymbol{\xi})$ of problem (12), it must hold that $\gamma_{i,j,t} = \min(\pi_{i,j,t} - \tau_t, 0)$, $\xi_{i,j,t} = \frac{(\gamma_{i,j,t})^2}{4\beta_t}$ for $\beta_t < 0$, and $\xi_{i,j,t} = \gamma_{i,j,t} = 0$, $\pi_{i,j,t} - \tau_t \ge 0$ for $\beta_t = 0$. Hence, the solution $(\mathbf{x}, \boldsymbol{\beta}, \boldsymbol{\tau}, \boldsymbol{\lambda})$ is feasible to problem (A.5) with the same objective value Π', so that $\Pi'' \ge \Pi'$. Combining with $\Pi' \ge \Pi''$, we conclude that $\Pi' = \Pi''$. $\square$

A.6. Proof of Proposition 4

Proof. By the shortest path formulatiounder the assumption that $d_i \in \mathbb{R}$, we know problem (3) is equivalent to the following problem:

$$\begin{aligned} \Pi'' = \max_{\mathbf{x}\geq \mathbf{0}, \boldsymbol{\beta}\leq \mathbf{0}, \boldsymbol{\tau}, \boldsymbol{\lambda}} \quad & -\lambda_0 + \lambda_{n+1} + \sum_{i\in[n]} \left(\tau_i \mu_i + \beta_i(\sigma_i^2 + \mu_i^2)\right) + \sum_{i\in[n]} (a_i - c_i) x_i \\ \text{s.t.} \quad & \lambda_i - \lambda_j + \sum_{t\in[i+1,\min\{j,n\}]} \min_{d_t\in\mathbb{R}} \left\{(d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2)\right\} \geq 0, \forall i \in [0,n], j \in [i+1, n+1]. \end{aligned} \tag{A.6}$$

First, it must hold that $\beta_t < 0$ for any $t \in [n]$ in an optimal solution to (A.6). Suppose, on the contrary that, $\beta_t = 0$ for some t. We notice that $\min_{d_t\in\mathbb{R}} ((d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2) + \lambda_i - \lambda_j) = -\infty$ if $\pi_{i,j,t} \neq \tau_t$. But by definition of the $\pi_{i,j,t}$, $\pi_{i,j,t} \neq \pi_{i,j,k}$ if $t \neq k$, and hence $\pi_{i,j,t} = \tau_t$ can hold for at most one t. Therefore, if $\beta_t = 0$, then $\min_{d_t\in\mathbb{R}} ((d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2) + \lambda_i - \lambda_j) = -\infty$ will hold except for at most one $t \in [i+1, \min\{j,n\}]$. This already implies infeasibility, and cannot be the optimal solution. Then we have

$$\min_{d_t\in\mathbb{R}} \left((d_t - x_t)\pi_{i,j,t} - (\tau_t d_t + \beta_t d_t^2) + \lambda_i - \lambda_j\right) = \frac{(\pi_{i,j,t} - \tau_t)^2}{4\beta_t} - x_t \pi_{i,j,t} + \lambda_i - \lambda_j.$$

By the definition of $\pi_{i,j,t}$, we know $\pi_{i,j,t} = \pi_{k,j,t}$, for $k < i$. We then define $\pi_{tj} = \pi_{i,j,t}$ for $t \leq j$. In (A.6), we observe that there exists one optimal solution satisfying $\lambda_n = \lambda_{n+1}$. To explain, consider when $i = n$ and $j = n+1$ so that the constraint $\lambda_n - \lambda_{n+1} \geq 0$ holds. Suppose there exists an optimal solution with $\lambda_n^* > \lambda_{n+1}^*$. We can construct a new solution by reducing λ_n^* to λ_{n+1}^* while keeping all other decision variables unchanged. We note that all constraints, including those involving λ_n, remain feasible with the reduced λ_n^*. Furthermore, the objective value does not change with this adjustment. Hence, the newly constructed solution with $\lambda_n^* = \lambda_{n+1}^*$ is also an optimal solution. Let $\alpha_i = -(\lambda_{i-1} - \lambda_i)$ for $i \in [n]$. Then we can reformulate (A.6) into the following equivalent problem:

$$\begin{aligned} \max_{\boldsymbol{\beta}<\mathbf{0}, \mathbf{x}\geq 0, \boldsymbol{\tau}, \boldsymbol{\alpha}} \quad & \sum_{i\in[n]} \alpha_i + \sum_{i\in[n]} \left(\tau_i \mu_i + \beta_i(\sigma_i^2 + \mu_i^2)\right) + \sum_{i\in[n]} (a_i - c_i) x_i \\ \text{s.t.} \quad & \sum_{i=k}^{\min\{j,n\}} \left(\frac{(\pi_{ij} - \tau_i)^2}{4\beta_i} - x_i \pi_{ij} - \alpha_i\right) \geq 0, \forall k \in [n], j \in [k, n+1]. \end{aligned}$$

Under the conditions of Proposition 4, we first ignore the constraint $\mathbf{x} \geq \mathbf{0}$. Then we solve the following problem:

$$\begin{aligned} \max_{\boldsymbol{\beta}<\mathbf{0}, \mathbf{x}, \boldsymbol{\tau}, \boldsymbol{\alpha}} \quad & \sum_{i\in[n]} \alpha_i + \sum_{i\in[n]} \left(\tau_i \mu_i + \beta_i(\sigma_i^2 + \mu_i^2)\right) + \sum_{i\in[n]} (a_i - c_i) x_i \\ \text{s.t.} \quad & \sum_{i=k}^{\min\{j,n\}} \left(\frac{(\pi_{ij} - \tau_i)^2}{4\beta_i} - x_i \pi_{ij} - \alpha_i\right) \geq 0, \forall k \in [n], j \in [k, n+1]. \end{aligned} \tag{A.7}$$

Let $\phi_{kj} \geq 0$ be the dual variable associated to the constraints in (A.7). The Lagrangian dual function is given by

$$\begin{aligned} L(\boldsymbol{\phi}) = & \max_{\boldsymbol{\beta}<\mathbf{0},\mathbf{x},\boldsymbol{\tau},\boldsymbol{\alpha}} \sum_{i\in[n]} \alpha_i + \sum_{i\in[n]} \left(\tau_i\mu_i + \beta_i(\sigma_i^2+\mu_i^2)\right) \\ & + \sum_{i\in[n]} (a_i - c_i)x_i + \sum_{k\in[n]} \sum_{j\in[k,n+1]} \phi_{kj} \sum_{i=k}^{\min\{j,n\}} \left(\frac{(\pi_{ij}-\tau_i)^2}{4\beta_i} - x_i\pi_{ij} - \alpha_i \right) \\ = & \max_{\boldsymbol{\beta}<\mathbf{0},\mathbf{x},\boldsymbol{\tau},\boldsymbol{\alpha}} \sum_{i\in[n]} \left(\tau_i\mu_i + \beta_i(\sigma_i^2+\mu_i^2)\right) + \sum_{i\in[n]}\sum_{k\in[i]}\sum_{j\in[i,n+1]} \phi_{kj} \frac{(\pi_{ij}-\tau_i)^2}{4\beta_i} \\ & + \sum_{i\in[n]} \left(1 - \sum_{k\in[i]}\sum_{j\in[i,n+1]} \phi_{kj} \right) \alpha_i + \sum_{i\in[n]} \left(a_i - c_i - \sum_{k\in[i]}\sum_{j\in[i,n+1]} \phi_{kj}\pi_{ij} \right) x_i, \end{aligned} \tag{A.8}$$

where the second equality in (A.8) is obtained by the fact that $\sum_{k=1}^{n}\sum_{j=k}^{n+1}\phi_{kj}\sum_{i=k}^{\min\{j,n\}}\left(\frac{(\pi_{ij}-\tau_i)^2}{4\beta_i} - x_i\pi_{ij} - \alpha_i\right) = \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{j=i}^{n+1}\phi_{kj}\left(\frac{(\pi_{ij}-\tau_i)^2}{4\beta_i} - x_i\pi_{ij} - \alpha_i\right)$. Note that $\boldsymbol{\alpha}$ and $\mathbf{x}$ are free. Therefore, to ensure $L(\boldsymbol{\phi})$ is bounded, we should let $\sum_{k=1}^{i}\sum_{j=i}^{n+1}\phi_{kj} = 1$ and $\sum_{k=1}^{i}\sum_{j=i}^{n+1}\phi_{kj}\pi_{ij} = a_i - c_i$, $\forall i \in [n]$, with which we obtain

$$L(\boldsymbol{\phi}) = \max_{\boldsymbol{\beta}<\mathbf{0},\boldsymbol{\tau}} \sum_{i\in[n]} \left(\tau_i\mu_i + \beta_i(\sigma_i^2+\mu_i^2) + \sum_{k\in[i]}\sum_{j\in[i,n+1]} \phi_{kj}\frac{(\pi_{ij}-\tau_i)^2}{4\beta_i} \right),$$

where

$$\boldsymbol{\phi} \in \Xi_0 \equiv \left\{ \begin{array}{ll} \boldsymbol{\phi} \geq 0, \sum_{k\in[i]}\sum_{j\in[i,n+1]} \phi_{kj} = 1, & \forall i \in [n] \\ \sum_{k\in[i]}\sum_{j\in[i,n+1]} \phi_{kj}\pi_{ij} = a_i - c_i, & \forall i \in [n] \end{array} \right\}.$$

For given $\boldsymbol{\tau}$ and $\boldsymbol{\phi}$, it is easy to verify that the objective function of (A.8) is concave in $\boldsymbol{\beta}$. By the first-order condition, we can get $\beta_i^* = -\frac{1}{2}\sqrt{\frac{\sum_{k=1}^{i}\sum_{j=i}^{n+1}\phi_{kj}(\pi_{ij}-\tau_i)^2}{\sigma_i^2+\mu_i^2}}$. Substituting β_i^* back into (A.8), we obtain the following formulation:

$$\begin{aligned} L(\boldsymbol{\phi}) &= \max_{\boldsymbol{\tau}} \sum_{i\in[n]} \left(\tau_i\mu_i - \sqrt{(\sigma_i^2+\mu_i^2)\left(\sum_{k\in[i]}\sum_{j\in[i,n+1]}\phi_{kj}(\pi_{ij}-\tau_i)^2\right)} \right) \\ &= \max_{\boldsymbol{\tau}} \sum_{i\in[n]} \left(\tau_i\mu_i - \sqrt{(\sigma_i^2+\mu_i^2)}\sqrt{\sum_{k\in[i]}\sum_{j\in[i,n+1]}\phi_{kj}\pi_{ij}^2 - 2(a_i-c_i)\tau_i + \tau_i^2} \right), \end{aligned} \tag{A.9}$$

where the second equality is derived based on the equality constraints of $\boldsymbol{\phi}$ in Ξ_0. Now it is easy to verify that (A.9) is also concave in $\boldsymbol{\tau}$. By the first-order condition, we can get $\tau_i^* = a_i - c_i + \frac{\mu_i}{\sigma_i}\sqrt{\sum_{k=1}^{i}\sum_{j=i}^{n+1}\phi_{kj}\pi_{ij}^2 - (a_i-c_i)^2}$. Substituting τ_i^* back into (A.9), we obtain

$$L(\boldsymbol{\phi}) = \sum_{i\in[n]} \left(\mu_i(a_i-c_i) - \sigma_i\sqrt{\sum_{k\in[i]}\sum_{j\in[i,n+1]}\phi_{kj}\pi_{ij}^2 - (a_i-c_i)^2} \right). \tag{A.10}$$

With (A.10), the Lagrangian dual problem can be formulated as:

$$\min_{\boldsymbol{\phi}\in\Xi_0} L(\boldsymbol{\phi}) = \min_{\boldsymbol{\phi}\in\Xi_0} \sum_{i\in[n]} \left(\mu_i(a_i-c_i) - \sigma_i\sqrt{\sum_{k\in[i]}\sum_{j\in[i,n+1]}\phi_{kj}\pi_{ij}^2 - (a_i-c_i)^2} \right). \tag{A.11}$$

It is clear that there exists an interior feasible solution to problem (A.7). Thus, according to Slater's condition, the strong duality holds, and problem (A.7) can be solved as problem (A.11). To solve problem (A.11), we first introduce new decision variables $v_{ij} = \sum_{k\in[i]} \phi_{kj}$. Problem (A.11) is then equivalent to the following problem:

$$\min_{\boldsymbol{\phi}\in\Xi_0} L(\boldsymbol{\phi}) = \min_{\mathbf{v}\in\Xi} \sum_{i\in[n]} \left(\mu_i(a_i - c_i) - \sigma_i \sqrt{\sum_{j\in[i+1,n+1]} v_{ij}\pi_{ij}^2 - (a_i - c_i)^2} \right), \tag{A.12}$$

where $v_{ii}\pi_{ii} = 0$ since $\pi_{ii} = 0$, and

$$\Xi = \left\{ \mathbf{v} \in \mathbb{R}^{n\times(n+1)} \,\middle|\, \begin{array}{ll} \sum_{j\in[i,n+1]} v_{ij} = 1, & \forall i \in [n] \\ v_{ij} \ge 0, & \forall i \in [n], j \in [i, n+1] \\ v_{ij} \ge v_{i-1,j}, & \forall i \in [2,n], j \in [i, n+1] \\ \sum_{j\in[i+1,n+1]} v_{ij}\pi_{ij} = a_i - c_i, & \forall i \in [n] \end{array} \right\}.$$

We start by optimizing a relaxed problem without the first and third set of constraints, i.e., $\sum_{j\in[i,n+1]} v_{ij} = 1$, for $i \in [n]$, and $v_{ij} \ge v_{i-1,j}$, for $i \in [2,n], j \in [i, n+1]$, in the feasible region Ξ and denote its optimal solution as v_{ij}^*. We then verify v_{ij}^* satisfies those constraints. Note that both the objective function and constraints can be decomposed by index i. We can thus equivalently solve a series of knapsack problems, i.e., for any $i \in [n]$, we solve the following problem:

$$\begin{aligned} \max_{v_{ij}\ge 0} \quad & \sum_{j\in[i+1,n+1]} v_{ij}\pi_{ij}^2 \\ \text{s.t.} \quad & \sum_{j\in[i+1,n+1]} v_{ij}\pi_{ij} = a_i - c_i. \end{aligned}$$

As π_{ij} is nondecreasing in j, the optimal solution is to allocate all the capacity to $v_{i,n+1}$, i.e., $v_{i,n+1}^* = \frac{a_i - c_i}{\pi_{i,n+1}} = \frac{a_i - c_i}{a_i} = p_i$, $v_{ij}^* = 0$, for $j \in [i+1, n]$. To satisfy the first set of constraints $\sum_{j\in[i,n+1]} v_{ij} = 1$, for $i \in [n]$, we further let $v_{ii}^* = 1 - p_i = \frac{c_i}{a_i}$. With the conditions that p_i is nondecreasing in i, the third set of constraints $v_{ij} \ge v_{i-1,j}$, for $i \in [2,n], j \in [i, n+1]$ are satisfied with the obtained v_{ij}^*, which implies that v_{ij}^* is the optimal solution to problem (A.12). Bring v_{ij}^* back into the objective function of (A.12), we get the optimal objective value as $\sum_{i\in[n]} \left(\mu_i(a_i - c_i) - \sigma_i \sqrt{c_i(a_i - c_i)} \right)$, which is the same as the summation of the optimal objective values of the n decomposed robust newsvendor problems, i.e., $\sum_{i\in[n]} \Pi_i^{\mathrm{nv}}$.

Note that under the condition $\sigma_i \le \frac{2\mu_i\sqrt{c_i(a_i - c_i)}}{(2c_i - a_i)^+}$, for $i \in [n]$, we have $\mathbf{x}^{\mathrm{nv}} \ge 0$. Therefore, $\mathbf{x}^{\mathrm{nv}}$ is a feasible solution to problem (A.6). Let $\Pi(\mathbf{x}^{\mathrm{nv}})$ be the optimal objective value with the decision $\mathbf{x}^{\mathrm{nv}}$ and $\Pi^{\mathrm{nv}}(\mathbf{x}^{\mathrm{nv}}) = \sum_{i\in[n]} \Pi_i^{\mathrm{nv}}$ be the profit when upgrading is not allowed and the firm chooses the decision $\mathbf{x}^{\mathrm{nv}}$. We have

$$\Pi^* \ge \Pi(\mathbf{x}^{\mathrm{nv}}) \ge \Pi^{\mathrm{nv}}(\mathbf{x}^{\mathrm{nv}}).$$

Since $\Pi^* = \Pi^{\mathrm{nv}}(\mathbf{x}^{\mathrm{nv}})$, we obtain $\Pi^* = \Pi(\mathbf{x}^{\mathrm{nv}})$, which shows that $\mathbf{x}^{\mathrm{nv}}$ is the optimal capacity decision and the optimal profits are the total profits of the n decomposed robust newsvendor problem.

By the definition of $\mathbb{P}^*(\mathbf{d})$, one can observe that it is indeed comonotonic. It is also easy to verify that $\mathbb{E}(\tilde{d}_i) = \mu_i$, $\mathbb{E}(\tilde{d}_i^2) = \mu_i^2 + \sigma_i^2$. From Proposition 3, we know $\mathbf{x}^* = \mathbf{x}^{\mathrm{nv}}$ and $\Pi^* = \sum_{i\in[n]} \Pi_i^{\mathrm{nv}}$.

Next, we prove that with $\mathbf{x}^{\mathrm{nv}}$ and $\mathbb{P}^*(\mathbf{d})$, the optimal objective value, denoted by $\Pi^*(\mathbf{x}^{\mathrm{nv}})$, is equal to $\sum_{i=1}^n \Pi_i^{\mathrm{nv}}$. Note that $\underline{\xi}_i \le x_i^{\mathrm{nv}} \le \bar{\xi}_i$. Therefore, by the definition of $\mathbb{P}^*$, we know no upgrading occurs no matter what scenarios are realized. Let ξ_{ij}, for $i \in [n+1], j \in [n]$ be the j^{th} element of scenario i. We have $\xi_{1j} = \underline{\xi}_j$ for $j \in [n]$. For $i \in [2, n+1]$, $\xi_{ij} = \underline{\xi}_j$ for $i \leq j$, and $\xi_{ij} = \bar{\xi}_j$ for $i > j$. Let $\chi_1 = \frac{a_1 - c_1}{a_1}$, $\chi_{k+1} = \frac{c_k}{a_k} - \frac{c_{k+1}}{a_{k+1}}$, for $k \in [2, n]$, and $\chi_{n+1} = \frac{c_n}{a_n}$. Then we obtain

$$
\begin{aligned}
\Pi^*(\mathbf{x}^{\text{nv}}) &= \sum_{i=1}^{n+1} \chi_i \sum_{j=1}^{n} a_j \min(x_j^{\text{nv}}, \xi_{ij}) - \sum_{j=1}^{n} c_j x_j^{\text{nv}} \\
&= \sum_{j=1}^{n} a_j \left(\frac{a_j - c_j}{a_j} \min(x_j^{nv}, \underline{\xi}_j) + \frac{c_j}{a_j} \min(x_j^{\text{nv}}, \bar{\xi}_j) \right) - \sum_{j=1}^{n} c_j x_j^{\text{nv}} \\
&= \sum_{j=1}^{n} \Pi_j^{\text{nv}}.
\end{aligned}
$$

Hence, we conclude that $\mathbb{P}^*(\mathbf{d})$ is the worst-case distribution. $\square$

A.7. Proof of Proposition 5

Proof. Let us define $\pi_{11} = 0$, $\pi_{12} = 0$, $\pi_{13} = a_1 - b_1$, $\pi_{14} = a_1$, $\pi_{15} = a_1$; $\pi_{21} = 0$, $\pi_{22} = a_2$, $\pi_{23} = 0$, $\pi_{24} = b_1$, $\pi_{25} = a_2$. First, we obtain the extreme points of Λ_0 as follows: (π_{11}, π_{21}), (π_{12}, π_{22}), (π_{13}, π_{23}), (π_{14}, π_{24}), (π_{15}, π_{25}). Then, based on Proposition 2, we aim to solve the following problem to obtain the optimal x_1^* and x_2^*, where we first ignore the constraint $\mathbf{x} \geq 0$:

$$
\begin{aligned}
\max_{\boldsymbol{\beta} \leq \mathbf{0}, \boldsymbol{\tau}, \mathbf{x}} \ & \left\{ \theta + \sum_{i \in [2]} \left(\tau_i \mu_i + \beta_i (\mu_i^2 + \sigma_i^2) \right) + \sum_{i \in [2]} (a_i - c_i) x_i \right\}, && \text{(A.13)} \\
\text{s.t.} \quad & \theta \leq h_1(\pi_{11}, \tau_1, \beta_1, x_1) + h_2(\pi_{21}, \tau_2, \beta_2, x_2) && (\cdots \text{ dual variable } \phi_1) \\
& \theta \leq h_1(\pi_{12}, \tau_1, \beta_1, x_1) + h_2(\pi_{22}, \tau_2, \beta_2, x_2) && (\cdots \text{ dual variable } \phi_2) \\
& \theta \leq h_1(\pi_{13}, \tau_1, \beta_1, x_1) + h_2(\pi_{23}, \tau_2, \beta_2, x_2) && (\cdots \text{ dual variable } \phi_3) \\
& \theta \leq h_1(\pi_{14}, \tau_1, \beta_1, x_1) + h_2(\pi_{24}, \tau_2, \beta_2, x_2) && (\cdots \text{ dual variable } \phi_4) \\
& \theta \leq h_1(\pi_{15}, \tau_1, \beta_1, x_1) + h_2(\pi_{25}, \tau_2, \beta_2, x_2). && (\cdots \text{ dual variable } \phi_5)
\end{aligned}
$$

Let ϕ_i, for $i \in [5]$, be the dual variables associated to the constraints in problem (A.13). The Lagrangian dual function is given by

$$
\begin{aligned}
L(\boldsymbol{\phi}) = & \max_{\boldsymbol{\beta} < \mathbf{0}, \mathbf{x}, \boldsymbol{\tau}, \theta} \theta + \sum_{i \in [2]} \left(\tau_i \mu_i + \beta_i (\sigma_i^2 + \mu_i^2) \right) \\
& \quad + \sum_{i \in [2]} (a_i - c_i) x_i + \sum_{k \in [5]} \left(\frac{(\pi_{1k} - \tau_1)^2}{4\beta_1} - x_1 \pi_{1k} + \frac{(\pi_{2k} - \tau_2)^2}{4\beta_2} - x_2 \pi_{2k} - \theta \right) \phi_k \\
= & \max_{\boldsymbol{\beta} < \mathbf{0}, \mathbf{x}, \boldsymbol{\tau}, \theta} \sum_{i \in [2]} \left(\tau_i \mu_i + \beta_i (\sigma_i^2 + \mu_i^2) \right) + \sum_{k \in [5]} \left(\frac{(\pi_{1k} - \tau_1)^2}{4\beta_1} + \frac{(\pi_{2k} - \tau_2)^2}{4\beta_2} \right) \phi_k \\
& \quad + \left(1 - \sum_{k \in [5]} \phi_k \right) \theta + \sum_{i \in [2]} \left(a_i - c_i - \sum_{k \in [5]} \pi_{ik} \phi_k \right) x_i.
\end{aligned} \tag{A.14}
$$

Note that θ and $\mathbf{x}$ are free in (A.14). Therefore, to ensure $L(\boldsymbol{\phi})$ is bounded, we should let $1 - \sum_{k \in [5]} \phi_k = 0$ and $a_i - c_i - \sum_{k \in [5]} \pi_{ik} \phi_k = 0$, for $i \in [2]$, with which we obtain

$$
L(\boldsymbol{\phi}) = \max_{\boldsymbol{\beta} < \mathbf{0}, \boldsymbol{\tau}} \sum_{i \in [2]} \left(\tau_i \mu_i + \beta_i (\sigma_i^2 + \mu_i^2) \right) + \sum_{k \in [5]} \left(\frac{(\pi_{1k} - \tau_1)^2}{4\beta_1} + \frac{(\pi_{2k} - \tau_2)^2}{4\beta_2} \right) \phi_k,
$$

where

$$\boldsymbol{\phi} \in \Xi_0 \equiv \left\{ \begin{array}{l} \phi_1+\phi_2+\phi_3+\phi_4+\phi_5=1 \\ \pi_{11}\phi_1+\pi_{12}\phi_2+\pi_{13}\phi_3+\pi_{14}\phi_4+\pi_{15}\phi_5=a_1-c_1 \\ \pi_{21}\phi_1+\pi_{22}\phi_2+\pi_{23}\phi_3+\pi_{24}\phi_4+\pi_{25}\phi_5=a_2-c_2 \end{array} \right\}.$$

For given $\boldsymbol{\tau}$ and $\boldsymbol{\phi}$, it is easy to verify that the objective function is concave in $\boldsymbol{\beta}$. By the first-order condition, we can get $\beta_i^*=-\frac{1}{2}\sqrt{\frac{\sum_{k=1}^5 \phi_k(\pi_{ik}-\tau_i)^2}{\sigma_i^2+\mu_i^2}}$. Substituting β_i^* back into $L(\boldsymbol{\phi})$, we obtain the following formulation:

$$\begin{aligned} L(\boldsymbol{\phi}) &= \max_{\boldsymbol{\tau}} \sum_{i\in[2]} \left(\tau_i\mu_i - \sqrt{(\sigma_i^2+\mu_i^2)\left(\sum_{k=1}^5 \phi_k(\pi_{ik}-\tau_i)^2\right)} \right) \\ &= \max_{\boldsymbol{\tau}} \sum_{i\in[2]} \left(\tau_i\mu_i - \sqrt{(\sigma_i^2+\mu_i^2)}\sqrt{\sum_{k=1}^5 \phi_k\pi_{ik}^2 - 2\tau_i(a_i-c_i)+\tau_i^2} \right). \end{aligned} \tag{A.15}$$

Now it is easy to verify that problem (A.15) is also concave in $\boldsymbol{\tau}$. By the first-order condition, we can get $\tau_i^*=a_i-c_i+\frac{\mu_i}{\sigma_i}\sqrt{\sum_{k=1}^5 \phi_k\pi_{ik}^2-(a_i-c_i)^2}$. Substituting τ_i^* back, we obtain

$$L(\boldsymbol{\phi}) = \sum_{i\in[n]} \left(\mu_i(a_i-c_i) - \sigma_i\sqrt{\sum_{k=1}^5 \phi_k\pi_{ik}^2-(a_i-c_i)^2} \right).$$

The Lagrangian dual problem can then be formulated as:

$$\min_{\boldsymbol{\phi}\in\Xi_0} L(\boldsymbol{\phi}) = \min_{\boldsymbol{\phi}\in\Xi_0} \quad \sum_{i\in[2]} \left(\mu_i(a_i-c_i) - \sigma_i\sqrt{\sum_{k=1}^5 \phi_k\pi_{ik}^2-(a_i-c_i)^2} \right). \tag{A.16}$$

Under the condition that $\frac{a_1-c_1}{a_1} > \frac{a_2-c_2}{a_2}$ and $c_1>c_2$, we obtain the optimal solution of (A.16) as $\phi_5=\frac{a_2-c_2}{a_2}$, $\phi_3=\frac{a_1c_2-a_2c_1}{(a_1-b_1)a_2}$, $\phi_1=1-\phi_5-\phi_3$, $\phi_2=\phi_4=0$. Then we obtain $\tau_1^*=a_1-c_1+\frac{\mu_1}{\sigma_1}\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}$, $\tau_2^*=a_2-c_2+\frac{\mu_2}{\sigma_2}\sqrt{(a_2-c_2)c_2}$, $\beta_1^*=-\frac{1}{2\sigma_1}\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}$, $\beta_2^*=-\frac{1}{2\sigma_2}\sqrt{(a_2-c_2)c_2}$. By complementary slackness in the optimality conditions, the constraints in (A.13) associated to ϕ_1, ϕ_3 and ϕ_5 must bind. Then we obtain $\theta^*=\frac{(\pi_{11}-\tau_1^*)^2}{4\beta_1^*}+\frac{(\pi_{21}-\tau_2^*)^2}{4\beta_2^*}=\frac{(\tau_1)^2}{4\beta_1^*}+\frac{(\tau_2^*)^2}{4\beta_2^*}$, and

$$\begin{aligned} x_1^* &= \frac{1}{\pi_{13}}\left(\frac{(\pi_{13}-\tau_1^*)^2}{4\beta_1^*}+\frac{(\tau_2^*)^2}{4\beta_2^*}-\theta^*\right) \\ &= \frac{1}{\pi_{13}}\left(\frac{(\pi_{13}-\tau_1^*)^2}{4\beta_1^*}-\frac{(\tau_1^*)^2}{4\beta_1^*}\right) \\ &= \mu_1+\sigma_1\frac{a_1-2c_1+b_1}{2\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}} \\ &= x_1^{nv}+\hat{\zeta}_1\sigma_1, \end{aligned}$$

where $\hat{\zeta}_1=\frac{1}{2}\left(\frac{a_1-2c_1+b_1}{\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}}+\frac{2c_1-a_1}{\sqrt{(a_1-c_1)c_1}}\right)$.

$$\begin{aligned} x_2^* &= \frac{1}{\pi_{25}}\left(\frac{(\pi_{15}-\tau_1^*)^2}{4\beta_1^*}-x_1^*\pi_{15}-\theta^*+\frac{(\pi_{25}-\tau_2^*)^2}{4\beta_2^*}\right) \\ &= \mu_2+\frac{\sigma_2(a_2-2c_2)}{2\sqrt{c_2(a_2-c_2)}}-\frac{a_1b_1\sigma_1}{2a_2\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}} \\ &= x_2^{nv}-\hat{\zeta}_2\sigma_1, \end{aligned}$$

where $\hat{\zeta}_2 = \frac{a_1 b_1}{2a_2\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}} > 0$.

To see $\hat{\zeta}_1 > 0$, we first observe that when $b_1 = 0$, $\hat{\zeta}_1 = 0$ holds. It is also easy to observe that $\hat{\zeta}_1$ is increasing in b_1. Hence, we have $\hat{\zeta}_1 > 0$.

By the expression of x_1^* and x_2^*, there must exist $\hat{\sigma}_1 > 0$ such that $x_1^* \geq 0$ and $x_2^* \geq 0$ when $\sigma_1 \leq \hat{\sigma}_1$, $\sigma_2 \leq \hat{\sigma}_2$ and $\sigma_1 \leq \hat{\zeta}_3\sigma_2 + \hat{\zeta}_4\mu_2$, where $\hat{\zeta}_3 = \frac{a_2(a_2-2c_2)\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}}{a_1b_1\sqrt{c_2(a_2-c_2)}}$, $\hat{\zeta}_4 = \frac{2a_2\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}}{a_1b_1}$.

To prove $\mathbb{P}^*(\mathbf{d})$ is comonotonic, it is sufficient to show $\xi_{13} \leq \xi_{12} \leq \xi_{11}$, $\xi_{23} \leq \xi_{22} \leq \xi_{21}$. The equalities indeed hold since $\pi_{11} \leq \pi_{13} \leq \pi_{15}$, $\pi_{21} = \pi_{23} \leq \pi_{25}$.

Next, we show that the distribution $\mathbb{P}^*(\mathbf{d})$ is the worst-case distribution. Let $\chi_1 = 1 - \frac{a_1c_2-a_2c_1}{(a_1-b_1)a_2} - \frac{a_2-c_2}{a_2}$, $\chi_2 = \frac{a_1c_2-a_2c_1}{(a_1-b_1)a_2}$, $\chi_3 = \frac{a_2-c_2}{a_2}$. We first prove $\mathbb{E}(\tilde{d}_1) = \mu_1$, $\mathbb{E}(\tilde{d}_1^2) \leq \mu_1^2 + \sigma_1^2$, $\mathbb{E}(\tilde{d}_2) = \mu_2$ and $\mathbb{E}(\tilde{d}_2^2) \leq \mu_2^2 + \sigma_2^2$, which are verified as follows,

$$\mathbb{E}(\tilde{d}_1) = \xi_{11}\chi_1 + \xi_{12}\chi_2 + \xi_{13}\chi_3 = \frac{\pi_{13}\chi_2 + \pi_{15}\chi_3}{2\beta_1^*} - \frac{\tau_1^*}{2\beta_1^*} = \frac{a_1 - c_1}{2\beta_1^*} - \frac{a_1 - c_1 - 2\mu_1\beta_1^*}{2\beta_1^*} = \mu_1.$$

$$\mathbb{E}(\tilde{d}_1 - \mu_1)^2 = (\xi_{11} - \mu_1)^2\chi_1 + (\xi_{12} - \mu_1)^2\chi_2 + (\xi_{13} - \mu_1)^2\chi_3 = \xi_{11}^2\chi_1 + \xi_{12}^2\chi_2 + \xi_{13}^2\chi_3 - \mu_1^2 = \sigma_1^2.$$

Therefore, we have $\mathbb{E}(\tilde{d}_1^2) \leq \mu_1^2 + \sigma_1^2$. Similarly, we can prove $\mathbb{E}(\tilde{d}_2) = \mu_2$, $\mathbb{E}(\tilde{d}_2^2) \leq \mu_2^2 + \sigma_2^2$. Next, we prove the objective value under the worst-case distribution is equal to the following optimal objective value of problem (A.13):

$$\begin{aligned}
\Pi^{\text{opt}} &= \theta^* + \tau_1^*\mu_1 + \beta_1^*(\sigma_1^2 + \mu_1^2) + \tau_2^*\mu_2 + \beta_2^*(\sigma_2^2 + \mu_2^2) + (a_1 - c_1)x_1 + (a_2 - c_2)x_2 \\
&= \frac{(a_1 - c_1)^2 - 4(a_1 - c_1)\mu_1\beta_1 + 4\beta_1^2\mu_1^2}{4\beta_1} + (a_1 - c_1)\mu_1 - 2\mu_1^2\beta_1 + \beta_1(\mu_1^2 + \sigma_1^2) \\
&+ \frac{(a_2 - c_2)^2 - 4(a_2 - c_2)\mu_2\beta_2 + 4\beta_2^2\mu_2^2}{4\beta_2} + (a_2 - c_2)\mu_2 - 2\mu_2^2\beta_2 + \beta_2(\sigma_2^2 + \mu_2^2) + (a_1 - c_1)x_1 + (a_2 - c_2)x_2 \\
&= \frac{(a_1 - c_1)^2}{4\beta_1} + \sigma_1^2\beta_1 + \frac{(a_2 - c_2)^2}{4\beta_2} + \sigma_2^2\beta_2 + (a_1 - c_1)x_1 + (a_2 - c_2)x_2 \\
&= \frac{\chi_2\pi_{13}^2 + \chi_3\pi_{15}^2}{4\beta_1} + \frac{\chi_3\pi_{25}^2}{4\beta_2} + (a_1 - c_1)x_1 + (a_2 - c_2)x_2.
\end{aligned}$$

We first prove the following results: $x_2^* - \xi_{21} = \frac{\pi_{15}^2 - \pi_{15}\pi_{13}}{4\beta_1\pi_{25}} + \frac{\pi_{25}-2\tau_2}{4\beta_2} - \frac{\pi_{21}-\tau_2^*}{2\beta_2^*} = \frac{\pi_{15}^2-\pi_{15}\pi_{13}}{4\beta_1\pi_{25}} + \frac{\pi_{25}}{4\beta_2} \leq 0$, $x_2^* - \xi_{22} = \frac{\pi_{15}^2-\pi_{15}\pi_{13}}{4\beta_1\pi_{25}} + \frac{\pi_{25}-2\tau_2}{4\beta_2} - \frac{\pi_{23}-\tau_2^*}{2\beta_2^*} = \frac{\pi_{15}^2-\pi_{15}\pi_{13}}{4\beta_1\pi_{25}} + \frac{\pi_{25}}{4\beta_2} \leq 0$, $x_2^* - \xi_{23} = \frac{\pi_{15}^2-\pi_{15}\pi_{13}}{4\beta_1\pi_{25}} + \frac{\pi_{25}-2\tau_2}{4\beta_2} - \frac{\pi_{25}-\tau_2^*}{2\beta_2^*} = \frac{\pi_{15}^2-\pi_{15}\pi_{13}}{4\beta_1\pi_{25}} - \frac{\pi_{25}}{4\beta_2} = \frac{\sigma_2 a_2}{2\sqrt{c_2(a_2-c_2)}} - \frac{a_1b_1\sigma_1}{2a_2\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}} \geq 0$ when $\sigma_1 \leq \hat{\zeta}_3\sigma_2$, where $\hat{\zeta}_3 = \frac{a_2^2\sqrt{a_1^2\frac{a_2-c_2}{a_2}+(a_1-b_1)\frac{a_1c_2-a_2c_1}{a_2}-(a_1-c_1)^2}}{a_1b_1\sqrt{c_2(a_2-c_2)}}$. $x_1^* - \xi_{11} = \frac{\pi_{13}-2\tau_1^*}{4\beta_1^*} - \frac{\pi_{11}-\tau_1^*}{2\beta_1^*} \leq 0$, $x_1^* - \xi_{12} = \frac{\pi_{13}-2\tau_1^*}{4\beta_1^*} - \frac{\pi_{13}-\tau_1^*}{2\beta_1^*} \geq 0$, $x_1^* - \xi_{13} = \frac{\pi_{13}-2\tau_1^*}{4\beta_1^*} - \frac{\pi_{15}-\tau_1^*}{2\beta_1^*} \geq 0$.

Furthermore, we have

$$\begin{aligned}
x_1^* - \min(\xi_{12}, x_1^*) + x_2^* = x_1^* - \xi_{12} + x_2^* &= \frac{-\pi_{13}\pi_{25}}{4\beta_1\pi_{25}} + \frac{\pi_{15}^2 - \pi_{15}\pi_{13}}{4\beta_1\pi_{25}} + \frac{\pi_{25} - 2\tau_2}{4\beta_2} \\
&= \frac{a_1a_2 - b_1(a_1 - a_2)}{4\beta_1} + \frac{\pi_{25} - 2\tau_2}{4\beta_2} \leq \frac{\pi_{25} - 2\tau_2}{4\beta_2} \leq \frac{\pi_{23} - \tau_2}{2\beta_2} = \xi_{22}.
\end{aligned}$$

$$x_1^* - \min(\xi_{11}, x_1^*) + x_2^* = x_1^* - x_1^* + x_2^* = x_2^* \leq \xi_{21}.$$

Let Π^{worse} be the optimal objective value under $\mathbb{P}^*(\mathbf{d})$. Then we have

$$\begin{aligned}\Pi^{\text{worse}} &= (\xi_{13}a_1+\xi_{23}a_2)\chi_3+(\xi_{12}a_1+(x_1^*-\xi_{12}+x_2^*)b_1)\chi_2+(x_1^*a_1+x_2^*a_2)\chi_1-c_1x_1^*-c_2x_2^* \\ &= ((\xi_{13}-x_1^*)a_1+(\xi_{23}-x_2^*)a_2)\chi_3+(\xi_{12}-x_1^*)(a_1-b_1)\chi_2+(a_1-c_1)x_1^*+(a_2-c_2)x_2^* \\ &= (\frac{\pi_{15}-\tau_1}{2\beta_1}-\frac{\pi_{13}-2\tau_1}{4\beta_1})a_1\chi_3+(\frac{\pi_{25}-\tau_2}{2\beta_2}-\frac{\pi_{15}^2-\pi_{15}\pi_{13}}{4\beta_1\pi_{25}}-\frac{\pi_{25}-2\tau_2}{4\beta_2})a_2\chi_3+\frac{\pi_{13}}{4\beta_1}(a_1-b_1)\chi_2 \\ &\quad+(a_1-c_1)x_1^*+(a_2-c_2)x_2^* \\ &= \frac{\chi_2\pi_{13}^2+\phi_3\pi_{15}^2}{4\beta_1}+\frac{\chi_3\pi_{25}^2}{4\beta_2}+(a_1-c_1)x_1^*+(a_2-c_2)x_2^* \\ &= \Pi^{\text{opt}},\end{aligned}$$

which shows that the distribution $\mathbb{P}^*(\mathbf{d})$ is indeed the worst-case distribution. □

A.8. Proof of Lemma 2

Proof. We can follow a similar procedure in proving Lemma 1 and thus omit the details here. □

A.9. Proof of Theorem 1

Proof. We can follow a similar procedure in proving Proposition 3 and thus omit the details here. □

A.10. Proof of Lemma 3

Proof. First, following **?**, we introduce the following lifted ambiguity set, which involves the auxiliary random matrices $\tilde{\mathbf{U}}_i \in \mathbb{R}^{2\times 2}$, $\forall i \in \mathcal{I}_1$:

$$\mathbb{Q} = \left\{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^n \times \mathbb{R}^{n\times n}) \;\middle|\; \begin{array}{l} \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu} \\ \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{U}}_i) = \boldsymbol{\Sigma}_i, \quad \forall i \in \mathcal{I}_1 \\ \mathbb{P}\left(\tilde{\mathbf{d}} \in \mathcal{W}, \tilde{\mathbf{d}}_i^T\tilde{\mathbf{d}}_i \preceq \tilde{\boldsymbol{U}}_i, i \in \mathcal{I}_1\right) = 1 \end{array} \right\}.$$

Then the problem (17) is equivalent to the following problem:

$$\max_{\mathbf{x}\in\mathbb{R}_+^n} \quad \left\{ -\sum_{j\in[n]} c_j x_j + \inf_{\mathbb{P}\in\mathbb{Q}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\mathbf{d}},\mathbf{x})] \right\}. \tag{A.17}$$

The inner optimization problem $\inf_{\mathbb{P}\in\mathbb{Q}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\mathbf{d}},\mathbf{x})]$ in (A.17) is equivalent to the following problem:

$$\begin{aligned} \inf \quad & \mathbb{E}_{\mathbb{P}}[f(\tilde{\mathbf{d}},\mathbf{x})] && \text{(A.18)} \\ \text{s.t.} \quad & \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu} && (\cdots \text{dual variable } \boldsymbol{\tau}\in\mathbb{R}^n) \\ & \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{U}}_i) = \boldsymbol{\Sigma}_i, \quad \forall i \in \mathcal{I}_1 && (\cdots \text{dual variable } \boldsymbol{\Gamma}_i\in\mathbb{R}^{2\times 2}) \\ & \mathbb{P}\left(\tilde{\mathbf{d}} \in \mathcal{W}, \tilde{\mathbf{d}}_i^T\tilde{\mathbf{d}}_i \preceq \tilde{\boldsymbol{U}}_i, i \in \mathcal{I}_1\right) = 1. && (\cdots \text{dual variable } \theta\in\mathbb{R}) \end{aligned}$$

With the dual variables defined above, by strong duality, problem (A.18) is equivalent to the following dual formulation:

$$\begin{aligned} \max_{\boldsymbol{\Gamma}_i,\theta,\boldsymbol{\tau}} \quad & \theta + \sum_{i\in[n]} \tau_i\mu_i + \sum_{i\in\mathcal{I}_1} \langle \boldsymbol{\Sigma}_i \,,\, \boldsymbol{\Gamma}_i\rangle && \text{(A.19)} \\ \text{s.t.} \quad & \theta + \sum_{i\in[n]} \tau_i d_i + \sum_{i\in\mathcal{I}_1} \langle \mathbf{U}_i \,,\, \boldsymbol{\Gamma}_i\rangle \le f(\mathbf{d},\mathbf{x}), \forall \mathbf{d}\in\mathcal{W}, \mathbf{d}_i^T\mathbf{d}_i \preceq \boldsymbol{U}_i, i\in\mathcal{I}_1. \end{aligned}$$

Note that the constraint in problem (A.19) is equivalent to

$$\theta \le \min_{\substack{\mathbf{d}\in\mathcal{W} \\ \mathbf{d}_i^T\mathbf{d}_i \preceq \boldsymbol{U}_i, i\in\mathcal{I}_1}} \left\{ f(\mathbf{d},\mathbf{x}) - \sum_{i\in[n]} \tau_i d_i - \sum_{i\in\mathcal{I}_1} \langle \mathbf{U}_i \,,\, \boldsymbol{\Gamma}_i\rangle \right\},$$

which can be further written as

$$\begin{aligned}
\theta \le & \min_{\substack{\mathbf{d}\in\mathcal{W} \\ \mathbf{d}_i^T\mathbf{d}_i\preceq \boldsymbol{U}_i, i\in\mathcal{I}_1}} \min_{\mathbf{z}\in\Lambda_0} \left\{ \sum_{i\in[n]}(d_i-x_i)z_i + \sum_{i\in[n]} a_i x_i - \sum_{i\in[n]}\tau_i d_i - \sum_{i\in\mathcal{I}_1}\langle \mathbf{U}_i\,,\,\boldsymbol{\Gamma}_i\rangle \right\} \\
= & \min_{\mathbf{z}\in\Lambda_0} \min_{\substack{\mathbf{d}\in\mathcal{W} \\ \mathbf{d}_i^T\mathbf{d}_i\preceq \boldsymbol{U}_i, i\in\mathcal{I}_1}} \left\{ \sum_{i\in[n]}(d_i-x_i)z_i + \sum_{i\in[n]} a_i x_i - \sum_{i\in[n]}\tau_i d_i - \sum_{i\in\mathcal{I}_1}\langle \mathbf{U}_i\,,\,\boldsymbol{\Gamma}_i\rangle \right\} \\
= & \sum_{i\in[n]} a_i x_i + \min_{\mathbf{z}\in\Lambda_0}\sum_{i\in\mathcal{I}_1} \min_{\substack{d_i\in\mathcal{W}_i, d_{i+1}\in\mathcal{W}_{i+1} \\ \mathbf{d}_i^T\mathbf{d}_i\preceq \boldsymbol{U}_i}} \{(d_i-x_i)z_i + (d_{i+1}-x_{i+1})z_{i+1} - \tau_i d_i - \tau_{i+1}d_{i+1} - \langle \mathbf{U}_i\,,\,\boldsymbol{\Gamma}_i\rangle\} \\
= & \sum_{i\in[n]} a_i x_i + \min_{\mathbf{z}\in\Lambda_0}\sum_{i\in\mathcal{I}_1} h_i(\mathbf{z}_i,\mathbf{x}_i,\boldsymbol{\tau}_i,\boldsymbol{\Gamma}_i).
\end{aligned}$$

In the optimal solution, there must be $\theta = \sum_{i\in[n]} a_i x_i + \min_{\mathbf{z}\in\Lambda_0}\sum_{i\in\mathcal{I}_1} h_i(\mathbf{z}_i,\mathbf{x}_i,\boldsymbol{\tau}_i,\boldsymbol{\Gamma}_i)$. We thus obtain the formulation (18). $\square$

A.11. Proof of Proposition 6

Proof. We first obtain the dual formulation of (20) as follows:

$$\begin{aligned}
\max_{\boldsymbol{\lambda}} \quad & -\lambda_0 + \lambda_{n+1} \qquad\qquad \text{(A.20)} \\
\text{s.t.} \quad & \lambda_{l_i} - \lambda_{l_j} + \sum_{t\in\mathcal{I}_{ij}} h_t(\pi_{l_i,l_j,\kappa,t}, \pi_{l_i,l_j,\kappa,t+1}, \mathbf{x}_t, \boldsymbol{\tau}_t, \boldsymbol{\Gamma}_t) \ge 0, \forall i\in[0,n], j\in[i+1,n+1], \kappa\in\mathcal{K}_{l_i,l_j} \\
& \lambda_{u_i} - \lambda_{l_j} + \sum_{t\in\mathcal{I}_{ij}} h_t(\pi_{u_i,l_j,\kappa,t}, \pi_{u_i,l_j,\kappa,t+1}, \mathbf{x}_t, \boldsymbol{\tau}_t, \boldsymbol{\Gamma}_t) \ge 0, \forall i\in[n], j\in[i+1,n+1], \kappa\in\mathcal{K}_{u_i,l_j} \\
& \lambda_{l_i} - \lambda_{u_j} + \sum_{t\in\mathcal{I}_{ij}} h_t(\pi_{l_i,u_j,\kappa,t}, \pi_{l_i,u_j,\kappa,t+1}, \mathbf{x}_t, \boldsymbol{\tau}_t, \boldsymbol{\Gamma}_t) \ge 0, \forall i\in[0,n-1], j\in[i+1,n], \kappa\in\mathcal{K}_{l_i,u_j} \\
& \lambda_{u_i} - \lambda_{u_j} + \sum_{t\in\mathcal{I}_{ij}} h_t(\pi_{u_i,u_j,\kappa,t}, \pi_{u_i,u_j,\kappa,t+1}, \mathbf{x}_t, \boldsymbol{\tau}_t, \boldsymbol{\Gamma}_t) \ge 0, \forall i\in[n-1], j\in[i+1,n], \kappa\in\mathcal{K}_{u_i,u_j}.
\end{aligned}$$

Using Schurs complement, we have $\mathbf{d}_t^T\mathbf{d}_t \preceq \boldsymbol{U}_t$ is equivalent to $\begin{pmatrix} \mathbf{U}_t & \mathbf{d}_t^T \\ \mathbf{d}_t & 1 \end{pmatrix} \succeq 0$. Then $h_t(\mathbf{z}_t,\mathbf{x}_t,\boldsymbol{\tau}_t,\boldsymbol{\Gamma}_t)$ is equivalent to the following optimization problem:

$$\begin{aligned}
\min \quad & (d_t-x_t)z_t + (d_{t+1}-x_{t+1})z_{t+1} - \tau_t d_t - \tau_{t+1}d_{t+1} - \langle \mathbf{U}_t\,,\,\boldsymbol{\Gamma}_t\rangle \\
\text{s.t.} \quad & d_t \ge 0, \\
& d_{t+1} \ge 0, \\
& \begin{pmatrix} \mathbf{U}_t & \mathbf{d}_t^T \\ \mathbf{d}_t & 1 \end{pmatrix} \succeq 0, \qquad \left(\cdots \text{dual variable } \begin{pmatrix} \boldsymbol{\Psi}_t & \boldsymbol{g}_t^T \\ \boldsymbol{g}_t & \phi_t \end{pmatrix} \succeq 0 \in \mathbb{R}^{3\times 3}\right)
\end{aligned}$$

where $\boldsymbol{g}_t = (g_t, g_{t+1})$. The dual of the above problem is as follows:

$$\begin{aligned}
\max \quad & -x_t z_t - x_{t+1}z_{t+1} - \phi_t \\
\text{s.t.} \quad & z_t - \tau_t - 2g_t \ge 0, \\
& z_{t+1} - \tau_{t+1} - 2g_{t+1} \ge 0, \\
& \begin{pmatrix} \boldsymbol{\Psi}_t & \boldsymbol{g}_t^T \\ \boldsymbol{g}_t & \phi_t \end{pmatrix} \succeq 0, \\
& \boldsymbol{\Gamma}_t = -\boldsymbol{\Psi}_t.
\end{aligned}$$

Substituting with the above dual back into (A.20), we obtain the following problem:

$$
\begin{aligned}
\max_{\boldsymbol{\lambda}} \quad & -\lambda_0 + \lambda_{n+1} \\
\text{s.t.} \quad & \lambda_{v_i} - \lambda_{v_j} + \sum_{t\in\mathcal{I}_{ij}} \sum_{l=t}^{t+1} -x_l \pi_{v_i,v_j,\kappa,l} - \phi_{v_i,v_j,\kappa,t} \geq 0, \forall i\in[0,n], j\in[i+1,n+1], v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j, \kappa\in\mathcal{K}_{v_i,v_j} \\
& \pi_{v_i,v_j,\kappa,t} - \tau_t - 2g_{v_i,v_j,\kappa,t} \geq 0, \forall t\in\mathcal{I}_{ij}, i\in[0,n], j\in[i+1,n+1], v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j, \kappa\in\mathcal{K}_{v_i,v_j} \\
& \pi_{v_i,v_j,\kappa,t+1} - \tau_{t+1} - 2g_{v_i,v_j,\kappa,t+1} \geq 0, \forall t\in\mathcal{I}_{ij}, i\in[0,n], j\in[i+1,n+1], v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j, \kappa\in\mathcal{K}_{v_i,v_j} \\
& \begin{pmatrix} \boldsymbol{\Psi}_{v_i,v_j,\kappa,t} & \boldsymbol{g}^T_{v_i,v_j,\kappa,t} \\ \boldsymbol{g}_{v_i,v_j,\kappa,t} & \phi_{v_i,v_j,\kappa,t} \end{pmatrix} \succeq 0, \forall t\in\mathcal{I}_{ij}, i\in[0,n], j\in[i+1,n+1], v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j, \kappa\in\mathcal{K}_{v_i,v_j} \\
& \boldsymbol{\Gamma}_t = -\boldsymbol{\Psi}_{v_i,v_j,\kappa,t}, \forall t\in\mathcal{I}_{ij}, i\in[0,n], j\in[i+1,n+1], v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j, \kappa\in\mathcal{K}_{v_i,v_j}.
\end{aligned}
$$

Substituting $\min_{\mathbf{z}\in\Lambda_0} \sum_{i\in\mathcal{I}_1} h_i(\mathbf{z}_i, \mathbf{x}_i, \boldsymbol{\tau}_i, \boldsymbol{\Gamma}_i)$ with the above problem in (18) and combining with the first-stage decision, we obtain the formulation (21) in the proposition. ☐

A.12. Proof of Proposition 7

Proof. Refer to Theorem 1 in **?**, we obtain

$$
\begin{aligned}
\inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\mathbf{d}},\mathbf{x})] &= \max_{\beta\geq 0} \left\{ \frac{1}{K}\sum_{k=1}^{K} \min_{\mathbf{d}\in\mathcal{W}} \{ f(\mathbf{d},\mathbf{x}) + \beta||\mathbf{d}-\boldsymbol{\zeta}^k||_2^2 \} - \theta^2\beta \right\} \\
&= \max_{\beta\geq 0} \left\{ \frac{1}{K}\sum_{k=1}^{K} \min_{\mathbf{d}\in\mathcal{W}} \{ \min_{\mathbf{z},\mathbf{w}\in\Lambda_0} \{ \sum_{i\in[n]} (d_i - x_i) z_i \} + \sum_{i\in[n]} a_i x_i + \beta||\mathbf{d}-\boldsymbol{\zeta}^k||_2^2 \} - \theta^2\beta \right\}
\end{aligned}
$$

Refer to our Theorem 1, for given k, we can reformulate $\min_{\mathbf{d}\in\mathcal{W}} \left\{ \min_{\mathbf{z},\mathbf{w}\in\Lambda_0} \{ \sum_{i\in[n]} (d_i - x_i) z_i \} + \sum_{i\in[n]} a_i x_i + \beta||\mathbf{d}-\boldsymbol{\zeta}^k||_2^2 \right\}$ into the following problem,

$$
\begin{aligned}
\max_{\boldsymbol{\xi}^k\leq 0, \boldsymbol{\gamma}^k\leq 0, \boldsymbol{\lambda}^k} & \left\{ -\lambda_0^k + \lambda_{n+1}^k + \beta \sum_{i\in[n]} (\zeta_i^k)^2 + \sum_{i\in[n]} a_i x_i \right\}, \qquad \text{(A.21)} \\
\text{s.t.} \quad & \lambda_{v_i}^k - \lambda_{v_j}^k + \sum_{t\in\mathcal{T}_{ij}} (\xi_{v_i,v_j,\kappa,t}^k - x_t \pi_{v_i,v_j,\kappa,t}) \geq 0, \forall i\in[0,n], j\in[i+1,n+1], \kappa\in\mathcal{K}_{v_i,v_j}, v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j \\
& \sqrt{(\beta+\xi_{v_i,v_j,\kappa,t}^k)^2 + (\gamma_{v_i,v_j,\kappa,t}^k)^2} \leq \beta - \xi_{v_i,v_j,\kappa,t}^k, \forall t\in\mathcal{T}_{ij}, i\in[0,n], j\in[i+1,n+1], \kappa\in\mathcal{K}_{v_i,v_j}, v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j \\
& \gamma_{v_i,v_j,\kappa,t}^k \leq \pi_{v_i,v_j,\kappa,t} - 2\beta\zeta_t^k, \forall t\in\mathcal{T}_{ij}, i\in[0,n], j\in[i+1,n+1], \kappa\in\mathcal{K}_{v_i,v_j}, v_i\in\mathcal{V}_i, v_j\in\mathcal{V}_j.
\end{aligned}
$$

Introducing the decision variables τ^k and combining (A.21) with the out-layer maximization with respect to β and $\mathbf{x}$, we obtain the SOCP formulation in Proposition 7.

Appendix B: Additional Numerical Experiments

B.1. Comparison between Partial Covariance Information and Full Covariance Information

In this section, we investigate the effectiveness of the partial correlation information, by conducting a numerical comparison between the PCM model and the robust model, denoted by FCM, that specifies the full covariance matrix information. Specifically, the FCM model considers the following ambiguity set $\tilde{\mathcal{F}}$ that specifies the full covariance matrix information $\boldsymbol{\Sigma}$:

$$
\tilde{\mathcal{F}} = \left\{ \mathbb{P}\in\mathcal{P}_0(\mathbb{R}^n) \;\middle|\; \begin{array}{l} \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}) = \boldsymbol{\mu} \\ \mathbb{E}_{\mathbb{P}}(\tilde{\mathbf{d}}^T\tilde{\mathbf{d}}) \preceq \boldsymbol{\Sigma} \\ \mathbb{P}(\tilde{\mathbf{d}}\in\mathcal{W}) = 1 \end{array} \right\}.
$$

Then, we aim to solve the following FCM model:

$$\max_{\mathbf{x}\in\mathbb{R}^n_+} \left\{ -\sum_{j\in[n]} c_j x_j + \inf_{\mathbb{P}\in\tilde{\mathcal{F}}} \mathbb{E}_{\mathbb{P}}[f(\tilde{\mathbf{d}},\mathbf{x})] \right\}. \tag{B.1}$$

Suppose we have obtained the extreme points of Λ_0, denoted by $\mathbf{z}^1$, $\mathbf{z}^2$,..., $\mathbf{z}^K$. Then, we can reformulate problem (B.1) into the following SDP,

$$\begin{aligned} \max_{\substack{\boldsymbol{\Gamma},\boldsymbol{\tau},t^k \\ \boldsymbol{Q}^k,\boldsymbol{g}^k,\mathbf{x}\geq\mathbf{0}}} \quad & -\sum_{j\in[n]} c_j x_j + \theta + \sum_{i\in[n]} \tau_i \mu_i + \langle \boldsymbol{\Sigma}\,,\,\boldsymbol{\Gamma}\rangle \\ \text{s.t.} \quad & \theta \leq \alpha + (\mathbf{a}-\mathbf{z}^k)'\mathbf{x} - t^k + 2(\boldsymbol{g}^k)'\boldsymbol{\mu}, \forall k\in[K], \\ & \begin{pmatrix} \boldsymbol{Q}^k & \boldsymbol{g}^k \\ (\boldsymbol{g}^k)' & t^k \end{pmatrix} \succeq \mathbf{0}, \forall k\in[K] \\ & \boldsymbol{\Gamma} = \boldsymbol{Q}^k, \forall k\in[K] \\ & \mathbf{z}^k - \boldsymbol{\tau} - 2\boldsymbol{g}^k \geq \mathbf{0}, \forall k\in[K]. \end{aligned} \tag{B.2}$$

We note that the SDP (B.2) grows exponentially with n, as the number of extreme points in Λ_0 increases exponentially in n. Given the computational challenge, we compare SDP reformulation (B.2) to our SDP reformulation (21) for $n\in\{4,6,8,10\}$, in terms of both the mean of out-of-sample profits and the computational time. We set the parameters p_i, c_i, and b_i to be the same as in Subsection 6.1. We set $b_i = a_{i+1}$, and the variation of coefficient $\text{CV}=0.6$. We generate the training and testing data from a truncated multivariate Normal distribution with a covariance matrix where the correlation coefficient between class-i demand and class-j demand is set identically as ρ, and we vary $\rho\in\{0.5,-0.5\}$.

Table 1 presents the comparison result. We observe that the profit loss from our SDP formulation, when considering only partial correlation information, is minimal, with the difference in the mean out-of-sample earnings being less than 0.3%. In some cases, our PCM may even outperform the FCM, possibly due to the estimation error of the full covariance matrix. Additionally, we find that our PCM demonstrates significantly higher computational efficiency than FCM in terms of CPU time, especially when n is large.

Table 1 Percentage (in %) Change in the Mean of Out-of-Sample Profit from FCM over PCM and the Computational Times

n	Percentage Change		CPU Time (in seconds)	
	$\rho=0.5$	$\rho=-0.5$	PCM	FCM
4	-0.05	-0.42	0.48	1.61
6	0.04	0.28	1.94	8.31
8	-0.06	-0.16	6.30	88.27
10	-0.10	0.13	18.47	2606.19

B.2. Computational Comparison between MMM, PCM and WAS

We compare in Table 2 the computational time of MMM, PCM and WAS under various unit upgrading values and numbers of products. We use the parameters $c_1=1$, $c_n=0.8$, $p_1=0.5$, $p_n=0.3$, $\text{CV}=0.4$ and iterate η over $\{0,1,0.5,0.9\}$, and n over $\{2,4,6,8\}$. We sample 100 in-sample data to solve WAS.

We observe that MMM exhibits the highest computational efficiency, while WAS requires much more computational effort, especially when n is large. The efficiency of MMM over PCM is easy to understand. Although the number of constraints in both models is around the same magnitude, MMM only involves second-order cone constraints, while PCM involves semi-definite constraints, which are more difficult to handle. WAS, although only involving the second-order cone constraints, the number of such constraints also scales linearly with the number of in-sample data. The 100 in-sample data in our experiments then significantly slows down its computational time.

Table 2 Comparison of the Computational Time in Solving MMM, PCM and WAS

Number of products	η	CPU time (in seconds)		
		MMM	PCM	WAS
2	0.1	0.239	0.460	30.574
	0.5	0.232	0.452	31.006
	0.9	0.232	0.445	31.193
4	0.1	0.485	1.118	62.321
	0.5	0.421	1.100	68.131
	0.9	0.625	1.988	196.418
6	0.1	0.663	2.079	105.367
	0.5	0.618	2.069	117.801
	0.9	1.492	8.689	599.790
8	0.1	0.870	3.397	172.073
	0.5	0.816	3.368	185.611
	0.9	2.875	29.899	***

*** denotes that MOSEK fails to solve the case.